Notre Dame Journal of Formal Logic
Volume 58, Number 4, 2017

Forking and Dividing in Henson Graphs

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Abstract For $n \geq 3$, define T_n to be the theory of the generic K_n-free graph, where K_n is the complete graph on n vertices. We prove a graph-theoretic characterization of dividing in T_n and use it to show that forking and dividing are the same for complete types. We then give an example of a forking and nondividing formula. Altogether, T_n provides a counterexample to a question of Chernikov and Kaplan.

1 Introduction

Classification in model theory, beginning with stability theory, is strongly fueled by the study of abstract notions of independence, the frontrunners of which are forking and dividing. These notions have proved useful in the abstract treatment of independence and dimension in the stable setting and initiated a quest to understand when they are useful in the unstable context. For example, significant success has been achieved in the class of simple theories (see Kim and Pillay [7]). Meaningful results have also been found for NIP theories and, more generally, NTP_2 theories, which include both simple and NIP. A notable example is the following result from Chernikov and Kaplan [3].

Theorem (Chernikov and Kaplan) *Suppose that $\mathbb{M}$ is a sufficiently saturated monster model of an NTP_2 theory. Given $C \subset \mathbb{M}$, the following are equivalent.*

(i) *A partial type forks over C if and only if it divides over C.*
(ii) *C is an extension base for nonforking; that is, if $\pi(\bar{x})$ is a partial type with parameters from C, then $\pi(\bar{x})$ does not fork over C.*

In general, if condition (i) holds for a set C, then condition (ii) does as well. In fact, condition (ii) should be thought of as the minimal requirement for nonforking to be meaningful for types over C. In particular, if C is *not* an extension base for

Received July 23, 2014; accepted July 10, 2015
First published online June 6, 2017
2010 Mathematics Subject Classification: Primary 03C45; Secondary 03C68
Keywords: forking and dividing, Henson graphs, TP2
 10.1215/00294527-2017-0016

nonforking, then there are types with no nonforking extensions. There are few known examples where condition (ii) fails and, in most of them, this happens due to some kind of circular ordering (see, e.g., Tent and Ziegler [10, Exercise 7.1.6]). A priori, condition (i) is harder to achieve and, even before the theorem above, was known to hold for all sets in certain well-behaved classes of theories (e.g., simple theories [7]). This leads to the natural question, which is asked in [3], of whether the above theorem extends to classes of theories other than NTP_2 (e.g., NSOP). In this paper, we give an example of an NSOP theory in which condition (ii) holds for all sets (in a rather strong way), while condition (i) fails in general.

We will consider the theory of a well-known structure: the generic K_n-free graph, also known as the *Henson graph*. Our main goal is to characterize forking and dividing in this theory, which is known to be TP_2 and NSOP (in fact, NSOP_4). We will first show that dividing independence has a meaningful graph-theoretic interpretation, which reflects the combinatorics of the structure in a precise way. We will then show that, despite the complexity of the theory, forking and dividing are the same for *complete types*, which implies every set is an extension base for nonforking. On the other hand, we will show that there are *formulas* which fork, but do not divide.

2 Model-Theoretic Preliminaries

This section contains definitions and basic facts concerning forking and dividing. We first specify some conventions that will be maintained throughout the paper. If T is a complete first-order theory and $\mathbb{M}$ is a monster model of T, we write $C \subset \mathbb{M}$ to mean that C is a "small" subset of $\mathbb{M}$; that is, $C \subseteq \mathbb{M}$ and $\mathbb{M}$ is $|C|^+$-saturated. We use the letters $a, b, c, \ldots$ to denote singletons, and $\bar{a}, \bar{b}, \bar{c}, \ldots$ to denote tuples (of possibly infinite length). We let $|\bar{a}|$ denote the length of a tuple. Given $C \subseteq \mathbb{M}$, we will abuse notation and write $\bar{a} \in C$ to mean that $\bar{a}$ is a tuple of elements from C. We will also frequently identify tuples with their sets of coordinates; for example, $\bar{a} \cap \bar{b}$ denotes the set of elements occurring as coordinates of both $\bar{a}$ and $\bar{b}$. When working with subsets of $\mathbb{M}$, we omit the union symbol (e.g., AB denotes $A \cup B$).

Definition 2.1 Suppose that $C \subset \mathbb{M}$, $\pi(\bar{x}, \bar{y})$ is a partial type with parameters from C, and $\bar{b} \in \mathbb{M}$.

(1) The type $\pi(\bar{x}, \bar{b})$ *divides over* C if there is a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$, with $\bar{b}^0 = \bar{b}$, such that $\bigcup_{l<\omega} \pi(\bar{x}, \bar{b}^l)$ is inconsistent.

(2) The type $\pi(\bar{x}, \bar{b})$ *forks over* C if there is some $D \supseteq \bar{b}C$ such that, for any $p \in S_n(D)$, if p extends $\pi(\bar{x}, \bar{b})$, then p divides over C.

A formula $\varphi(\bar{x}, \bar{b})$ *forks* (*divides*) over C if $\{\varphi(\bar{x}, \bar{b})\}$ forks (divides) over C.

The following basic facts can be found in [10, Chapter 7].

Proposition 2.2 *Let* $C \subset \mathbb{M}$.

(a) *If a complete type forks (divides) over* C*, then it contains some formula that forks (divides) over* C.

(b) *If* $\pi(\bar{x})$ *is a consistent type over* C*, then* $\pi(\bar{x})$ *does not divide over* C.

(c) *A partial type* $\pi(\bar{x}, \bar{b})$ *forks over* C *if and only if there are finitely many formulas* $\varphi_1(\bar{x}, \bar{b}^1), \ldots, \varphi_m(\bar{x}, \bar{b}^m)$ *such that* $\pi(\bar{x}, \bar{b}) \vdash \bigvee_{i=1}^m \varphi_i(\bar{x}, \bar{b}^i)$ *and, for all* $1 \leq i \leq m$, $\varphi_i(\bar{x}, \bar{b}^i)$ *divides over* C.

Nondividing and nonforking are used to define the following ternary relations on small subsets of $\mathbb{M}$:

(1) $A \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} B$ if and only if $\operatorname{tp}(A/BC)$ does not divide over C;

(2) $A \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B$ if and only if $\operatorname{tp}(A/BC)$ does not fork over C.

These relations were originally defined to abstractly capture notions of independence and dimension in stable theories and have been found to still be meaningful in more complicated theories as well. In particular, we will consider the interpretation of these notions in the unstable theories of certain ultrahomogeneous graphs.

3 Graphs

Throughout the paper, all graphs are undirected and without loops or multiple edges. We say a graph G is *complete* if any two distinct points are connected by an edge and G is *independent* if no two points are connected by an edge. Recall that a countable graph G is *ultrahomogeneous* if any graph isomorphism between finite subsets of G extends to an automorphism of G.

The canonical example of an ultrahomogeneous graph is the countable *random graph*, which contains (an isomorphic copy of) any finite graph as an induced subgraph. We let $\mathcal{G}$ denote the random graph. Henson [5] introduced another family of ultrahomogeneous graphs: the generic K_n-free graphs, where $n \geq 3$ and K_n is the complete graph on n vertices. These countable graphs are often called the *Henson graphs*. For a particular $n \geq 3$, there is a unique such graph up to isomorphism.

Definition 3.1 Fix $n \geq 3$, and let K_n be the complete graph on n vertices. The *generic K_n-free graph*, denoted $\mathcal{H}_n$, is the unique countable K_n-free graph such that

(i) any finite K_n-free graph is isomorphic to an induced subgraph of $\mathcal{H}_n$,

(ii) any graph isomorphism between finite subsets of $\mathcal{H}_n$ extends to an automorphism of $\mathcal{H}_n$.

The graphs $\mathcal{G}$ and $\mathcal{H}_n$, for some $n \geq 3$, can be directly constructed as the Fraïssé limits of the classes of, respectively, all finite graphs and all finite K_n-free graphs.

We study graphs as structures in the language $\mathcal{L} = \{R\}$, where R is interpreted as the binary edge relation. We set $T_0 = \operatorname{Th}(\mathcal{G})$ and, for $n \geq 3$, $T_n = \operatorname{Th}(\mathcal{H}_n)$.

The following is a well-known fact and informative exercise (see, e.g., Hodges [6, Chapter 7] or [10, Exercise 4.4.3]).

Fact 3.2 For any $n \in \{0\} \cup \{3, 4, \ldots\}$, T_n is an $\aleph_0$-categorical theory with quantifier elimination.

Fix $n \geq 3$, and fix $\mathbb{H}_n \models T_n$, a sufficiently saturated monster model of T_n. As $\mathbb{H}_n$ is a graph, we can embed it in a *larger* monster model $\mathbb{G} \models T_0$. Note that $\mathbb{H}_n$ is a substructure (induced subgraph) of $\mathbb{G}$, but not an elementary substructure. Let $\kappa(\mathbb{H}_n) = \sup\{\kappa : \mathbb{H}_n \text{ is } \kappa\text{-saturated}\}$.

For the rest of the paper, $n \geq 3$, $\mathbb{H}_n$, and $\mathbb{G}$ are fixed. By saturation, we have the following fact.

Proposition 3.3 *Suppose that $C \subset \mathbb{H}_n$ and $X \subseteq \mathbb{G}$ are such that X is K_n-free, $C \subseteq X$, and $|X| \leq \kappa(\mathbb{H}_n)$. Then there is a graph embedding $f : X \longrightarrow \mathbb{H}_n$ such that $f|_C = \mathrm{id}_C$.*

The remainder of this section is devoted to specifying notation and conventions concerning the language $\mathcal{L}$. First, we consider types.

Definition/Convention 3.4 Suppose that $C \subset \mathbb{G}$, with $|C| < \kappa(\mathbb{H}_n)$.

(1) We only consider partial types $\pi(\bar{x})$ such that $|\bar{x}| \leq \kappa(\mathbb{H}_n)$. Furthermore, we will assume types are "symmetrically closed." For example, $R(x_i, x_j) \in \pi(\bar{x})$ if and only if $R(x_j, x_i) \in \pi(\bar{x})$.

(2) An *R-type over C* is a collection $\pi(\bar{x})$ of atomic and negated atomic $\mathcal{L}$-formulas, none of which is of the form $x_i = c$, where $c \in C$.

(3) Suppose that $\pi(\bar{x})$ is an R-type over C. An *optimal solution* of $\pi(\bar{x})$ is a tuple $\bar{a} \models \pi(\bar{x})$ such that
   (i) $a_i \neq a_j$ for all $i \neq j$ and $a_i \notin C$ for all i,
   (ii) $R(a_i, a_j)$ if and only if $R(x_i, x_j) \in \pi(\bar{x})$,
   (iii) given $c \in C$, $R(a_i, c)$ if and only if $R(x_i, c) \in \pi(\bar{x})$.

We will frequently use the following fact, which says that we can always find optimal solutions of R-types.

Proposition 3.5 *Suppose that $C \subset \mathbb{H}_n$, and $\pi(\bar{x})$ is an R-type over C.*

(a) *$\pi(\bar{x})$ is consistent with T_0 if and only if it has an optimal solution in $\mathbb{G}$.*
(b) *$\pi(\bar{x})$ is consistent with T_n if and only if it has an optimal solution in $\mathbb{H}_n$.*

This is a straightforward exercise, which we leave to the reader. The idea is that a type cannot prove that an edge exists in a graph, without explicitly saying so. Moreover, removing extra edges to "optimize" the solution of a consistent type is always possible and, in the case of T_n, will not conflict with the requirement that the solution be K_n-free.

Next, we specify notation and conventions concerning $\mathcal{L}$-formulas.

Definition/Convention 3.6 Suppose that $C \subset \mathbb{G}$.

(1) Let $\mathcal{L}_0(C)$ be the collection of conjunctions of atomic and negated atomic $\mathcal{L}$-formulas, with parameters from C, such that no conjunct is of the form $x = c$, where x is a variable and $c \in C$. When we write $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_0(C)$, we will assume further that no conjunct of $\varphi(\bar{x}, \bar{y})$ is of the form $x_i = x_j$ or $y_i = y_j$, for some $i \neq j$. When we write $\varphi(\bar{x}, \bar{b})$, we assume that $\bar{b}$ is a tuple of pairwise distinct coordinates.

(2) Given $\varphi(\bar{x}) \in \mathcal{L}_0(C)$ and $\theta(\bar{x})$, an atomic or negated atomic formula, we write "$\varphi(\bar{x}) \rhd \theta(\bar{x})$" if $\theta(\bar{x})$ is a conjunct of $\varphi(\bar{x})$.

(3) We will assume that $\mathcal{L}_0(C)$-formulas are "symmetrically closed." For example $\varphi(\bar{x}) \rhd R(x, c)$ if and only if $\varphi(\bar{x}) \rhd R(c, x)$.

(4) Let $\mathcal{L}_R(C)$ be the collection of formulas $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_0(C)$ such that no conjunct is of the form $x_i = y_j$.

The main result of this paper will be a characterization of forking and dividing in T_n. We will make use of the following characterization of dividing in T_0, which is a standard exercise (see, e.g., [10, Exercise 7.3.14]).

Fact 3.7 Fix $C \subset \mathbb{G}$ and $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_0(C)$. Suppose that $\bar{b} \in \mathbb{G} \backslash C$ is such that $\varphi(\bar{x}, \bar{b})$ is consistent. Then $\varphi(\bar{x}, \bar{b})$ divides over C (with respect to T_0) if and only if $\varphi(\bar{x}, \bar{b}) \rhd x_i = b$ for some $b \in \bar{b}$. Consequently, if $A, B, C \subset \mathbb{G}$, then $A \mathop{\smile\!\!\!\!\!\!\;|}^d_C B$ if and only if $A \cap B \subseteq C$.

The theory T_0 is a standard example of a *simple theory*, and so the previous fact also gives a characterization of forking. On the other hand, T_n is nonsimple. Indeed, the Henson graph is a canonical example where $\mathop{\smile\!\!\!\!\!\!|}^f$ fails *amalgamation over models* (see [7]). A direct proof of this (for $n = 3$) can be found in Hart [4, Example 2.11(4)]. The precise classification of T_n is well known and summarized by the following fact.

Fact 3.8 The theory T_n is TP_2, SOP_3, and NSOP_4.

See Chernikov [2] and Shelah [9] for definitions of these properties. The proof of TP_2 can be found in [2] for $n = 3$. The properties SOP_3 and NSOP_4 are demonstrated in [9] for $n = 3$. The generalizations of these arguments to $n \geq 3$ are fairly straightforward. Moreover, NSOP_4 for all $n \geq 3$ also follows from a more general result in Patel [8].

4 Dividing in T_n

The goal of this section is to find a graph-theoretic characterization of dividing independence in T_n. Therefore, unless otherwise stated, when we say that a partial type divides or is consistent, we mean with respect to T_n.

We first define a graph-theoretic binary relation, denoted $K_n(B/C)$, on pairs (B, C) of disjoint graphs.

Definition 4.1

(1) Given $A, B \subset \mathbb{G}$, we write $R(A, B)$ if $R(a, b)$ holds for all $a \in A$ and $b \in B$.

(2) Suppose that $B, C \subset \mathbb{G}$ are disjoint. We say B is *K_n-bound to C*, written $K_n(B/C)$, if there is $B_0 \subseteq BC$ such that
  (i) $|B_0| = n$ and $B_0 \cap C \neq \emptyset \neq B_0 \cap B$,
  (ii) $B_0 \cap C$ is complete and $R(B_0 \cap B, B_0 \cap C)$.
  We say B_0 *witnesses* $K_n(B/C)$. Note that B_0 is "almost isomorphic" to K_n, in the sense that the only possible missing edges are between points in $B_0 \cap B$.

(3) Suppose that $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$ and $\bar{b} \in \mathbb{H}_n \backslash C$ such that $\varphi(\bar{x}, \bar{b})$ is consistent. We say $\bar{b}$ is *φ-K_n-bound to C*, written $K_n^{\varphi}(\bar{b}/C)$, if there is $B \subseteq \bar{b}$, with $0 < |B| < n$ such that
  (i) $\neg K_n(B/C)$,
  (ii) $K_n(B/\bar{a}C)$ for all $\bar{a} \in \mathbb{H}_n$ such that $\bar{a} \models \varphi(\bar{x}, \bar{b})$.
  We say B *witnesses* $K_n^{\varphi}(\bar{b}/C)$.

The main result of this section (Theorem 4.4) will show that K_n^{φ} is the graph-theoretic interpretation of dividing. In particular, for $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$ and $\bar{b} \in \mathbb{H}_n \backslash C$ with $\varphi(\bar{x}, \bar{b})$ consistent, we will show that $\varphi(\bar{x}, \bar{b})$ divides over C if and only if $K_n^{\varphi}(\bar{b}/C)$. The reverse direction of the proof of this will use the following recipe for constructing indiscernible sequences.

Construction 4.2 Fix $C \subset \mathbb{H}_n$ and $\bar{b} \in \mathbb{H}_n \backslash C$ such that $\bar{b}$ has pairwise distinct coordinates. Given $B \subseteq \bar{b}$, we construct an induced subgraph $\Gamma(C\bar{b}, B)$ of $\mathbb{G}$ as follows.

The vertex set of $\Gamma(C\bar{b}, B)$ is $C \cup \bigcup_{l<\omega} \bar{b}^l$, where $\bar{b}^0 = \bar{b}$ and, for all $0 < l < \omega$, $|\bar{b}^l| = |\bar{b}|$ and $\bar{b}^l$ is disjoint from $C \cup \bigcup_{m<l} \bar{b}^m$. We define edges in $\Gamma(C\bar{b}, B)$ so that

(1) $\bar{b}^l \equiv_C \bar{b}$ for all $l < \omega$,

(2) given $i < j \leq |\bar{b}|$, if $b_i, b_j \in B$, then $R(b_i^l, b_j^m)$ for all $l < m < \omega$.

We let $\mathcal{J}(C\bar{b}, B)$ denote the infinite sequence $(\bar{b}^l)_{l<\omega}$. Note that $\mathcal{J}(C\bar{b}, B)$ is C-indiscernible. If the graph $\Gamma(C\bar{b}, B)$ is K_n-free, then, after possibly replacing $\Gamma(C\bar{b}, B)$ by some isomorphic copy, we may assume that $\Gamma(C\bar{b}, B)$ is an induced subgraph of $\mathbb{H}_n$. In this case, $\mathcal{J}(C\bar{b}, B)$ is a C-indiscernible sequence in $\mathbb{H}_n$.

Lemma 4.3 *Fix $C \subset \mathbb{H}_n$ and $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$. Suppose that $\bar{b} \in \mathbb{H}_n \backslash C$ is such that $\varphi(\bar{x}, \bar{b})$ is consistent and $K_n^\varphi(\bar{b}/C)$, witnessed by $B \subseteq \bar{b}$.*

(a) *$\Gamma(C\bar{b}, B)$ is K_n-free, and so $\mathcal{J}(C\bar{b}, B)$ is a C-indiscernible sequence in $\mathbb{H}_n$.*
(b) *If $\mathcal{J}(C\bar{b}, B) = (\bar{b}^l)_{l<\omega}$, then $\{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$ is $(n-1)$-inconsistent with T_n.*

Proof Part (a). Suppose that $K_n \cong W \subseteq \Gamma(C\bar{b}, B)$. Then $W \cap \bigcup \mathcal{J}(C\bar{b}, B) \neq \emptyset$, since C is K_n-free. Say $W \cap \bigcup \mathcal{J}(C\bar{b}, B) = \{b_{i_1}^{l_1}, \dots, b_{i_r}^{l_r}\}$ with $l_1 \leq \dots \leq l_r$. Note that, by construction of $\Gamma(C\bar{b}, B)$, $i_s \neq i_t$ for all $1 \leq s < t \leq r$. Define $V = (W \cap C) \cup \{b_{i_1}, \dots, b_{i_r}\}$. If $l_1 = l_r$, then, since $\bar{b}^{l_1} \equiv_C \bar{b}$, it follows that $V \cong K_n$, which is a contradiction. Therefore $l_1 < l_r$. By construction of $\Gamma(C\bar{b}, B)$, it follows that $b_{i_1}, b_{i_r} \in B$. If $1 \leq s \leq r$, then we have either $l_1 < l_s$ or $l_s < l_r$, and in either case it follows that $b_{i_s} \in B$. Therefore $r \leq |B| \leq n-1$; in particular $C \cap W \neq \emptyset$. But then V witnesses $K_n(B/C)$, which contradicts the definition of $K_n^\varphi(\bar{b}/C)$.

Part (b). By indiscernibility of $\mathcal{J}(C\bar{b}, B)$, it suffices to show that the R-type $\pi(\bar{x}) = \{\varphi(\bar{x}, \bar{b}^l) : l < n-1\}$ is unsatisfiable in $\mathbb{H}_n$. So suppose, toward a contradiction, that $\pi(\bar{x})$ is satisfiable, and let $\bar{a} \in \mathbb{H}_n$ be an optimal solution. Then $\bar{a} \models \varphi(\bar{x}, \bar{b})$, so, by assumption, there is $D \subseteq BC\bar{a}$ witnessing $K_n(B/C\bar{a})$. By definition, $D \cap B \neq \emptyset$. Suppose that $D \cap B = \{b_{i_0}, \dots, b_{i_k}\}$, with $i_0 < \dots < i_k$. Note that $k < n-1$. Let $B_0 = \{b_{i_0}^0, \dots, b_{i_k}^k\}$. We make the following observations.

(1) $D \cap C\bar{a}$ is complete, since D witnesses $K_n(B/C\bar{a})$.
(2) B_0 is complete, by construction of $\Gamma(C\bar{b}, B)$.
(3) $R(B_0, D \cap C)$, since $R(D \cap B, D \cap C)$, and $\mathcal{J}(C\bar{b}, B)$ is C-indiscernible.
(4) $R(B_0, D \cap \bar{a})$: if $a_j \in D \cap \bar{a}$ and $b_{i_t}^t \in B_0$, then, since $R(D \cap B, D \cap \bar{a})$ and $\bar{a}$ is an optimal solution of $\pi(\bar{x})$, we have

$$R(a_j, b_{i_t}) \Rightarrow \big(\varphi(\bar{x}, \bar{b}) \rhd R(x_j, b_{i_t})\big) \Rightarrow \big(\varphi(\bar{x}, \bar{b}^t) \rhd R(x_j, b_{i_t}^t)\big) \Rightarrow R(a_j, b_{i_t}^t).$$

These observations imply $(D \cap C\bar{a})B_0 \cong K_n$, which is a contradiction. □

Theorem 4.4 *Fix $C \subset \mathbb{H}_n$, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$, and $\bar{b} \in \mathbb{H}_n \backslash C$ such that $\varphi(\bar{x}, \bar{b})$ is consistent. Then $\varphi(\bar{x}, \bar{b})$ divides over C if and only if $K_n^\varphi(\bar{b}/C)$.*

Proof ($\Leftarrow$): Suppose that $B \subseteq \bar{b}$ witnesses $K_n^\varphi(\bar{b}/C)$. By Lemma 4.3, $\Gamma(C\bar{b}, B) \subseteq \mathbb{H}_n$ and $\{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$ is $(n-1)$-inconsistent. So $\varphi(\bar{x}, \bar{b})$ divides over C.

($\Rightarrow$): Suppose that $\varphi(\bar{x}, \bar{b})$ divides over C. Then there is a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$, with $\bar{b}^0 = \bar{b}$, such that $\pi(\bar{x}) := \{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$ is inconsistent.

Let $F = C \cup \bigcup_{l<\omega} \bar{b}^l$. Consider F as a subgraph of $\mathbb{G}$, and note that $(\bar{b}^l)_{l<\omega}$ is still C-indiscernible in $\mathbb{G}$. Since $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$ and $\varphi(\bar{x}, \bar{b})$ is still consistent with respect to T_0, it follows from Fact 3.7 that $\varphi(\bar{x}, \bar{b})$ does not divide over C with respect to T_0. Therefore, there is an optimal realization $\bar{d} \in \mathbb{G}$ of $\pi(\bar{x})$. If $F\bar{d}$ is K_n-free, then $F\bar{d}$ embeds in $\mathbb{H}_n$ over F, which is a contradiction. Therefore there is

$K_n \cong W \subseteq F\bar{d}$. Note that $W \cap \bar{d} \neq \emptyset$ since F is K_n-free. To ease notation, we assume that $W \cap \bar{d} = (d_1, \dots, d_m)$.

Suppose, toward a contradiction, that $W \cap F \subseteq C$. Let $\bar{a} \in \mathbb{H}_n$ be a solution to $\varphi(\bar{x}, \bar{b})$, and set $A = \{a_1, \dots, a_m\}$. Since $\bar{d}$ is an optimal realization of $\pi(\bar{x})$, we can make the following observations.

(1) If $1 \leq i \neq j \leq m$, then $R(d_i, d_j) \Rightarrow (\varphi(\bar{x}, \bar{b}) \rhd R(x_i, x_j)) \Rightarrow R(a_i, a_j)$.

(2) If $1 \leq i \leq m$ and $c \in W \cap F$, then $R(d_i, c) \Rightarrow (\varphi(\bar{x}, \bar{b}) \rhd R(x_i, c)) \Rightarrow R(a_i, c)$.

Altogether, $K_n \cong (W \cap F)A$, which is a contradiction. Therefore $W \cap \bigcup_{l<\omega} \bar{b}^l \neq \emptyset$.

Let $W \cap \bigcup_{l<\omega} \bar{b}^l = \{b_{j_1}^{l_1}, \dots, b_{j_k}^{l_k}\}$, and note that $1 \leq k \leq n-1$. Note also that $s \mapsto j_s$ is injective. Indeed, if $s \neq t$ and $j_s = j_t$, then $R(b_{j_s}^{l_s}, b_{j_s}^{l_t})$, and so $\{b_{j_s}^l : l < \omega\}$ is an infinite complete graph by indiscernibility, which is a contradiction. Therefore, to ease notation, we can assume $W \cap \bigcup_{l<\omega} \bar{b}^l = \{b_1^{l_1}, \dots, b_k^{l_k}\}$. Let $B = \{b_1, \dots, b_k\}$.

Claim 1 $\neg K_n(B/C)$.

Proof Suppose that $X \subseteq BC$ witnesses $K_n(B/C)$. Set $B_0 = \{b_s^{l_s} : b_s \in B \cap X\}$. By indiscernibility, $(X \cap C)B_0$ witnesses $K_n(B_0/C)$. Note also that $B_0 \subseteq W$ and W is complete. Altogether, $(X \cap C)B_0 \cong K_n$, which is a contradiction. $\dashv_{\text{claim}}$

Claim 2 *If $\bar{a} \in \mathbb{H}_n$ is a solution of $\varphi(\bar{x}, \bar{b})$, then $K_n(B/C\bar{a})$.*

Proof Fix $\bar{a} \models \varphi(\bar{x}, \bar{b})$, and let $A = \{a_1, \dots, a_m\}$. We show $(W \cap C)AB$ witnesses $K_n(B/C\bar{a})$, which means verifying all of the necessary relations. Recall that $\bar{d}$ is an optimal solution to $\pi(\bar{x})$. Arguing as in (1) and (2) above, we have that $(W \cap C)A$ is complete. By indiscernibility, we have $R(B, W \cap C)$, and so it remains to show $R(B, A)$. For this, if $a_i \in A$ and $b_s \in B$, then, since $d_i, b_s^{l_s} \in W$, we have

$$\begin{aligned} R(d_i, b_s^{l_s}) &\Rightarrow \big(\varphi(\bar{x}, \bar{b}^{l_s}) \rhd R(x_i, b_s^{l_s})\big) \Rightarrow \big(\varphi(\bar{x}, \bar{b}) \rhd R(x_i, b_s)\big) \\ &\Rightarrow R(a_i, b_s). \end{aligned}$$

$\dashv_{\text{claim}}$

By Claims 1 and 2, we have $K_n^{\varphi}(\bar{b}/C)$, as desired. $\square$

We can now give the full characterization of nondividing formulas in T_n and the ternary relation $\mathop{\smile\!\!\!\!\!\!|}\,^d$ on subsets of $\mathbb{H}_n$, which gives the analogy of Fact 3.7 for T_n.

Theorem 4.5

(a) *Suppose that $C \subset \mathbb{H}_n$, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_0(C)$, and $\bar{b} \in \mathbb{H}_n \backslash C$ are such that $\varphi(\bar{x}, \bar{b})$ is consistent. Then $\varphi(\bar{x}, \bar{b})$ divides over C if and only if $\varphi(\bar{x}, \bar{b}) \rhd x_i = b$ for some $b \in \bar{b}$, or $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$ and $K_n^{\varphi}(\bar{b}/C)$.*

(b) *Suppose that $A, B, C \subset \mathbb{H}_n$. Then $A \mathop{\smile\!\!\!\!\!\!|}_C^d B$ if and only if*

 (i) *$A \cap B \subseteq C$, and*

 (ii) *for all $\bar{b} \in B \backslash C$, if $K_n(\bar{b}/AC)$, then $K_n(\bar{b}/C)$.*

Proof Part (a) follows immediately from Theorem 4.4.

Part (b). ($\Rightarrow$): If (i) fails, then we clearly have $A \not\mathop{\smile\!\!\!\!\!\!|}_C^d B$. Suppose that (ii) fails, and fix $\bar{b} \in B \backslash C$ such that $\neg K_n(\bar{b}/C)$ and $K_n(\bar{b}/AC)$. Fix $W \subseteq AC\bar{b}$ witnessing $K_n(\bar{b}/AC)$. Without loss of generality, let $W \cap A = \{a_1, \dots, a_m\}$, and let $W \cap \bar{b} = (b_1, \dots, b_k) =: \bar{b}_*$. Let $\bar{x} = (x_1, \dots, x_m)$ and $\bar{y} = (y_1, \dots, y_k)$, and define an $\mathcal{L}_R(C)$-formula $\varphi(\bar{x}, \bar{y})$ expressing that $(W \cap C)\bar{x}$ is complete and $R((W \cap C)\bar{x}, \bar{y})$. Then $K_n^\varphi(\bar{b}_*/C)$ and $\varphi(\bar{x}, \bar{b}_*) \in \text{tp}(A/BC)$. Therefore $A \not\downarrow^d_C B$ by Theorem 4.4.

($\Leftarrow$): Suppose that $A \not\downarrow^d_C B$. Then there is some $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_0(C)$ and $\bar{b} \in B\backslash C$ such that $\varphi(\bar{x}, \bar{b})$ divides over C and $\varphi(\bar{x}, \bar{b}) \in \text{tp}(A/BC)$. If $\varphi(\bar{x}, \bar{y}) \rhd x_i = y_j$ for some i, j, then $a_i = b_j \in (A \cap B)\backslash C$, and (i) fails. Otherwise, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$ and so, by Theorem 4.4, we may fix $\bar{b}_* \subseteq \bar{b}$ witnessing $K_n^\varphi(\bar{b}/C)$. By definition, we have $\neg K_n(\bar{b}_*/C)$ and $K_n(\bar{b}_*/AC)$, and so (ii) fails. $\square$

The theorem translates the model-theoretic notion of dividing to the graph-theoretic notion $K_n^\varphi(\bar{b}/C)$. Although the definition of $K_n^\varphi(\bar{b}/C)$ implies that we must check all solutions of φ, it suffices to check an optimal one.

Corollary 4.6 *Fix $C \subset \mathbb{H}_n$, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$, and $\bar{b} \in \mathbb{H}_n\backslash C$ such that $\varphi(\bar{x}, \bar{b})$ is consistent. Let $\bar{a}$ be an optimal solution. Then $\varphi(\bar{x}, \bar{b})$ divides over C if and only if there is $B \subseteq \bar{b}$ such that $\neg K_n(B/C)$ and $K_n(B/C\bar{a})$.*

Proof By Theorem 4.4, we need to show $K_n^\varphi(\bar{b}/C)$ if and only if there is $B \subseteq \bar{b}$ such that $\neg K_n(B/C)$ and $K_n(B/C\bar{a})$. The forward direction is trivial.

Conversely, suppose that $B \subseteq \bar{b}$ is such that $\neg K_n(B/C)$ and $K_n(B/C\bar{a})$. Let $\bar{d}$ be any solution to $\varphi(\bar{x}, \bar{b})$. We want to show $K_n(B/C\bar{d})$. Let $B_0 \subseteq BC\bar{a}$ witness $K_n(B/C\bar{a})$. Define $B_1 = B_0 \cap BC$ and $D = \{d_i : a_i \in B_0 \cap \bar{a}\}$. Since $\bar{a}$ is optimal, arguments similar to those in Lemma 4.3 and Theorem 4.4 show that $B_1 D$ witnesses $K_n(B/C\bar{d})$. $\square$

We end this section by giving some examples and traits of dividing formulas in T_n.

Corollary 4.7 *Suppose that $C \subset \mathbb{H}_n$ and $b_1, \ldots, b_{n-1} \in \mathbb{H}_n\backslash C$ are pairwise distinct. Then the formula*

$$\varphi(x, \bar{b}) := \bigwedge_{i=1}^{n-1} R(x, b_i)$$

divides over C if and only if $\neg K_n(\bar{b}/C)$.

Proof First, if $\varphi(x, \bar{b})$ is inconsistent, then $\bar{b} \cong K_{n-1}$, and so $\neg K_n(\bar{b}/C)$. So we may assume that $\varphi(\bar{x}, \bar{b})$ is consistent.

($\Rightarrow$): If $\varphi(x, \bar{b})$ divides over C, then, by Theorem 4.4, there is some $B \subseteq \bar{b}$ such that $\neg K_n(B/C)$ and $K_n(B/Ca)$ for any $a \models \varphi(x, \bar{b})$. Let $a \models \varphi(x, \bar{b})$ such that $\neg R(a, c)$ for all $c \in C$. Let $X \subseteq CBa$ witness $K_n(B/Ca)$. Then $\neg K_n(B/C)$ implies $a \in X$, and so $X \cap C = \emptyset$ by choice of a. Therefore $X \subseteq Ba \subseteq \bar{b}a$, $|X| = n$, and $|\bar{b}| = n - 1$. It follows that $B = \bar{b}$, and so $\neg K_n(\bar{b}/C)$.

($\Leftarrow$): Note that if a realizes $\varphi(x, \bar{b})$, then $K_n(\bar{b}/a)$. So if $\neg K_n(\bar{b}/C)$, then $\bar{b}$ itself witnesses $K_n^\varphi(\bar{b}/C)$. By Theorem 4.4, $\varphi(x, \bar{b})$ divides over C. $\square$

Corollary 4.8 *Fix $C \subset \mathbb{H}_n$ and $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_R(C)$. Suppose that $\bar{b} \in \mathbb{H}_n\backslash C$ is such that $\varphi(\bar{x}, \bar{b})$ is consistent and divides over C. Define R^φ to be the set*

$$\left\{b \in C\bar{b} : \varphi(\bar{x}, \bar{b}) \rhd R(x_i, b) \text{ for some } i\right\}$$
$$\cup \left\{x_i : \varphi(\bar{x}, \bar{b}) \rhd R(x_i, b) \text{ for some } b \in C\bar{b}\right\}.$$

Then $|R^\varphi| \geq n$ and $|\bar{b} \cap R^\varphi| > 1$.

Proof By assumption, we have $K_n^{\varphi}(\bar{b}/C)$. If $\bar{a}$ is an optimal solution of $\varphi(\bar{x},\bar{b})$, then there is some $X \subseteq C\bar{b}\bar{a}$ witnessing $K_n(\bar{b}/C\bar{a})$. Note that $X \cap \bar{a} \neq \emptyset$ since $\neg K_n(\bar{b}/C)$. Set $B = (X \cap C\bar{b}) \cup \{x_i : a_i \in X\}$. Then $|B| \geq n$ and $B \subseteq R^{\varphi}$ since $\bar{a}$ is optimal. Finally, $|\bar{b} \cap B| = |\bar{b} \cap X| > 1$ since otherwise $X \cong K_n$. □

Corollary 4.8 says that if a formula from $\mathcal{L}_R(C)$ divides, then it needs to mention edges between at least n vertices (and more than one parameter). This is not surprising since no consistent formula from $\mathcal{L}_R(C)$ will divide in T_0, and so dividing in T_n should come from the creation of a graph that is too close to K_n.

5 Forking for Complete Types

In this section, we use our characterization of $\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}$ in T_n to show that forking and dividing are the same for complete types. The proof takes two steps, the first of which is *full existence* for the following ternary relation on graphs. We take the following definition from Adler [1].

Definition 5.1 Given $A, B, C \subset \mathbb{H}_n$, define *edge independence* by

$$A \mathop{\smile\!\!\!\!\!\!\!\;|}^{R}_{C} B \Leftrightarrow A \cap B \subseteq C \text{ and there is no edge from } A\backslash C \text{ to } B\backslash C.$$

In other words, $A \mathop{\smile\!\!\!\!\!\!\!\;|}^{R}_{C} B$ asserts that ABC is isomorphic to the free amalgamation of AC and BC over C. It is easy to see that the free amalgamation of two K_n-free graphs (over some common induced subgraph) is still K_n-free. Therefore, we have the following fact, the details of which are left to the reader.

Lemma 5.2 *For all $A, B, C \subset \mathbb{H}_n$, there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\!\;|}^{R}_{C} B$.*

Using this, we can prove that $\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}$ and $\mathop{\smile\!\!\!\!\!\!\!\;|}^{f}$ coincide in T_n, which yields a full characterization of forking and dividing for complete types.

Theorem 5.3 *Suppose that $A, B, C \subset \mathbb{H}_n$. Then $A \mathop{\smile\!\!\!\!\!\!\!\;|}^{f}_{C} B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_{C} B$ if and only if $A \cap B \subseteq C$ and, for all $\bar{b} \in B\backslash C$, $K_n(\bar{b}/AC)$ implies $K_n(\bar{b}/C)$.*

Proof The second equivalence is by Theorem 4.5, and dividing implies forking in any theory. Therefore we only need to show $A \mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!\!\;|}^{f}_{C} B$. Suppose that $A \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{f}_{C} B$. Then there is some $D \subset \mathbb{H}_n\backslash BC$ such that $A' \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_{C} BD$ for any $A' \equiv_{BC} A$. By Lemma 5.2, let $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\!\;|}^{R}_{BC} D$. By assumption, we have $A' \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_{C} BD$.

Case 1: $A' \cap BD \not\subseteq C$.

We have $A' \cap BD \subseteq BC$ by assumption, so this means there is $b \in (A' \cap B)\backslash C$. But $A' \equiv_{BC} A$ and so $b \in (A \cap B)\backslash C$. Therefore $A \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_{C} B$, as desired.

Case 2: $A' \cap BD \subseteq C$.

Then, since $A' \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_{C} BD$, it follows from Theorem 4.5 that there is $\bar{b} \in BD\backslash C$ such that $\neg K_n(\bar{b}/C)$ and $K_n(\bar{b}/A'C)$. Let $X \subseteq A'C\bar{b}$ witness $K_n(\bar{b}/A'C)$. Note that $X \subseteq A'BCD$. Moreover, note also that if $X \cap (A'\backslash BC) = \emptyset$, then $X \subseteq BCD$, and so X witnesses $K_n(\bar{b}/C)$, which is a contradiction.

Therefore $X \cap (A'\backslash BC) \neq \emptyset$. Then we claim that $X \subseteq A'BC$. Indeed, otherwise there is $u \in X \cap (A'\backslash BC)$ and $v \in X \cap (D\backslash A'BC)$. Therefore $u \neq v$, $u \in A'$, and

$v \in \bar{b}$, and so, since X witnesses $K_n(\bar{b}/A'C)$, we have $R(u,v)$. But this contradicts that there is no edge from $A' \backslash BC$ to $D \backslash BC$.

So we have $X \subseteq A'BC$. Let $\bar{b}_* = X \cap \bar{b} \in B \backslash C$. Then $\neg K_n(\bar{b}/C)$ implies $\neg K_n(\bar{b}_*/C)$, and X witnesses $K_n(\bar{b}_*/A'C)$. Therefore $A' \not\mathop{\smile\!\!\!\!\mid}^d_C B$. Since $A' \equiv_{BC} A$, we have $A \not\mathop{\smile\!\!\!\!\mid}^d_C B$, as desired. □

It is a general fact that, if $\mathop{\smile\!\!\!\!\mid}^d = \mathop{\smile\!\!\!\!\mid}^f$ in some theory T, then all sets are extension bases for nonforking. Indeed, if a partial type forks over C, then it can be extended to a complete type that forks (and therefore divides) over C. Therefore, by Proposition 2.2(b), no partial type forks over its own set of parameters. So we have the following corollary.

Corollary 5.4 *If $C \subset \mathbb{H}_n$, then C is an extension base for nonforking.*

6 A Forking and Nondividing Formula in T_n

We have shown that forking and dividing are the same for complete types in T_n. In this section, we show that the same result cannot be obtained for partial types, by demonstrating an example of a formula in T_n that forks, but does not divide.

Lemma 6.1 *Fix distinct points $b_1, b_2, b_3, b_4 \in \mathbb{G}$ such that $\bar{b} := (b_1, b_2, b_3, b_4)$ is independent. Suppose that $(\bar{b}^l)_{l<\omega}$ is an indiscernible sequence in $\mathbb{G}$, with $\bar{b}^0 = \bar{b}$. If $\bigcup_{l<\omega} \bar{b}^l$ is K_3-free, then there are $i < j$ such that $\{b_i^l, b_j^l : l < \omega\}$ is independent.*

Proof Let $B = \bigcup_{l<\omega} \bar{b}^l$, and assume that B is K_3-free. Note that, for any $i \leq 4$ and $l < m < \omega$, we have $\neg R(b_i^l, b_i^m)$ by indiscernibility and the assumption that B is K_3-free. Moreover, for any $i < j \leq 4$ and $l < \omega$, we have $\neg R(b_i^l, b_j^l)$ by indiscernibility and the assumption that $\bar{b}$ is independent. Therefore, to prove the result, it suffices by indiscernibility to find $i < j$ such that $\neg R(b_i^0, b_j^1)$ and $\neg R(b_i^1, b_j^0)$. Suppose, toward a contradiction, that there are no such $i < j$. Define the function $f : \{(i,j) : 1 \leq i < j \leq 4\} \longrightarrow \{0,1\}$ such that $f(i,j) = 0$ if and only if $R(b_i^0, b_j^1)$. In particular, for any $i < j$, if $f(i,j) = 1$, then $R(b_i^1, b_j^0)$.

Claim *For all $i < j < k$, $f(i,j) = f(j,k)$.*

Proof Suppose not, and fix $i < j < k$ with $f(i,j) \neq f(j,k)$.
Case 1: $f(i,j) = 1$ and $f(j,k) = 0$.

If $f(i,k) = 0$, then, by indiscernibility, we have $R(b_i^1, b_j^0)$, $R(b_j^0, b_k^2)$ and $R(b_i^1, b_k^2)$. Then $\{b_i^1, b_j^0, b_k^2\} \cong K_3$, which is a contradiction. If $f(i,k) = 1$, then, by indiscernibility, we have $R(b_i^2, b_j^0)$, $R(b_j^0, b_k^1)$ and $R(b_i^2, b_k^1)$. Then $\{b_i^2, b_j^0, b_k^1\} \cong K_3$, which is a contradiction.
Case 2: $f(i,j) = 0$ and $f(j,k) = 1$.

Similar to Case 1, we see that $f(i,k) = 0$ implies $\{b_i^0, b_j^2, b_k^1\} \cong K_3$, and $f(i,k) = 1$ implies $\{b_i^1, b_j^2, b_k^0\} \cong K_3$. $\dashv_{\text{claim}}$

From the claim, we have $f(1,2) = f(2,3) = f(3,4) = f(1,3)$. Then $f(1,2) = 0$ implies $\{b_1^0, b_2^1, b_3^2\} \cong K_3$ and $f(1,2) = 1$ implies $\{b_1^2, b_2^1, b_3^0\} \cong K_3$, which gives the desired contradiction. □

Theorem 6.2 *Fix $C \subset \mathbb{H}_n$ and $\bar{b} = (b_1, b_2, b_3, b_4) \in \mathbb{H}_n$ such that $C \cong K_{n-3}$, $\bar{b}$ is independent, and $R(\bar{b}, C)$. Let $\varphi(x, \bar{b}) := \bigvee_{i<j} R(x, b_i b_j C)$. Then $\varphi(x, \bar{b})$ forks over C, but does not divide over C.*

Proof For any $i < j$, we have $|b_i b_j C| = n - 1$, and so $\neg K_n(b_i, b_j / C)$. Moreover, we clearly have that, for any $a \in \mathbb{H}_n$, $R(a, b_i b_j C)$ implies $K_n(b_i, b_j / Ca)$. By Theorem 4.4, $R(x, b_i b_j C)$ divides over C, and so $\varphi(x, \bar{b})$ forks over C by Proposition 2.2(c).

Let $(\bar{b}^l)_{l<\omega}$ be C-indiscernible, with $\bar{b}^0 = \bar{b}$. If there is some $K_3 \cong W \subset \bigcup_{l<\omega} \bar{b}^l$, then $K_n \cong CW$, since $R(\bar{b}^l, C)$ for all $l < \omega$. Therefore $\bigcup_{l<\omega} \bar{b}^l$ is K_3-free and so, by Lemma 6.1, we obtain $i < j$ such that $B := \{b_i^l, b_j^l : l < \omega\}$ is independent. Since $|C| = n - 3$, it follows that BC is K_{n-1}-free, and so there is some $a \in \mathbb{H}_n$ such that $R(a, BC)$. In particular, $a \models \{\varphi(x, \bar{b}^l) : l < \omega\}$. By Definition 2.1, $\varphi(x, \bar{b})$ does not divide over C. $\square$

7 Final Remarks

We have shown that in the theory T_n (which is NSOP$_4$ and TP$_2$) all sets are extension bases for nonforking, but forking and dividing are not always the same. However, this only partially addresses the extent to which the results of [3] apply to theories with TP$_2$. In particular, forking is the same as dividing for complete types in T_n, which means there is good behavior of nonforking beyond just the fact that all sets are extension bases. This leads to the following amended version of the main question.

Question 7.1 Suppose that, in some complete first-order theory, all sets are extension bases for nonforking.

(1) Does NSOP$_3$ imply forking and dividing are the same for partial types?
(2) For what classes of theories do we have $\mathop{\smile\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!|}^{d}$?

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Acknowledgments

I would like to thank Lynn Scow, John Baldwin, Artem Chernikov, Dave Marker, Caroline Terry, and Phil Wesolek for their part in the development of this project. I also thank the anonymous referee for many helpful suggestions.

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