# Decidable Fragments of the Simple Theory of Types with Infinity and NF

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**Abstract**    We identify complete fragments of the simple theory of types with infinity (TSTI) and Quine's new foundations (NF) set theory. We show that TSTI decides every sentence $\phi$ in the language of type theory that is in one of the following forms:

(A)  $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^{s_1} \cdots \exists y_l^{s_l} \theta$ where the superscripts denote the types of the variables, $s_1 > \cdots > s_l$, and $\theta$ is quantifier-free,

(B)  $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^s \cdots \exists y_l^s \theta$ where the superscripts denote the types of the variables and $\theta$ is quantifier-free.

This shows that NF decides every stratified sentence $\phi$ in the language of set theory that is in one of the following forms:

(A′)  $\phi = \forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_l \theta$ where $\theta$ is quantifier-free and $\phi$ admits a stratification that assigns distinct values to all of the variables $y_1, \ldots, y_l$,

(B′)  $\phi = \forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_l \theta$ where $\theta$ is quantifier-free and $\phi$ admits a stratification that assigns the same value to all of the variables $y_1, \ldots, y_l$.

## 1  Introduction

Roland Hinnion [3] showed that every consistent $\exists^*$ sentence in the language of set theory is a theorem of new foundations (NF) or, equivalently, every finite binary structure can be embedded in every model of NF. Both these formulations invite generalizations. On the one hand we find results like every countable binary structure can be embedded in every model of NF (this is Forster's [1, Theorem 4]), and on the other we can ask about the status of sentences with more quantifiers: $\forall^*\exists^*$ sentences in the first instance; it is the latter that will be our concern here.

It is elementary to check that NF does not decide all $\forall^*\exists^*$ sentences, since the existence of Quine atoms ($x = \{x\}$) is consistent with, and independent of, NF.

However "$(\forall x)(x \neq \{x\})$" is not stratified, and this invites the conjecture that (i) NF decides all stratified $\forall^*\exists^*$ sentences and that (ii) all unstratified $\forall^*\exists^*$ sentences can be proved both relatively consistent and independent by means of Rieger–Bernays permutation methods. We consider limb (i) of this conjecture here.

The foregoing is all about NF; the connection with the simple theory of types with infinity (TSTI) arises because of work of Specker [7], [8]: NF decides all stratified $\forall^*\exists^*$ sentences of the language of set theory if and only if TSTI + ambiguity decides all $\forall^*\exists^*$ sentences of the language of type theory.

**Conjecture**    *All models of* TSTI *agree on all* $\forall^*\exists^*$ *sentences.*

It is toward a proof of this conjecture that our efforts in this paper are directed.

Observe that the statement that there is a total order of $V$ is consistent with and independent of TST, and it can be said with three blocks of quantifiers:

$$(\exists O)\big[(\forall xy \in O)(x \subseteq y \lor y \subseteq x) \land (\forall uv)$$
$$\times \big(u \neq v \to (\exists x \in O)(u \in x \iff v \notin x)\big)\big],$$

making it $\exists^1\forall^6\exists^1$.

## 2  Background and Definitions

The simple theory of types is the simplification of the ramified theory of types, the underlying system of Russell and Whitehead [6], that was independently discovered by Frank Ramsey and Leon Chwistek. Following Mathias [4] we use TSTI and TST to abbreviate the simple theory of types with and without the axiom of infinity, respectively. These theories are naturally axiomatized in a many-sorted language with sorts for each $n \in \mathbb{N}$.

**Definition 2.1**    We use $\mathcal{L}_{\text{TST}}$ to denote the $\mathbb{N}$-sorted language endowed with binary relation symbols $\in_n$ for each sort $n \in \mathbb{N}$. There are variables $x^n, y^n, z^n, \ldots$ for each sort $n \in \mathbb{N}$, and well-formed $\mathcal{L}_{\text{TST}}$-formulae are built up inductively from atomic formulae of the form $x^n \in_n y^{n+1}$ and $x^n = y^n$ by using the connectives quantifiers of first-order logic.

We refer to sorts of $\mathcal{L}_{\text{TST}}$ as types. We will attempt to stick to the convention of denoting $\mathcal{L}_{\text{TST}}$-structures by using calligraphy letters ($\mathcal{M}, \mathcal{N}, \ldots$). An $\mathcal{L}_{\text{TST}}$-structure $\mathcal{M}$ consists of domains $M_n$ for each type $n \in \mathbb{N}$ and interpretations of the relations $\in_n^{\mathcal{M}} \subseteq M_n \times M_{n+1}$ for each type $n \in \mathbb{N}$; we write $\mathcal{M} = \langle M_0, M_1, \ldots, \in_0^{\mathcal{M}}, \in_1^{\mathcal{M}}, \ldots\rangle$. If $\mathcal{M} = \langle M_0, M_1, \ldots, \in_0^{\mathcal{M}}, \in_1^{\mathcal{M}}, \ldots\rangle$ is an $\mathcal{L}_{\text{TST}}$-structure, then we call the elements of $M_0$ atoms.

**Definition 2.2**    We use TST to denote the $\mathcal{L}_{\text{TST}}$-theory with the following axioms:
   (Extensionality) for all $n \in \mathbb{N}$,

$$\forall x^{n+1}\forall y^{n+1}\big(x^{n+1} = y^{n+1} \iff \forall z^n(z^n \in_n x^{n+1} \iff z^n \in_n y^{n+1})\big);$$

   (Comprehension) for all $n \in \mathcal{N}$ and for all well-formed $\mathcal{L}_{\text{TST}}$-formulae $\phi(x^n, \vec{z})$,

$$\forall\vec{z}\exists y^{n+1}\forall x^n\big(x^n \in_n y^{n+1} \iff \phi(x^n, \vec{z})\big).$$

Comprehension ensures that every successor type is closed under the set-theoretic operations: union ($\cup$), intersection ($\cap$), difference ($\backslash$), and symmetric difference ($\triangle$). For all $n \in \mathbb{N}$, we use $\emptyset^{n+1}$ to denote the point at type $n + 1$ which contains no

points from type $n$, and we use $V^{n+1}$ to denote the point at type $n + 1$ that contains every point from type $n$. The Wiener–Kuratowski ordered pair allows us to code ordered pairs, denoted $\langle x, y \rangle$, as objects in TST which have type two higher than the type of $x$ and $y$. Functions, as usual, are thought of as collections of ordered pairs. This means that a function $f : X \longrightarrow Y$ will be coded by an object in TST that has type two higher than the type of $X$ and $Y$. The theory TSTI is obtained from TST by asserting the existence of a Dedekind infinite collection at type 1.

**Definition 2.3**    We use TSTI to denote the $\mathcal{L}_{\mathrm{TST}}$-theory obtained from TST by adding the axiom

$$\exists x^1 \exists f^3 (f^3 : x^1 \longrightarrow x^1 \text{ is injective but not surjective}).$$

Let $X$ be a set. If the $\mathcal{L}_{\mathrm{TST}}$-structure $\mathcal{M} = \langle M_0, M_1, \ldots, \in_0, \in_1, \ldots \rangle$ is defined by $M_n = \mathcal{P}^n(X)$ and $\in_n^{\mathcal{M}} = \in \restriction \mathcal{P}^n(X) \times \mathcal{P}^{n+1}(X)$ for all $n \in \mathbb{N}$, then $\mathcal{M} \models$ TST. If $m \in \mathbb{N}$ and $|X| = m$, then $\mathcal{M}$ is the unique, up to isomorphism, model of TST with exactly $m$ atoms and we say that $\mathcal{M}$ is finitely generated by $m$ atoms. Alternatively, if $X$ is Dedekind infinite, then $\mathcal{M} \models$ TSTI. This shows that ZFC proves the consistency of TSTI. In fact, in [4] it is shown that TSTI is equiconsistent with Mac Lane set theory.

We say that an $\mathcal{L}'$-theory $T$ decides an $\mathcal{L}'$-sentence $\phi$ if and only if $T \vdash \phi$ or $T \vdash \neg\phi$. The completeness theorem implies that $T$ decides $\phi$ if and only if $\phi$ holds in all $\mathcal{L}'$-structures $\mathcal{M} \models T$ or $\neg\phi$ holds in all $\mathcal{L}'$-structures $\mathcal{M} \models T$.

**Definition 2.4**    We say that an $\mathcal{L}_{\mathrm{TST}}$-sentence $\phi$ is $\exists^*\forall^*$ if and only if $\phi = \exists x_1^{r_1} \cdots \exists x_k^{r_k} \forall y_1^{s_1} \cdots \forall y_l^{s_l} \theta$, where $\theta$ is quantifier-free.

**Definition 2.5**    We say that an $\mathcal{L}_{\mathrm{TST}}$-sentence $\phi$ is $\forall^*\exists^*$ if and only if $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^{s_1} \cdots \exists y_l^{s_l} \theta$, where $\theta$ is quantifier-free.

We will show that TSTI decides a significant fragment of the $\forall^*\exists^*$ sentences. (Thus, it also decides the $\exists^*\forall^*$ sentences that are logically equivalent to the negation of these $\forall^*\exists^*$ sentences.) We achieve this result by showing that every sentence or negation of a sentence in this fragment that is true in some model of TSTI is true in all models of TST that are finitely generated by sufficiently many atoms.

**Definition 2.6**    We say that an $\mathcal{L}_{\mathrm{TST}}$-sentence $\phi$ has the finitely generated model property if and only if, if there exists an $\mathcal{N} \models$ TSTI $+ \phi$, then there exists a $k \in \mathbb{N}$ such that, for all $m \geq k$, if $\mathcal{M} \models$ TST is finitely generated by $m$ atoms, then $\mathcal{M} \models \phi$.

Note that if $\Gamma$ is a class of $\mathcal{L}_{\mathrm{TST}}$-sentences that have the finitely generated model property and $\Gamma$ is closed under negations, then TST decides every sentence in $\Gamma$.

Quine [5] introduces a set theory by identifying a syntactic condition on formulae in the single sorted language of set theory that captures the restricted comprehension available in TST. This set theory has been dubbed "new foundations" (NF) after the title of [5]. We will use $\mathcal{L}$ to denote the language of set theory—the language of first-order logic endowed with a binary relation symbol $\in$ whose intended interpretation is membership. Before giving the axioms of NF we first recall Quine's definition of the class of stratified formulae. If $\phi$ is an $\mathcal{L}$-formula, then we use $\mathbf{Var}(\phi)$ to denote the set of variables (both free and bound) which appear in $\phi$.

**Definition 2.7**    Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula.    We say that $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$ is a stratification of $\phi$ if and only if

(i) "$x \in y$" is a subformula of $\phi$, then $\sigma("y") = \sigma("x") + 1$;
(ii) "$x = y$" is a subformula of $\phi$, then $\sigma("y") = \sigma("x")$.

If there exists a stratification of $\phi$, then we say that $\phi$ is stratified.

Let $\phi$ be an $\mathcal{L}$-formula. Note that $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$ is a stratification of $\phi$ if and only if the formula obtained by decorating every variable appearing in $\phi$ with the type given by $\sigma$ yields a well-formed $\mathcal{L}_{\mathrm{TST}}$-formula. Conversely, let $\theta$ be a well-formed $\mathcal{L}_{\mathrm{TST}}$-formula, and let $\phi$ an $\mathcal{L}$-formula obtained for $\theta$ by deleting the types from the variables appearing in $\theta$ while ensuring (by relabeling variables) that no two distinct variables in $\theta$ become the same variable in $\phi$. Then the $\mathcal{L}$-formula $\phi$ is stratified, and the function which sends a variable in $\phi$ to the type index of the corresponding variable in $\theta$ is a stratification.

**Definition 2.8**     Let $\phi$ be an $\mathcal{L}$-formula with stratification $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$. We use $\phi^{(\sigma)}$ to denote the $\mathcal{L}_{\mathrm{TST}}$-formula obtained by assigning each variable "$x$" appearing in $\phi$ the type $\sigma("x")$.

NF is the $\mathcal{L}$-theory with the axiom of extensionality and comprehension for all stratified $\mathcal{L}$-formulae.

**Definition 2.9**     We use NF to denote the $\mathcal{L}$-theory with the following axioms:
(Extensionality) $\forall x \forall y (x = y \iff \forall z(z \in x \iff z \in y))$,
(Stratified comprehension) for all stratified $\phi(x, \vec{z})$,

$$\forall \vec{z} \exists y \forall x (x \in y \iff \phi(x, \vec{z})).$$

We direct the interested reader to Forster [2] for a detailed treatment of NF. One interesting feature of NF is that it refutes the axiom of choice and so proves the axiom of infinity (see [7]). There is a strong connection between the theories NF and TSTI. Specker [8] shows that models of NF can be obtained from models of TSTI plus the scheme $\phi \iff \phi^+$, for all $\mathcal{L}_{\mathrm{TST}}$-sentences $\phi$, where $\phi^+$ is obtained from $\phi$ by incrementing the types of all the variables appearing in $\phi$. Conversely, let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be an $\mathcal{L}$-structure with $\mathcal{M} \models$ NF. The $\mathcal{L}_{\mathrm{TST}}$-structure $\mathcal{N} = \langle N_0, N_1, \ldots, \in_0^{\mathcal{N}}, \in_1^{\mathcal{N}}, \ldots \rangle$ defined by $N_n = M$ and $\in_n^{\mathcal{N}} = \in^{\mathcal{M}}$ is such that $\mathcal{N} \models$ TSTI. Moreover, if $\phi$ is an $\mathcal{L}$-sentence with stratification $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$ and $\mathcal{M} \models \phi$, then $\mathcal{N} \models \phi^{(\sigma)}$. This immediately shows that a decidable fragment of TSTI yields a decidable fragment of NF.

**Theorem 2.1**     Let $\phi$ be an $\mathcal{L}$-sentence with stratification $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$. If TSTI decides $\phi^{(\sigma)}$, then NF decides $\phi$.

## 3   ∃*∀* Sentences Have the Finitely Generated Model Property

In this section we prove that all $\exists^*\forall^*$ sentences have the finitely generated model property. This result follows from the fact that if $\mathcal{N}$ is a model of TSTI, $a_1^{r_1}, \ldots, a_k^{r_k} \in \mathcal{N}$ with $r_1 \leq \cdots \leq r_k$, and $\mathcal{M}$ is a model of TST that is finitely generated by sufficiently many atoms, then there is an embedding of $\mathcal{M}$ into $\mathcal{N}$ with $a_1^{r_1}, \ldots, a_k^{r_k}$ in the range. Given $k \in \mathbb{N}$ we define the function $\mathbf{G}_k : \mathbb{N} \longrightarrow \mathbb{N}$ by recursion:

$$\mathbf{G}_k(0) = k \qquad \text{and} \qquad \mathbf{G}_k(n+1) = \binom{\mathbf{G}_k(n)}{2} + k. \qquad (1)$$

**Lemma 3.1**    *Let $\mathcal{N} \models$ TSTI, and let $a_1^{r_1}, \dots, a_k^{r_k} \in \mathcal{N}$ with $r_1 \leq \cdots \leq r_k$. If $\mathcal{M} \models$ TST is finitely generated by at least $\mathbf{G}_k(r_k)$ atoms, then there exists a sequence $\langle f_n \mid n \in \mathbb{N} \rangle$ such that, for all $n \in \mathbb{N}$,*

(i) *$f_n : M_n \longrightarrow N_n$ is injective,*
(ii) *for all $x \in M_n$ and for all $y \in M_{n+1}$,*

$$\mathcal{M} \models x \in_n y \qquad \text{if and only if} \qquad \mathcal{N} \models f_n(x) \in_n f_{n+1}(y),$$

*(iii)*

$$a_1^{r_1}, \dots, a_k^{r_k} \in \bigcup_{m \in \mathbb{N}} \text{rng}(f_m).$$

**Proof**    Let $\mathcal{N} = \langle N_0, N_1, \dots, \in_0^{\mathcal{N}}, \in_1^{\mathcal{N}}, \dots \rangle$ be such that $\mathcal{N} \models$ TSTI, and let $a_1^{r_1}, \dots, a_k^{r_k} \in \mathcal{N}$ with $r_1 \leq \cdots \leq r_k$. Let $\mathcal{M} = \langle M_0, M_1, \dots, \in_0^{\mathcal{M}}, \in_1^{\mathcal{M}}, \dots \rangle$ be such that $\mathcal{M} \models$ TST is finitely generated and $|M_0| \geq \mathbf{G}_k(r_k)$. We begin by defining $C \subseteq \mathcal{N}$ such that $|C \cap N_0| \leq \mathbf{G}_k(r_k)$ and, for any two points $x \neq y$ in $C$ that are not atoms, there exists a point $z$ in $C$ which $\mathcal{N}$ believes is in the symmetric difference of $x$ and $y$. Define $C_0 = \{a_1^{r_1}, \dots, a_k^{r_k}\} \subseteq \mathcal{N}$. Note that $|C_0 \cap N_{r_k}| \leq \mathbf{G}_k(0) = k$, and for all $0 \leq m < r_k$, $|C_0 \cap N_m| \leq k$. For $0 < n \leq r_k$ we recursively define $C_n \subseteq \mathcal{N}$ which satisfies

(I) $|C_n \cap N_{r_k - n}| \leq \mathbf{G}_k(n)$,
(II) for all $0 \leq m < r_k - n$, $|C_n \cap N_m| \leq k$.

Suppose that $n < r_k$ and that $C_n \subseteq \mathcal{N}$ has been defined and satisfies (I) and (II). For all $y, z \in N_{r_k - n}$ with $y \neq z$, let $\gamma_{\{y,z\}} \in N_{r_k - (n+1)}$ be such that

$$\mathcal{N} \models \gamma_{\{y,z\}} \in_{r_k - (n+1)} y \triangle z.$$

Define

$$C_{n+1} = C_n \cup \{\gamma_{\{y,z\}} \mid \{y, z\} \in [N_{r_k - n} \cap C_n]^2\}.$$

It follows from (I) and (II) that

$$|C_{n+1} \cap N_{r_k - (n+1)}| \leq |C_n \cap N_{r_k - (n+1)}| + \binom{|C_n \cap N_{r_k - n}|}{2} \leq k + \binom{\mathbf{G}_k(n)}{2}$$

$$= \mathbf{G}_k(n + 1),$$

and for all $0 \leq m < r_k - (n+1)$, $|C_{n+1} \cap N_m| \leq k$. Now, let $C = C_{r_k}$. This recursion ensures that $|C \cap N_0| \leq \mathbf{G}_k(r_k)$.

We now turn to defining the family of maps $\langle f_n \mid n \in \mathbb{N} \rangle$ which embed $\mathcal{M}$ into $\mathcal{N}$. We define the sequence $\langle f_n \mid n \in \mathbb{N} \rangle$ by induction. Let $C' = C \cap N_0$. Let $f_0 : M_0 \longrightarrow N_0$ be an injection such that $C' \subseteq \text{rng}(f_0)$. Suppose that $\langle f_0, \dots, f_n \rangle$ has been defined such that

(I′) for all $0 \leq j \leq n$, $f_j : M_j \longrightarrow N_j$ is injective,
(II′) for all $0 \leq j < n$, for all $x \in M_j$, and for all $y \in M_{j+1}$,

$$\mathcal{M} \models x \in_j y \qquad \text{if and only if} \qquad \mathcal{N} \models f_j(x) \in_j f_{j+1}(y),$$

(III′) for all $0 \leq j \leq n$, $C \cap N_j \subseteq \text{rng}(f_j)$.

If $0 \leq j \leq n$ and $x \in M_{j+1}$, then we use $f_j``x$ to denote the point in $N_{j+1}$ such that $\mathcal{N} \models f_j``x = \{f_j(y) \mid \mathcal{M} \models y \in_j x\}$. Note that, since $\mathcal{M}$ is finitely generated,

for all $x \in M_{j+1}$, $f_j``x$ exists in $\mathcal{N}$. We define $f_{n+1} : M_{n+1} \longrightarrow N_{n+1}$ by

$$f_{n+1}(x) = \begin{cases} \gamma & \text{if } \gamma \in C \cap N_{n+1} \text{ and } \mathcal{N} \models (f_n``x = \gamma \cap f_n``(V^{n+1})^{\mathcal{M}}). \\ f_n``x & \text{otherwise.} \end{cases}$$

We first need to show that the map $f_{n+1}$ is well defined. Suppose that $\xi_1, \xi_2 \in C \cap N_{n+1}$ with $\xi_1 \neq \xi_2$, and suppose that $x \in M_{n+1}$ are such that

$$\mathcal{N} \models \left( f_n``x = \xi_1 \cap f_n``(V^{n+1})^{\mathcal{M}} \right) \qquad \text{and} \qquad \mathcal{N} \models \left( f_n``x = \xi_2 \cap f_n``(V^{n+1})^{\mathcal{M}} \right).$$

Now, there is a $\gamma \in C \cap N_n$ such that $\mathcal{N} \models (\gamma \in_n \xi_1 \triangle \xi_2)$. By (III'), $\gamma \in \text{rng}(f_n)$, which is a contradiction. Therefore, $f_{n+1}$ is well defined. The fact that $f_n$ is injective ensures that $f_{n+1}$ is injective.

We now turn to showing that the sequence $\langle f_0, \ldots, f_{n+1} \rangle$ satisfies (II'). Let $x \in M_n$, and let $y \in M_{n+1}$. There are two cases. First, suppose that $f_{n+1}(y) = \gamma \in C$. Therefore, $\mathcal{N} \models (f_n``y = \gamma \cap f``(V^{n+1})^{\mathcal{M}})$. If $\mathcal{M} \models x \in_n y$, then $\mathcal{N} \models f_n(x) \in_n f_n``y$ and so $\mathcal{N} \models f_n(x) \in_n f_{n+1}(y)$. Conversely, if $\mathcal{N} \models f_n(x) \in_n \gamma$, then $\mathcal{N} \models f_n(x) \in_n f_n``y$ and so $\mathcal{M} \models x \in_n y$. The second case is when $f_{n+1}(y) = f_n``y$. In this case it is clear that

$$\mathcal{M} \models x \in_n y \qquad \text{if and only if} \qquad \mathcal{N} \models f_n(x) \in_n f_{n+1}(y).$$

This shows that the sequence $\langle f_0, \ldots, f_{n+1} \rangle$ satisfies (II'). This concludes the induction step of the construction and shows that we can construct a sequence $\langle f_n \mid n \in \mathbb{N} \rangle$ that satisfies (i)–(iii). $\qquad \square$

This embedding property allows us to show that every $\exists^* \forall^*$ sentence has the finitely generated model property.

**Theorem 3.2** *Let $\phi = \exists x_1^{r_1} \cdots \exists x_k^{r_k} \forall y_1^{s_1} \cdots \forall y_l^{s_l} \theta$, where $r_1 \leq \cdots \leq r_k$ and $\theta$ is quantifier-free. If $\mathcal{N} \models \text{TSTI} + \phi$ and $\mathcal{M} \models \text{TST}$ is finitely generated by at least $\mathbf{G}_k(r_k)$ atoms, then $\mathcal{M} \models \phi$.*

**Proof** Let $\mathcal{N} = \langle N_0, N_1, \ldots, \in_0^{\mathcal{N}}, \in_1^{\mathcal{N}}, \ldots \rangle$ be such that $\mathcal{N} \models \text{TSTI} + \phi$. Let $\mathcal{M} = \langle M_0, M_1, \ldots, \in_0^{\mathcal{M}}, \in_1^{\mathcal{M}}, \ldots \rangle$ be such that $\mathcal{M} \models \text{TST}$ and $\mathcal{M}$ is finitely generated by at least $\mathbf{G}_k(r_k)$ atoms. Let $a_1^{r_1}, \ldots, a_k^{r_k} \in \mathcal{N}$ be such that

$$\mathcal{N} \models \forall y_1^{s_1} \cdots \forall y_l^{s_l} \theta[a_1^{r_1}, \ldots, a_k^{r_k}].$$

Using Lemma 3.1 we can find a sequence $\langle f_n \mid n \in \mathbb{N} \rangle$ such that

  (i) $f_n : M_n \longrightarrow N_n$ is injective,
  (ii) for all $x \in M_n$ and for all $y \in M_{n+1}$,

$$\mathcal{M} \models x \in_n y \qquad \text{if and only if} \qquad \mathcal{N} \models f_n(x) \in f_{n+1}(y),$$

  (iii)

$$a_1^{r_1}, \ldots, a_k^{r_k} \in \bigcup_{m \in \mathbb{N}} \text{rng}(f_m).$$

Let $b_1^{r_1}, \ldots, b_k^{r_k} \in \mathcal{M}$ be such that, for all $1 \leq j \leq k$, $f_{r_j}(b_j^{r_j}) = a_j^{r_j}$. Let $c_1^{s_1}, \ldots, c_l^{s_l} \in \mathcal{M}$. Since $\mathcal{N} \models \theta[a_1^{r_1}, \ldots, a_k^{r_k}, f_{s_1}(c_1^{s_1}), \ldots, f_{s_l}(c_l^{s_l})]$, it follows that

$$\mathcal{M} \models \theta[b_1^{r_1}, \ldots, b_k^{r_k}, c_1^{s_1}, \ldots, c_l^{s_l}].$$

Therefore,

$$\mathcal{M} \models \forall y_1^{s_1} \cdots \forall y_l^{s_l} \theta[b_1^{r_1}, \ldots, b_k^{r_k}],$$

which proves the theorem. $\qquad \square$

## 4 Decidable Fragments of the ∀*∃* Sentences

In this section we will show that TSTI decides every $\forall^*\exists^*$ sentence $\phi$ that is in one of the following forms:

(A) $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^{s_1} \cdots \exists y_l^{s_l} \theta$, where $s_1 > \cdots > s_l$ and $\theta$ is quantifier-free;

(B) $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^{s} \cdots \exists y_l^{s} \theta$, where $\theta$ is quantifier-free.

By applying Theorem 2.1 it then follows that NF decides every stratified $\mathcal{L}$-sentence $\phi$ that is in one of the following forms:

(A′) $\phi = \forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_l \theta$, where $\theta$ is quantifier-free and $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$ is a stratification of $\phi$ that assigns distinct values to all of the variables $y_1, \ldots, y_l$;

(B′) $\phi = \forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_l \theta$, where $\theta$ is quantifier-free and $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$ is a stratification of $\phi$ that assigns the same value to all of the variables $y_1, \ldots, y_l$.

Throughout this section we will fix $k, l \in \mathbb{N}$ and a sequence $r_1 \leq \cdots \leq r_k$ that will represent the types of the universally quantified variables in a $\forall^*\exists^*$ sentence. Let $k'$ be the number of distinct elements in the list $r_1, \ldots, r_k$. Let $K_1, \ldots, K_{k'}$ be the multiplicities of the elements in the list $r_1, \ldots, r_k$, so $k = \sum_{1 \leq i \leq k'} K_i$, and let $K = \max\{K_1, \ldots, K_{k'}, l\}$. We also fix structures $\mathcal{N} = \langle N_0, N_1, \ldots, \in_0^{\mathcal{N}}, \in_1^{\mathcal{N}}, \ldots \rangle$ with $\mathcal{N} \models \mathrm{TSTI}$ and $\mathcal{M} = \langle M_0, M_1, \ldots, \in_0^{\mathcal{M}}, \in_1^{\mathcal{M}}, \ldots \rangle$ with $\mathcal{M} \models \mathrm{TST}$ finitely generated by at least $(2^K)^{k'+2}$ atoms. Let $a_1^{r_1}, \ldots, a_k^{r_k} \in \mathcal{M}$.

Our approach will be to define color classes $\mathcal{C}_{i,j}$, the elements of which we will call colors, and functions $c_{i,j}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,j}$ and $c_{i,j}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,j}$, which we will call colorings, for all $i \in \mathbb{N}$ and for all $0 \leq j \leq k'$. For all $0 < j \leq k'$, the colorings $c_{i,j}^{\mathcal{M}}$ will be defined using the elements $a_1^{r_1}, \ldots, a_{j'}^{r_{j'}}$ where $j' = \sum_{1 \leq m \leq j} K_m$, and in the process of defining the colorings $c_{i,j}^{\mathcal{N}}$ we will construct corresponding elements $b_1^{r_1}, \ldots, b_{j'}^{r_{j'}} \in \mathcal{N}$. The colorings will be designed with the following properties.

(i) For a fixed color $\alpha$ in some $\mathcal{C}_{i,j}$, the property of being an element of $\mathcal{N}$ that is given color $\alpha$ by $c_{i,j}^{\mathcal{N}}$ will be definable by an $\mathcal{L}_{\mathrm{TST}}$-formula, $\Phi_{i,j,\alpha}$, with parameters over $\mathcal{N}$.

(ii) The color given to an element $x$ in $\mathcal{M}$ (or $\mathcal{N}$) by the coloring $c_{i,j}^{\mathcal{M}}$ (resp., $c_{i,j}^{\mathcal{N}}$) will tell us which quantifier-free $\mathcal{L}_{\mathrm{TST}}$-formulae with parameters $a_1^{r_1}, \ldots, a_{j'}^{r_{j'}}$ (resp., $b_1^{r_1}, \ldots, b_{j'}^{r_{j'}}$), where $j' = \sum_{1 \leq m \leq j} K_m$, are satisfied by $x$ in $\mathcal{M}$ (resp., $\mathcal{N}$).

(iii) For every color $\beta$ in $\mathcal{C}_{i,j}$, the color given to an element $x$ in $\mathcal{M}$ (or $\mathcal{N}$) by the coloring $c_{i+1,j}^{\mathcal{M}}$ (resp., $c_{i+1,j}^{\mathcal{N}}$) will tell us whether or not there is an element $y$ in $\mathcal{M}$ (resp., $\mathcal{N}$) such that $\mathcal{M} \models y \in_i x$ (resp., $\mathcal{N} \models y \in_i x$) and $y$ is given color $\beta$ by $c_{i,j}^{\mathcal{M}}$ (resp., $c_{i,j}^{\mathcal{N}}$).

(iv) For every color $\beta$ in $\mathcal{C}_{i,j}$, the color given to an element $x$ in $\mathcal{M}$ (or $\mathcal{N}$) by the coloring $c_{i+1,j}^{\mathcal{M}}$ (resp., $c_{i+1,j}^{\mathcal{N}}$) will tell us whether or not there is an element $y$ in $\mathcal{M}$ (resp., $\mathcal{N}$) such that $\mathcal{M} \models y \notin_i x$ (resp., $\mathcal{N} \models y \notin_i x$) and $y$ is given color $\beta$ by $c_{i,j}^{\mathcal{M}}$ (resp., $c_{i,j}^{\mathcal{N}}$).

Note that, since $\mathcal{M}$ is finitely generated, the analogue of condition (i) automatically holds for $\mathcal{M}$.

Before defining the color classes $\mathcal{C}_{i,j}$ and the colorings $c_{i,j}^{\mathcal{M}}$ and $c_{i,j}^{\mathcal{N}}$ we first introduce the following definitions.

**Definition 4.1**      Let $m \in \mathbb{N}$. We say that a color $\alpha \in \mathcal{C}_{i,j}$ is $m$-special with respect to a coloring $f : X \longrightarrow \mathcal{C}_{i,j}$ if and only if

$$\big|\{x \in X \mid f(x) = \alpha\}\big| = m.$$

If $\alpha \in \mathcal{C}_{i,j}$ is 0-special, then we say that $\alpha$ is forbidden.

**Definition 4.2**      Let $m \in \mathbb{N}$. We say that a color $\alpha \in \mathcal{C}_{i,j}$ is $m$-abundant with respect to a coloring $f : X \longrightarrow \mathcal{C}_{i,j}$ if and only if

$$\big|\{x \in X \mid f(x) = \alpha\}\big| \geq m.$$

**Definition 4.3**      Let $J \in \mathbb{N}$. We say that colorings $f : X \longrightarrow \mathcal{C}_{i,j}$ and $g : Y \longrightarrow \mathcal{C}_{i,j}$ are $J$-similar if and only if, for all $0 \leq m < J$ and for all $\alpha \in \mathcal{C}_{i,j}$,

$\alpha$ is $m$-special w.r.t. $f$      if and only if      $\alpha$ is $m$-special w.r.t. $g$.

The color classes $\mathcal{C}_{i,j}$ and colorings $c_{i,j}^{\mathcal{M}}$ and $c_{i,j}^{\mathcal{N}}$ for all $i \in \mathbb{N}$ and for all $0 \leq j \leq k'$ will be defined by a 2-dimensional recursion. At each stage of the construction we will ensure that $c_{i,j}^{\mathcal{M}}$ and $c_{i,j}^{\mathcal{N}}$ are $(2^K)^{k'-j+2}$-similar.
   Let $\mathcal{C}_{0,0} = \{0\}$. Define $c_{0,0}^{\mathcal{M}} : M_0 \longrightarrow \mathcal{C}_{0,0}$ by

$$c_{0,0}^{\mathcal{M}}(x) = 0 \quad \text{for all } x \in M_0.$$

Define $c_{0,0}^{\mathcal{N}} : N_0 \longrightarrow \mathcal{C}_{0,0}$ by

$$c_{0,0}^{\mathcal{N}}(x) = 0 \quad \text{for all } x \in N_0.$$

Let $\Phi_{0,0,0}(x^0)$ be the $\mathcal{L}_{\text{TST}}$-formula $x^0 = x^0$. Note that, for all $x \in N_0$,

$$\mathcal{N} \models \Phi_{0,0,0}[x] \qquad \text{if and only if} \qquad c_{0,0}^{\mathcal{N}}(x) = 0.$$

**Lemma 4.1**      The colorings $c_{0,0}^{\mathcal{M}}$ and $c_{0,0}^{\mathcal{N}}$ are $(2^K)^{k'+2}$-similar.

**Proof**      This follows immediately from the fact that $|M_0| \geq (2^K)^{k'+2}$.      □

We now turn to defining the color classes $\mathcal{C}_{i,0}$ and colorings $c_{i,0}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,0}$ and $c_{i,0}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,0}$ for all $i \in \mathbb{N}$. Suppose that we have defined the color class $\mathcal{C}_{n,0}$ with a canonical ordering, colorings $c_{n,0}^{\mathcal{M}} : M_n \longrightarrow \mathcal{C}_{n,0}$ and $c_{n,0}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{n,0}$, and $\mathcal{L}_{\text{TST}}$-formulae $\Phi_{n,0,\alpha}(x^n)$ for all $\alpha \in \mathcal{C}_{n,0}$ with the following properties:
   (I) $c_{n,0}^{\mathcal{M}}$ and $c_{n,0}^{\mathcal{N}}$ are $(2^K)^{k'+2}$-similar,
   (II) for all $\alpha \in \mathcal{C}_{n,0}$ and for all $x \in N_n$,

$$\mathcal{N} \models \Phi_{n,0,\alpha}[x] \qquad \text{if and only if} \qquad c_{n,0}^{\mathcal{N}}(x) = \alpha.$$

Let $\mathcal{C}_{n,0} = \{\alpha_1, \ldots, \alpha_q\}$ be the enumeration obtained from the canonical ordering. Define $\mathcal{C}_{n+1,0} = 2^{2 \cdot q}$—the set of all 0–1 sequences of length $2 \cdot q$. Define $c_{n+1,0}^{\mathcal{M}} : M_{n+1} \longrightarrow \mathcal{C}_{n+1,0}$ such that, for all $x \in M_{n+1}$,

$$c_{n+1,0}^{\mathcal{M}}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle,$$

where

$$f_i = \begin{cases} 0 & \text{if, for all } y \in M_n, \text{ it holds that if } c_{n,0}^{\mathcal{M}}(y) = \alpha_i, \text{ then } \mathcal{M} \models y \notin_n x, \\ 1 & \text{if there exists } y \in M_n \text{ such that } c_{n,0}^{\mathcal{M}}(y) = \alpha_i \text{ and } \mathcal{M} \models y \in_n x, \end{cases}$$

and

$$g_i = \begin{cases} 0 & \text{if, for all } y \in M_n, \text{ it holds that if } c_{n,0}^{\mathcal{M}}(y) = \alpha_i, \text{ then } \mathcal{M} \models y \in_n x, \\ 1 & \text{if there exists } y \in M_n \text{ such that } c_{n,0}^{\mathcal{M}}(y) = \alpha_i \text{ and } \mathcal{M} \models y \notin_n x. \end{cases}$$

**Example 4.1**    Using this definition we get $\mathcal{C}_{1,0} = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$. There are no $x \in M_1$ which are given the color $\langle 0,0\rangle$ by $c_{1,0}^{\mathcal{M}}$. The only point in $M_1$ which is given the color $\langle 1,0\rangle$ by $c_{1,0}^{\mathcal{M}}$ is $(V^1)^{\mathcal{M}}$. Similarly, the only point in $M_1$ which is given the color $\langle 0,1\rangle$ by $c_{1,0}^{\mathcal{M}}$ is $(\emptyset^1)^{\mathcal{M}}$. Every other point in $M_1$ is given the color $\langle 1,1\rangle$ by $c_{1,0}^{\mathcal{M}}$.

We define the coloring $c_{n+1,0}^{\mathcal{N}} : N_{n+1} \longrightarrow \mathcal{C}_{n+1,0}$ identically. Define $c_{n+1,0}^{\mathcal{N}} :$ $N_{n+1} \longrightarrow \mathcal{C}_{n+1,0}$ such that, for all $x \in N_{n+1}$,

$$c_{n+1,0}^{\mathcal{N}}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle,$$

where

$$f_i = \begin{cases} 0 & \text{if, for all } y \in N_n, \text{ it holds that if } c_{n,0}^{\mathcal{N}}(y) = \alpha_i, \text{ then } \mathcal{N} \models y \notin_n x, \\ 1 & \text{if there exists } y \in N_n \text{ such that } c_{n,0}^{\mathcal{N}}(y) = \alpha_i \text{ and } \mathcal{N} \models y \in_n x, \end{cases}$$

and

$$g_i = \begin{cases} 0 & \text{if, for all } y \in N_n, \text{ it holds that if } c_{n,0}^{\mathcal{N}}(y) = \alpha_i, \text{ then } \mathcal{N} \models y \in_n x, \\ 1 & \text{if there exists } y \in N_n \text{ such that } c_{n,0}^{\mathcal{N}}(y) = \alpha_i \text{ and } \mathcal{N} \models y \notin_n x. \end{cases}$$

We first show that there are $\mathcal{L}_{\mathrm{TST}}$-formulae $\Phi_{n+1,0,\beta}$, for all $\beta \in \mathcal{C}_{n+1,0}$, that satisfy condition (II) above for the coloring $c_{n+1,0}^{\mathcal{N}}$.

**Lemma 4.2**    *For all $\beta \in \mathcal{C}_{n+1,0}$, there is an $\mathcal{L}_{\mathrm{TST}}$-formula $\Phi_{n+1,0,\beta}(x^{n+1})$ such that, for all $x \in N_{n+1}$,*

$$\mathcal{N} \models \Phi_{n+1,0,\beta}[x] \qquad \textit{if and only if} \qquad c_{n+1,0}^{\mathcal{N}}(x) = \beta.$$

**Proof**    For all $1 \le i \le q$, let $\Phi_{n,0,\alpha_i}(x^n)$ be such that, for all $x \in N_n$,

$$\mathcal{N} \models \Phi_{n,0,\alpha_i}[x] \qquad \text{if and only if} \qquad c_{n,0}^{\mathcal{N}}(x) = \alpha_i.$$

Let $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle \in \mathcal{C}_{n+1,0}$. For all $1 \le i \le q$ and $j \in \{0,1\}$ define the $\mathcal{L}_{\mathrm{TST}}$-formula $\Theta_{i,j}^{\beta}(x^{n+1})$ by

$$\Theta_{i,0}^{\beta}(x^{n+1}) \text{ is } \begin{cases} \forall y^n (\Phi_{n,0,\alpha_i}(y^n) \Rightarrow y^n \notin_n x^{n+1}) & \text{if } f_i = 0, \\ \exists y^n (y^n \in_n x^{n+1} \wedge \Phi_{n,0,\alpha_i}(y^n)) & \text{if } f_i = 1, \end{cases}$$

$$\Theta_{i,1}^{\beta}(x^{n+1}) \text{ is } \begin{cases} \forall y^n (\Phi_{n,0,\alpha_i}(y^n) \Rightarrow y^n \in_n x^{n+1}) & \text{if } g_i = 0, \\ \exists y^n (y^n \notin_n x^{n+1} \wedge \Phi_{n,0,\alpha_i}(y^n)) & \text{if } g_i = 1. \end{cases}$$

Define $\Phi_{n+1,0,\beta}(x^{n+1})$ to be the $\mathcal{L}_{\mathrm{TST}}$-formula

$$\bigwedge_{1 \le i \le q} \bigwedge_{j \in \{0,1\}} \Theta_{i,j}^{\beta}(x^{n+1}).$$

It follows from the definition of $c_{n+1,0}^{\mathcal{N}}$ that, for all $x \in N_{n+1}$,

$$\mathcal{N} \models \Phi_{n+1,0,\beta}[x] \qquad \text{if and only if} \qquad c_{n+1,0}^{\mathcal{N}}(x) = \beta. \qquad \square$$

We now turn to showing that $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ are $(2^K)^{k'+2}$-similar. To prove this we introduce the following sets:

$$\mathrm{FOR}_n = \left\{ i \in [q] \mid \alpha_i \text{ is forbidden w.r.t. } c_{n,0}^{\mathcal{M}} \text{ and } c_{n,0}^{\mathcal{N}} \right\},$$

$$m\text{-}\mathrm{SPC}_n = \left\{ i \in [q] \mid \alpha_i \text{ is } m\text{-special w.r.t. } c_{n,0}^{\mathcal{M}} \text{ and } c_{n,0}^{\mathcal{N}} \right\} \quad \text{for } 1 \le m < (2^K)^{k'+2},$$

$$\mathrm{ABN}_n = \left\{ i \in [q] \mid \alpha_i \text{ is } (2^K)^{k'+2}\text{-abundant w.r.t. } c_{n,0}^{\mathcal{M}} \text{ and } c_{n,0}^{\mathcal{N}} \right\}.$$

We classify the colors in $\mathcal{C}_{n+1,0}$ which are forbidden, 1-special, and abundant with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$.

**Lemma 4.3**     *Let $\beta \in \mathcal{C}_{n+1,0}$ with $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$. The color $\beta$ is forbidden with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ if and only if either*

    *(i) there exists an $i \in [q]$ with $i \notin \mathrm{FOR}_n$ such that $f_i = g_i = 0$, or*
    *(ii) there exists an $i \in 1\text{-}\mathrm{SPC}_n$ such that $f_i = g_i = 1$, or*
    *(iii) there exists an $i \in \mathrm{FOR}_n$ such that $f_i = 1$ or $g_i = 1$.*

**Proof**     It is clear that if any of the conditions (i)–(iii) hold, then the color $\beta$ is forbidden. Conversely, suppose that none of the conditions (i)–(iii) hold. We need to show that $\beta$ is not forbidden with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$. We first construct a point in $\mathcal{N}$ that is given color $\beta$ by $c_{n+1,0}^{\mathcal{N}}$. For all $1 \le i \le q$, let $\Phi_{n,0,\alpha_i}(x^n)$ be such that, for all $x \in N_n$,

$$\mathcal{N} \models \Phi_{n,0,\alpha_i}[x] \qquad \text{if and only if} \qquad c_{n,0}^{\mathcal{N}}(x) = \alpha_i.$$

Let $\Theta_1(x^n)$ be the $\mathcal{L}_{\mathrm{TST}}$-formula

$$\bigvee_{g_i = 0} \Phi_{n,0,\alpha_i}(x^n).$$

We work inside $\mathcal{N}$. Let $X_1 = \{x^n \mid \Theta_1(x^n)\}$. Note that comprehension ensures that $X_1$ exists. Let

$$B = \mathrm{ABN}_n \cup \bigcup_{2 \le m < (2^K)^{k'+2}} m\text{-}\mathrm{SPC}_n,$$

and let $A = \{i \in B \mid f_i = g_i = 1\}$. Let $\Theta_2(x^n)$ be the $\mathcal{L}_{\mathrm{TST}}$-formula

$$\bigvee_{i \in A} \Phi_{n,0,\alpha_i}(x^n).$$

Let $X_2 = \{x^n \mid \Theta_2(x^n)\}$. Again, comprehension ensures that $X_2$ exists. For all $i \in A$, let $x_i \in N_n$ be such that $c_{n,0}^{\mathcal{N}}(x_i) = \alpha_i$. Now, let $X = X_1 \cup (X_2 \setminus \{x_i \mid i \in A\})$. Comprehension guarantees that $X$ exists in $\mathcal{N}$, and our construction ensures that $c_{n+1,0}^{\mathcal{N}}(X) = \beta$. An identical construction shows that if none of the conditions (i)–(iii) hold, then there is a point $X$ in $\mathcal{M}$ such that $c_{n+1,0}^{\mathcal{M}}(X) = \beta$.     $\square$

**Lemma 4.4**     *Let $\beta \in \mathcal{C}_{n+1,0}$ with $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$. The color $\beta$ is 1-special with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ if and only if $\beta$ is not forbidden with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ and, for all $i \in [q]$ with $i \notin \mathrm{FOR}_n$, $f_i = 0$ or $g_i = 0$.*

**Proof**     Suppose that $\beta$ is not forbidden with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$, and suppose that, for all $i \in [q]$ with $i \notin \mathrm{FOR}_n$, $f_i = 0$ or $g_i = 0$. If $x$ is a point that is given color $\beta$ by $c_{n+1,0}^{\mathcal{M}}$ or $c_{n+1,0}^{\mathcal{N}}$, then $x$ is completely determined in $\mathcal{M}$ or $\mathcal{N}$, respectively. Therefore, $\beta$ is 1-special.

Conversely, suppose that $\beta$ is not forbidden, and suppose that there exists an $i \in [q]$ with $i \notin \mathrm{FOR}_n$ such that $f_i = g_i = 1$. We will show that $\beta$ is not 1-special with respect to $c_{n+1,0}^{\mathcal{M}}$ or $c_{n+1,0}^{\mathcal{N}}$. We first construct two distinct points of $\mathcal{N}$ that are given color $\beta$ by $c_{n+1,0}^{\mathcal{N}}$. For all $1 \leq i \leq q$, let $\Phi_{n,0,\alpha_i}(x^n)$ be such that, for all $x \in N_n$,

$$\mathcal{N} \models \Phi_{n,0,\alpha_i}[x] \qquad \text{if and only if} \qquad c_{n,0}^{\mathcal{N}}(x) = \alpha_i.$$

We work inside $\mathcal{N}$. Let $A = \{i \in [q] \mid f_i = g_i = 1\}$. Since $\beta$ is not forbidden, for all $i \in A$, we can find $x_i, y_i \in N_n$ such that $c_{n,0}^{\mathcal{N}}(x_i) = c_{n,0}^{\mathcal{N}}(y_i) = \alpha_i$ and $x_i \neq y_i$. Let $\Theta_1(x^n)$ be the $\mathscr{L}_{\mathrm{TST}}$-formula

$$\bigvee_{g_i = 0} \Phi_{n,0,\alpha_i}(x^n).$$

Let $\Theta_2(x^n)$ be the $\mathscr{L}_{\mathrm{TST}}$-formula

$$\bigvee_{i \in A} \Phi_{n,0,\alpha_i}(x^n).$$

Let $X_1 = \{x^n \mid \Theta_1(x^n)\}$, and let $X_2 = \{x^n \mid \Theta_2(x^n)\}$. Comprehension guarantees that both $X_1$ and $X_2$ exist. Let $X = X_1 \cup (X_2 \backslash \{x_i \mid i \in A\})$, and let $Y = X_1 \cup (X_2 \backslash \{y_i \mid i \in A\})$. Now, this construction ensures that $c_{n+1,0}^{\mathcal{N}}(X) = c_{n+1,0}^{\mathcal{N}}(Y) = \beta$ and $X \neq Y$. Therefore, $\beta$ is not 1-special with respect to $c_{n+1,0}^{\mathcal{N}}$. An identical construction shows that $\beta$ is not 1-special with respect to $c_{n+1,0}^{\mathcal{M}}$. □

**Lemma 4.5** *Let $\beta \in \mathscr{C}_{n+1,0}$ with $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$. If $\beta$ is not forbidden with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ and there exists an $i \in \mathrm{ABN}_n$ such that $f_i = g_i = 1$, then $\beta$ is $(2^K)^{k'+2}$-abundant with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$.*

**Proof** Suppose that $\beta$ is not forbidden with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$, and suppose that there exists an $i \in \mathrm{ABN}_n$ such that $f_i = g_i = 1$. We first construct $(2^K)^{k'+2}$ distinct points in $\mathcal{N}$ that are given color $\beta$ by $c_{n+1,0}^{\mathcal{N}}$. For all $1 \leq i \leq q$, let $\Phi_{n,0,\alpha_i}(x^n)$ be such that, for all $x \in N_n$,

$$\mathcal{N} \models \Phi_{n,0,\alpha_i}[x] \qquad \text{if and only if} \qquad c_{n,0}^{\mathcal{N}}(x) = \alpha_i.$$

We work inside $\mathcal{N}$. Let $u \in \mathrm{ABN}_n$ be such that $f_u = g_u = 1$. Let $A = \{i \in [q] \mid f_i = g_i = 1\}$. For all $i \in A$ with $i \neq u$, let $x_i \in N_n$ be such that $c_{n,0}^{\mathcal{N}}(x_i) = \alpha_i$. Let $y_1, \ldots, y_{(2^K)^{k'+2}} \in N_n$ be such that, for all $1 \leq v \leq (2^K)^{k'+2}$, $c_{n,0}^{\mathcal{N}}(y_v) = \alpha_u$ and, for all $1 \leq v_1 < v_2 \leq (2^K)^{k'+2}$, $y_{v_1} \neq y_{v_2}$. Let $\Theta_1(x^n)$ be the $\mathscr{L}_{\mathrm{TST}}$-formula

$$\bigvee_{g_i = 0} \Phi_{n,0,\alpha_i}(x^n).$$

Let $\Theta_2(x^n)$ be the $\mathscr{L}_{\mathrm{TST}}$-formula

$$\bigvee_{i \in A} \Phi_{n,0,\alpha_i}(x^n).$$

Let $X_1 = \{x^n \mid \Theta_1(x^n)\}$, and let $X_2 = \{x^n \mid \Theta_2(x^n)\}$. Comprehension guarantees that $X_1$ and $X_2$ exist. For all $1 \leq v \leq (2^K)^{k'+2}$, let

$$Y_v = X_1 \cup \big( X_2 \backslash (\{x_i \mid i \in A \wedge i \neq u\} \cup \{y_v\}) \big).$$

This construction ensures that, for all $1 \leq v_1 < v_2 \leq (2^K)^{k'+2}$, $Y_{v_1} \neq Y_{v_2}$ and, for all $1 \leq v \leq (2^K)^{k'+2}$, $c_{n+1,0}^{\mathcal{N}}(Y_v) = \beta$. Therefore, $\beta$ is $(2^K)^{k'+2}$-abundant with respect to $c_{n+1,0}^{\mathcal{N}}$. An identical construction shows that $\beta$ is $(2^K)^{k'+2}$-abundant with respect to $c_{n+1,0}^{\mathcal{M}}$.                                                            □

This allows us to show that the colorings $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ are $(2^K)^{k'+2}$-similar.

**Lemma 4.6**     *The colorings $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ are $(2^K)^{k'+2}$-similar.*

**Proof**     Lemma 4.3 shows that, for all $\beta \in \mathcal{C}_{n+1,0}$,

> $\beta$ is forbidden w.r.t. $c_{n+1,0}^{\mathcal{M}}$        if and only if        $\beta$ is forbidden w.r.t. $c_{n+1,0}^{\mathcal{N}}$.

Lemma 4.4 shows that, for all $\beta \in \mathcal{C}_{n+1,0}$,

> $\beta$ is 1-special w.r.t. $c_{n+1,0}^{\mathcal{M}}$        if and only if        $\beta$ is 1-special w.r.t. $c_{n+1,0}^{\mathcal{N}}$.

Let $\beta \in \mathcal{C}_{n+1,0}$ with $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$. Lemma 4.5 shows that if $\beta$ is not forbidden with respect to $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ and there is an $i \in \mathrm{ABN}_n$ such that $f_i = g_i = 1$, then $\beta$ is $(2^K)^{k'+2}$-abundant with respect to both $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$. The remaining case is if $\beta$ is not forbidden or 1-special and, for all $i \in \mathrm{ABN}_n$, $f_i = 0$ or $g_i = 0$. Let

$$B = \bigcup_{2 \leq m < (2^K)^{k'+2}} m\text{-SPC}.$$

In this case the number of $x \in M_{n+1}$ ($\in N_{n+1}$, resp.) with color $\beta$ is completely determined by the number of $y \in M_n$ ($\in N_n$, resp.) with color $\alpha_i$ such that $i \in B$ and $f_i = g_i = 1$. Therefore, the colorings $c_{n+1,0}^{\mathcal{M}}$ and $c_{n+1,0}^{\mathcal{N}}$ are $(2^K)^{k'+2}$-similar.     □

Therefore, by induction, for all $i \in \mathbb{N}$, the colorings $c_{i,0}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,0}$ and $c_{i,0}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,0}$ are $(2^K)^{k'+2}$-similar.

We now turn to defining the color classes $\mathcal{C}_{i,j}$ and the colorings $c_{i,j}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,j}$ and $c_{i,j}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,j}$ for $1 \leq j \leq k'$ and $i \in \mathbb{N}$. Let $0 \leq n < k'$. Suppose that the color classes $\mathcal{C}_{i,n}$ have been defined for all $i \in \mathbb{N}$, and suppose that each of these color classes has a canonical ordering. Let $j' = \sum_{1 \leq m \leq n} K_m$, and suppose that $b_1^{r_1}, \ldots, b_{j'}^{r_{j'}} \in \mathcal{N}$ have been chosen. Moreover, suppose that, for all $i \in \mathbb{N}$ and for all $\alpha \in \mathcal{C}_{i,n}$, the colorings $c_{i,n}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,n}$ and $c_{i,n}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,n}$ and the $\mathcal{L}_{\mathrm{TST}}$-formulae $\Phi_{i,n,\alpha}(x^i, \vec{z})$ have been defined with the following properties:

(I′)  $c_{i,n}^{\mathcal{M}}$ and $c_{i,n}^{\mathcal{N}}$ are $(2^K)^{k'-n+2}$-similar;
(II′)  for all $x \in N_i$,

$$\mathcal{N} \models \Phi_{i,n,\alpha}[x, b_1^{r_1}, \ldots, b_{j'}^{r_{j'}}] \qquad \text{if and only if} \qquad c_{i,n}^{\mathcal{N}}(x) = \alpha.$$

Observe that $r_{j'+1} = \cdots = r_{j'+K_{n+1}}$, and let $r = r_{j'+1}$. We will define the color classes $\mathcal{C}_{i,n+1}$ and colorings $c_{i,n+1}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,n+1}$ and $c_{i,n+1}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,n+1}$ such that, for all $i \in \mathbb{N}$, $c_{i,n+1}^{\mathcal{M}}$ and $c_{i,n+1}^{\mathcal{N}}$ are $(2^K)^{k'-n+1}$-similar and the coloring $c_{i,n+1}^{\mathcal{N}}$ is definable in $\mathcal{N}$. In the process of achieving this goal we will identify points $b_{j'+1}^r, \ldots, b_{j'+K_{n+1}}^r \in N_r$.

For all $0 \leq i < r - 1$, define

$$\mathcal{C}_{i,n+1} = \mathcal{C}_{i,n},$$
$$c_{i,n+1}^{\mathcal{M}} = c_{i,n}^{\mathcal{M}},$$
$$c_{i,n+1}^{\mathcal{N}} = c_{i,n}^{\mathcal{N}}.$$

We now define the color class $\mathcal{C}_{r-1,n+1}$ and the colorings $c_{r-1,n+1}^{\mathcal{M}} : M_{r-1} \longrightarrow$
$\mathcal{C}_{r-1,n+1}$ and $c_{r-1,n+1}^{\mathcal{N}} : N_{r-1} \longrightarrow \mathcal{C}_{r-1,n+1}$. Let $\mathcal{C}_{r-2,n+1} = \mathcal{C}_{r-2,n} = $
$\{\alpha_1, \ldots, \alpha_q\}$ be obtained from the canonical ordering. Consider $a_{j'+1}^r, \ldots,$
$a_{j'+K_{n+1}}^r \in M_r$, and use $\bar{a}_1, \ldots, \bar{a}_{K_{n+1}}$ to denote this sequence of elements. Define
$\mathcal{C}_{r-1,n+1} = 2^{K_{n+1}} \times \mathcal{C}_{r-1,n}$—the set of all 0–1 sequences of length $K_{n+1} + 2 \cdot q$.
Define $c_{r-1,n+1}^{\mathcal{M}} : M_{r-1} \longrightarrow \mathcal{C}_{r-1,n+1}$ such that, for all $x \in M_{r-1}$,

$$c_{r-1,n+1}^{\mathcal{M}}(x) = \langle F_1, \ldots, F_{K_{n+1}}, f_1, \ldots, f_q, g_1, \ldots, g_q \rangle,$$

where

$$c_{r-1,n}^{\mathcal{M}}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$$

and

$$F_p = \begin{cases} 0 & \text{if } \mathcal{M} \models x \notin_{r-1} \bar{a}_p, \\ 1 & \text{if } \mathcal{M} \models x \in_{r-1} \bar{a}_p, \end{cases} \quad \text{for all } 1 \leq p \leq K_{n+1}.$$

**Lemma 4.7**    *There exists $\bar{b}_1, \ldots, \bar{b}_{K_{n+1}} \in N_r$ such that $c_{r-1,n+1}^{\mathcal{M}}$ and the coloring*
$c_{r-1,n+1}^{\mathcal{N}} : N_{r-1} \longrightarrow \mathcal{C}_{r-1,n+1}$, *defined such that, for all $x \in N_{r-1}$,*

$$c_{r-1,n+1}^{\mathcal{N}}(x) = \langle F_1, \ldots, F_{K_{n+1}}, f_1, \ldots, f_q, g_1, \ldots, g_q \rangle,$$
$$\text{where } c_{r-1,n}^{\mathcal{N}}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$$
$$\text{and } F_p = \begin{cases} 0 & \text{if } \mathcal{N} \models x \notin_{r-1} \bar{b}_p, \\ 1 & \text{if } \mathcal{N} \models x \in_{r-1} \bar{b}_p, \end{cases} \quad \text{for all } 1 \leq p \leq K_{n+1}, \qquad (2)$$

*are $(2^K)^{k'-n+1}$-similar.*

**Proof**    Let $\mathcal{C}_{r-1,n} = \{\alpha_1, \ldots, \alpha_{q'}\}$ be obtained from the canonical ordering. For
all $1 \leq i \leq q'$ and for all $\sigma \in 2^{K_{n+1}}$, define $X_\sigma^i \subseteq M_{r-1}$ by

$$X_\sigma^i = \{x \in M_{r-1} \mid \left(c_{r-1,n}^{\mathcal{M}}(x) = \alpha_i\right)$$
$$\wedge \left(\forall v \in K_{n+1}\right)\left(\sigma(v) = 1 \iff \mathcal{M} \models x \in_{r-1} \bar{a}_v\right)\}.$$

Note that, for all $1 \leq i \leq q'$, the sets $\langle X_\sigma^i \mid \sigma \in 2^{K_{n+1}} \rangle$ partition the elements of
$M_{r-1}$ that are given color $\alpha_i$ by $c_{r-1,n}^{\mathcal{M}}$ into $2^{K_{n+1}}$ pieces. For each $1 \leq i \leq q'$
choose a sequence $\langle Z_\sigma^i \mid \sigma \in 2^{K_{n+1}} \rangle$ such that, for all $\sigma \in 2^{K_{n+1}}$,

(i) $Z_\sigma^i \in N_r$;
(ii) for all $z \in N_{r-1}$ with $\mathcal{N} \models (z \in_{r-1} Z_\sigma^i)$, $c_{r-1,n}^{\mathcal{N}}(z) = \alpha_i$;
(iii) if $|X_\sigma^i| < (2^K)^{k'-n+1}$, then $|\{z \in \mathcal{N} \mid \mathcal{N} \models z \in_{r-1} Z_\sigma^i\}| = |X_\sigma^i|$;
(iv) if $|X_\sigma^i| \geq (2^K)^{k'-n+1}$, then $|\{z \in \mathcal{N} \mid \mathcal{N} \models z \in_{r-1} Z_\sigma^i\}| \geq (2^K)^{k'-n+1}$.

To see that we can make this choice, we work inside $\mathcal{N}$. For all $1 \leq i \leq q'$, let
$\Phi_{r-1,n,\alpha_i}(x^{r-1}, \vec{z})$ be such that, for all $x \in N_{r-1}$,

$$\mathcal{N} \models \Phi_{r-1,n,\alpha_i}[x, b_1^{r_1}, \ldots, b_{j'}^{r_{j'}}] \quad \text{if and only if} \quad c_{r-1,n}^{\mathcal{N}}(x) = \alpha_i.$$

For all $1 \leq i \leq q'$, let $W_i = \{x^{r-1} \mid \Phi_{r-1,n,\alpha_i}(x^{r-1}, b_1^{r_1}, \ldots, b_{j'}^{r_{j'}})\}$. Comprehension ensures that the $W_i$'s exist. For all $1 \leq i \leq q'$ and for all $\sigma \in 2^{K_{n+1}}$, $Z_\sigma^i$ can be chosen to be a finite or cofinite subset of $W_i$. Moreover, the fact that $c_{r-1,n}^{\mathcal{M}}$ and $c_{r-1,n}^{\mathcal{N}}$ are $(2^K)^{k'-n+2}$-similar ensures that for all $1 \leq i \leq q'$ we can choose the sequence $\langle Z_\sigma^i \mid \sigma \in 2^{K_{n+1}} \rangle$ to satisfy condition (iii) above.

Now, for all $1 \leq p \leq K_{n+1}$, let $\bar{b}_p \in N_r$ be such that

$$\mathcal{N} \models \bar{b}_p = \bigcup_{1 \leq i \leq q'} \bigcup_{\substack{\sigma \in 2^{K_{n+1}} \\ \text{s.t. } \sigma(p)=1}} Z_\sigma^i.$$

This construction ensures that the colorings $c_{r-1,n+1}^{\mathcal{M}}$ and $c_{r-1,n+1}^{\mathcal{N}}$ defined by (2) are $(2^K)^{k'-n+1}$-similar. $\qquad\square$

Let $b_{j'+1}^r, \ldots, b_{j'+K_{n+1}}^r \in \mathcal{N}$ be the points $\bar{b}_1, \ldots, \bar{b}_{K_{n+1}}$ produced in the proof of Lemma 4.7, and let $c_{r-1,n+1}^{\mathcal{N}}$ be defined by (2). Therefore, $c_{r-1,n+1}^{\mathcal{M}}$ and $c_{r-1,n+1}^{\mathcal{N}}$ are $(2^K)^{k'-n+1}$-similar. We can immediately observe that the coloring $c_{r-1,n+1}^{\mathcal{N}}$ is definable in $\mathcal{N}$ by an $\mathscr{L}_{\text{TST}}$-formula with parameters $b_1^{r_1}, \ldots, b_{j'+K_{n+1}}^{r_{j'+K_{n+1}}}$.

**Lemma 4.8** *For all $\alpha \in \mathcal{C}_{r-1,n+1}$, there exists an $\mathscr{L}_{\text{TST}}$-formula $\Phi_{r-1,n+1,\alpha}(x^{r-1}, \vec{z})$ such that, for all $x \in N_{r-1}$,*

$$\mathcal{N} \models \Phi_{r-1,n+1,\alpha}[x, b_1^{r_1}, \ldots, b_{j'+K_{n+1}}^{r_{j'+K_{n+1}}}] \qquad \text{if and only if} \qquad c_{r-1,n+1}^{\mathcal{N}}(x) = \alpha.$$

Let $t = \sum_{1 \leq m \leq n+1} K_m$. Lemmas 4.7 and 4.8 show that we can define colorings $c_{r-1,n+1}^{\mathcal{M}}$ and $c_{r-1,n+1}^{\mathcal{N}}$ and $\mathscr{L}_{\text{TST}}$-formulae $\Phi_{r-1,n+1,\alpha}(x^{r-1}, \vec{z})$ for all $\alpha \in \mathcal{C}_{r-1,n+1}$ which satisfy the following properties:

(I″) $c_{r-1,n+1}^{\mathcal{M}}$ and $c_{r-1,n+1}^{\mathcal{N}}$ are $(2^K)^{k'-n+1}$-similar;

(II″) for all $x \in N_{r-1}$,

$$\mathcal{N} \models \Phi_{r-1,n+1,\alpha}[x, b_1^{r_1}, \ldots, b_t^{r_t}] \qquad \text{if and only if} \qquad c_{r-1,n+1}^{\mathcal{N}}(x) = \alpha.$$

We now turn to defining the color classes $\mathcal{C}_{i,n+1}$ and the colorings $c_{i,n+1}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,n+1}$ and $c_{i,n+1}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,n+1}$ for all $i \geq r$. Let $i \geq r-1$. Suppose that the color class $\mathcal{C}_{i,n+1}$ has been defined with a canonical ordering. Suppose, also, that the colorings $c_{i,n+1}^{\mathcal{M}} : M_i \longrightarrow \mathcal{C}_{i,n+1}$ and $c_{i,n+1}^{\mathcal{N}} : N_i \longrightarrow \mathcal{C}_{i,n+1}$ and the $\mathscr{L}_{\text{TST}}$-formulae $\Phi_{i,n+1,\alpha}(x^i, \vec{z})$ have been defined and satisfy the following properties:

(I‴) $c_{i,n+1}^{\mathcal{M}}$ and $c_{i,n+1}^{\mathcal{N}}$ are $(2^K)^{k'-n+1}$-similar;

(II‴) for all $x \in N_i$,

$$\mathcal{N} \models \Phi_{i,n+1,\alpha}[x, b_1^{r_1}, \ldots, b_t^{r_t}] \qquad \text{if and only if} \qquad c_{i,n+1}^{\mathcal{N}}(x) = \alpha.$$

We "lift" the color class $\mathcal{C}_{i,n+1}$ and the colorings $c_{i,n+1}^{\mathcal{M}}$ and $c_{i,n+1}^{\mathcal{N}}$ in the same way that we "lifted" the color classes $\mathcal{C}_{i,0}$ and the colorings $c_{i,0}^{\mathcal{M}}$ and $c_{i,0}^{\mathcal{N}}$ above. Let $\mathcal{C}_{i,n+1} = \{\alpha_1, \ldots, \alpha_q\}$ be obtained from the canonical ordering. Define $\mathcal{C}_{i+1,n+1} = 2^{2 \cdot q}$—the set of all 0–1 sequences of length $2 \cdot q$. Define $c_{i+1,n+1}^{\mathcal{M}} : M_{i+1} \longrightarrow \mathcal{C}_{i+1,n+1}$ such that, for all $x \in M_{i+1}$,

$$c_{i+1,n+1}^{\mathcal{M}}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle,$$

where

$$f_p = \begin{cases} 0 & \text{if, for all } y \in M_i, \text{ it holds that if } c^{\mathcal{M}}_{i,n+1}(y) = \alpha_p, \text{ then } \mathcal{M} \models y \notin_i x, \\ 1 & \text{if there exists } y \in M_i \text{ such that } c^{\mathcal{M}}_{i,n+1}(y) = \alpha_p \text{ and } \mathcal{M} \models y \in_i x, \end{cases}$$

and

$$g_p = \begin{cases} 0 & \text{if, for all } y \in M_i, \text{ it holds that if } c^{\mathcal{M}}_{i,n+1}(y) = \alpha_p, \text{ then } \mathcal{M} \models y \in_i x, \\ 1 & \text{if there exists } y \in M_i \text{ such that } c^{\mathcal{M}}_{i,n+1}(y) = \alpha_p \text{ and } \mathcal{M} \models y \notin_i x. \end{cases}$$

Again, we define $c^{\mathcal{N}}_{i+1,n+1}$ identically. Define $c^{\mathcal{N}}_{i+1,n+1} : N_{i+1} \longrightarrow \mathcal{C}_{i+1,n+1}$ such that, for all $x \in N_{i+1}$,

$$c^{\mathcal{N}}_{i+1,n+1}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle,$$

where

$$f_p = \begin{cases} 0 & \text{if, for all } y \in N_i, \text{ it holds that if } c^{\mathcal{N}}_{i,n+1}(y) = \alpha_p, \text{ then } \mathcal{N} \models y \notin_i x, \\ 1 & \text{if there exists } y \in N_i \text{ such that } c^{\mathcal{N}}_{i,n+1}(y) = \alpha_p \text{ and } \mathcal{N} \models y \in_i x, \end{cases}$$

and

$$g_p = \begin{cases} 0 & \text{if, for all } y \in N_i, \text{ it holds that if } c^{\mathcal{N}}_{i,n+1}(y) = \alpha_p, \text{ then } \mathcal{N} \models y \in_i x, \\ 1 & \text{if there exists } y \in N_i \text{ such that } c^{\mathcal{N}}_{i,n+1}(y) = \alpha_p \text{ and } \mathcal{N} \models y \notin_i x. \end{cases}$$

We first observe that there exist $\mathcal{L}_{\text{TST}}$-formulae $\Phi_{i+1,n+1,\beta}(x^{i+1}, \vec{z})$ for each $\beta \in \mathcal{C}_{i+1,n+1}$ which witness the fact that the coloring $c^{\mathcal{N}}_{i+1,n+1}$ satisfies condition (II‴).

**Lemma 4.9** *For all $\beta \in \mathcal{C}_{i+1,n+1}$, there is an $\mathcal{L}_{\text{TST}}$-formula $\Phi_{i+1,n+1,\beta}(x^{i+1}, \vec{z})$ such that, for all $x \in N_{i+1}$,*

$$\mathcal{N} \models \Phi_{i+1,n+1,\beta}[x, b_1^{r_1}, \ldots, b_t^{r_t}] \qquad \text{if and only if} \qquad c^{\mathcal{N}}_{i+1,n+1}(x) = \beta.$$

**Proof** This is identical to the proof of Lemma 4.2 using the fact that $c^{\mathcal{N}}_{i,n+1}$ satisfies condition (II‴). □

We now turn to showing that $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$ are $(2^K)^{k'-n+1}$-similar. To do this we prove analogues of Lemmas 4.3, 4.4, and 4.5:

$$\text{FOR}^{n+1}_i = \{ v \in [q] \mid \alpha_v \text{ is forbidden w.r.t. } c^{\mathcal{M}}_{i,n+1} \text{ and } c^{\mathcal{N}}_{i,n+1} \},$$

$$m\text{-SPC}^{n+1}_i = \{ v \in [q] \mid \alpha_v \text{ is } m\text{-special w.r.t. } c^{\mathcal{M}}_{i,n+1} \text{ and } c^{\mathcal{N}}_{i,n+1} \}$$

$$\text{for } 1 \leq m < (2^K)^{k'-n+1},$$

$$\text{ABN}^{n+1}_i = \{ v \in [q] \mid \alpha_v \text{ is } (2^K)^{k'-n+1}\text{-abundant w.r.t. } c^{\mathcal{M}}_{i,n+1} \text{ and } c^{\mathcal{N}}_{i,n+1} \}.$$

**Lemma 4.10** *Let $\beta \in \mathcal{C}_{i+1,n+1}$ with $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$. The color $\beta$ is forbidden with respect to $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$ if and only if either*

   (i) *there exists a $v \in [q]$ with $v \notin \text{FOR}^{n+1}_i$ such that $f_v = g_v = 0$, or*
   (ii) *there exists a $v \in 1\text{-SPC}^{n+1}_i$ such that $f_v = g_v = 1$, or*
   (iii) *there exists a $v \in \text{FOR}^{n+1}_i$ such that $f_v = 1$ or $g_v = 1$.*

**Proof** This is identical to the proof of Lemma 4.3. □

**Lemma 4.11**    *Let $\beta \in \mathcal{C}_{i+1,n+1}$ with $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$. The color $\beta$ is 1-special with respect to $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$ if and only if $\beta$ is not forbidden with respect to $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$ and, for all $v \in [q]$ with $v \notin \mathrm{FOR}^{n+1}_i$, $f_v = 0$ or $g_v = 0$.*

**Proof**    This is identical to the proof of Lemma 4.4.                    □

**Lemma 4.12**    *Let $\beta \in \mathcal{C}_{i+1,n+1}$ with $\beta = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$. If $\beta$ is not forbidden with respect to $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$ and there exists a $v \in \mathrm{ABN}^{n+1}_i$ with $f_v = g_v = 1$, then $\beta$ is $(2^K)^{k'-n+1}$-abundant with respect to $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$.*

**Proof**    This is identical to the proof of Lemma 4.5.                    □

These results allow us to show that $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$ are $(2^K)^{k'-n+1}$-similar.

**Lemma 4.13**    *The colorings $c^{\mathcal{M}}_{i+1,n+1}$ and $c^{\mathcal{N}}_{i+1,n+1}$ are $(2^K)^{k'-n+1}$-similar.*

**Proof**    This is identical to the proof of Lemma 4.6 when using Lemmas 4.10, 4.11, and 4.12.                    □

This recursion allows us to define the color classes $\mathcal{C}_{n,k'}$ and colorings $c^{\mathcal{M}}_{n,k'}$ and $c^{\mathcal{N}}_{n,k'}$, for all $n \in \mathbb{N}$, and elements $b^{r_1}_1, \ldots, b^{r_k}_1 \in \mathcal{N}$. The above arguments show that, for all $n \in \mathbb{N}$, $c^{\mathcal{M}}_{n,k'}$ and $c^{\mathcal{N}}_{n,k'}$ are $2^K$-similar. We have constructed the colorings $c^{\mathcal{M}}_{n,k'}$ and $c^{\mathcal{N}}_{n,k'}$ so that the color assigned to a point $x \in \mathcal{M}$ (or $\mathcal{N}$) completely captures the set of quantifier-free formulae with parameters $a^{r_1}_1, \ldots, a^{r_k}_k$ (resp., $b^{r_1}_1, \ldots, b^{r_k}_k$) that are satisfied by $x$.

**Lemma 4.14**    *Let $n \in \mathbb{N}$, and let $\theta(x^{r_1}_1, \ldots, x^{r_k}_k, x^n)$ be a quantifier-free $\mathcal{L}_{\mathrm{TST}}$-formula. If $x \in M_n$ and $y \in N_n$ are such that $c^{\mathcal{M}}_{n,k'}(x) = c^{\mathcal{M}}_{n,k'}(y)$, then*

$$\mathcal{M} \models \theta[a^{r_1}_1, \ldots, a^{r_k}_k, x] \qquad \text{if and only if} \qquad \mathcal{N} \models \theta[b^{r_1}_1, \ldots, b^{r_k}_k, y].$$

**Proof**    This follows immediately from the definition of the colorings $c^{\mathcal{M}}_{n,k'}$ and $c^{\mathcal{N}}_{n,k'}$.                    □

Our construction also ensures that if $x \in M_{n+1}$ (or $N_{n+1}$), then the color assigned to $x$ by $c^{\mathcal{M}}_{n+1,k'}$ (resp., $c^{\mathcal{N}}_{n+1,k'}$) tells us, for all $\alpha \in \mathcal{C}_{n,k'}$, whether there exists a point $y \in M_n$ (resp., $N_n$) such that $c^{\mathcal{M}}_{n,k'}(y) = \alpha$ (resp., $c^{\mathcal{N}}_{n,k'}(y) = \alpha$) and $y$ is in the relationship $\in_n$ or $\notin_n$ to $x$ in $\mathcal{M}$ (resp., $\mathcal{N}$).

**Lemma 4.15**    *Let $x \in M_{n+1}$ and $y \in N_{n+1}$, and let $\alpha \in \mathcal{C}_{n,k'}$. If $c^{\mathcal{M}}_{n+1,k'}(x) = c^{\mathcal{N}}_{n+1,k'}(y)$, then*

$$(\exists z \in M_n)\big(c^{\mathcal{M}}_{n,k'}(z) = \alpha \wedge \mathcal{M} \models z \in_n x\big) \qquad \text{if and only if}$$
$$(\exists z \in N_n)\big(c^{\mathcal{N}}_{n,k'}(z) = \alpha \wedge \mathcal{N} \models z \in_n y\big),$$

*and*

$$(\exists z \in M_n)\big(c^{\mathcal{M}}_{n,k'}(z) = \alpha \wedge \mathcal{M} \models z \notin_n x\big) \qquad \text{if and only if}$$
$$(\exists z \in N_n)\big(c^{\mathcal{N}}_{n,k'}(z) = \alpha \wedge \mathcal{N} \models z \notin_n y\big).$$

**Proof**    This follows immediately from the definition of the colorings $c_{n+1,k'}^{\mathcal{M}}$ and $c_{n+1,k'}^{\mathcal{N}}$.    □

This allows us to show that an $\mathcal{L}_{\mathrm{TST}}$-sentence of form (A) or (B) which is true in $\mathcal{N}$ is also true in $\mathcal{M}$.

**Theorem 4.16**    Let $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^s \cdots \exists y_l^s \theta$ be an $\mathcal{L}_{\mathrm{TST}}$-formula with $\theta$ quantifier-free. If $\mathcal{N} \models \phi$, then $\mathcal{M} \models \phi$.

**Proof**    Suppose that $\mathcal{N} \models \phi$. Let $a_1^{r_1}, \ldots, a_k^{r_k} \in \mathcal{M}$. Using $a_1^{r_1}, \ldots, a_k^{r_k}$ and the construction we presented above we can define the color classes $\mathcal{C}_{n,k'}$ and colorings $c_{n,k'}^{\mathcal{M}}$ and $c_{n,k'}^{\mathcal{N}}$, for all $n \in \mathbb{N}$, and elements $b_1^{r_1}, \ldots, b_k^{r_k} \in \mathcal{N}$. The colorings $c_{n,k'}^{\mathcal{M}}$ and $c_{n,k'}^{\mathcal{N}}$ are $2^K$-similar and satisfy Lemma 4.14. Let $e_1, \ldots, e_l \in N_s$ be such that

$$\mathcal{N} \models \theta[b_1^{r_1}, \ldots, b_k^{r_k}, e_1, \ldots, e_l].$$

For all $1 \le i \le l$, let $d_i \in M_s$ be such that $c_{s,k'}^{\mathcal{M}}(d_i) = c_{s,k'}^{\mathcal{N}}(e_i)$ and, for all $1 \le j < i$, $d_j \ne d_i$ if and only if $e_i \ne e_j$. The fact that $l < 2^K$ and the fact that $c_{s,k'}^{\mathcal{M}}$ and $c_{s,k'}^{\mathcal{N}}$ are $2^K$-similar ensure we can find $d_1, \ldots, d_l \in M_s$ satisfying these conditions. Now, since the variables $y_1^s, \ldots, y_l^s$ all have the same type in $\theta$, the only atomic or negatomic subformulae of $\theta$ are of the form $y_i^s = y_j^s$, $y_i^s \in_s x_j^{r_j}$ if $r_j = s + 1$, $x_i^{r_i} \in_{r_i} y_j^s$ if $s = r_i + 1$, $x_i^{r_i} \in_{r_i} x_j^{r_j}$ if $r_j = r_i + 1$, or one of negations of these. Therefore, by Lemma 4.14,

$$\mathcal{M} \models \theta[a_1^{r_1}, \ldots, a_k^{r_k}, d_1, \ldots, d_l].$$

Since the $a_1^{r_1}, \ldots, a_k^{r_k} \in \mathcal{M}$ were arbitrary, this shows that $\mathcal{M} \models \phi$.    □

**Theorem 4.17**    Let $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^{s_1} \cdots \exists y_l^{s_l} \theta$ be an $\mathcal{L}_{\mathrm{TST}}$-sentence with $s_1 > \cdots > s_l$ and $\theta$ quantifier-free. If $\mathcal{N} \models \phi$, then $\mathcal{M} \models \phi$.

**Proof**    Suppose that $\mathcal{N} \models \phi$. Let $a_1^{r_1}, \ldots, a_k^{r_k} \in \mathcal{M}$. Using $a_1^{r_1}, \ldots, a_k^{r_k}$ and the construction we presented above, we can define the color classes $\mathcal{C}_{n,k'}$ and colorings $c_{n,k'}^{\mathcal{M}}$ and $c_{n,k'}^{\mathcal{N}}$, for all $n \in \mathbb{N}$, and elements $b_1^{r_1}, \ldots, b_k^{r_k} \in \mathcal{N}$. The colorings $c_{n,k'}^{\mathcal{M}}$ and $c_{n,k'}^{\mathcal{N}}$ are $2^K$-similar and satisfy Lemma 4.14. Let $e_1^{s_1}, \ldots, e_l^{s_l} \in \mathcal{N}$ be such that

$$\mathcal{N} \models \theta[b_1^{r_1}, \ldots, b_k^{r_k}, e_1^{s_1}, \ldots, e_l^{s_l}].$$

We inductively choose $d_1^{s_1}, \ldots, d_l^{s_l} \in \mathcal{M}$. Let $d_1^{s_1} \in \mathcal{M}$ be such that $c_{s_1,k'}^{\mathcal{M}}(d_1^{s_1}) = c_{s_1,k'}^{\mathcal{N}}(e_1^{s_1})$. Suppose that $1 \le i < l$, and suppose that we have chosen $d_i^{s_i} \in \mathcal{M}$ with $c_{s_i,k'}^{\mathcal{M}}(d_i^{s_i}) = c_{s_i,k'}^{\mathcal{N}}(e_i^{s_i})$. If $s_i \ne s_{i+1} + 1$, then let $d_{i+1}^{s_{i+1}} \in \mathcal{M}$ be such that $c_{s_{i+1},k'}^{\mathcal{M}}(d_{i+1}^{s_{i+1}}) = c_{s_{i+1},k'}^{\mathcal{N}}(e_{i+1}^{s_{i+1}})$. If $s_i = s_{i+1} + 1$ and $\mathcal{N} \models e_{i+1}^{s_{i+1}} \in_{s_{i+1}} e_i^{s_i}$, then let $d_{i+1}^{s_{i+1}} \in \mathcal{M}$ be such that $c_{s_{i+1},k'}^{\mathcal{M}}(d_{i+1}^{s_{i+1}}) = c_{s_{i+1},k'}^{\mathcal{N}}(e_{i+1}^{s_{i+1}})$ and $\mathcal{M} \models d_{i+1}^{s_{i+1}} \in_{s_{i+1}} d_i^{s_i}$. If $s_i = s_{i+1} + 1$ and $\mathcal{N} \models e_{i+1}^{s_{i+1}} \notin_{s_{i+1}} e_i^{s_i}$, then let $d_{i+1}^{s_{i+1}} \in \mathcal{M}$ be such that $c_{s_{i+1},k'}^{\mathcal{M}}(d_{i+1}^{s_{i+1}}) = c_{s_{i+1},k'}^{\mathcal{N}}(e_{i+1}^{s_{i+1}})$ and $\mathcal{M} \models d_{i+1}^{s_{i+1}} \notin_{s_{i+1}} d_i^{s_i}$. Lemma 4.15, the fact that $1 < 2^K$, and the fact that $c_{s_{i+1},k'}^{\mathcal{M}}$ and $c_{s_{i+1},k'}^{\mathcal{N}}$ are $2^K$-similar ensure that we can find $d_{i+1}^{s_{i+1}} \in \mathcal{M}$ satisfying these conditions. Now, since the variables $y_1^{s_1}, \ldots, y_l^{s_l}$ all have distinct types in $\theta$, the only atomic or negatomic subformulae of $\theta$ are of the form $y_{i+1}^{s_{i+1}} \in_{s_{i+1}} y_i^{s_i}$ if $s_i = s_{i+1} + 1$,

$y_i^{s_i} \in_{s_i} x_j^{r_j}$ if $r_j = s_i + 1$, $x_i^{r_i} \in_{r_i} y_j^{s_j}$ if $s_j = r_i + 1$, $x_i^{r_i} \in_{r_i} x_j^{r_j}$ if $r_j = r_i + 1$, or one of the negations of these. Therefore, by Lemma 4.14,

$$\mathcal{M} \models \theta[a_1^{r_k}, \dots, a_k^{r_k}, d_1^{s_1}, \dots, d_l^{s_l}].$$

Since the $a_1^{r_1}, \dots, a_k^{r_k} \in \mathcal{M}$ were arbitrary, this shows that $\mathcal{M} \models \phi$.                    □

Since $\mathcal{N}$ is an arbitrary model of TSTI and $\mathcal{M}$ is an arbitrary sufficiently large finitely generated model of TST, Theorems 4.16 and 4.17 show that any $\mathscr{L}_{\mathrm{TST}}$-sentence of form (A) or (B) has the finitely generated model property. Combining this with Theorem 3.2 shows that TSTI decides any sentence of form (A) or (B).

**Corollary 4.18**      *If $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^{s_1} \cdots \exists y_l^{s_l} \theta$ is an $\mathscr{L}_{\mathrm{TST}}$-sentence with $s_1 > \cdots > s_l$ and $\theta$ quantifier-free, then* TST *decides $\phi$.*

**Corollary 4.19**      *If $\phi = \forall x_1^{r_1} \cdots \forall x_k^{r_k} \exists y_1^{s} \cdots \exists y_l^{s} \theta$ is an $\mathscr{L}_{\mathrm{TST}}$-sentence with $\theta$ quantifier-free, then* TST *decides $\phi$.*

Combining these results with Theorem 2.1 shows that sentences of form (A′) or (B′) are decided by NF.

**Corollary 4.20**      *If $\phi = \forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_l \theta$ is an $\mathscr{L}$-formula with $\theta$ quantifier-free and $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$ is a stratification of $\phi$ that assigns the same value to all of the variables $y_1, \dots, y_l$, then* NF *decides $\phi$.*

**Corollary 4.21**      *If $\phi = \forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_l \theta$ is an $\mathscr{L}$-formula with $\theta$ quantifier-free and $\sigma : \mathbf{Var}(\phi) \longrightarrow \mathbb{N}$ is a stratification of $\phi$ that assigns distinct values to all of the variables $y_1, \dots, y_l$, then* NF *decides $\phi$.*

It is interesting to note that the only use of the axiom of infinity in the above arguments is to ensure that the bottom type is externally infinite. Thus, our arguments show that all models of TST with infinite bottom type agree on all sentences of form (A) and all sentences of form (B).

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