An Order-Theoretic Account of Some Set-Theoretic Paradoxes

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Abstract   We present an order-theoretic analysis of set-theoretic paradoxes. This analysis will show that a large variety of purely set-theoretic paradoxes (including the various Russell paradoxes as well as all the familiar implementations of the paradoxes of Mirimanoff and Burali-Forti) are all instances of a single limitative phenomenon.

1 Set-Theoretical Paradoxes

"*Logical* Paradoxes"? The paradoxes of the Age of the Crisis in Foundations were divided by Ramsey—in a robust classification that endures to this day—into two bundles called the *Semantic paradoxes* and the *Logical paradoxes*. The semantic paradoxes (paradox of the liar, paradox of the smallest number not definable in fewer than nineteen syllables, . . .) are so called because they involve semantic notions such as *truth* and *definability*; the logical paradoxes do not. We incline to the view that—since the semantic paradoxes involve notions additional to those involved in the logical paradoxes—it is necessary to understand the logical paradoxes before tackling the semantic paradoxes.

The logical paradoxes fall naturally into two classes:

 (i) those that involve set-theoretic notions only; and

(ii) those that involve other mathematical notions, such as cardinals or ordinals or other data structures such as lists or streams.

In the spirit of the last paragraph we suggest that the best way to understand (ii) is to first master (i). Furthermore, the paradoxes of bundle (ii) give rise to paradoxes involving set-theoretic notions only once we decide on implementations in set theory for the non-set-theoretic notions that they concern. It is therefore good policy to

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identify those paradoxes of naïve set theory that show themselves in this way as paradoxes in their own right and to distinguish them from the paradoxes of which they are implementations.

We take a very literal view here of what the language of set theory is: it is the language whose sole nonlogical gadget is '$\in$'.[1] On this reading, the concepts of *function, cardinal, stream, well-ordering,...* are not part of set theory, though of course they can be implemented in it. We need to distinguish the semiformal logical paradoxes (that use incompletely formalized mathematical notions like ordered pair, function, and well-ordering) from the formalized counterparts that are their implementations in set theory. Accordingly, we shall insist that the paradoxes of naïve set theory are a proper subset of the collection of logical paradoxes. The paradoxes of Cantor and Burali-Forti are logical paradoxes all right but they are not paradoxes of naïve set theory because they involve the notions of *cardinal* and *ordinal*.

Typically one associates a paradox of naïve set theory with a specific set existence axiom and, in fact, with a specific *closed* set abstract '$\{x \mid \varphi(x)\}$'. Comprehension axioms are usually thought of as coming equipped *with parameters*, but it is worth recording here that all paradoxical set abstracts can actually be set up without them and within a first-order language only. Moreover, hardly any of them seem to require extensionality, and this fact tells us that the set-theoretic paradoxes are not *au fond* set-theoretic phenomena.

Some set existence axioms are refutable in first-order logic. Some, as we shall see, are not paradoxical on their own, but give rise to contradiction in conjunction with set existence axioms (with parameters) that are so uncontroversial and so weak that most people don't even know their names. There are two weak principles which we will study below:

Adjunction '$x \cup \{y\}$ exists for all x and y', that is,

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w \in x \vee w = y);$$

Subcission '$x \backslash \{y\}$ exists for all x and y', that is,

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w \in x \wedge w \neq y).$$

We are indebted to Hazen for introducing us to these names; the second is his coinage. The operations themselves are to be found in [10].

The best-known paradoxes of pure set theory are Russell's paradox and its generalizations with exponent n, that is, those associated with the following set abstract, for each $n \in \mathbb{N}_0$:

$$'\{x \mid \neg \exists y_1 \ldots \exists y_{n-1} (x \in y_{n-1} \in \cdots \in y_1 \in x)\}'.$$

It is worth emphasizing that the nonexistence of these classes is a theorem of first-order predicate logic: no set-theoretic axioms are needed. Indeed (although we will not develop this point) these nonexistence results can be proved purely constructively. A (constructive) proof of the nonexistence of '$\{x \mid \forall y (x \in y \rightarrow y \notin x)\}$' (supplied by Grice) can be found in the Appendix.

All the paradoxes of naïve set theory seem to have the special character that we notice in Russell's paradox, namely, that the way one arrives at a contradiction is by asking whether the corresponding set abstract belongs to itself or not; so self-membership seems to play a key role. However, not every paradox of naïve set theory is associated with a set abstract as simple as Russell's.

The paradox of Mirimanoff—in one of its many forms—can be set up as a kind of generalization of Russell with exponent ∞, and to look like a purely set-theoretic paradox, but a paradox in an *infinitary* language, in the sense that it seems that we can prove by logic alone the nonexistence of

$$`\{x \mid \neg \exists y_1 \ldots \exists y_n \ldots (\cdots \in y_n \in \cdots \in y_1 \in x)\}`.$$

One has to be careful in saying things like this, since one does not want to put too much weight on the idea of a proof in infinitary languages. Without resorting to such languages the present version of Mirimanoff does not appear as a paradox of naïve set theory because it involves the notion of *stream*.

The more elaborate the non-set-theoretic notions involved in a logical paradox, the more complicated will be the set abstract associated with any implementation of that paradox in set theory. In the case of the paradoxes of Mirimanoff and Burali-Forti, the two corresponding collections of sets, namely, the collection of *well-founded sets* and the collection of *von Neumann ordinals*, admit *inductive* definitions. Not all paradoxical classes have inductive definitions (the Russell class is not inductively defined, for example); this is nevertheless a straw in the wind. There is a tendency for inductively defined families of sets to be paradoxical—at least in those cases where the operations that construct them have some infinitary, higher-order, or unbounded character (see [3], Section 5, for a general result along these lines). Such families of sets can be thought of in *naïve* set theory as least fixpoints (lfps) of functions that are *monotone* on the poset of all sets under inclusion; for example, the cumulative hierarchy appears as the lfp for the function that sends a set to the set of all its subsets—namely, the power set operation; the collection of von Neumann ordinals appears as the lfp for the function that sends a set to the set of its *transitive* subsets. We are greatly struck by the observation that, in contrast, the corresponding *greatest* fixpoints (gfps) seem not to be paradoxical.

This contrast between the lfps and the gfps might seem odd: If the lfp is paradoxical (and is therefore a proper class) then surely the gfp must also be a proper class...? To think like this is to resort to ZF-style ways of thinking—and therefore ZF-style ways of foreclosing exploration—before we are compelled to. The gfp of the power set operation is not a paradoxical object: we know this because there are *consistent* set theories with a universal set.

On the other hand, to think *naïvely* is not to think *carefully*—naïve set theory is inconsistent after all. As a matter of fact, it is not true that the collection of well-founded sets—however this is implemented—coincides with the lfp of the power set operation acting on sets, because (as we shall see in Section 5) there are consistent set theories in which this latter object can be proved to be a set.

These thoughts about fixpoints prompted us to have a look at set-theoretic paradoxes from an order-theoretic viewpoint. And—since classes that are paradoxical in one system may well be entirely harmless in another—we have opted for an *external* analysis through what we call *membership structures*, rather than any analysis *internal* to a particular set theory. Not only are we going to prove in this paper that lfps of certain monotone functions associated with the power set operation are intrinsically paradoxical, but more generally that what we shall call *quasi* lfps are. This will enable us to see that a large variety of purely set-theoretic paradoxes (including all the familiar implementations of the paradoxes of Mirimanoff and Burali-Forti, as well

as the various Russell paradoxes) are actually one and all manifestations of the one phenomenon—and one not even specifically set-theoretic.

2 Membership Structures, Galois Connections

We work in an informal meta-set-theory, which will not be spelled out, since not much depends on it; readers can mostly please themselves which system they wish to reason it. However, we are emphatically not going to assume the axiom of choice.

We define a *membership structure* as a pair $\langle V, s \rangle$ where V is a set and $s : V \longrightarrow \mathcal{P}(V)$. Elements of V are *containers* and the *contents* of a container a is precisely the subset $s(a)$ of V. We will use boldface letters for contents, writing $\boldsymbol{a}$ for $s(a)$ when convenient. Accordingly, given a membership structure $\langle V, s \rangle$, we say that a is a *member* of b, and write $a \in_s b$, if $a \in \boldsymbol{b}$. We will refer to $\in_s$ as the *membership relation* associated with s. Of course, membership structures could equally be defined by relational structures of type $\langle V, R \rangle$, where $V \neq \varnothing$ and $R \subseteq V \times V$.

That way of thinking is reminiscent of di Giorgi's view on structures for the language $\mathcal{L}$ of set theory, namely, as a set V together with an *injection* $s : V \hookrightarrow \mathcal{P}(V)$.[2] The injectivity of s clearly compels the membership relation $\in_s$ to be extensional, as indeed befits a membership relation in set theory. What is striking about this picture is its *simplicity*: all the information about the model $\langle V, \in_s \rangle$ is encoded in the choice of the injection s, and the whole transfinite structure of $\in$ has been compressed into two levels. We will use common set-theoretic terminology even in those cases where s is not 1-1.

Given a membership structure $\langle V, s \rangle$, the language $\mathcal{L}_V$ is $\mathcal{L}$ extended by the elements of V as constants, and then, given a closed $\mathcal{L}_V$-formula φ, we write φ_s for the interpretation of φ in $\langle V, \in_s \rangle$, that is, φ where $\in$ is interpreted by $\in_s$, and where all quantifiers are taken to range over V. We will make use of the common set-theoretic abbreviations in expressing formulas of $\mathcal{L}$ or $\mathcal{L}_V$. In particular, we might be using operations such as power set '$\mathcal{P}\_$', union '$\bigcup\_$', and other set abstracts while these might not be defined, the only proviso being that they can be eliminated from the formula in which they appear. We let $\mathcal{P}_1(V)$ denote the set of *(first-order) definable subsets* of V, that is, $A \in \mathcal{P}_1(V)$ if and only if there exists an $\mathcal{L}_V$-formula $\varphi(x)$ (with at most x as free variable) such that $A = \{a \in V \mid \varphi(a/x)_s\}$. The notation $\mathcal{P}_{1*}(V)$ is when parameters are proscribed, that is to say, when defining formulas $\varphi(x)$ are restricted to $\mathcal{L}$.

Clearly, $s[V] \subseteq \mathcal{P}_1(V)$, given that $s(b) = \{a \in V \mid a \in_s b\}$ for each $b \in V$, by definition of $\in_s$. In contrast the discovery that $\mathcal{P}_{1*}(V) \nsubseteq s[V]$, for example, $\{a \in V \mid a \notin_s a\} \notin s[V]$ was rather a shock: containers cannot have just any $\mathcal{L}$-definable subset of V as contents. In other words, there are limitations on the kind of contents available to elements of membership structures.

Remark 2.1   Subsets of V that can be proved not to belong to $s[V]$ in a given membership structure $\langle V, s \rangle$ will be said to be *problematic* rather than "paradoxical." There is clearly nothing actually *paradoxical* about the theorem of Cantor's that says that $s : V \longrightarrow \mathcal{P}(V)$ can never be surjective.[3]

Cantor's diagonal construction provides one particular answer to the following some-what combinatorial question.

Question 2.2 *Given a set V together with a function $s : V \longrightarrow \mathcal{P}(V)$, how can one* generate *a $W \subseteq V$ such that $W \notin s[V]$?*

More generally, any paradoxical set abstract '$\{x \mid \varphi(x)\}$', when interpreted in $\langle V, s \rangle$, will provide us with a *particular* answer to that question. What we are going to do in this paper is to provide answers—which we call *limitative results*—that will account for multiple set-theoretic paradoxes simultaneously.

Our limitative results will involve some order-theoretic machinery, which will be introduced in Section 3. For the moment we will illuminate a few basic set-theoretic notions from an order-theoretic viewpoint, which can easily be done by adopting di Giorgi's view on set-theoretic structures.

Given a membership structure $\langle V, s \rangle$, we regard $\langle \mathcal{P}(V), \subseteq \rangle$ as a complete upper-semilattice and define σ as the unique endomorphism such that, for all $a \in V$,

$$\sigma(\{a\}) := s(a).$$

In other words, we regard $\langle \mathcal{P}(V), \subseteq \rangle$ as the *free* complete upper-semilattice on V, with the inclusion map $\iota : V \longrightarrow \mathcal{P}(V) : a \longmapsto \{a\}$, and we look at s as a *substitution*,[4] its action on every $A \subseteq V$ being just defined by

$$\sigma(A) := \bigcup \{s(b) \mid b \in A\}.$$

Remark 2.3 In the set-theoretic case, an object $a \in V$ such that $\boldsymbol{a} = \{a\}$—that is, $s(a) = \iota(a)$—is called a *Quine atom*. Such objects have no substitutive action: they just fix the corresponding "variables," namely, *atoms* in $\mathcal{P}(V)$.

The theory of Galois connections on posets tells us that to any function g preserving sups on a complete lattice L we can associate a *unique* function d preserving infs such that for all x, y in L, $g(x) \leqslant y$ if and only if $x \leqslant d(y)$—in which case $\langle g, d \rangle$ is called a *Galois connection* (or *an adjunction*) on L; g is the *left (or lower) adjoint*, d the *right (or upper)* one. It is also standard that any left (respectively, right) adjoint preserves all existing sups (respectively, infs).

Now, since σ preserves sups in $\langle \mathcal{P}(V), \subseteq \rangle$, it must have a right adjoint, which we denote by π, and which is actually defined on every $B \subseteq V$ by

$$\pi(B) := \{a \in V \mid s(a) \subseteq B\},$$

for it is readily seen that $\sigma(A) \subseteq B \iff A \subseteq \pi(B)$, for all A, $B \subseteq V$. We shall call $\langle \sigma, \pi \rangle$ the *canonical Galois connection* associated with $\langle V, s \rangle$. General properties of Galois connections ensure that $(\sigma \circ \pi)(A) \subseteq A \subseteq (\pi \circ \sigma)(A)$ for all $A \subseteq V$. It is actually the case here that $(\sigma \circ \pi)(\boldsymbol{a}) = \boldsymbol{a}$ for each $a \in V$.

Remark 2.4 We note that any Galois connection $\langle \rho, \theta \rangle$ on $\langle \mathcal{P}(V), \subseteq \rangle$ can be seen as the canonical Galois connection of some membership structure $\langle V, s \rangle$, namely, $s : V \longrightarrow \mathcal{P}(V)$ defined by $s(b) := (\rho \circ \iota)(b)$ for each $b \in V$. This is of course because $\rho(A) = \bigcup \{\rho(\{b\}) \mid b \in A\}$ for every $A \subseteq V$, given that ρ preserves $\bigcup$. All that is to say that membership structures on V and Galois connections on $\langle \mathcal{P}(V), \subseteq \rangle$ are synonymous, mathematically speaking.[5]

In the set-theoretic case, σ and π correspond exactly to the union '$\bigcup\,$_' and power set '$\mathcal{P}\,$_' operations, respectively. And saying that a set-theoretic structure $\langle V, s \rangle$ satisfies the axiom of union (respectively, power set) just amounts to saying that $s[V]$ is closed under σ (respectively, π). Most axioms of set-theory are indeed intended to provide closure properties of $s[V]$ with respect to natural or simply useful operations. After all, comprehension and its restricted versions are some forms of completeness. Basic axioms that are going to play a role later are the axiom of *singleton*, which holds in $\langle V, s \rangle$ precisely when $\iota[V] \subseteq s[V]$, and the axiom of *self-subcission*, which holds in $\langle V, s \rangle$ when $(s - \iota)[V] \subseteq s[V]$, where $s - \iota : V \longrightarrow \mathcal{P}(V) : a \longmapsto s(a) \backslash \iota(a)$. These two are restricted versions of (respectively) the axioms of adjunction and subcission mentioned in Section 1.

3   Inductive and Transitive Subsets

Given a $\subseteq$-monotone operator θ on $\mathcal{P}(V)$, we recall that the set of fixpoints $\mathrm{fix}(\theta)$ is a complete sublattice of $\langle \mathcal{P}(V), \subseteq \rangle$. In particular, θ has an lfp and a gfp, which we denote by $\perp_\theta$ and $\top^\theta$, respectively, and which are given by

$$\perp_\theta = \bigcap \mathrm{ind}(\theta) \quad \& \quad \top^\theta = \bigcup \mathrm{trans}(\theta),$$

where

$$\mathrm{ind}(\theta) := \{A \subseteq V \mid \theta(A) \subseteq A\} \quad \& \quad \mathrm{trans}(\theta) := \{A \subseteq V \mid A \subseteq \theta(A)\}.$$

Cued by the notation, we shall refer to members of $\mathrm{ind}(\theta)$ (respectively, $\mathrm{trans}(\theta)$) as θ-*inductive* (respectively, θ-*transitive*) subsets of V. Clearly these notions are dual, and we note that for any Galois connection $\langle \rho, \theta \rangle$ in $\langle \mathcal{P}(V), \subseteq \rangle$, $\mathrm{trans}(\theta) = \mathrm{ind}(\rho)$. Our choice of terminology is in part guided here by the set-theoretic case where $a \in V$ is said to be *transitive* precisely when $\boldsymbol{a} \subseteq \pi(\boldsymbol{a})$, or equally $\sigma(\boldsymbol{a}) \subseteq \boldsymbol{a}$— whereas it is sometimes said to be *fat* (instead of inductive) when $\pi(\boldsymbol{a}) \subseteq \boldsymbol{a}$. We shall also use the terminology $\in$-*inductive* (respectively, $\in$-*transitive*) instead of π-inductive (respectively, π-transitive).

We are going to be interested in what we shall call *quasi-least-fixpoints* (qlfps) and *quasi-greatest-fixpoints* (qgfps) of certain monotone operators—but the reader is warned that these so-called qlfps and qgfps might, in fact, not be fixpoints at all!

Definition 3.1   Given a $\subseteq$-monotone operator θ on $\mathcal{P}(V)$ and $\mathcal{S} \subseteq \mathcal{P}(V)$, we define the *qlfp* and *qgfp* of θ *generated by* $\mathcal{S}$, respectively, by

$$\perp_\theta^{\mathcal{S}} := \bigcap (\mathrm{ind}(\theta) \cap \mathcal{S}) \quad \& \quad \top_{\mathcal{S}}^\theta := \bigcup (\mathrm{trans}(\theta) \cap \mathcal{S}).$$

Thus $\perp_\theta$ and $\top^\theta$ just correspond to $\mathcal{S} := \mathcal{P}(V)$, and we note that $\top_{\mathcal{S}}^\theta$ is monotone in $\mathcal{S}$, that is,

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \Rightarrow \top_{\mathcal{S}_1}^\theta \subseteq \top_{\mathcal{S}_2}^\theta,$$

whereas $\perp_\theta^{\mathcal{S}}$ is antimonotone in $\mathcal{S}$, that is,

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \Rightarrow \perp_\theta^{\mathcal{S}_2} \subseteq \perp_\theta^{\mathcal{S}_1}.$$

However, both are monotone in θ, that is,

$$\theta_1 \sqsubseteq \theta_2 \Rightarrow \perp_{\theta_1}^{\mathcal{S}} \subseteq \perp_{\theta_2}^{\mathcal{S}} \;\&\; \top_{\mathcal{S}}^{\theta_1} \subseteq \top_{\mathcal{S}}^{\theta_2},$$

where $\theta_1 \sqsubseteq \theta_2$ if and only if $\forall A \subseteq V, \theta_1(A) \subseteq \theta_2(A)$. Besides, consider the following fact.

Fact 3.2 For any $\subseteq$-monotone operator θ on $\mathcal{P}(V)$ and $\mathcal{S} \subseteq \mathcal{P}(V)$, we have

(i) $\perp_\theta^{\mathcal{S}} \in \mathrm{ind}(\theta)$ (respectively, $\top_{\mathcal{S}}^\theta \in \mathrm{trans}(\theta)$);

(ii) $\perp_\theta^{\mathcal{S}} \in \mathrm{fix}(\theta)$ (respectively, $\top_{\mathcal{S}}^\theta \in \mathrm{fix}(\theta)$) whenever $\mathcal{S}$ is closed under θ and θ is a right adjoint (respectively, left adjoint).

(iii) $\perp_\theta^{\mathcal{S}} \in \mathrm{fix}(\theta)$ (respectively, $\top_{\mathcal{S}}^\theta \in \mathrm{fix}(\theta)$) whenever $\mathcal{S}$ is closed under θ and $\mathcal{S}$ is closed under arbitrary $\bigcap$ (respectively, $\bigcup$).[6]

Proof (i) For every $B \in \mathrm{ind}(\theta) \cap \mathcal{S}$, $\perp_\theta^{\mathcal{S}} \subseteq B$, so that $\theta(\perp_\theta^{\mathcal{S}}) \subseteq \theta(B) \subseteq B$. It follows that $\theta(\perp_\theta^{\mathcal{S}}) \subseteq \perp_\theta^{\mathcal{S}}$, that is, $\perp_\theta^{\mathcal{S}} \in \mathrm{ind}(\theta)$.

(ii) Let $\mathcal{I} := \mathrm{ind}(\theta) \cap \mathcal{S}$. Assuming that $\mathcal{S}$ is closed under θ, we have $\theta[\mathcal{I}] \subseteq \mathcal{I}$. On the other hand, assuming that θ preserves $\bigcap$, we have $\theta(\perp_\theta^{\mathcal{S}}) = \bigcap \theta[\mathcal{I}]$. Then it follows from both assumptions that $\perp_\theta^{\mathcal{S}} = \bigcap \mathcal{I} \subseteq \bigcap \theta[\mathcal{I}] = \theta(\perp_\theta^{\mathcal{S}})$.

(iii) Assuming $\mathcal{S}$ is closed under $\bigcap$, $\perp_\theta^{\mathcal{S}} \in \mathcal{S}$; hence $\perp_\theta^{\mathcal{S}} \in \mathrm{ind}(\theta) \cap \mathcal{S}$ by (i). Then $\theta(\perp_\theta^{\mathcal{S}}) \in \mathrm{ind}(\theta) \cap \mathcal{S}$ too, since $\mathcal{S}$ is closed under θ. So $\perp_\theta^{\mathcal{S}} \subseteq \theta(\perp_\theta^{\mathcal{S}})$.

The proof for $\top_{\mathcal{S}}^\theta$ is dual. $\qquad\square$

We now turn to basic facts about duality.

Definition 3.3 Given a $\subseteq$-monotone operator θ on $\mathcal{P}(V)$, we define its *dual* $\widehat{\theta}$ by $\widehat{\theta}(A) := \theta(A^-)^-$, where $A^- := V \backslash A$, for each $A \subseteq V$.

Clearly, $\widehat{\widehat{\theta}} = \theta$, $\widehat{\rho \circ \theta} = \widehat{\rho} \circ \widehat{\theta}$, and we note that $\langle \rho, \theta \rangle$ is a Galois connection on $\langle \mathcal{P}(V), \subseteq \rangle$ if and only if $\langle \widehat{\theta}, \widehat{\rho} \rangle$ is. Furthermore, we have the following.

Fact 3.4 For any $\subseteq$-monotone operator θ on $\mathcal{P}(V)$ and $\mathcal{S} \subseteq \mathcal{P}(V)$, we have

(i) $\mathrm{ind}(\widehat{\theta}) = -\mathrm{trans}(\theta)$ & $\mathrm{trans}(\widehat{\theta}) = -\mathrm{ind}(\theta)$,

(ii) $\perp_{\widehat{\theta}}^{\mathcal{S}} = (\top_{-\mathcal{S}}^\theta)^-$ & $\top_{\mathcal{S}}^{\widehat{\theta}} = (\perp_\theta^{-\mathcal{S}})^-$,

where $-\mathcal{S} := \{A^- \mid A \in \mathcal{S}\}$, for each $\mathcal{S} \subseteq \mathcal{P}(V)$.

Proof Immediate. $\qquad\square$

As our main concern will be with the π operator or variants of it, it is worth making explicit the definition of its dual: for every $B \subseteq V$,

$$\widehat{\pi}(B) := \{a \in V \mid s(a) \between B\},$$

where '$\between$' is defined for all $A, B \subseteq V$ by

$$A \between B \iff A \cap B \neq \varnothing.$$

4 The First Limitative Theorem

The production of set-theoretic paradoxes always involves self-membership somehow. The following proposition precisely shows what kind of constraints this can induce in the case of qlfps of a monotone operator θ bounded by π.

Proposition 4.1 *Suppose $\theta \sqsubseteq \pi$ and $a \subseteq \perp_\theta^{\mathcal{S}}$ for some $a \in V$, $\mathcal{S} \subseteq \mathcal{P}(V)$. Then, $a \in_s a$ implies $B \backslash \{a\} \notin \mathcal{S}$ for every $B \in \mathrm{ind}(\theta)$.*

Proof Let $B \in \mathrm{ind}(\theta)$ and $B' := B\backslash\{a\}$; so $a \notin B'$. Suppose $a \in \boldsymbol{a}$ and $B' \in \mathcal{S}$. To get a contradiction, we show that $\theta(B') \subseteq B'$, which would imply $\boldsymbol{a} \subseteq B'$ since $\boldsymbol{a} \subseteq \perp_\theta^\mathcal{S}$, and then $a \in B'$. As $B' \subseteq B$, we have $\theta(B') \subseteq \theta(B) \subseteq B$, by monotonicity and the fact that $B \in \mathrm{ind}(\theta)$. On the other hand, $a \notin \theta(B')$. Otherwise, using our assumption on θ, we would have $a \in \pi(B')$, that is, $\boldsymbol{a} \subseteq B'$, and again $a \in B'$. Hence $\theta(B') \subseteq B'$. $\square$

Accordingly, self-membership will be blocked for qlfps whose generating families are *replete*, in the following sense.

Definition 4.2 Given a membership structure $\langle V, s \rangle$ and a $\subseteq$-monotone operator θ on $\mathcal{P}(V)$, we say that $\mathcal{S} \subseteq \mathcal{P}(V)$ is θ-*replete* if

$$\forall a \in V, \boldsymbol{a} \in \mathrm{ind}(\theta) \implies \exists B \in \mathrm{ind}(\theta) : B\backslash\{a\} \in \mathcal{S}.$$

Remark 4.3 Some families are intrinsically θ-replete for *every* θ; for example, $-\iota[V]$ or $(s - \iota)[V]$, and a fortiori $\mathcal{P}_1(V)$ or $\mathcal{P}(V)$. Others are if we assume $\langle V, s \rangle$ satisfies basic assumptions; for example, $-s[V]$ if $\langle V, s \rangle \models$ singleton, or $s[V]$ if $\langle V, s \rangle \models$ self-subcission. Of course, whenever $\mathcal{S}_1$ is θ-replete, so is $\mathcal{S}_2$ with $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

On the other hand, self-membership is forced for distinguished subsets of V, notably the $\in$-inductive ones.

Lemma 4.4 *If $\boldsymbol{a} \in \mathrm{ind}(\pi)$ for some a, then $a \in_s a$.*

Proof We simply observe that for every $a \in V$, $a \in \pi(\boldsymbol{a})$. Therefore, if $\boldsymbol{a}$ is π-inductive, that is, $\pi(\boldsymbol{a}) \subseteq \boldsymbol{a}$, then $a \in \boldsymbol{a}$. $\square$

So we can deduce the following limitative result from Proposition 4.1.

Theorem 4.5 *Assume that $\mathcal{S}$ is π-replete. Then, whenever $A \in \mathrm{ind}(\pi)$ and $A \subseteq \perp_\pi^\mathcal{S}$, we have $A \notin s[V]$. In particular, $\perp_\pi^\mathcal{S} \notin s[V]$.*

Proof Suppose that $A \in \mathrm{ind}(\pi)$, $A \subseteq \perp_\pi^\mathcal{S}$, and $A = \boldsymbol{a}$ for some $a \in V$. As $\mathcal{S}$ is π-replete and $\boldsymbol{a} \in \mathrm{ind}(\pi)$, it is the case that $B\backslash\{a\} \in \mathcal{S}$ for some $B \in \mathrm{ind}(\pi)$, so $a \notin_s a$ by Proposition 4.1. But $a \in_s a$ by Lemma 4.4. $\square$

It should be stressed that Theorem 4.5 is applicable not only to any qlfp of π whose generating family $\mathcal{S}$ is π-replete, but also to any $\in$-inductive subset below such a qlfp—see Example 4.6. That said, we recall that $\perp_\pi^\mathcal{S}$ is antimonotone in $\mathcal{S}$, so that any consequence of Theorem 4.5 with $\mathcal{S}_2 \supseteq \mathcal{S}_1$, for some π-replete $\mathcal{S}_1$, is already a consequence of Theorem 4.5 with $\mathcal{S}_1$.

   Now, to turn an application of that limitative result into a paradox of *first-order* naïve set theory, one needs the corresponding problematic subset of V to be in $\mathcal{P}_{1*}(V)$. In view of Definition 3.1, that will be the case of $\perp_\pi^\mathcal{S}$ provided the generating family $\mathcal{S}$ is quantifiable in $\mathscr{L}$. To give an illustration, let us identify here a variety of set-theoretic paradoxes that are associated with one possible choice for $\mathcal{S}$ in Theorem 4.5, namely, $\mathcal{S} := -\iota[V]$.

Example 4.6 (Russell's paradox with exponent n)   For each $n \in \mathbb{N}_0$, let

$$\varphi_n(\mathrm{x}) :\equiv \neg \exists \mathrm{y}_1 \ldots \exists \mathrm{y}_{n-1}(\mathrm{x} \in \mathrm{y}_{n-1} \in \cdots \in \mathrm{y}_1 \in \mathrm{x}),$$

with the obvious convention that $\varphi_1(\mathrm{x}) :\equiv \neg(\mathrm{x} \in \mathrm{x})$, and let

$$\varphi_\infty(\mathrm{x}) :\equiv \neg \exists \mathrm{y}_1 \ldots \exists \mathrm{y}_n \ldots (\cdots \in \mathrm{y}_n \in \cdots \in \mathrm{y}_1 \in \mathrm{x}),$$

though this is not an $\mathscr{L}$-formula. Then, for any $n \in \mathbb{N}_0 \cup \{\infty\}$, let

$$R_n := \{a \in V \mid [\varphi_n(a)]_s\}.$$

Clearly, $R_\infty \subseteq R_{n+1} \subseteq R_n$ for each $n \in \mathbb{N}_0$, and we remark that $R_1 = \perp_\pi^{-\iota[V]}$. Indeed, we have

$$
\begin{aligned}
(\perp_\pi^{-\iota[V]})^- = \top_{\iota[V]}^{\widehat{\pi}} &= \bigcup\{B \in \iota[V] \mid B \subseteq \widehat{\pi}(B)\} \\
&= \{a \in V \mid \exists b \in V : a \in \iota(b) \ \& \ \iota(b) \subseteq \widehat{\pi}(\iota(b))\} \\
&= \{a \in V \mid \exists b \in V : a = b \ \& \ b \in s(b)\} \\
&= \{a \in V \mid [a \in a]_s\}.
\end{aligned}
$$

Now we observe that R_n is π-inductive, or equally that R_n^- is $\widehat{\pi}$-transitive, for each $n \in \mathbb{N}_0 \cup \{\infty\}$. For $n = 1$ this follows from Fact 3.2(i) since $R_1^- = \top_{\iota[V]}^{\widehat{\pi}}$. Now suppose $n > 1$, finite, and that $a \in R_n^-$. Then there exist $b_1, \ldots, b_{n-1}$ in V such that $a \in_s b_{n-1} \in_s \ldots \in_s b_1 \in_s a$, or equally $b_1 \in_s a \in_s \ldots \in_s b_2 \in_s b_1$, which shows that $b_1 \in s(a) \cap R_n^-$, so $s(a) \between R_n^-$, that is, $a \in \widehat{\pi}(R_n^-)$. The proof for $n = \infty$ is similar. Thus we have shown that $R_n \in \mathrm{ind}(\pi)$ and $R_n \subseteq \perp_\pi^{-\iota[V]}$, so it follows from Theorem 4.5 that R_n never belongs to $s[V]$ for all $n \in \mathbb{N}_0 \cup \{\infty\}$. In other words, '$\{x \mid \varphi_n(x)\}$' is always a paradoxical set abstract, as is widely known for $n = 1$, and this does not rely on any set-theoretic assumption other than the existence of the incriminated set-abstract—even in the case $n = \infty$, though this is not expressible in $\mathscr{L}$. Note also that it is not hard to concoct examples of membership structures in which $R_\infty \subsetneq R_{n+1} \subsetneq R_n$ for each $n \in \mathbb{N}_0$. For example, the membership structure given by $V := \{a, b\}$ with $a := \{b\}$, $b := \{a\}$ is such that $\varnothing = R_2 \subsetneq R_1 = V$.

So Russell's paradox and its generalizations with exponent n all appear as manifestations of the above limitative result. We show in the next section that so are the *inductive* and *regular* versions of Mirimanoff's paradox.

5 Well-foundedness and Regularity

Mirimanoff's paradox concerns the set of all *well-founded* sets, that is, as commonly stated, the set of all those sets such that there is no infinite descending $\in$-chain starting at them. Chains are not purely set-theoretic objects, though they can of course be implemented as sets. However, we are not going to define well-foundedness in terms of descending chains here. That would require us to assume the axiom of dependent choice in order to show that the well-founded part of a membership structure satisfies $\in$-induction, which can arguably be regarded as the main motivation for well-foundedness, independently of any form of choice.[7] Then another source of complications is the second-order nature of induction principles. These can only be approximated in first-order theories by axiom schemes, which can nevertheless be replaced in some set-theoretic systems by a single axiom.

By Definition 3.1, $\perp_\pi^{\mathscr{S}}$ is the largest subset A of V such that

$$\forall B \in \mathscr{S}, \pi(B) \subseteq B \implies A \subseteq B.$$

In other words, $\perp_\pi^{\mathscr{S}}$ is the largest subset for which $\in$-*induction over* $\mathscr{S}$ works. We therefore identify $\perp_\pi^{\mathscr{P}(V)}$ as the *(standard) well-founded part* of $\langle V, s \rangle$, which will be denoted by Wf^2. The superscript '2' is to remind us of the *second-order* nature of that object and to distinguish it from its *first-order* approximation Wf^1, which we

then recognize as $\perp_\pi^{\mathcal{P}_1(V)}$. In set-theoretical contexts it seems natural to also consider the *zero-order, grounded* version Wf^0 that we define as $\perp_\pi^{s[V]}$. It follows from Theorem 4.5 that Wf^2 and Wf^1 are *always* problematic, whereas Wf^0 is problematic only when $s[V]$ is π-replete, as for instance when $\langle V, s \rangle \models$ self-subcission (cf. Remark 4.3). These limitations are what we call the *inductive* versions of Mirimanoff's paradox.

Remark 5.1   Unless the membership structure is finite, and as pointed out above, the axiom of dependent choice is needed in the metatheory in order to show that $\mathrm{Wf}^2 = \mathrm{R}_\infty$. We shall not pay much attention in the sequel to second-order classes as we want our discussion to be transposable—as far as possible—into *first-order* set theory.

The zero-order version gives rise directly to a paradox of first-order naïve set theory, since the definition of Wf^0 involves quantification over $s[V]$ only, so that $\mathrm{Wf}^0 \in \mathcal{P}_{1*}(V)$:

$$
\begin{aligned}
\mathrm{Wf}^0 := \perp_\pi^{s[V]} \;&=\; \bigcap \{ B \in s[V] \mid \pi(B) \subseteq B \} \\
&=\; \{ a \in V \mid \forall b \in V, \pi(\boldsymbol{b}) \subseteq \boldsymbol{b} \Rightarrow a \in \boldsymbol{b} \} \\
&=\; \{ \mathrm{x} \mid [\forall \mathrm{y}(\mathcal{P}\mathrm{y} \subseteq \mathrm{y} \rightarrow a \in \mathrm{y})]_s \}.
\end{aligned}
$$

However, we emphasize that the nonexistence of the corresponding set abstract

$$
\text{`}\{ \mathrm{x} \mid \forall \mathrm{y}(\mathcal{P}\mathrm{y} \subseteq \mathrm{y} \rightarrow \mathrm{x} \in \mathrm{y}) \}\text{'}
$$

is *not* a theorem of first-order predicate logic: some (other) instances of comprehension are needed, such as self-subcission. As a matter of fact, there are set-theoretic structures in which $\mathrm{Wf}^0 \in s[V]$ (see Examples 5.6 and 5.8 below).

On the other hand, one would also remark that $\mathrm{Wf}^0 = V$ in any model of a set theory that outlaws fat sets, for example, any set theory that has *unrestricted* separation, as does Zermelo's system (among others).

Fact 5.2   Assume $\langle V, s \rangle \models$ separation.[8] Then there is no $b \in V$ such that $\pi(\boldsymbol{b}) \subseteq \boldsymbol{b}$.

Proof   Suppose there was $b \in V$ such that $\pi(\boldsymbol{b}) \subseteq \boldsymbol{b}$. By separation, there would exist $r \in V$ such that $\boldsymbol{r} = \boldsymbol{b} \cap \mathrm{R}_1$, where $\mathrm{R}_1 := \{ a \in V \mid a \notin_s a \}$ (cf. Example 4.6). But then $r \in_s b$, so one would have $r \in_s r$ if and only if $r \notin_s r$.   □

So Wf^0 is unlikely to collect well-founded sets only. At the first-order level that should be the role of Wf^1, though this might not be first-order definable. One way of showing that $\mathrm{Wf}^1 \in \mathcal{P}_{1*}(V)$, precisely, goes by showing that Wf^1 coincides with the collection of *regular* sets, to which we now turn.

We write Rg for $\perp_\pi^{-s[V]}$, which we call the *regular part* of $\langle V, s \rangle$ since

$$
\begin{aligned}
\mathrm{Rg} := \perp_\pi^{-s[v]} \;&=\; \bigcap \{ B \in -s[V] \mid \pi(B) \subseteq B \} \\
&=\; \{ a \in V \mid \forall b \in V, \pi(\boldsymbol{b}^-) \subseteq \boldsymbol{b}^- \Rightarrow a \in \boldsymbol{b}^- \} \\
&=\; \{ a \in V \mid \forall b \in V, a \in \boldsymbol{b} \Rightarrow \boldsymbol{b} \not\subseteq \widehat{\pi}(\boldsymbol{b}) \} \\
&=\; \{ a \in V \mid [\forall \mathrm{y}(a \in \mathrm{y} \rightarrow \exists \mathrm{z}(\mathrm{z} \in \mathrm{y} \wedge \mathrm{z} \cap \mathrm{y} = \varnothing))]_s \},
\end{aligned}
$$

and this indeed corresponds to the notion of *regularity* that one uses in set theory.

Remark 5.3   The regularity condition is not to be confused with the $\in$-*minimality* requirement that appears in the usual formulation of the foundation axiom in ZF, which asserts that $V = F$, where

$$F := \{a \in V \mid [a = \varnothing \vee \exists x(x \in a \wedge x \cap a = \varnothing))]_s\}.$$

Although the assertion $V = F$ is logically equivalent to $V = \mathrm{Rg}$, it should be clear that in general $F \neq \mathrm{Rg}$; for example, in any $\langle V, s \rangle$ with $a, b \in V$ such that $\boldsymbol{a} = V$, $\boldsymbol{b} = \varnothing$, and $\{a\} \in s[V]$, we have $a \in F \backslash \mathrm{Rg}$. On the other hand, a basic set-theoretic assumption suffices to ensure that $\mathrm{Rg} \subseteq F$.

Fact 5.4   Assume $\langle V, s \rangle \models$ self-adjunction.[9] Then $\mathrm{Rg} \subseteq F$.

Proof   Let $a \in \mathrm{Rg}$ with $\boldsymbol{a} \neq \varnothing$. Now, consider $b \in V$ such that $\boldsymbol{b} = \boldsymbol{a} \cup \{a\}$. As $a \in_s b$ and $a \in \mathrm{Rg}$, there is some $c \in V$ such that $c \in_s b$ and $\boldsymbol{c} \cap \boldsymbol{b} = \varnothing$. It is not the case that $c = a$, since $\boldsymbol{a} \neq \varnothing$. Hence $c \in_s a$. On the other hand, $\boldsymbol{c} \cap \boldsymbol{a} = \varnothing$, because $\boldsymbol{c} \cap \boldsymbol{b} = \varnothing$. This shows that $a \in F$. $\square$

It follows from Theorem 4.5 that Rg is problematic whenever $-s[V]$ is π-replete, as for instance when $\langle V, s \rangle \models$ singleton (cf. Remark 4.3). And this limitation is what we call the *regular* version of the paradox of Mirimanoff. Again we stress that the nonexistence of the corresponding set abstract

$$\text{`}\{x \mid \forall y(x \in y \rightarrow \exists z(z \in y \wedge z \cap y = \varnothing))\}\text{'}$$

requires other instances of comprehension, such as singleton. Or else self-subscission and complement, for we note that $\mathrm{Rg} = \mathrm{Wf}^1$ whenever $-s[V] = s[V]$, that is, when $\langle V, s \rangle \models$ complement. But most set theorists would consider the existence of complement as more controversial than the one of singleton or even self-subscission. Note by the way that singleton and self-subscission are relatively independent of each other in the presence of complement; for example, the membership structure made of two Quine atoms—that is, $V := \{a, b\}$ with $\boldsymbol{a} = \{a\}$, $\boldsymbol{b} = \{b\}$—satisfies complement and singleton but not self-subscission.

Clearly, $\mathrm{Wf}^1 \subseteq \mathrm{Rg}$, since $-s[V] \subseteq \mathcal{P}_1(V)$ and $\perp_\pi^s$ is antimonotone in s. Any ZF-ist would tell us that $\mathrm{Wf}^1 = \mathrm{Rg}$, but this is due to the axiomatic cocktail of ZF. The key ingredients are in fact separation and transitive containment, which is a consequence of replacement in ZF.

Fact 5.5   Assume that $\langle V, s \rangle \models$ separation $+$ transitive containment[10] and let $B \in \mathcal{P}_1(V)$ such that $\pi(B) \subseteq B$. Then $\mathrm{Rg} \subseteq B$. So it follows that $\mathrm{Rg} \subseteq \mathrm{Wf}^1$.

Proof   Suppose $a \in \mathrm{Rg}$ but $a \notin B$. Then take any $b \in V$ such that $a \in_s b$ and $\boldsymbol{b} \subseteq \pi(\boldsymbol{b})$, and let $c \in V$ such that $\boldsymbol{c} := \boldsymbol{b} \cap B^c$ (separation). As $a \in_s c$ and $a \in \mathrm{Rg}$, there exists $d \in_s c$ such that $\boldsymbol{d} \cap \boldsymbol{c} = \varnothing$. But as $d \in_s b$ and $\boldsymbol{b} \subseteq \pi(\boldsymbol{b})$, we have $\boldsymbol{d} \subseteq \boldsymbol{b}$, and thus $\boldsymbol{d} \subseteq B$ since $\boldsymbol{d} \cap \boldsymbol{b} \cap B^c = \varnothing$. But then $d \in B$, because $\pi(B) \subseteq B$, which is impossible as $d \in B^c$. $\square$

It is not true in Zermelo's system that $\mathrm{Wf}^1 = \mathrm{Rg}$; that is essentially why the axiom of foundation asserting that $V = \mathrm{Wf}^1$ can only be expressed by a first-order axiom scheme in Z (see [1]). On the other hand, there are situations where separation fails but nevertheless $\mathrm{Wf}^1 = \mathrm{Rg}$.

Example 5.6 (GPK$^+$)   In the topological set theory GPK$^+$ (see [2])—where separation fails—it is also the case that Wf^1 = Rg but the reason there is merely that Wf$^1 \in -s[V] \cap \mathrm{ind}(\pi)$, so that Rg $\subseteq$ Wf^1. The situation in GPK$^+$ is interesting in that Wf$^1 \subsetneq$ Wf$^0 \subsetneq V$, where Wf^1 = Rg $\notin s[V]$, but Wf$^0 \in s[V]$, which does not contradict Theorem 4.5 yet because self-subcission is prohibited in GPK$^+$. We have not been able to prove in the theory that the topological closure of Wf^1 is precisely Wf^0, although this is in fact the case in all natural models of GPK$^+$—the so-called hyperuniverses—as one would expect. Besides, we note that in both ZF and GPK$^+$ the first-order well-founded part of the universe Wf^1 just coincides with the familiar von Neumann cumulative hierarchy. And this is defined as usual, by transfinite iteration of the power set operation '$\mathscr{P}\_$' from $\varnothing$, which of course requires the prior internal development of the machinery of von Neumann ordinals. It might be surprising then that what we get in GPK$^+$ by iterating '$\mathscr{P}\_$' from $\varnothing$ is not the least *set* fixpoint of '$\mathscr{P}\_$', namely, Wf^0, but Wf^1, which is a proper class contained in Wf^0! To understand the situation, we recall that in topological models of GPK$^+$ $s[V]$ appears to be the set of *closed* subsets with respect to some suitable topology on V. So in the complete lattice $\langle s[V], \subseteq \rangle$, the sup operator is *not* the union operator but its closure. As a matter of fact, in topological models of GPK$^+$, $s[V]$ is not closed under unions unless the 'index set' itself is in $s[V]$, which cannot be the case if this is taken to be the class of all von Neumann ordinals, because of Burali-Forti paradox.

Whether there are general assumptions that force Rg $\subseteq$ Wf^1 is not clear. Note that whenever this is the case, Rg must be a fixpoint of π, for Wf^1 is by Fact 3.2(ii), given that $\mathscr{P}_1(V)$ is closed under π. Here it would not be sensible to ask $-s[V]$ to be itself closed under π. It turns out that a simple set-theoretic assumption would ensure at least that Rg $\in \mathrm{fix}(\pi)$.

Fact 5.7   Assume that $\langle V; s \rangle \models$ adjunction. Then $\pi(\mathrm{Rg}) = \mathrm{Rg}$.

Proof   It suffices to show that Rg $\subseteq \pi(\mathrm{Rg})$, as Rg $\in \mathrm{ind}(\pi)$ by Fact 3.2(i). So let $a \in \mathrm{Rg}$, $b \in_s a$ (we may of course assume $\boldsymbol{a} \neq \varnothing$) and $b \in_s c$. We want to show that there is $\boldsymbol{d} \in_s c$ such $\boldsymbol{d} \cap \boldsymbol{c} = \varnothing$. If $a \in_s c$, we are done since $a \in \mathrm{Rg}$. Assume then that $a \notin_s c$ and let $e \in V$ such that $\boldsymbol{e} := \boldsymbol{c} \cup \{a\}$ (adjunction). As $a \in_s e$ and $a \in \mathrm{Rg}$, there is $\boldsymbol{d} \in_s e$ with $\boldsymbol{d} \cap \boldsymbol{e} = \varnothing$. Since $b \in \boldsymbol{a} \cap \boldsymbol{c}$, it is not the case that $d = a$. Therefore, $\boldsymbol{d} \in_s c$ and $\boldsymbol{d} \cap \boldsymbol{c} = \varnothing$.   □

Yet there are set-theoretic structures where Rg is not a fixpoint of π, not even Wf^0, though this latter *naïvely* appears by definition as the lfp of '$\mathscr{P}\_$'!

Example 5.8 (Skala [9])   Take $V := \{a, b, c, d\}$ with $\boldsymbol{a} := \varnothing$, $\boldsymbol{b} := \{a, b, c\}$, $\boldsymbol{c} := \{d\}$, and $\boldsymbol{d} := V$. Clearly $-s[V] = s[V]$, and one can see that $\pi(\boldsymbol{b}) = \{a, b\} \subseteq \boldsymbol{b}$, whereas $\pi(\boldsymbol{a}) = \{a\} \not\subseteq \boldsymbol{a}$ and $\pi(\boldsymbol{c}) = \{a, c\} \not\subseteq \boldsymbol{c}$, so that Wf^0 = Rg = $\boldsymbol{b}$, but $\pi(\boldsymbol{b}) \notin s[V]$. It is also easy to see that Wf^1(= Wf^2 = R$_\infty$) = $\{a\}$, so that Wf$^1 \in \mathscr{P}_{1*}(V)$ here, though Wf$^1 \subsetneq$ Rg. That set-theoretic structure is proved to be a model of Skala's topological set theory (see [6], Example 8.1), in which system Wf^0 and Rg are *sets*, indeed one and the same set, for the axiom of complement holds. That Wf^0 cannot be proved to be a fixpoint of '$\mathscr{P}\_$' in Skala's theory lies in the fact that—unlike GPK$^+$—the power set axiom does not hold; that is, $s[V]$ may not be closed under π (cf. Fact 3.2(ii)).

One could of course refrain from using the terminology "set theory" for such set-theoretic systems as Skala's, but we feel this would miss the point. The point is that the set-theoretic paradoxes are not peculiar to set theory. Such finite examples of set-theoretic structures as the one here above bear witness to the purely combinatorial nature of so-called set-theoretic paradoxes—which is the message (or one of them) of this paper.

6 The Second Limitative Theorem

We are now going to extend the limitative result of Section 4 to fixpoints of monotone operators θ with $\theta \sqsubseteq \pi$ as suggested by Proposition 4.1. We shall in fact restrict ourselves to the operators π_D defined for each $D \subseteq V$ by $\pi_D(A) := D \cap \pi(A)$, for all $A \subseteq V$. It turns out that suitable choices of D can then lead directly to results of the kind we are seeking.

Define

$$\mathrm{T} := \{a \in V \mid \boldsymbol{a} \subseteq \pi(\boldsymbol{a})\}.$$

Lemma 6.1   *Suppose $D \supseteq \mathrm{T}$. If $\boldsymbol{a} \in \mathrm{fix}(\pi_D)$ for some a, then $a \in_s a$.*

Proof   Suppose $D \supseteq \mathrm{T}$ and $\boldsymbol{a} = D \cap \pi(\boldsymbol{a})$. Then $\boldsymbol{a} \subseteq \pi(\boldsymbol{a})$, that is, $a \in \mathrm{T}$, so $a \in D$ by assumption. Thus, since $a \in \pi(\boldsymbol{a})$, we have $a \in D \cap \pi(\boldsymbol{a}) = \boldsymbol{a}$.   □

So we deduce the following limitative result.

Theorem 6.2   *Assume that $\mathcal{S}$ is π_D-replete and let D be such that $D \supseteq \mathrm{T}$. Then, whenever $A \in \mathrm{fix}(\pi_D)$ and $A \subseteq \perp_{\pi_D}^{\mathcal{S}}$, we have $A \notin s[V]$. In particular, $\perp_{\pi_D}^{\mathcal{S}} \notin s[V]$ provided $\mathcal{S}$ is closed under π_D.*

Proof   Suppose that $A \in \mathrm{fix}(\pi_D)$, $A \subseteq \perp_{\pi}^{\mathcal{S}}$, and $A = \boldsymbol{a}$ for some $a \in V$. As $\mathcal{S}$ is π_D-replete and $\boldsymbol{a} \in \mathrm{ind}(\pi_D)$, it is the case that $B \backslash \{a\} \in \mathcal{S}$ for some $B \in \mathrm{ind}(\pi_D)$, so $a \notin_s a$ by Proposition 4.1. But $a \in_s a$ by Lemma 6.1. Now, that $\perp_{\pi_D}^{\mathcal{S}} \notin s[V]$ whenever $\mathcal{S}$ is closed under π_D follows from Fact 3.2 given that π_D preserves $\bigcap$—this is a right adjoint too.   □

It is worth stressing that Theorem 6.2 is applicable only to *fixpoints* of π_D that are below $\perp_{\pi_D}^{\mathcal{S}}$, and this latter may well *not* be such a fixpoint. Accordingly, Theorem 4.5 is *not*—strictly speaking—a special case of Theorem 6.2, although $\pi_D = \pi$ for $D = V$. The restriction to fixpoints of π_D comes from Lemma 6.1 and appears to be necessary in the proof, as well as the condition $D \supseteq \mathrm{T}$. That said, given that $\perp_{\pi_D}^{\mathcal{S}} \subseteq \perp_{\pi}^{\mathcal{S}}$, we remark that Theorem 6.2 brings nothing new when D is $\in$-inductive, because of the following simple observation.

Fact 6.3   Suppose $\pi(D) \subseteq D$. Then $\mathrm{fix}(\pi_D) \subseteq \mathrm{fix}(\pi)$.

Proof   Suppose $\pi_D(A) = A$. Then $A \subseteq \pi(A)$ and $A \subseteq D$. From the latter and our assumption on D, we get $\pi(A) \subseteq D$. So $\pi(A) \subseteq D \cap \pi(A) = A$, and then $\pi(A) = A$, showing that $\mathrm{fix}(\pi_D) \subseteq \mathrm{fix}(\pi)$.   □

We shall see in the next section that for some non-$\in$-inductive $D \supseteq \mathrm{T}$ there are indeed problematic subsets provided by Theorem 6.2 that are themselves *not* $\in$-inductive—so Theorem 6.2 brings something genuinely new. This happens most notably in the case where $D = \mathrm{T}$ and the corresponding limitation is just related

to the Burali-Forti paradox. Other choices for D will provide us with lesser-known paradoxical classes of naïve set theory.

7 Well-foundedness and Transitivity

The Burali-Forti paradox concerns the set of all *ordinals*, thought of as concretizations of order types of well-ordered sets. In ordinary set theory we usually use the *von Neumann* ordinals for this purpose. However, this requires nontrivial set-theoretic assumptions such as replacement. Nevertheless, people do things like (i) implement ordinals *à la von Neumann*, and then (ii) argue that the collection of all von Neumann ordinals cannot be a set because of Burali-Forti. But that is a bit misleading. The collection of von Neumann ordinals is a paradoxical object quite independently of it being the destination of the collection of all ordinals under the von Neumann implementation.

Fact 7.1   The von Neumann ordinals are precisely the well-founded hereditarily transitive sets.[11]

Informal proof   That every von Neumann ordinal is hereditarily transitive follows from the facts that every von Neumann ordinal (i) is transitive and (ii) is the set of all its predecessors—all of which are transitive. So by $\in$-induction every von Neumann ordinal is not only transitive but hereditarily transitive.

For the other direction we remark that a transitive set of von Neumann ordinals is itself a von Neumann ordinal. So it follows by $\in$-induction on the hereditarily transitive sets that they are all von Neumann ordinals. $\square$

The point is that the pure set-theoretic paradox of the collection of well-founded hereditarily transitive sets is not only the von Neumann implementation of the Burali-Forti paradox, but also a manifestation of Theorem 6.2.

Given a set-theoretic structure $\langle V, s \rangle$ and a property $\varphi(\mathsf{x})$ in $\mathscr{L}_V$, we write H^1_φ for $\perp^{\mathcal{P}_1(V)}_{\pi_{V_\varphi}}$ where $V_\varphi := \{a \in V \mid \langle V, s \rangle \models \varphi(a)\}$. Clearly $\mathrm{H}^1_\varphi \subseteq \mathrm{H}^1_{\mathtt{true}} = \mathrm{Wf}^1$, and by Definition 3.1, H^1_φ can be characterized as the *largest* subset A of Wf^1 satisfying the following principle of $\in$-induction:

$$\forall B \in \mathcal{P}_1(V), \quad \pi(B) \cap V_\varphi \subseteq B \Longrightarrow A \subseteq B. \tag{1}$$

Note there can be at most one $A \in \mathrm{ind}(\pi_{V_\varphi}) \cap \mathcal{P}_1(V)$ satisfying (1). On the other hand, since $\mathcal{P}_1(V)$ is closed under π_{V_φ}, H^1_φ not only belongs to $\mathrm{ind}(\pi_{V_\varphi})$, but also to $\mathrm{fix}(\pi_{V_\varphi})$ by Fact 3.2(ii). In other words, $A := \mathrm{H}^1_\varphi$ is a solution to the reflexive equation $A = \pi_{V_\varphi}(A)$, which we expand into

$$\forall a \in V, \quad a \in A \iff \varphi_s(a) \,\&\, \boldsymbol{a} \subseteq A. \tag{2}$$

Now, assuming H^1_φ is definable, it would be entirely characterized by (2), because of the following.

Fact 7.2   There is at most one *definable* $A \subseteq \mathrm{Wf}^1$ satisfying (2) above.

Proof   The proof goes by $\in$-induction. Suppose $A_1, A_2 \in \mathcal{P}_1(V)$ satisfy (2). Then let $B := \{a \in V \mid a \in A_1 \Leftrightarrow a \in A_2\}$. Clearly, $B \in \mathcal{P}_1(V)$, and we show that $B \in \mathrm{ind}(\pi)$. Let $a \in V$ such that $\boldsymbol{a} \subseteq B$. If $a \in A_1$, then $\varphi_s(a)$ and $\boldsymbol{a} \subseteq A_1$ by (2)[$\Rightarrow$], so $\boldsymbol{a} \subseteq A_2$ by the definition of B, since $\boldsymbol{a} \subseteq B$. Hence $a \in A_2$ by (2)[$\Leftarrow$]. Likewise, $a \in A_2$ implies $a \in A_1$. Thus $a \in B$. Now, given that $\pi(B) \subseteq B$, we get

$\mathrm{Wf}^1 \subseteq B$ by (1)[with $\varphi :\equiv \mathtt{true}$] and it follows therefrom that $A_1 = A_2$ whenever $A_1, A_2 \subseteq \mathrm{Wf}^1$. □

As it is, (2) could only serve as a *recursive* definition of H^1_φ. As in the particular case of Wf^1 discussed in Section 5, some set-theoretic assumptions might be required in order to show that $\mathrm{H}^1_\varphi \in \mathcal{P}_{1*}(V)$. It is well known, for instance, that in any set theory where the operation of transitive closure '$\mathcal{TC}$_' is available, and where foundation holds, H^1_φ can be defined as

$$`\{x \mid \forall y(y \in \mathcal{TC}\{x\} \to \varphi(y))\}`.$$

On the other hand, in ZF *without* foundation as well as in GPK$^+$ (cf. Example 5.6), H^1_φ can be defined as the sharpened cumulative hierarchy one obtains by transfinite iteration of the power set operation restricted to φ-subsets. Accordingly, and in view of (2), we shall refer to H^1_φ as *the collection of well-founded hereditarily φ-sets*, which is a natural set-theoretical object.

Remark 7.3   As we did in Section 5 for well-founded sets, one could also consider the zero-order version H^0_φ, which at least always belongs to $\mathcal{P}_{1*}(V)$. However, the particular case of Wf^0 has revealed that this really seems to be of minor interest from a *classical* set-theoretic point of view, say.

It follows from Theorem 6.2 that H^1_φ is intrinsically problematic whenever $\mathrm{T} \subseteq V_\varphi$, that is, whenever the property φ holds for *transitive* sets. Such a property is *unbounded* in the sense that it is true in particular for all (von Neumann) ordinals. When φ is *inductive*, $\mathrm{H}^1_\varphi = \mathrm{H}^1_{\mathtt{true}} = \mathrm{Wf}^1$ by Fact 6.3; when φ is *hereditary*, it is worth noting that H^1_φ has a simple form.

Fact 7.4   Suppose φ is hereditary, that is, $V_\varphi \subseteq \pi(V_\varphi)$. Then $\mathrm{H}^1_\varphi = \mathrm{Wf}^1 \cap V_\varphi$.

Proof   We are tacitly assuming here that both H^1_φ and Wf^1 are definable. Then, in view of Fact 7.2, we are going to show that $\mathrm{Wf}^1 \cap V_\varphi \in \mathrm{fix}(\pi_{V_\varphi})$. We have $\pi_{V_\varphi}(\mathrm{Wf}^1 \cap V_\varphi) = \pi(\mathrm{Wf}^1 \cap V_\varphi) \cap V_\varphi = \pi(\mathrm{Wf}^1) \cap \pi(V_\varphi) \cap V_\varphi = \mathrm{Wf}^1 \cap V_\varphi$, since $\mathrm{Wf}^1 \in \mathrm{fix}(\pi)$ and $V_\varphi \subseteq \pi(V_\varphi)$. □

The property of being transitive, that is, $\mathtt{trans}(x) :\equiv x \subseteq \mathcal{P}x$, or equally $\mathtt{trans}(x) \equiv \bigcup x \subseteq x$, is neither inductive nor hereditary, and is the least choice for φ. It gives rise to the collection $\mathrm{H}^1_{\mathtt{trans}}$ of hereditarily well-founded transitive sets, which we identified as the collection of von Neumann ordinals. Here are *natural* examples of φ for which $\mathrm{H}^1_{\mathtt{trans}} \subsetneqq \mathrm{H}^1_\varphi \subsetneqq \mathrm{Wf}^1$ is consistent.

Example 7.5   Call a set x *extensional* if the membership relation restricted to the members of x is extensional, that is, if the following property holds:

$$\mathtt{ext}(x) :\equiv \forall y \forall z(y \in x \wedge z \in x \wedge y \cap x = z \cap x \to y = z).$$

In any set theory, transitive sets are extensional—so are transitive *classes*, more generally, which is a mere consequence of the axiom of extensionality. Therefore, the collection $\mathrm{H}^1_{\mathtt{ext}}$ of hereditarily well-founded extensional sets is genuinely a paradoxical object. Basic set-theoretic assumptions would ensure that $\mathrm{H}^1_{\mathtt{trans}} \subsetneqq \mathrm{H}^1_{\mathtt{ext}} \subsetneqq \mathrm{Wf}^1$. For instance, assuming pairing and the existence of the empty set only, one can see that $\mathrm{H}^1_{\mathtt{ext}} \models$ singleton, but $\mathrm{H}^1_{\mathtt{ext}} \nvDash$ pairing.

Example 7.6   Call a set x *normal* if it has a transitive closure, that is, if the following property holds:

$$\texttt{norm}(x) :\equiv \exists y(x \subseteq y \wedge y \subseteq \mathcal{P}y \wedge \forall z(x \subseteq z \wedge z \subseteq \mathcal{P}z \rightarrow y \subseteq z)).$$

Obviously, any transitive set is normal, so the collection $H^1_{\texttt{norm}}$ of hereditarily well-founded normal sets is a paradoxical object. Note that assuming power set and Δ_0-separation, a set is normal if and only if it belongs to a transitive set;[12] that is,

$$\texttt{norm}(x) \equiv \exists y(x \in y \wedge y \subseteq \mathcal{P}y).$$

Thus formulated, normality is plainly a hereditarily property, so that $H^1_{\texttt{norm}} = \text{Wf}^1 \cap V_{\texttt{norm}}$ by Fact 7.4. Now, Zermelo's system (including foundation) is consistent with $H^1_{\texttt{trans}} \subsetneq H^1_{\texttt{norm}} \subsetneq \text{Wf}^1 (= V)$, for transitive containment is not provable: it is consistent to have sets without transitive closure in Z (see [1] or [7] for other references).

Remark 7.7 (Hinnion)   One can concoct many other examples from the mere definition of transitivity if one assumes that some (definable) notion of cardinality ($|_-|, \leqslant$) is available. For example, the following choices for $\varphi(x)$ will do:

- $|\bigcup x| \leqslant |x|$;
- $\forall y(y \in x \rightarrow |y| \leqslant |x|)$;
- $\forall y(y \in x \rightarrow |\mathcal{P}y| \leqslant |\mathcal{P}x|)$.

It might be interesting to find an axiomatic characterization of particular paradoxical classes H^1_φ such as those in Examples 7.5 and 7.6. After all, the axioms of Zermelo-Fraenkel—including foundation—are just meant to characterize $H^1_{\texttt{true}}$. We leave this task to workers who will come after us.

Appendix

A constructive proof of the nonexistence of '{x | ∀y(x ∈ y → y ∉ x)}':

The derivation (3) is a sequent-calculus proof tree, displayed rotated on the page. Its nodes, read from the leaves down to the conclusion, are:

$$
\dfrac{b \in a \vdash b \in a}
{a \in b, b \in a, b \in a \to (\forall z)(z \in b \to b \notin z), (\forall z)(z \in b \to b \notin z) \to b \in a \vdash}
$$

Left branch:

$$
a \in b \vdash a \in b
\qquad
\dfrac{b \in a \vdash b \in a \qquad b \in a, b \notin a \vdash}{\ }\ \neg L
$$

$$
\dfrac{\ }{b \in a, a \in b, (\forall z)(z \in b \to b \notin z) \vdash}\ \to L
$$

$$
\dfrac{\ }{a \in b, b \in a \to (\forall z)(z \in b \to b \notin z) \vdash b \in a}\ \forall L
$$

$$
\dfrac{\ }{b \in a, b \in a \to (\forall z)(z \in b \to b \notin z) \vdash a \notin b}\ \to L
$$

$$
\dfrac{\ }{b \in a \to (\forall z)(z \in b \to b \notin z) \vdash b \in a \to a \notin b}\ \neg R
$$

$$
\dfrac{\ }{b \in a \leftrightarrow (\forall z)(z \in b \to b \notin z) \vdash b \in a \to a \notin b}\ \wedge L_1 \quad \to R
$$

$$
\dfrac{\ }{(\forall y)(y \in a \leftrightarrow (\forall z)(z \in y \to y \notin z)) \vdash b \in a \to a \notin b}\ \forall L
$$

$$
(\forall y)(y \in a \leftrightarrow (\forall z)(z \in y \to y \notin z)) \vdash (\forall z)(z \in a \to a \notin z)\ \forall R
$$

Right branch:

$$
a \in a \vdash a \in a
\qquad
\dfrac{a \in a \vdash a \in a \qquad a \in a, a \notin a \vdash}{\ }\ \neg L
$$

$$
\dfrac{\ }{a \in a, a \in a \to a \notin a, a \in a \vdash}\ \to L
$$

$$
\dfrac{\ }{(\forall z)(z \in a \to a \notin z), a \in a \vdash}\ \forall L
$$

$$
\dfrac{\ }{a \in a \to (\forall z)(z \in a \to a \notin z), a \in a \vdash}\ \to L
$$

$$
\dfrac{\ }{a \in a \leftrightarrow (\forall z)(z \in a \to a \notin z), a \in a \vdash}\ \wedge L_1
$$

$$
\dfrac{\ }{(\forall y)(y \in a \leftrightarrow (\forall z)(z \in y \to y \notin z)), a \in a \vdash}\ \forall L
$$

Combining and concluding:

$$
\dfrac{\ }{(\forall y)(y \in a \leftrightarrow (\forall z)(z \in y \to y \notin z)),\ (\forall y)(y \in a \leftrightarrow (\forall z)(z \in y \to y \notin z)) \vdash}\ \to L
$$

$$
\dfrac{\ }{(\forall y)(y \in a \leftrightarrow (\forall z)(z \in y \to y \notin z)) \vdash}\ \text{Contract-L}
$$

$$
\dfrac{\ }{(\exists x)(\forall y)(y \in x \leftrightarrow (\forall z)(z \in y \to y \notin z)) \vdash}\ \exists L
$$

(3)

Notes

1. There is a tradition, going back to the earliest days of set theory, of having a more in-
   clusive view of what set theory is. There may be a more inclusive version for which it
   is possible to supply both a clear definition and a motivation, but we have never encoun-
   tered one. We quote Zermelo here ([11] p. 200):

   > Set theory is that branch of mathematics whose task is to investigate math-
   > ematically the fundamental notions 'number', 'order' and 'function', taking
   > them in their pristine simple form, and to develop thereby the logical foun-
   > dations of all of arithmetic and analysis; thus it constitutes an indispensable
   > component of the science of mathematics.

2. Di Giorgi never wrote it up, but the idea is explained and exploited in [4].

3. It is true that in some proof-theoretic presentations the *proof* of Cantor's theorem can be
   pathological (lacks a normal form) but these features are not our concern here.

4. We are referring here to the *algebraic* notion of substitution in free algebras, which gen-
   eralizes the syntactic notion of substitution in term algebras—these being the *absolutely*
   free algebras on the corresponding signature. The reader not familiar with that notion
   may just think here of members of $\iota[V]$ (or simply V) as "variables" and members of
   $\mathcal{P}(V)$ as "terms" built up from variables with the operation "$\bigcup$" (of unbounded arity).

5. See also [8].

6. It follows in particular from (iii)—not from (ii)—that $\perp_{\theta}^{\mathcal{P}(V)} = \perp_{\theta} \in \text{fix}(\theta)$, for *any* θ.

7. Recall that we are not assuming choice in the metatheory.

8. That is, for all $a \in V$ and $B \in \mathcal{P}_1(V)$, $\boldsymbol{a} \cap B \in s[A]$.

9. That is, for all $a \in V$, $\boldsymbol{a} \cup \{a\} \in s[V]$.

10. That is, for each $a \in V$, there is b such that $a \in_s b$ and $\boldsymbol{b} \subseteq \pi(\boldsymbol{b})$. Assuming Δ_0-
    separation and power set, transitive containment is equivalent to the axiom of transitive
    closure.

11. This fact (assuming foundation) has been known for some time; for example, it was
    stated in a letter from Bernays to Gödel of 3 May 1931. See [5] p. 112.

12. Cf. note attached to Fact 5.5.

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