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A Covering Lemma for HOD of $K(\mathbb{R})$

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Abstract Working in ZF + AD alone, we prove that every set of ordinals with cardinality at least Θ can be covered by a set of ordinals in HOD of $K(\mathbb{R})$ of the same cardinality, when there is no inner model with an $\mathbb{R}$-complete measurable cardinal. Here $\mathbb{R}$ is the set of reals and Θ is the supremum of the ordinals which are the surjective image of $\mathbb{R}$.

1 Covering Lemmas in a Playful Universe

The following theorem, established in [4], demonstrates that interesting covering lemmas are indeed possible in a "choiceless" universe in which the axiom of determinacy (AD) holds.

Theorem 1.1 (The Covering Lemma for $L(\mathbb{R})$) *Assume* ZF + AD *and suppose that* $\mathbb{R}^\#$ *does not exist.*

(i) *If X is a set of reals, then $X \in L(\mathbb{R})$.*
(ii) *If X is a set of ordinals with $|X| \geq \Theta$, then there is a set of ordinals $Y \in L(\mathbb{R})$ such that $X \subseteq Y$ and $|X| = |Y|$.*

Thus, assuming ZF + AD and $\mathbb{R}^\#$ does not exist, we can conclude that all sets of reals are in $L(\mathbb{R})$ and thus, $\Theta^{L(\mathbb{R})} = \Theta$. Furthermore, as a corollary, we have that V and $L(\mathbb{R})$ have exactly the same cardinals and virtually all the same cofinalities. Let $|\lambda|$ and $\mathrm{cf}(\lambda)$ denote the cardinality and cofinality, respectively, of an ordinal λ.

Corollary 1.2 *Assume* ZF + AD *and that* $\mathbb{R}^\#$ *does not exist. For all ordinals λ, we have the following:*

(1) $|\lambda|^{L(\mathbb{R})} = |\lambda|$,
(2) $\mathrm{cf}(\lambda)^{L(\mathbb{R})} = \mathrm{cf}(\lambda)$ *when* $\lambda \leq \Theta$ *or* $\mathrm{cf}(\lambda) \geq \Theta$.

In particular, Θ is regular in the universe.

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Proof Since all sets of reals are in $L(\mathbb{R})$, the Moschovakis Coding Lemma [10] implies that all subsets of $\lambda < \Theta$ are in $L(\mathbb{R})$. Statements (1) and (2) now follow from Theorem 1.1. The proof in [7, see 28.19] showing that Θ is regular in $L(\mathbb{R})$ also implies that Θ is regular in V. □

Of course, part (ii) of the above Theorem 1.1 is an analogue of Jensen's Covering Lemma which asserts that if $0^{\#}$ does not exist, then every uncountable set of ordinals can be covered by a constructible set of ordinals of the same cardinality. Jensen's proof of his covering lemma makes liberal use of the axiom of choice.[1] Our proof of Theorem 1.1, however, explicitly refrains from employing any choice principles which are not provable, or are unknown to be provable, in ZF + AD. In particular, our proof makes no appeal to $\mathrm{DC}_{\mathbb{R}}$, the axiom of dependent choices restricted to relations on the reals.

Definition 1.3 Let μ be a normal measure on κ. We say that μ is an $\mathbb{R}$-*complete measure* on κ whenever $\langle A_x : x \in \mathbb{R}\rangle$ is any sequence such that $A_x \in \mu$ for all $x \in \mathbb{R}$, then $\bigcap_{x\in\mathbb{R}} A_x \in \mu$.

Definition 1.4 If μ is an $\mathbb{R}$-complete measure on κ in $L[\mu](\mathbb{R})$, then the inner model $L[\mu](\mathbb{R})$ is said to be a $\rho(\mathbb{R})$-model.

In [5], again assuming only the axiom of determinacy and no weak forms of the axiom of choice, we generalize the above covering lemma for $L(\mathbb{R})$ to an inner model which we call $K(\mathbb{R})$ to obtain the following theorem.

Theorem 1.5 (The Covering Lemma for $K(\mathbb{R})$) *Assume* ZF + AD. *Suppose no $\rho(\mathbb{R})$-model exists.*

(i) *If X is a set of reals, then $X \in K(\mathbb{R})$.*
(ii) *If X is a set of ordinals with $|X| \geq \Theta$, then there is a set of ordinals $Y \in K(\mathbb{R})$ such that $X \subseteq Y$ and $|X| = |Y|$.*

Thus, assuming ZF + AD and no $\rho(\mathbb{R})$-model exists, we conclude that all sets of reals and all subsets of $\lambda < \Theta$ are in $K(\mathbb{R})$. Furthermore, V and $K(\mathbb{R})$ have the same cardinals and many of the same cofinalities.

Corollary 1.6 *Assume* ZF + AD *and that no $\rho(\mathbb{R})$-model exists. For all ordinals λ, we have the following:*

(1) $|\lambda|^{K(\mathbb{R})} = |\lambda|$,
(2) $\mathrm{cf}(\lambda)^{K(\mathbb{R})} = \mathrm{cf}(\lambda)$ *when* $\lambda \leq \Theta$ *or* $\mathrm{cf}(\lambda) \geq \Theta$.

In particular, Θ is regular in the universe.

Our proof of part (i) of Theorem 1.5 in [5] relies on an analysis of the fine structure of $K(\mathbb{R})$ and its descriptive set theory (see [3]). Moreover, assuming ZF + AD and no $\rho(\mathbb{R})$-model exists, one can prove that any inner model satisfying parts (i) and (ii) of Theorem 1.5 must contain $K(\mathbb{R})$ as an inner model. So, $K(\mathbb{R})$ is the least inner model satisfying conclusions (i) and (ii).

Part (ii) of Theorem 1.5 closely resembles the classic covering lemma for K, the Dodd-Jensen core model. The Dodd-Jensen Covering Lemma states that every uncountable set of ordinals can be covered by a set of ordinals in K of the same cardinality, when there is no inner model with a measurable cardinal. Under the ambient theory ZF + AD, it is easy to show that K is a proper inner model of $K(\mathbb{R})$;

that is, $K \subsetneq K(\mathbb{R})$. One can also show that conclusion (ii) in the above theorem fails to hold when $K(\mathbb{R})$ is replaced with K. In this paper we shall address the following question.

Question 1.7 *Assume* ZF + AD *and suppose no $\rho(\mathbb{R})$-model exists. Can conclusion (ii) in Theorem 1.5 hold for a proper inner model of $K(\mathbb{R})$? An inner model of* ZFC*?*

In Section 6 we shall establish the following positive answer to Question 1.7.

Theorem 6.1 (The Covering Lemma for HOD of $K(\mathbb{R})$) Assume ZF + AD and that no $\rho(\mathbb{R})$-model exists. If X is a set of ordinals with $|X| \geq \Theta$, then there is a set of ordinals $Y \in \mathrm{HOD}^{K(\mathbb{R})}$ such that $X \subseteq Y$ and $|X| = |Y|$.

Thus, assuming ZF + AD and that no $\rho(\mathbb{R})$-model exists, V and $\mathrm{HOD}^{K(\mathbb{R})}$ possess the same cardinals and cofinalities greater than or equal to Θ.

Corollary 6.2 Assume ZF+AD and that no $\rho(\mathbb{R})$-model exists. Let $H = \mathrm{HOD}^{K(\mathbb{R})}$. For all ordinals λ,

(1) $|\lambda|^H = |\lambda|$ when $\lambda \geq \Theta$,
(2) $\mathrm{cf}(\lambda)^H = \mathrm{cf}(\lambda)$ when $\mathrm{cf}(\lambda) \geq \Theta$.

The above Theorem 6.1 provokes another question. Assuming ZF + AD, is there a proper inner model of $\mathrm{HOD}^{K(\mathbb{R})}$ satisfying the conclusion of Theorem 6.1? In fact, one can prove that any inner model of $\mathrm{HOD}^{K(\mathbb{R})}$ satisfying the conclusion of Theorem 6.1 (and containing a relevant partial order) must equal $\mathrm{HOD}^{K(\mathbb{R})}$.

We end this section with the motivation behind our proof of Theorem 6.1. The following unpublished result of Woodin (see [8, p. 278]) asserts that $L(\mathbb{R})$ is a symmetric generic extension of its version of HOD.

Theorem 1.8 (Woodin) *Assume* ZF + AD *and* $V = L(\mathbb{R})$*. There is an* $S \subseteq \Theta$ *such that* HOD $= L(S)$*. Moreover, there is a partial order* $\mathbb{P}$ *in* HOD *such that*

(1) $\mathbb{P}$ *has cardinality* Θ *in* HOD*,*
(2) $\mathbb{P}$ *has the* Θ*-chain condition in* HOD*,*
(3) $L(\mathbb{R})$ *is a symmetric* $\mathbb{P}$*-generic extension of* HOD.

Using Theorem 1.1 and (1)–(3) of Theorem 1.8, we prove the following in [4].

Theorem 1.9 (The Covering Lemma for HOD of $L(\mathbb{R})$) *Assume* ZF + AD *and that* $\mathbb{R}^{\#}$ *does not exist. If X is a set of ordinals with $|X| \geq \Theta$, then there is a set of ordinals $Y \in \mathrm{HOD}^{L(\mathbb{R})}$ such that $X \subseteq Y$ and $|X| = |Y|$.*

The strategy behind our proof of Theorem 1.9 in [4] was to exploit Theorems 1.1 and 1.8. In the present paper, we shall show that this strategy can be extended to prove Theorem 6.1 as well.

Paper organization This paper is organized into six sections. Section 2 presents the basic notation and concepts that will be presumed throughout the paper. In the interest of making the paper self-contained, Section 3 presents a survey of the work in [2] that we will need here, including a comprehensive definition of $K(\mathbb{R})$. In Section 4 we present some theorems concerning ordinal definability and ∞-Borel codes in $K(\mathbb{R})$. These theorems will be used in Section 5 to show that $K(\mathbb{R})$ is a symmetric extension of $\mathrm{HOD}^{K(\mathbb{R})}$ (see Remark 3.11). Finally, in Section 6 we prove Theorem 6.1. Our proof ensures that there is no need for any weak forms of the axiom of choice which are not provable in ZF + AD.

2 Preliminaries and Notation

Let ω be the set of all natural numbers and let $\mathbb{R} = {}^{\omega}\omega$ be the set of all functions from ω to ω. We call $\mathbb{R}$ the set of reals and we let Θ be the supremum of the ordinals which are the surjective image of $\mathbb{R}$. Under the axiom of determinacy, Θ is a limit cardinal. Given an inner model M, we will write Θ^M to denote the interpretation of Θ within M. We shall always write Θ to denote Θ^V, the Θ of the universe. The smallest inner model of ZF containing the reals is denoted by $L(\mathbb{R})$.

We work in ZF and state our additional hypotheses as required. We do this to maintain a careful surveillance in our proofs on the use of determinacy and any principles of choice. We let OR denote the class of ordinals. We shall say that a proper class M is an *inner model* if and only if M is a transitive $\in$-model of ZF with $\mathrm{OR} \subseteq M$. We distinguish between the notations $L(A)$ and $L[A]$. The inner model $L(A)$ is defined to be the class of sets constructible *above* A; that is, one starts with a set A and iterates definability in the language of set theory. Thus, $L(A)$ is the smallest inner model M such that $A \in M$. The inner model $L[A]$ is defined to be the class of sets constructible *relative* to A; that is, one starts with the empty set and iterates definability in the language of set theory augmented by the predicate A. When A is a set, $L[A]$ is the smallest inner model M such that $A \cap M \in M$ (see p. 34 of [7]). Furthermore, one defines $L[A, B]$ to be the class of sets constructible *relative* to A and B, whereas $L[A](B)$ is defined as the class of sets constructible *relative* to A and *above* the set B.

Let M be any inner model of ZF and let $\mathbb{P} = (P, \leq)$ be a partial order in M. If G is $\mathbb{P}$-generic over M, we let $M[G]$ be the resulting generic extension of M. We let $\check{x}$ represent a canonical name for $x \in M$, and we shall write $p \in \mathbb{P}$ when we mean $p \in P$. For any forcing notation or terminology which we do not define, we refer the reader to Kunen [9].

Definition 2.1 Let M be a transitive model of ZF and let $\mathbb{P} \in M$ be a partial order. Let $\dot{\tau}$ be a $\mathbb{P}$-name in M. We shall say that $\mathbb{P}$ is *$\dot{\tau}$-homogeneous over M* if for all first-order formulas $\varphi(v, v_1, \ldots, v_n)$ and all $x_1, \ldots, x_n \in M$ we have that $p \Vdash \varphi(\dot{\tau}, \check{x}_1, \ldots, \check{x}_n)$ if and only if $q \Vdash \varphi(\dot{\tau}, \check{x}_1, \ldots, \check{x}_n)$, for all $p, q \in \mathbb{P}$.

Definition 2.2 Let M and N be transitive models of ZF and let $\mathbb{P} \in M$ be a partial order. We shall say that N is a symmetric $\mathbb{P}$-generic extension of M if there is a G which is $\mathbb{P}$-generic over M such that $M \subseteq N \subseteq M[G]$ and N is a symmetric submodel (see [6, p. 249]) of $M[G]$.

Let $\bigwedge \Psi$ be the conjunction of a finite set Ψ of sentences of set theory. If $\bigwedge \Psi$ has a transitive model (e.g., V_α), then Barwise [1, Theorem 8.10] proves in ZF that $\bigwedge \Psi$ has a transitive model in L, the constructible universe of sets. So if a sentence ψ is true in every countable transitive model in L of a sufficiently large finite fragment of ZF, then it follows (using the reflection principle) that $\mathrm{ZF} \vdash \psi$. There will be times when we want to prove that a certain statement ψ holds in V. To do this, we may implement the following strategy: we presume (without loss of generality) that V is a countable transitive set and then naïvely (as if V were a countable transitive model in L) construct a generic extension $V[G]$. After working in $V[G]$, we show that ψ holds in V.

Given a set model $\mathcal{M} = (M, c_1, \ldots, c_m, A_1, \ldots, A_N)$, where the A_i are predicates and the c_i are constants, let $X \subseteq M$. Then $\Sigma_n(\mathcal{M}, X)$ is the class of relations on M definable over $\mathcal{M}$ by a Σ_n formula from parameters in $X \cup \{c_1, \ldots, c_m\}$. We write "$\Sigma_n(\mathcal{M})$" for $\Sigma_n(\mathcal{M}, \varnothing)$ and "$\boldsymbol{\Sigma}_n(\mathcal{M})$" for the boldface class $\Sigma_n(\mathcal{M}, M)$. Let $H = \{a \in M : \{a\} \text{ is } \Sigma_n(\mathcal{M}, X)\}$. The Σ_n *hull* of X is the substructure $\mathrm{Hull}_n^{\mathcal{M}}(X) = (H, c_1, \ldots, c_m, H \cap A_1, \ldots, H \cap A_N)$. In addition, for any two models $\mathcal{M}$ and $\mathcal{N}$, we write $\pi : \mathcal{M} \underset{\Sigma_n}{\longrightarrow} \mathcal{N}$ to indicate that the map π is a Σ_n-elementary embedding; that is, $\mathcal{M} \models \varphi[a]$ if and only if $\mathcal{N} \models \varphi[\pi(a)]$, for all $a = \langle a_0, a_1, \ldots \rangle \in (M)^{<\omega}$ and for all Σ_n formulas φ, where $0 \leq n < \omega$ and $\pi(a) = \langle \pi(a_0), \pi(a_1), \ldots \rangle$. Finally, we may write $a \in \mathcal{M}$ when we mean $a \in M$.

2.1 Embedding the partial order $\mathbb{Q}$ into a Boolean algebra $\mathcal{B}$ We shall often refer to the following separative partial order.

Definition 2.3 Let $\mathbb{Q} = (Q, <)$ where $Q = {}^{<\omega}\mathbb{R} = \{s \in {}^{n}\mathbb{R} : n \in \omega\}$ and $p < q$ if and only if $q \subsetneq p$ for $p, q \in Q$.

We now briefly mention the key ingredients used in [6] to construct the unique Boolean algebra $\mathcal{B}$ which forms the "completion" of $\mathbb{Q}$.

Definition 2.4 Let $C \subseteq Q$. Then C is a *cut* if $q \in C$ and $p \leq q$ implies $p \in C$.

For every $p \in Q$, let U_p denote the cut $\{x \in Q : x \leq p\}$. We shall refer to any such U_p as a *basic open set*.

Definition 2.5 A cut C is *regular* if $q \notin C$ implies $(\exists r \leq q)(U_r \cap C = \varnothing)$.

For all $p \in Q$, it follows that U_p is a regular cut. Let C be a cut. Define $\overline{C} = \{p \in Q : (\forall q \leq p)(C \cap U_q \neq \varnothing)\}$ and $-C = \{p \in Q : U_p \cap C = \varnothing\}$ which are both regular cuts. Let $\mathcal{B}$ be the set of all regular cuts in $\mathbb{Q}$. For $C, D \in \mathcal{B}$ define

$$C \cdot D = C \cap D, C + D = \overline{C \cup D} \text{ and } -C = \{p \in Q : U_p \cap C = \varnothing\}.$$

It follows that $\mathcal{B}$ is a complete Boolean algebra. For $C, D \in \mathcal{B}$ we define $C \leq D$ if and only if $C \subseteq D$. Let $\mathfrak{o} = \varnothing$ represent the smallest element in $\mathcal{B}$. We now restate Lemma 3.3 of [9] in terms of $\mathbb{Q}$ and the Boolean algebra $\mathcal{B}$.

Lemma 2.6 *Let $\mathbb{Q} = (Q, <)$ be as above. Then the embedding $\sigma : \mathbb{Q} \to \mathcal{B} \setminus \{\mathfrak{o}\}$ defined by $\sigma(p) = U_p$ satisfies*

1. *$\sigma[Q]$, the image of σ, is dense in $\mathcal{B} \setminus \{\mathfrak{o}\}$;*
2. *for all $p, q \in Q$ if $p \leq q$, then $\sigma(p) \subseteq \sigma(q)$;*
3. *for all $p, q \in Q$ if $p \perp q$, then $\sigma(p) \perp \sigma(q)$.*

We will say that a partial order $\mathbb{P}$ satisfies the $\mathbb{R}$-*chain condition* if for every antichain A in $\mathbb{P}$, there is a surjective function $f : \mathbb{R} \to A$. Since every antichain A in $\mathbb{Q}$ is essentially a subset of $\mathbb{R}$, we see that $\mathbb{Q}$ satisfies the $\mathbb{R}$-chain condition. Thus, because $\mathbb{Q}$ is "dense" in $\mathcal{B}$, we have the following result.

Lemma 2.7 *The Boolean algebra $\mathcal{B}$ satisfies the $\mathbb{R}$-chain condition.*

We will now verify that $\mathcal{B}$ is *countably generated*; that is, there exists a countable set $X \subseteq \mathcal{B}$ such that $\mathcal{B}$ is the smallest complete Boolean algebra containing X.

Definition 2.8 For each $m, n \in \omega$ and $s \in {}^{n}\omega$, define

$$N_{m,s} = \{\langle x_0, x_1, \ldots x_k\rangle \in {}^{<\omega}\mathbb{R} : k \geq m \text{ and } x_m \upharpoonright n = s\}.$$

We shall refer to any such $N_{m,s}$ as a *generator*.

We observe that each $N_{m,s}$, in the above definition, is a regular cut. Let $m, n, k \in \omega$ and $p = \langle x_0, x_1, \ldots x_{k-1}\rangle \in {}^{k}\mathbb{R}$. When $m < k$ we define $p \upharpoonright \langle m, n\rangle = x_m \upharpoonright n$. Then

$$N_{m,s} = \bigcup\{U_p : p \text{ has length } m+1 \text{ and } p \upharpoonright \langle m, n\rangle = s\}.$$

Thus, each $N_{m,s}$ can be expressed as the union of basic open sets. Furthermore, for $p \in {}^{k}\mathbb{R}$, we have that $U_p = \bigcap\{N_{m,p\upharpoonright\langle m,n\rangle} : m < k \text{ and } n \in \omega\}$. So, each U_p can be expressed as an intersection of generators. It follows that $X = \{N_{m,s} : s \in {}^{n}\omega \text{ and } m, n \in \omega\}$ is a countable set of generators for $\mathcal{B}$ and thus, $\mathcal{B}$ is countably generated. We note that for any permutation $\sigma : \omega \to \omega$, the map $N_{m,s} \mapsto N_{\sigma(m),s}$ induces an automorphism of the Boolean algebra $\mathcal{B}$.

2.2 ∞-Borel codes A key idea in Woodin's proof of Theorem 1.8 is the notion of an ∞-Borel set. We now define by transfinite recursion the concept of an ∞-Borel code and its interpretation in the Boolean algebra $\mathcal{B}$.

Definition 2.9 Let $F = \omega \times {}^{<\omega}\omega$. Every $\langle m, s\rangle \in F$ is an ∞-Borel code with interpretation $B_{\langle m,s\rangle} = N_{m,s}$. If c is an ∞-Borel code with interpretation B_c, then the ordered pair $\langle 1, c\rangle$ is also an ∞-Borel code with interpretation $B_{\langle 1,c\rangle} = -B_c$. If $c = \langle c_\eta : \eta < \lambda\rangle$ is a sequence of ∞-Borel codes for some ordinal λ, then $\langle 2, c\rangle$ is an ∞-Borel code with interpretation $B_{\langle 2,c\rangle} = \overline{\bigcup_{\eta<\lambda} B_{c_\eta}}$. A set is ∞-Borel if it is the interpretation of an ∞-Borel code.

Remark 2.10 Consider the function $\mathfrak{g} : \mathrm{OR} \to V$ defined by the recursion

$$\begin{aligned} \mathfrak{g}(0) &= \omega \times {}^{<\omega}\omega, \\ \mathfrak{g}(\alpha+1) &= \{\langle 1, c\rangle : c \in \mathfrak{g}(\alpha)\} \cup \{\langle 2, c\rangle : c \in {}^{<\alpha}\mathfrak{g}(\alpha)\}, \\ \mathfrak{g}(\lambda) &= \bigcup_{\alpha<\lambda} \mathfrak{g}(\alpha) \text{ when } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Then c is an ∞-Borel code if and only if $c \in \mathfrak{g}(\alpha)$ for some ordinal α. Similarly, there is a recursively defined sequence $\langle \mathfrak{h}_\alpha : \alpha \in \mathrm{OR}\rangle$ of functions such that $\mathfrak{h}_\alpha : \mathfrak{g}(\alpha) \to \mathcal{B}$ which interprets all the codes in $\mathfrak{g}(\alpha)$. For any ∞-Borel code c we define the $\mathfrak{g}$-*rank* of c to be the least ordinal α such that $c \in \mathfrak{g}(\alpha)$, and we will write $\mathfrak{g}\text{-rank}(c) = \alpha$. We note that the notion of an ∞-Borel code is absolute, as is the $\mathfrak{g}$-rank function.

Since there is an absolute one-to-one function $f : \omega \times {}^{<\omega}\omega \to \omega$, every $\langle m, s\rangle \in F$ can be uniquely encoded as an element of ω. Using the Gödel pairing function $\Gamma : \mathrm{OR} \times \mathrm{OR} \to \mathrm{OR}$ [6, see Section 3], one can show that every ∞-Borel code can be canonically encoded as a set of ordinals. The definition of Γ is absolute, and Γ is one-to-one and onto. If λ is a cardinal, then $\Gamma : \lambda \times \lambda \to \lambda$. If λ is a regular cardinal, then the image $\Gamma[\alpha \times \alpha]$ is bounded in λ for all $\alpha < \lambda$.

3 An Overview of $K(\mathbb{R})$

In this section we summarize the fundamental notions presented in [2] which will be assumed in the subsequent sections of this paper. The language $\mathcal{L} = \{\in, \underset{\sim}{\mathbb{R}}, \underset{\sim}{\kappa}, \underset{\sim}{\mu}\}$ consists of the constant symbols $\underset{\sim}{\mathbb{R}}$ and $\underset{\sim}{\kappa}$ together with the membership relation $\in$ and the predicate symbol $\underset{\sim}{\mu}$. We shall write κ, $\mathbb{R}$ for the constants $\underset{\sim}{\kappa}$, $\underset{\sim}{\mathbb{R}}$, respectively, and write μ for the predicate symbol $\underset{\sim}{\mu}$.

Definition 3.1 A set model $\mathcal{M} = (M, \in, \mathbb{R}, \kappa, \mu)$ is called a *real premouse* if

1. $\mathcal{M}$ is a transitive model of $V = L[\mu](\mathbb{R})$, and
2. $\mathcal{M} \models$ "μ is an $\mathbb{R}$-complete measure on κ."

Definition 3.2 Let $\mathcal{M}$ be a real premouse. The *premouse iteration* of $\mathcal{M}$

$$\langle\langle \mathcal{M}_\alpha\rangle_{\alpha\in\mathrm{OR}}, \langle \pi_{\alpha\beta}\colon \mathcal{M}_\alpha \underset{\Sigma_1}{\longrightarrow} \mathcal{M}_\beta\rangle_{\alpha\leq\beta\in\mathrm{OR}}\rangle \tag{3.1}$$

is the commutative system satisfying the inductive definition:

1. $\mathcal{M}_0 = \mathcal{M}$.
2. $\pi_{\gamma\gamma} =$ identity map, and $\pi_{\beta\gamma} \circ \pi_{\alpha\beta} = \pi_{\alpha\gamma}$ for all $\alpha \leq \beta \leq \gamma$.
3. If $\lambda = \lambda' + 1$, then $\mathcal{M}_\lambda =$ ultrapower of $\mathcal{M}_{\lambda'}$, and $\pi_{\alpha\lambda} = \pi^{\mathcal{M}_{\lambda'}} \circ \pi_{\alpha\lambda'}$ for all $\alpha \leq \lambda'$.
4. If λ is limit, then $\langle \mathcal{M}_\lambda, \langle \pi_{\alpha\lambda}\colon \mathcal{M}_\alpha \to \mathcal{M}_\lambda\rangle_{\alpha<\lambda}\rangle$ is the direct limit of
$$\langle\langle \mathcal{M}_\alpha\rangle_{\alpha<\lambda}, \langle \pi_{\alpha\beta}\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta\rangle_{\alpha\leq\beta<\lambda}\rangle.$$

The maps in the commutative system (3.1) are cofinal Σ_1 embeddings.

Definition 3.3 A real premouse $\mathcal{M}$ is *iterable* provided that $\mathcal{M}_\lambda$ is well-founded for all ordinals λ.

Let $\mathcal{M}$ be an iterable real premouse. For each ordinal α we identify $\mathcal{M}_\alpha$ with its transitive collapse and write $\mathcal{M}_\alpha = (M_\alpha, \in, \mathbb{R}, \kappa_\alpha, \mu_\alpha)$. We call μ_α and κ_α the αth iterate of μ and κ, respectively.

Definition 3.4 An iterable real premouse $\mathcal{M} = (M, \in, \mathbb{R}, \kappa, \mu)$ is said to be a *real 1-mouse* if it satisfies $\mathcal{P}(\mathbb{R} \times \kappa) \cap \underset{\sim}{\mathbf{\Sigma}}_1(\mathcal{M}) \not\subseteq M$.

For any real 1-mouse $\mathcal{M}$ there is a canonical partial function $h_\mathcal{M}\colon \mathbb{R}\times M \to M$ which is a Σ_1 Skolem function for $\mathcal{M}$ and is Σ_1 definable over $\mathcal{M}$ without parameters. For $p \in \mathcal{M}$, $h^p_\mathcal{M}$ is defined by $h^p_\mathcal{M}(x, s) = h_\mathcal{M}(x, \langle s, p\rangle)$. In [2] we define the substructure $\mathcal{H} = \mathrm{Hull}_1^\mathcal{M}(\mathbb{R} \cup \{p\})$ and show that $\mathcal{H} \prec_1 \mathcal{M}$ and $H = h^{p\ \prime\prime}_\mathcal{M}(\mathbb{R} \times (\mathbb{R})^{<\omega})$ where H is the domain of the structure $\mathcal{H}$.

Lemma 3.5 *Suppose $\mathcal{M}$ is a real 1-mouse and $a \in \mathcal{M}$. Let $\mathcal{N}$ be the transitive collapse of* $\mathrm{Hull}_1^\mathcal{M}(\mathbb{R} \cup \{a\})$*. Then $\mathcal{N}$ is also a real 1-mouse and $\sigma\colon \mathcal{N} \underset{\Sigma_1}{\longrightarrow} \mathcal{M}$ where σ is the inverse of the collapse map.*

Proof This follows from the development presented in Sections 1 and 2 of [2]. In particular, see Lemmas 1.17 and 2.37 and Corollary 2.41 in [2]. □

Remark 3.6 In $L(\mathbb{R})$ the αth level of the Jensen hierarchy, $J_\alpha(\mathbb{R})$, is ordinal definable. If $a \in J_\alpha(\mathbb{R})$, then the transitive collapse of $\mathcal{H} = \mathrm{Hull}_1^{J_\alpha(\mathbb{R})}(\mathbb{R} \cup \{a\})$ has the form $J_\beta(\mathbb{R})$ which is again ordinal definable in $L(\mathbb{R})$. In the next section we will show that each real 1-mouse $\mathcal{M}$ is ordinal definable in $K(\mathbb{R})$. Lemma 3.5 will then imply that for any $a \in \mathcal{M}$, the collapse of $\mathrm{Hull}_1^{\mathcal{M}}(\mathbb{R} \cup \{a\})$ is also ordinal definable in $K(\mathbb{R})$. This is an important property we will use in our proof of Theorem 6.1.

When $\mathcal{M}$ is a real 1-mouse, we can also define the projectum $\rho_{\mathcal{M}} \leq \kappa^{\mathcal{M}}$ and standard parameter $p_{\mathcal{M}}$ (see Definition 2.33 of [2]). We then define a structure as follows: Let $\mathcal{H} = \mathrm{Hull}_1^{\mathcal{M}}(\mathbb{R} \cup \omega\rho_{\mathcal{M}} \cup \{p_{\mathcal{M}}\})$ and let $\mathcal{C}$ be the transitive collapse of $\mathcal{H}$. In [2] it is shown that $\mathcal{C}$ is a real 1-mouse. We write $\mathcal{C} = \mathcal{C}(\mathcal{M})$ and shall refer to $\mathcal{C}$ as the core of $\mathcal{M}$. We also show in [2] that every real 1-mouse is a premouse iterate of its core.

Definition 3.7 The class $K(\mathbb{R}) = \{a : \exists \mathcal{M}\, (\mathcal{M} \text{ is a real 1-mouse} \wedge a \in M)\}$ is called the *real core model.*

Remark 3.8 There exists an iterable real premouse if and only if $\mathbb{R}^{\#}$ exists (see [2, Lemma 5.2]). If $\mathbb{R}^{\#}$ does not exist, then we define $K(\mathbb{R}) = L(\mathbb{R})$. In this paper we implicitly assume $\mathbb{R}^{\#}$ exists. Thus, $K(\mathbb{R})$ will be defined as in Definition 3.7. It is shown in [2] that every real 1-mouse is in $K(\mathbb{R})$.

The following is a slight modification of Definition 2.23 of [2].

Definition 3.9 Let $\mathcal{N} = (J_\alpha^{\mathcal{N}}(\mathbb{R}), \in, \mathbb{R}, \kappa^{\mathcal{N}}, \nu)$ and $\mathcal{M} = (J_\beta^{\mathcal{M}}(\mathbb{R}), \in, \mathbb{R}, \kappa^{\mathcal{M}}, \mu)$ be real premice. Then $\mathcal{N}$ is a *proper initial segment* of $\mathcal{M}$ if $\alpha < \beta$, $\kappa^{\mathcal{N}} = \kappa^{\mathcal{M}}$, $J_\alpha^{\mathcal{N}}(\mathbb{R}) = J_\alpha^{\mathcal{M}}(\mathbb{R})$, and $\nu \cap J_\alpha^{\mathcal{N}}(\mathbb{R}) = \mu \cap J_\alpha^{\mathcal{M}}(\mathbb{R})$.

Lemma 3.10 *Assume $V = K(\mathbb{R})$ and let S be a set. Then there is a real 1-mouse $\mathcal{M}$ such that S is in a proper initial segment of $\mathcal{M}$.*

Proof Lemma 5.4 (and its proof) in [2] implies that there is a real 1-mouse $\mathcal{M}$ so that S is in a proper initial segment of $\mathcal{M}$. □

We first introduced $K(\mathbb{R})$ in [2] and showed that $K(\mathbb{R})$ is an inner model containing definable scales beyond those in $L(\mathbb{R})$. Real 1-mice suffice to define the real core model and to present our results in [2] on the existence of scales in $K(\mathbb{R})$. Furthermore, real 1-mice suffice to prove Theorem 1.5, our covering lemma for $K(\mathbb{R})$. In the next section we show that real 1-mice also allow us to establish the relevant properties of ordinal definability in $K(\mathbb{R})$ that we will employ in Section 5 to show that $K(\mathbb{R})$ is a symmetric generic extension of its version of HOD.

Remark 3.11 Steel [11] generalizes our notion of a real 1-mouse to obtain a more general notion of a mouse over the reals which we shall denote as $\mathcal{M}_S$. Steel constructs the mouse $\mathcal{M}_S$ using an *appropriate* sequence of extenders over $\mathcal{M}_S$. Steel then defines $\mathbf{K}(\mathbb{R})$ to be the union of these more general mice and extends our work in [2], on the analysis of scales in $K(\mathbb{R})$, to $\mathbf{K}(\mathbb{R})$. To construct new scales in $\mathbf{K}(\mathbb{R})$, Steel requires a version of part (3) in Woodin's Theorem 1.8 and thus affirms that each $\mathcal{M}_S$ is a symmetric generic extension of HOD, as interpreted in $\mathcal{M}_S$. One can conclude, as a special case of Steel's analysis, that $K(\mathbb{R})$ is also a symmetric generic extension of its version of HOD. Nevertheless, we provide our own proof of this latter result here (see Theorem 5.3). We do this for two reasons. First, in order to make this paper virtually self-contained, we will not presume that the reader is familiar with Woodin's unpublished proof of Theorem 1.8 or with Steel's work in [11]. Second, under ZF + AD we explicitly prove that $K(\mathbb{R})$ is a symmetric $\mathbb{P}$-generic extension of $\mathrm{HOD}^{K(\mathbb{R})}$ where the partial order $\mathbb{P}$ satisfies two relevant properties:

(1) $\mathbb{P}$ has cardinality $\Theta^{K(\mathbb{R})}$ in $\mathrm{HOD}^{K(\mathbb{R})}$,

(2) $\mathbb{P}$ satisfies the $\Theta^{K(\mathbb{R})}$-chain condition in $\mathrm{HOD}^{K(\mathbb{R})}$.

Conditions (1) and (2) are used in the proof of our main Theorem 6.1. One final note: The assumptions of Theorem 6.1 imply that $\mathbf{K}(\mathbb{R}) = K(\mathbb{R})$ because if $\mathbf{K}(\mathbb{R})$ properly extended $K(\mathbb{R})$, it then would contain an inner $\rho(\mathbb{R})$-model. Thus, under these assumptions, $K(\mathbb{R})$ can be viewed as the largest core model above the reals.

4 Ordinal Definability in $K(\mathbb{R})$

A set X is *ordinal definable* if there is a formula φ in the language of set theory such that $X = \{y : \varphi(y, \alpha_1, \ldots, \alpha_n)\}$ for a finite sequence of ordinals $\alpha_1, \ldots, \alpha_n$. From Gödel we know that a set X is ordinal definable if and only if there is an ordinal β such that $X = \{y : (V_\beta, \in) \models \varphi(y, \alpha_1, \ldots, \alpha_n)\}$ for some formula φ and ordinals $\alpha_1, \ldots, \alpha_n < \beta$.

Suppose that $\mathcal{M}$ is a real 1-mouse. Let $\mathcal{C} = \mathcal{C}(\mathcal{M})$ be the core of $\mathcal{M}$ with premouse iteration $\langle \pi_{\alpha\beta} : \mathcal{C}_\alpha \to \mathcal{C}_\beta \rangle_{\alpha \le \beta \in \mathrm{OR}}$ and let $\kappa_\alpha = \pi_{0\alpha}(\kappa^{\mathcal{C}})$ for each ordinal α. Define $i(\mathcal{M})$ to be the ordinal λ such that $\mathcal{C}_\lambda = \mathcal{M}$ (see Definition 2.36 and Theorem 2.39 of [2]) and define $I_{\mathcal{M}} = \{\kappa_\alpha : \alpha < i(\mathcal{M})\}$. We note that $I_{\mathcal{M}} \subseteq \kappa^{\mathcal{M}}$.

Definition 4.1 Let $\mathcal{M}$ be a real 1-mouse with core $\mathcal{C} = \mathcal{C}(\mathcal{M})$. Then $w(\mathcal{M})$ denotes the ordinal $\kappa^{\mathcal{C}_\omega}$.

Lemma 4.2 *Let $\mathcal{M}$ and $\mathcal{N}$ be real 1-mice. Then $\mathcal{N} = \mathcal{M}$ if and only if $i(\mathcal{M}) = i(\mathcal{N})$ and $w(\mathcal{M}) = w(\mathcal{N})$.*

Proof Let $\mathcal{M}$ and $\mathcal{N}$ be real 1-mice. Theorem 2.39 of [2] shows that $\mathcal{M}$ and $\mathcal{N}$ are iterates of their respective cores $\mathcal{C}(\mathcal{M})$ and $\mathcal{C}(\mathcal{N})$. Lemma 5.3 of [2] asserts that if $\kappa^{\mathcal{C}(\mathcal{M})_\omega} = \kappa^{\mathcal{C}(\mathcal{N})_\omega}$, then $\mathcal{C}(\mathcal{M}) = \mathcal{C}(\mathcal{N})$. □

Corollary 4.3 *A real 1-mouse $\mathcal{M}$ is definable with ordinal parameters $i(\mathcal{M})$ and $w(\mathcal{M})$.*

Definition 4.4 Let $\mathcal{M}$ be a real 1-mouse. We say that a set X is Σ_1 *ordinal definable over $\mathcal{M}$* if there is a Σ_1 formula $\varphi(v, v_1, \ldots, v_n)$ in the language $\mathcal{L}$ and ordinals $\alpha_1, \ldots, \alpha_n \in \mathcal{M}$ such that $X = \{v : \mathcal{M} \models \varphi(v, \alpha_1, \ldots, \alpha_n)\}$.

Theorem 4.5 *In $K(\mathbb{R})$, a set X is ordinal definable if and only if X is Σ_1 ordinal definable over a real 1-mouse.*

Proof Assume $V = K(\mathbb{R})$. Suppose X is ordinal definable over some V_β with ordinal parameters $\alpha_1, \alpha_2, \ldots, \alpha_n < \beta$. Consider the sequence $S = \langle V_\gamma : \gamma \le \beta \rangle$. Lemma 3.10 asserts that there is a real 1-mouse $\mathcal{M}$ so that S is in a proper initial segment of $\mathcal{M}$. Since S is definable over this initial segment, it follows that V_β is Σ_1 ordinal definable over $\mathcal{M}$. Therefore, X is Σ_1 ordinal definable over $\mathcal{M}$. Conversely, if X is Σ_1 ordinal definable over a real 1-mouse, then X is ordinal definable by Corollary 4.3. □

We shall say that a set A is the *image of* $\mathbb{R}$ if there is an onto function $f : \mathbb{R} \to A$.

Theorem 4.6 *Assume $V = K(\mathbb{R})$. Suppose that X is an ordinal definable set of reals. Then X is Σ_1 ordinal definable over a real 1-mouse $\mathcal{N}$ such that $\mathcal{N}$ is the image of $\mathbb{R}$. Therefore, X is ordinal definable with ordinal parameters strictly less than Θ.*

Proof Assume $V = K(\mathbb{R})$ and that X is an ordinal definable set of reals. By Theorem 4.5 we have that X is Σ_1 definable over a real 1-mouse $\mathcal{M}$ with ordinal parameters in $p = \{\alpha_1, \ldots, \alpha_n\}$. Let $\mathcal{N}$ be the transitive collapse of $\mathrm{Hull}_1^{\mathcal{M}}(\mathbb{R} \cup \{p\})$. Thus by Lemma 3.5, $\mathcal{N}$ is a real 1-mouse and X is Σ_1 ordinal definable over $\mathcal{N}$. Clearly, $\mathcal{N}$ is the image of $\mathbb{R}$. Let $\mathfrak{C} = \mathfrak{C}(\mathcal{N})$. Because $\mathfrak{C}_\omega$ is also the image of $\mathbb{R}$, we have that $\kappa^{\mathfrak{C}_\omega} = w(\mathcal{N}) < \Theta$. Since the ordinal $i(\mathcal{N})$ can be embedded into the ordinal $\kappa^{\mathcal{N}}$, it follows that $i(\mathcal{N}) < \Theta$. By Corollary 4.3 we conclude that X is ordinal definable in ordinals less that Θ. □

Lemma 4.7 *Suppose c is an ∞-Borel code. If c is ordinal definable, then $c \in \mathrm{HOD}$ and B_c is ordinal definable.*

Since any ∞-Borel code c can be canonically encoded as a set of ordinals, we may implicitly equate c with this set of ordinals.

Definition 4.8 An ∞-Borel code c is *bounded in* Θ if $c \subseteq \lambda$ for some $\lambda < \Theta$.

Any ∞-Borel code having $\mathfrak{g}$-rank (see 2.10) less than Θ will be bounded in Θ.

Lemma 4.9 *In $K(\mathbb{R})$ every ∞-Borel set has an ∞-Borel code with $\mathfrak{g}$-rank less than Θ.*

Proof Assume $V = K(\mathbb{R})$. Let c be an ∞-Borel code and let $X = B_c$. Thus there is an ordinal β such that $\mathfrak{h}_\beta(c) = X$ where $\mathfrak{h}_\beta$ is defined in Remark 2.10. Let $S = \langle \mathfrak{h}_\alpha : \alpha \le \beta \rangle$. By Lemma 3.10, there is a real 1-mouse $\mathcal{M}$ such that S is in a proper initial segment of $\mathcal{M}$. Since S is definable over this initial segment, we conclude that $\mathfrak{h}_\beta$ is Σ_1 ordinal definable over $\mathcal{M}$ using a finite set p of ordinals. Thus, $\mathcal{M} \models \exists c(\mathfrak{h}_\beta(c) = X)$; that is, $\mathcal{M} \models \exists c$("$X$ is the interpretation of c"). Let $\mathfrak{C}$ be the transitive collapse of $\mathrm{Hull}_1^{\mathcal{M}}(\mathbb{R} \cup \{X\} \cup \{p\})$. Since X is (essentially) a set of reals, this collapse is the identity on X. Moreover, the collapse of an ∞-Borel code in this hull is again an ∞-Borel code. Lemma 3.5 asserts that $\mathfrak{C}$ is a real 1-mouse and $\mathfrak{C} \models \exists d$("$X$ is the interpretation of d"). Let $d \in \mathfrak{C}$ be an ∞-Borel code so that $X = B_d$. Because $\mathfrak{C}$ is the image of $\mathbb{R}$, it follows that $\mathfrak{g}$-rank$(d) < \Theta$. □

Lemma 4.10 *Assume $V = K(\mathbb{R})$. For every ∞-Borel code $c \in \mathrm{HOD}$, there is an ∞-Borel code $d \in \mathrm{HOD}$ such that $B_c = B_d$ and $(\mathfrak{g}\text{-rank}(d))^{\mathrm{HOD}} < \Theta$.*

Proof Assume $V = K(\mathbb{R})$. Let $c \in \mathrm{HOD}$ be an ∞-Borel code and let $X = B_c$. Suppose the ordinal β satisfies $c \in \mathfrak{g}(\beta)$ and $\mathfrak{h}_\beta(c) = X$ where $\mathfrak{g}, \mathfrak{h}$ are defined in Remark 2.10. As in the proof of Theorem 4.5, there is a real 1-mouse $\mathcal{M}$ such that

1. $\langle \mathfrak{h}_\alpha : \alpha \le \beta \rangle$, $\mathfrak{g} \restriction \beta + 1$ are in a proper initial segment of $\mathcal{M}$,
2. $\mathcal{M} \models \varphi(c, p)$ and $\mathcal{M} \models \forall x \forall y((\varphi(x, p) \wedge \varphi(y, p)) \rightarrow x = y)$,
3. $\mathcal{M} \models (\exists e \in \mathfrak{g}(\beta))(\varphi(e, p) \wedge \mathfrak{h}_\beta(e) = X)$

for some Σ_1 formula φ where p is a finite set of ordinals in $\mathcal{M}$. Since $\mathfrak{h}_\beta$ and $\mathfrak{g}(\beta)$ are Σ_1 ordinal definable over $\mathcal{M}$ using a finite set of ordinals q, let $\mathfrak{C}$ be the transitive collapse of $\mathcal{H} = \mathrm{Hull}_1^{\mathcal{M}}(\mathbb{R} \cup \{X\} \cup \{p, q\})$. Lemma 3.5 implies that $\mathfrak{C}$ is a real 1-mouse. Note that $c \in \mathcal{H}$ and so, let $d \in \mathfrak{C}$ be the collapse of c. It now follows that d is Σ_1 ordinal definable over $\mathfrak{C}$ and thus $d \in \mathrm{HOD}$, by Theorem 4.5 and Lemma 4.7. It also follows that $\mathfrak{C} \models$ ("X is the interpretation of d") and therefore, $X = B_d$. Since $\mathfrak{C}$ is the image of $\mathbb{R}$, we can also conclude that $(\mathfrak{g}\text{-rank}(d))^{\mathrm{HOD}} < \Theta$. □

5 $K(\mathbb{R})$ Is a Symmetric Extension of HOD

The following are restatements of Definition 1.15 and Lemma 1.16 of [3].

Definition 5.1 Define the class D of ordinal pairs to be

$$D = \left\{(\xi, \kappa) : \exists \mathcal{M}\ (\mathcal{M} \text{ is a real 1-mouse} \wedge \kappa = \kappa^{\mathcal{M}} \wedge i(\mathcal{M}) = \omega \wedge \xi \in I_{\mathcal{M}})\right\}.$$

Lemma 5.2 $K(\mathbb{R}) = L[D](\mathbb{R})$.

We shall now prove that $K(\mathbb{R})$ is our desired symmetric extension of $\text{HOD}^{K(\mathbb{R})}$.

Theorem 5.3 *Assume* ZF + AD *and* $V = K(\mathbb{R})$. *There is an* $S \subseteq \Theta$ *such that* $\text{HOD} = L[D](S)$. *Moreover, there is a partial order* $\mathbb{P}$ *in* HOD *such that*

(1) $\mathbb{P}$ *has cardinality* Θ *in* HOD,
(2) $\mathbb{P}$ *has the* Θ*-chain condition in* HOD,
(3) $K(\mathbb{R})$ *is a symmetric* $\mathbb{P}$*-generic extension of* HOD.

Our proof of Theorem 5.3 is motivated by [12] which outlines a proof of Theorem 1.8. We shall provide a proof of Theorem 5.3 that can be easily understood by anyone who is familiar with the basics of forcing as presented, say, by Kunen in [9]. As a result, we will have clearly shown that Theorem 5.3 requires no consequences of the axiom of choice that are not available in $K(\mathbb{R})$. Moreover, we only appeal to AD in our proof of part (1). More specifically, we prove that $|\mathbb{P}| \leq \Theta$ in ZF and then, given that Θ is a limit cardinal (see [7, 28.16]), we prove that $|\mathbb{P}| = \Theta$.

In his proof of Theorem 1.8, Woodin uses the following definable equivalence relation E on ∞-Borel codes.

Definition 5.4 For all ∞-Borel codes c, d we write $E(c, d)$ if and only if $B_c = B_d$.

For the remainder of this section we shall be working in ZF + AD. Furthermore, we shall assume that $V = K(\mathbb{R})$. Consider the inner model $N = L[D, E]$ where D and E are as in Definitions 5.1 and 5.4, respectively. Let BC denote the class of ∞-Borel codes. Note that there exists an ordinal α such that for all $c \in \text{BC}^N$ there is $d \in \text{BC}^N$ so that $d \in L_\alpha[D, E]$ and $E(c, d)$. Clearly, $N \subseteq \text{HOD}$ and N satisfies the axiom of choice. In N, let P be a set of ∞-Borel codes such that

(i) for all $c, d \in P$ if $c \neq d$, then $\neg E(c, d)$,
(ii) for all $c \in \text{BC}$ there is a $d \in P$ of minimal $\mathfrak{g}$-rank such that $E(c, d)$.

In N, the set P is a maximal collection of E-inequivalent ∞-Borel codes such that every ∞-Borel code c has an E-equivalent representative $d \in P$ of minimal $\mathfrak{g}$-rank. It follows, in particular, that P contains all of the codes $\langle m, s\rangle$ for the generators $N_{m,s}$ of $\mathcal{B}$.

For $c \in P$ define $-c$ to be the unique element $d \in P$ such that $E(d, \langle 1, c\rangle)$. For $c, d \in P$ define $c + d$ to be the unique element $e \in P$ such that $E(e, \langle 2, \langle c, d\rangle\rangle)$. We let $\mathbb{0}$ and $\mathbb{1}$ be the unique elements in P that are respectively E-equivalent to $c - c$ and $c + (-c)$ for any $c \in P$. As usual, the partial order on P is defined by $c \leq d$ if and only if $c - d = \mathbb{0}$. Thus, we obtain a Boolean algebra $\mathbb{P} = (P, +, \cdot, -)$. In addition, $\mathbb{P}$ is a complete Boolean algebra in N and the map $h \colon \mathbb{P} \to \mathcal{B}$ defined by $h(c) = B_c$ is a complete embedding (see [6, p. 83]) with respect to the subsets of $\mathbb{P}$ in N. Define $M = L[D](\mathbb{P})$, an inner model of N. Hence, $\mathbb{P}$ is a complete Boolean algebra in M and h is a complete embedding on the subsets of $\mathbb{P}$ in M.

In some of our proofs, as noted in Section 2, we shall implicitly assume that V is a countable transitive set. Thus, for every $p \in \mathbb{Q}$ there is a G such that $p \in G$ and G is $\mathbb{Q}$-generic over V. We now investigate the $\mathbb{P}$-forcing relation in the inner model M. First we introduce some notation. Whenever G is $\mathbb{Q}$-generic over V, let $s_G \in V[G]$ denote the unique ω sequence of reals such that $s_G \restriction n \in G$ for all $n \in \omega$. Given s_G define $\overline{G} = \{C \in \mathcal{B} : s_G \restriction n \in C \text{ for all } n \in \omega\}$. It follows that $\overline{G}$ is $\mathcal{B}$-generic over V and that $F_G = \{f \in P : B_f \in \overline{G}\}$ is $\mathbb{P}$-generic over M. Let $\dot{F}$ be a canonical name in M for any F which is $\mathbb{P}$-generic over M.

Lemma 5.5 *$c \Vdash_{\mathbb{P}} (\exists! s \in {}^{\omega}\mathbb{R})(\forall n \in \omega)(s \restriction (n+1) \in \bigcap_{f\in\dot{F}} B_f)$, for all $c \in \mathbb{P} \setminus \{\mathbb{O}\}$.*

Proof Let $c \in \mathbb{P}$ be nonzero. We show that the set

$$X = \{d \in \mathbb{P} : d \Vdash_{\mathbb{P}} (\exists! s \in {}^{\omega}\mathbb{R})(\forall n \in \omega)(s \restriction (n+1) \in \bigcap_{f\in\dot{F}} B_f)\}$$

is dense below c. Let B_c be the interpretation of c in V and $p \in \mathbb{Q}$ be such that $U_p \subseteq B_c$. Let G be $\mathbb{Q}$-generic over V with $p \in G$. Working in $V[G]$, let s_G be the unique ω sequence of reals such that $s_G \restriction n \in G$ for all $n \in \omega$. So, $F_G = \{f \in P : s_G \restriction n \in B_f \text{ for all } n \in \omega\}$ is $\mathbb{P}$-generic over M. For each $m \in \omega$, we have $s_G(m) = \bigcup\{s : \langle m, s\rangle \in F_G\}$. So, $s_G \in M[F_G]$ and it is the unique element satisfying $M[F_G] \models (\forall n \in \omega)(s_G \restriction (n+1) \in \bigcap_{f\in F_G} B_f)$. Thus, $e \Vdash_{\mathbb{P}} (\exists! s \in {}^{\omega}\mathbb{R})(\forall n \in \omega)(s \restriction (n+1) \in \bigcap_{f\in\dot{F}} B_f)$ for some $e \in F_G$. Since $c \in F_G$, we see that $d = c \cdot e \in X$. □

Corollary 5.6 *Suppose that G is $\mathbb{Q}$-generic over $K(\mathbb{R})$. Let*

$$F_G = \{f \in P : s_G \restriction n \in B_f \text{ for all } n \in \omega\}.$$

Then F_G is $\mathbb{P}$-generic over $L[D](\mathbb{P})$ and $K(\mathbb{R})[G] = L[D](\mathbb{P})[F_G]$.

Let $\dot{r}$ be a canonical name so that for any F which is $\mathbb{P}$-generic over M, the interpretation $\dot{r}_F$ in $M[F]$ consists of the set of reals in the sequence $s \in M[F]$ as asserted by Lemma 5.5.

Lemma 5.7 *$\mathbb{P}$ is $\dot{r}$-homogeneous over M.*

Proof Let $\varphi(v, v_1, \ldots, v_n)$ be a formula of set theory and let $x_1, \ldots, x_n \in M$. We shall use ψ to denote the formula $\varphi(\dot{r}, \check{x}_1, \ldots, \check{x}_n)$. Let $c, d \in \mathbb{P}$ be nonzero. We show that $c \Vdash_{\mathbb{P}} \psi$ if and only if $d \Vdash_{\mathbb{P}} \psi$. Suppose that $c \Vdash_{\mathbb{P}} \psi$. To show that $d \Vdash_{\mathbb{P}} \psi$, suppose for a contradiction that $e \leq d$ is such that $e \Vdash_{\mathbb{P}} \neg\psi$. Let $p \in \mathbb{Q}$ be such that $U_p \subseteq B_c$ and let $q \in \mathbb{Q}$ be such that $U_q \subseteq B_e$. Let $x = p^\frown q$ and $y = q^\frown p$ be the concatenation of these finite sequences. Let $D_1, D_2, \ldots$ be a countable listing of all the $\mathbb{Q}$-dense sets in V. Construct two generic sets G and H sequentially, as follows:

1. Let $x \in G$ and $y \in H$.
2. Find $x_1, y_1 \in \mathbb{R}^{<\omega}$ such that $x^\frown x_1 \in D_1$ and $y^\frown x_1{}^\frown y_1 \in D_1$. Let $x^\frown x_1 \in G$ and $y^\frown x_1{}^\frown y_1 \in H$.
3. Find $x_2, y_2 \in \mathbb{R}^{<\omega}$ so that $x^\frown x_1{}^\frown y_1{}^\frown x_2 \in D_2$ and $x^\frown x_1{}^\frown y_1{}^\frown x_2{}^\frown y_2 \in D_2$. Let $x^\frown x_1{}^\frown y_1{}^\frown x_2 \in G$ and $y^\frown x_1{}^\frown y_1{}^\frown x_2{}^\frown y_2 \in H$.

Continuing in this manner, we obtain $\mathbb{Q}$-generic filters G and H such that $V[G] = V[H]$. Hence F_G and F_H are both $\mathbb{P}$-generic over M with $\dot{r}_{F_G} = \dot{r}_{F_H}$ and $M[F_G] = M[F_H]$. Since $c \in F_G$ and $e \in F_H$, this contradicts the truth lemma of forcing (see [9, Theorem 3.5]). Therefore, $d \Vdash_{\mathbb{P}} \psi$. □

Before we prove Theorem 5.3, we need to observe that $\mathbb{Q}$-forcing "preserves" $K(\mathbb{R})$. Lemma 1.9 of [3] implies that if G is $\mathbb{Q}$-generic over $K(\mathbb{R})$, then $K(\mathbb{R}) = K(\mathbb{R}^V)^{K(\mathbb{R})[G]}$ where $\mathbb{R}^V$ is the set of reals occurring in the ground model and $K(\mathbb{R}^V)$ is defined in $K(\mathbb{R})[G]$ just as in Definition 3.7 except that the set of reals must be interpreted as $\mathbb{R}^V$ (see the paragraph prior to the statement of Lemma 1.9 in [3]). Thus, we have the following crucial tool that will be used in our proof of Theorem 5.3.

Lemma 5.8 *Let G be $\mathbb{Q}$-generic over $K(\mathbb{R})$. Then $K(\mathbb{R}) = K(\mathbb{R}^V)^{K(\mathbb{R})[G]}$ and $\mathbb{R}^V$ is the set of reals occurring in s_G. Thus, $K(\mathbb{R}) = K(\dot{r}_{F_G})^{M[F_G]}$.*

Proof of Theorem 5.3 Assume ZF + AD and $V = K(\mathbb{R})$. Let $\mathbb{P}$ be as defined prior to Lemma 5.5. To show that HOD $= L[D](\mathbb{P})$, we shall show that every set of ordinals which is ordinal definable in $K(\mathbb{R})$ belongs to $L[D](\mathbb{P})$. Let $\alpha, \alpha_1, \ldots \alpha_n$ be a finite list of ordinals and let $\varphi(v, v_1, \ldots, v_n)$ be a formula of set theory. Suppose that $X \in K(\mathbb{R})$ is such that $X = \{\beta \in \alpha : \varphi(\beta, \alpha_1, \ldots \alpha_n)^{K(\mathbb{R})}\}$. Lemma 5.7 and Lemma 5.8 now imply that for all β we have

$$\varphi(\beta, \alpha_1, \ldots \alpha_n)^{K(\mathbb{R})} \text{ if and only if } L[D](\mathbb{P}) \models \left(\mathbb{1} \Vdash_{\mathbb{P}} \varphi(\check{\beta}, \check{\alpha}_1, \ldots \check{\alpha}_n)^{K(\dot{r})}\right).$$

Hence, $X \in L[D](\mathbb{P})$. Therefore, HOD $\subseteq L[D](\mathbb{P})$. Since $L[D](\mathbb{P}) \subseteq$ HOD, we conclude that HOD $= L[D](\mathbb{P})$.

To see that $\mathbb{P}$ has the Θ-chain condition in HOD, let $\mathcal{K} \in$ HOD be an antichain in $\mathbb{P}$. Then $\overline{\mathcal{K}} = \{B_c : c \in \mathcal{K}\}$ is an antichain in $\mathcal{B}$. Because $\mathcal{K}$ has a well-ordering and the map $c \mapsto B_c$ is one-to-one for $c \in \mathcal{K}$, we see that $\overline{\mathcal{K}}$ must have cardinality strictly less than Θ (see Lemma 2.7). Therefore, $|\mathcal{K}| < \Theta$ in $K(\mathbb{R})$ and thus, $|\mathcal{K}| < \Theta$ in HOD as well.

We now have that HOD $= L[D](\mathbb{P})$, $\mathbb{P}$ has the Θ-chain condition in HOD, and Θ is regular. Therefore, $\mathbb{P}$-forcing over HOD preserves cardinals $\geq \Theta$ (HOD is a model of ZFC and so, Lemma 6.9 of [9, p. 213] applies). Let G be $\mathbb{Q}$-generic over $K(\mathbb{R})$ and let $F_G = \{f \in \mathbb{P} : B_f \in \overline{G}\}$. We see that F_G is $\mathbb{P}$-generic over HOD and Corollary 5.6 implies that HOD$[F_G] = K(\mathbb{R})[G]$. We infer that HOD and $K(\mathbb{R})[G]$ have the same cardinals $\geq \Theta$. Therefore, HOD and $K(\mathbb{R})$ also have the same cardinals $\geq \Theta$. Since $\mathbb{P} \in$ HOD, we know that the cardinal $|\mathbb{P}|$ exists. Moreover, because the map $c \mapsto \mathcal{B}_c$ for all $c \in \mathbb{P}$ is one-to-one and each such B_c is ordinal definable, Theorem 4.6 implies that $|\mathbb{P}| \leq \Theta$ in $K(\mathbb{R})$. Thus, $|\mathbb{P}| \leq \Theta$ in HOD as well.

Since Θ is a limit cardinal, we see that Θ is also a limit cardinal in HOD. We now prove that $|\mathbb{P}| = \Theta$ in HOD. Suppose, for a contradiction, that $|\mathbb{P}| < \Theta$ holds in HOD. Thus, because Θ is a limit cardinal, the set of cardinals in HOD, below Θ, that are preserved under $\mathbb{P}$-forcing must be cofinal in Θ. Therefore, Θ^V is a limit cardinal in HOD$[F_G]$. On the other hand, $\Theta^V = \omega_1$ in $K(\mathbb{R})[G]$ (see [5, Theorem 4.4]). Because HOD$[F_G] = K(\mathbb{R})[G]$, we conclude that HOD$[F_G] \models$ (ω_1 is a limit cardinal). This contradiction forces us to conclude $|\mathbb{P}| = \Theta$ holds in HOD.

Because HOD $= L[D](\mathbb{P})$, Lemma 4.10 implies that the Boolean algebra $\mathbb{P}$ consists of ∞-Borel codes that are all bounded in Θ. Since Θ is a regular cardinal in HOD, we can canonically encode $\mathbb{P}$ as a subset S of Θ. Thus, HOD $= L[D](S)$. Finally, since $K(\mathbb{R}) = K(\dot{r}_{F_G})^{\mathrm{HOD}[F_G]}$ where G and F_G are as above, it follows that $K(\mathbb{R})$ is a symmetric $\mathbb{P}$-generic extension of HOD. □

6 The Covering Lemma for HOD of $K(\mathbb{R})$

Theorem 6.1 *Assume* ZF + AD *and that no* $\rho(\mathbb{R})$*-model exists. If* X *is a set of ordinals with* $|X| \geq \Theta$*, then there is a set of ordinals* $Y \in \mathrm{HOD}^{K(\mathbb{R})}$ *such that* $X \subseteq Y$ *and* $|X| = |Y|$.

Proof Assume ZF + AD and that no $\rho(\mathbb{R})$-model exists. By (i) of Theorem 1.5, $\Theta = \Theta^{K(\mathbb{R})}$. Suppose that X is a set of ordinals with $|X| \geq \Theta$. We prove that there is a set of ordinals $Y \in \mathrm{HOD}^{K(\mathbb{R})}$ such that $X \subseteq Y$ and $|X| = |Y|$. Since no $\rho(\mathbb{R})$-model exists, Theorem 1.5 implies there is a set of ordinals $W \in K(\mathbb{R})$ such that $X \subseteq W$ and $|X| = |W|$. Because $|X| \geq \Theta$, it follows that $K(\mathbb{R}) \models |W| \geq \Theta$. Let $M = \mathrm{HOD}^{K(\mathbb{R})}$ and let $\mathbb{P} \in M$ be the partial order which satisfies conclusions (1) and (2) of Theorem 5.3. In addition, let G be $\mathbb{P}$-generic over M so that conclusion (3) of Theorem 5.3 also holds. Thus, $M \subseteq K(\mathbb{R}) \subseteq M[G]$. Since Θ is a regular cardinal in $K(\mathbb{R})$, it is also a regular cardinal in M. Consequently, $\mathbb{P}$-forcing over M preserves cardinals $\geq \Theta$ by (2) of Theorem 5.3. Furthermore, for any ordinal $\xi \geq \Theta$ the following are equivalent:

1. ξ is a cardinal in M,
2. ξ is a cardinal in $K(\mathbb{R})$,
3. ξ is a cardinal in $M[G]$.

Thus, in particular, Θ is a cardinal in $M[G]$. Since $K(\mathbb{R}) \models |W| \geq \Theta$, it follows that the order-type of W is $\geq \Theta$. Therefore, $M[G] \models |W| \geq \Theta$.

Claim There is a set of ordinals Y such that $Y \in M$, $W \subseteq Y$, and $M[G] \models |W| = |Y|$.

Proof We know $W \in M[G]$ is a set of ordinals such that $M[G] \models |W| \geq \Theta$. Let $\dot{W}$ be a $\mathbb{P}$-name for $W \in M[G]$, and let $\check{\lambda}$ be a canonical $\mathbb{P}$-name for the ordinal $\lambda \in M$ such that $M[G] \models |W| = \lambda$. We observe that $W = \dot{W}_G$ where $\dot{W}_G$ denotes the interpretation of $\dot{W}$ in the generic extension $M[G]$. Let $\beta \in \mathrm{OR}$ be such that $W \subseteq \beta$ and let $p \in G$ be so that

$$M \models \text{“}p \Vdash_{\mathbb{P}} (\dot{W} \subseteq \check{\beta} \wedge |\dot{W}| = \check{\lambda})\text{”}. \tag{6.1}$$

Since $\lambda \geq \Theta$ and λ is a cardinal in $M[G]$, we see that λ is also a cardinal in M. Let $B_p = \{q \in P : q \leq p\}$ be the set of conditions below p. For each $q \in B_p$ define $Y_q \in M$ by $Y_q = \{\alpha \in \beta : q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{W}\}$. Let $Y \in M$ be defined by $Y = \bigcup_{q \in B_p} Y_q$.

Now we prove that $M \models |Y| \leq \lambda$. First we show that $M \models |Y_q| \leq \lambda$, for each $q \in B_p$. Let $q \in B_p$ be arbitrary. Suppose, for a contradiction, that $M \models (\nu = |Y_q| > \lambda)$. Let G' be $\mathbb{P}$-generic over M such that $q \in G'$. Thus, $p \in G'$ and $Y_q \subseteq \dot{W}_{G'}$. Because $M \models (\nu > \lambda \geq \Theta)$ and such cardinality is preserved, it follows that ν is also a cardinal in $M[G']$. Therefore, $M[G'] \models |\dot{W}_{G'}| > \lambda$, contradicting (6.1). Thus, $M \models |Y_q| \leq \lambda$. Since $M \models (|\mathbb{P}| = \Theta \leq \lambda)$ (by (1) of Theorem 5.3), it follows that $M \models |Y| \leq \lambda$. Since $M[G] \models |Y| \leq \lambda$ and because $M[G] \models (W \subseteq Y \wedge |W| = \lambda)$, we conclude that $M[G] \models |W| = |Y|$. (Claim) □

Proof of Theorem 6.1 continued Let Y be as stated in the Claim. Thus, $Y \in M$, $W \in K(\mathbb{R})$, and $M[G] \models (W \subseteq Y \wedge |W| = |Y| \geq \Theta)$. Now, since $W, Y \in K(\mathbb{R})$, we can conclude that $X \subseteq W \subseteq Y$ (in V). In addition, since $M[G] \models (|W| = |Y| \geq \Theta)$, it follows that $K(\mathbb{R}) \models (|W| = |Y| \geq \Theta)$. Hence, $|X| = |W| = |Y|$ (in V). Therefore, $Y \in \mathrm{HOD}^{K(\mathbb{R})}$, $X \subseteq Y$, and $|X| = |Y|$. □

Corollary 6.2 *Assume* ZF + AD *and that there is no $\rho(\mathbb{R})$-model. Let* $H = \mathrm{HOD}^{K(\mathbb{R})}$. *For all ordinals λ,*

(1) $|\lambda|^H = |\lambda|$ *when* $\lambda \geq \Theta$,
(2) $\mathrm{cf}(\lambda)^H = \mathrm{cf}(\lambda)$ *when* $\mathrm{cf}(\lambda) \geq \Theta$.

Since one can prove in ZF (via $\mathbb{Q}$-forcing) that a $\rho(\mathbb{R})$-model is iterable, Theorem 6.1 yields a seemingly weak condition for the existence of a $\rho(\mathbb{R})$-model.

Corollary 6.3 *Assume* ZF + AD. *A $\rho(\mathbb{R})$-model exists if and only if there is a set of ordinals X such that* $|X| \geq \Theta^{K(\mathbb{R})}$ *and X has no covering set in* $\mathrm{HOD}^{K(\mathbb{R})}$.

Note

1. Jensen's covering lemma is provable in ZF, since his proof can be employed in an inner model of ZFC.

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