

Justification by Infinite Loops

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Abstract In an earlier paper we have shown that a proposition can have a well-defined probability value, even if its justification consists of an infinite linear chain. In the present paper we demonstrate that the same holds if the justification takes the form of a closed loop. Moreover, in the limit that the size of the loop tends to infinity, the probability value of the justified proposition is always well-defined, whereas this is not always so for the infinite linear chain. This suggests that infinitism sits more comfortably with a coherentist view of justification than with an approach in which justification is portrayed as a linear process.

1 Introduction

Present-day epistemologists routinely assume that justification is probabilistic in character: a proposition E_n may be justified by another proposition E_{n+1} , even if the latter only partially supports the former. Recently we have shown that this apparently innocent assumption flies in the face of classical foundationalism [1]. For if we take seriously that justification comes in degrees, then there is in general no need for a foundation from which the justification springs. The target proposition E_n may have a perfectly well-defined probability value, even though its support consists of an infinite linear chain in which E_n is probabilistically justified by E_{n+1} , which in turn is probabilistically justified by E_{n+2} , and so on, ad infinitum.¹

In the present paper we extend this research on infinite linear chains by exploring the viability of infinite epistemic loops. We show that, once justification is interpreted probabilistically, the prospects for infinite loops are even brighter than those for infinite linear chains. If a proposition is justified by an infinite loop, it always has a well-defined unconditional probability, whereas this is not always so for infinite linear chains. An infinite linear chain normally confers upon the target proposition an unconditional probability that is well-defined, but there are exceptional cases in which it fails to do so.

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Here is how we plan to make our case. In Section 2 we start with a discussion of finite and infinite linear probabilistic chains, briefly summarizing the results that we derived in [1]. We explain that infinite linear probabilistic chains always converge and that they normally yield a well-defined probability value for the target proposition. In addition to the results in [1], we also delineate a class of exceptional cases, in which the target proposition lacks a unique value. After this analysis of justification by probabilistic chains, we turn our attention to justification by probabilistic loops. We first discuss in Section 3 loops that are finite, describing and analyzing several examples. We show that justification by a finite loop is often nontrivial, yielding a definite value for the target proposition where we would not immediately expect it. In Section 4 we extend our study to loops of infinite size. We demonstrate that justification by an infinite linear chain is usually indistinguishable from justification by an infinite loop. The only cases in which an infinite chain and an infinite loop differ are the exceptional situations that we had already identified in Section 2. In those situations, the infinite loop does, whereas the infinite chain does not yield a well-defined unconditional probability for the target proposition. Finally, in Section 5, we sum up our results.

2 Finite and Infinite Linear Chains

Let E_0, E_1, E_2, \dots be a sequence of propositions, finite or infinite in number. We say that E_n is probabilistically justified by E_{n+1} if and only if the conditional probability of E_n , given that E_{n+1} is true, is greater than the conditional probability of E_n , given that E_{n+1} is false:

$$P(E_n|E_{n+1}) > P(E_n|\neg E_{n+1}). \quad (1)$$

The unconditional probabilities $P(E_n)$ and $P(E_{n+1})$ are related by the rule of total probability,

$$P(E_n) = P(E_n|E_{n+1})P(E_{n+1}) + P(E_n|\neg E_{n+1})[1 - P(E_{n+1})]. \quad (2)$$

With the abbreviations

$$\begin{aligned} \alpha_n &= P(E_n|E_{n+1}), \\ \beta_n &= P(E_n|\neg E_{n+1}), \\ \gamma_n &= \alpha_n - \beta_n, \end{aligned}$$

rule (2) becomes

$$P(E_n) = \beta_n + \gamma_n P(E_{n+1}). \quad (3)$$

Clearly, $\gamma_n > 0$ is equivalent to the condition of probabilistic support as expressed in (1).

If each member of the sequence $E_0, E_1, E_2, \dots, E_{s+1}$, except E_{s+1} , is probabilistically justified by its successor, we speak of a *finite* linear chain of probabilistic support. The linear chain is then grounded in the ultimate link E_{s+1} , which is unsupported by any of the other links. We can consider a finite chain in which the number of links, $s + 1$, is *fixed*, for example when there are just three of them, or we might be interested in a situation in which the length of the finite chain is allowed to *vary*, thus turning s into a variable. In both cases we must find some reason for the veridicality or plausibility of E_{s+1} , on pain of leaving the entire chain hanging in the air.

Equation (3) can be iterated from $n = 0$ up to $n = s$, with the result

$$P(E_0) = \beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \dots + \gamma_0\gamma_1\dots\gamma_{s-1}\beta_s + \gamma_0\gamma_1\dots\gamma_s P(E_{s+1}). \quad (4)$$

Equation (4) is the most general formulation of a finite linear chain.² It applies not only to cases where both α_n and β_n are uniform, that is, where they remain constant from one link to the next throughout the chain, but also to cases where both α_n and β_n are nonuniform, taking on different values that depend on n . In [1] three examples were given of a finite linear chain with variable s : one in which α_n and β_n do not depend on n , one in which they do depend on n although γ_n does not, and finally one in which α_n , β_n , and γ_n all depend on n .

The structure of (4) can be represented as

$$P(E_0) = X + YZ, \tag{5}$$

where X is a finite sum of conditional probabilities only and YZ is a remainder term. The factor Y is the finite product of γ factors, while Z is the unconditional probability of the alleged ground of the chain, E_{s+1} :

$$\begin{aligned} X &= \beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \cdots + \gamma_0\gamma_1 \cdots \gamma_{s-1}\beta_s \\ Y &= \gamma_0\gamma_1 \cdots \gamma_s \\ Z &= P(E_{s+1}). \end{aligned} \tag{6}$$

In the limit that s goes to infinity, all the members of the sequence E_0, E_1, E_2, \dots are probabilistically justified by their successors. In this case the chain of support is *infinite* and its structure becomes

$$P(E_0) = X' + Y'Z', \tag{7}$$

where X' , Y' , and Z' are

$$\begin{aligned} X' &= \lim_{s \rightarrow \infty} \{\beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \cdots + \gamma_0\gamma_1 \cdots \gamma_{s-1}\beta_s\} \\ Y' &= \lim_{s \rightarrow \infty} \{\gamma_0\gamma_1 \cdots \gamma_s\} \\ Z' &= \lim_{s \rightarrow \infty} P(E_{s+1}). \end{aligned} \tag{8}$$

We have proved that X' always converges to a unique and well-defined number ([2], Appendix A). In addition, we showed that Y' usually tends to zero ([2], Appendix B). This means that, in general, the product $Y'Z'$ vanishes and thus that, as a rule, $P(E_0)$ in (7) takes on the well-defined value X' . It is only in very exceptional cases that Y' does not tend to zero, and then the value of $P(E_0)$ cannot be determined (see the [Appendix](#) of the present paper for the condition under which this happens). Surprisingly enough, even in these exceptional cases the value of $P(E_0)$ can still be determined if the probabilistic justification has the form of an infinite loop rather than an infinite linear chain. We will come back to this point in Section 4, where we discuss justification by infinite loops. First, in Section 3, we consider loops of finite size. We show that justification by a finite probabilistic loop is nontrivial in most cases. For usually the conditional probabilities on the loop vary, and then the value of $P(E_0)$ is a nontrivial function of the length of the loop.

3 Finite Loops

We have seen that Equation (4) is the general formulation of a finite linear chain. The general formulation of a finite loop has a similar form, but for some finite s it

is so that $E_{s+1} = E_0$. Mathematically, there is no problem whatsoever if we insert $E_{s+1} = E_0$ into Equation (4):

$$P(E_0) = \beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \cdots + \gamma_0\gamma_1 \cdots \gamma_{s-1}\beta_s + \gamma_0\gamma_1 \cdots \gamma_s P(E_0), \quad (9)$$

for this yields

$$P(E_0) = \frac{\beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \cdots + \gamma_0\gamma_1 \cdots \gamma_{s-1}\beta_s}{1 - \gamma_0\gamma_1 \cdots \gamma_s}, \quad (10)$$

which is well-defined, on condition that $\gamma_0\gamma_1 \cdots \gamma_s$ is not equal to unity.³ With that proviso, the solution furnishes a generic justification of the viability of the coherentist scenario in its simplest form, that of a finite one-dimensional ring.

So mathematically speaking a self-supporting finite loop or ring is certainly possible. But the fact that something makes good mathematical sense is, of course, not enough. Can a loop that closes upon itself really occur? A temporal example of such a loop is difficult to come by in the real world, but the science fiction of time travel can provide one. E_0 could be the event that young Bif decides in 1958 to use the 2018 edition of the sports almanac, E_1 the event that he continues his successful career as bettor until 2018, and E_2 the event that old Bif succeeds in stealing Doc Brown's time machine in 2018, returning to 1958 in order to give the almanac to his younger self. $E_3 = E_0$ could be the event that young Bif decides in 1958 to use the 2018 edition of the sports almanac... and so on.

However, the events need not follow one another in time. For example, consider the following three propositions:

C: "Peter read parts of the *Critique of Pure Reason*."

P: "Peter is a philosopher."

S: "Peter knows that Kant defended the synthetic a priori."

Assuming that all philosophers read at least parts of the *Critique of Pure Reason* as undergraduates, if Peter is a philosopher, then he read parts of the *Critique*. Of course, even if he is not a philosopher, he may still have read Kant's magnum opus. If Peter knows that Kant defended the synthetic a priori, he very likely is a philosopher, whereas if he does not, he is probably not a philosopher, although of course he might be an exceptionally incompetent one, not having understood anything of Kant or the *Critique*. Finally, if he read the *Critique*, he quite likely knows that Kant defended the synthetic a priori, whereas this is rather less likely if he never opened the book. Here then is a simple finite loop, consisting of a fixed number of links, namely three:

$$C \leftarrow P \leftarrow S \leftarrow C, \quad (11)$$

where the arrow indicates that the proposition at the right-hand side probabilistically justifies the one at the left.

We can make loop (11) nonuniform by investing the three propositions C, P, and S with, for example, the following dissimilar values for the the conditional probabilities:

$$\begin{array}{lll} \text{C:} & \alpha_0 = P(C|P) = 1; & \beta_0 = P(C|\neg P) = \frac{1}{10}; & \gamma_0 = \alpha_0 - \beta_0 = \frac{9}{10} \\ \text{P:} & \alpha_1 = P(P|S) = \frac{9}{10}; & \beta_1 = P(P|\neg S) = \frac{1}{5}; & \gamma_1 = \alpha_1 - \beta_1 = \frac{7}{10} \\ \text{S:} & \alpha_2 = P(S|C) = \frac{4}{5}; & \beta_2 = P(S|\neg C) = \frac{2}{5}; & \gamma_2 = \alpha_2 - \beta_2 = \frac{2}{5}. \end{array}$$

Then the unconditional probabilities⁴ are

$$\begin{aligned}
 P(C) &= \frac{\beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2}{1 - \gamma_0\gamma_1\gamma_2} = 0.711 \\
 P(P) &= \frac{\beta_1 + \gamma_1\beta_2 + \gamma_1\gamma_2\beta_0}{1 - \gamma_0\gamma_1\gamma_2} = 0.679 \\
 P(S) &= \frac{\beta_2 + \gamma_2\beta_0 + \gamma_2\gamma_0\beta_1}{1 - \gamma_0\gamma_1\gamma_2} = 0.684.
 \end{aligned}$$

The number of links in the above nonuniform loop is fixed—there are exactly three propositions. Often, however, we are dealing with cases in which there is a variable s . Let us therefore look at an example of a nonuniform loop in which the number of links varies. We will see that, in this example, the value of $P(E_0)$ depends on the length of the loop in a nontrivial way. Consider

$$\beta_n = \frac{1}{n + 3} \qquad \gamma_n = \frac{n + 1}{n + 2} = 1 - \frac{1}{n + 2}. \tag{12}$$

Then

$$\begin{aligned}
 \gamma_0\gamma_1 \dots \gamma_s &= \frac{1}{2} \times \frac{2}{3} \times \dots \times \frac{s}{s+1} \times \frac{s+1}{s+2} = \frac{1}{s+2} \\
 \gamma_0\gamma_1 \dots \gamma_{s-1}\beta_s &= \frac{1}{s+1} \times \frac{1}{s+3} = \frac{1}{2} \left(\frac{1}{s+1} - \frac{1}{s+3} \right),
 \end{aligned}$$

so Equation (10) reduces to

$$\begin{aligned}
 P(E_0) &= \frac{\frac{1}{3} + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \frac{1}{2} \left(\frac{1}{s+1} - \frac{1}{s+3} \right)}{1 - \frac{1}{s+2}} \\
 &= \frac{s + 2}{s + 1} \left[\frac{1}{3} + \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{s + 2} - \frac{1}{s + 3} \right) \right] = \frac{3s + 8}{4(s + 3)}. \tag{13}
 \end{aligned}$$

Here $P(E_0)$ does depend on the number of sites on the finite loop (and, moreover, there is a definite limit as the number of sites tends to infinity, namely $\frac{3}{4}$). What is more, since in this example the conditional probabilities, $P(E_n|E_{n+1})$ and $P(E_n|\neg E_{n+1})$, are not the same for different n , the unconditional probabilities, $P(E_n)$, also are not the same for different n . In fact, one finds

$$P(E_n) = 1 - \frac{1}{2} \frac{1}{n + 2} - \frac{1}{4} \frac{n + 1}{s + 3}, \tag{14}$$

which indeed depends nontrivially on n , as well as on s , the length of the loop.⁵

Cases like (13), in which the value of $P(E_0)$ varies with the number of links, form in fact the generic situation. There also exist special cases, where the value of $P(E_0)$ does not depend on s . Such nongeneric cases arise when the conditional probabilities are uniform (that is, all the β_n are the same, and all the γ_n are the same, independent of n). Intuitively, it is clear that loops for which the conditional probabilities are uniform will yield unconditional probabilities that are independent of s . Here is a formal proof. In the uniform case, Equation (10) becomes

$$P(E_0) = \frac{\beta(1 + \gamma + \gamma^2 + \dots + \gamma^s)}{1 - \gamma^{s+1}}. \tag{15}$$

The finite geometrical series $1 + \gamma + \gamma^2 + \dots + \gamma^s$ is equal to $(1 - \gamma^{s+1})/(1 - \gamma)$, and on substituting this we find

$$P(E_0) = \frac{\beta}{1 - \gamma}. \quad (16)$$

Indeed, this does not depend on s at all. The value of $P(E_0)$ is the same, irrespective of s , that is, however long or short the loop may be. Moreover, in this uniform case $P(E_0)$, $P(E_1)$, $P(E_2)$, and so on, are all equal, since they are all determined by the same cyclic expression (15). The number of links is completely irrelevant to the value of the unconditional probabilities in the uniform case; moreover, this holds whether s is finite or infinite.

4 Infinite Loops

We consider now an infinite loop of probabilistic support, that is, one where s in (9) and (10) goes to infinity. We first look at what happens when the product $\gamma_0\gamma_1\dots\gamma_s$ tends to zero as s goes to infinity, and then what happens when it doesn't. The former represents the typical case, the latter the atypical one.

If $\gamma_0\gamma_1\dots\gamma_s$ tends to zero, (10) yields the infinite series

$$P(E_0) = \beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \gamma_0\gamma_1\gamma_2\beta_3 \dots \quad (17)$$

and we have seen that such an infinite series is always convergent. In fact, Equation (17) is what one obtains by letting s in Equation (4) go to infinity. So in the typical case there is no difference between the infinite chain and the infinite loop.

The above equation covers both the uniform and the nonuniform cases. In the uniform infinite situation, in which β_n and γ_n are constant, that is, independent of n , Equation (17) reduces to

$$P(E_0) = \beta(1 + \gamma + \gamma^2 + \dots) = \frac{\beta}{1 - \gamma}. \quad (18)$$

This infinite uniform loop yields the same unconditional probability as does the *finite* uniform loop, a fact that is intuitively easy to grasp. After all, in the latter case, propositions are uniformly connected round and round ad infinitum.

The nonuniform case is more interesting. Here (17) can take many different forms, dependent on the values of β_n and γ_n . If we choose for these values the ones that were given in (12), then we obtain the expression (14) in the limit that s is taken to infinity, namely,

$$P(E_n) = 1 - \frac{1}{2} \frac{1}{n+2} = \frac{2n+3}{2(n+2)}. \quad (19)$$

In particular, $P(E_0) = \frac{3}{4}$, as we have already noted—see (13) and the lines following that equation.

So much for the typical case. What of the atypical situation in which the infinite product of the γ s is not zero? Here the linear chain fails, in the infinite limit, to produce a definite answer for the probability, whereas the infinite loop gives a unique value. To illustrate this, consider the specific example

$$\beta_n = \frac{1}{(n+2)(n+3)} \quad \gamma_n = \frac{(n+1)(n+3)}{(n+2)^2} = 1 - \frac{1}{(n+2)^2}.$$

The crucial difference is that here $1 - \gamma_n$ tends to zero as fast as $1/n^2$, whereas this difference had the slower asymptotic behavior $1/n$ in Equation (12). We find now

$$\begin{aligned} \gamma_0 \gamma_1 \dots \gamma_s &= \left[\frac{1}{2} \cdot \frac{3}{2}\right] \times \left[\frac{2}{3} \cdot \frac{4}{3}\right] \times \dots \times \left[\frac{s+1}{s+2} \cdot \frac{s+3}{s+2}\right] = \frac{1}{2} \frac{s+3}{s+2} \\ \gamma_0 \gamma_1 \dots \gamma_{s-1} \beta_s &= \frac{1}{2} \frac{s+2}{s+1} \times \frac{1}{(s+2)(s+3)} = \frac{1}{4} \left(\frac{1}{s+1} - \frac{1}{s+3}\right), \end{aligned}$$

so Equation (4) becomes

$$\begin{aligned} P(E_0) &= \frac{1}{6} - \frac{1}{4} \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \dots + \right. \\ &\quad \left. \frac{1}{s} - \frac{1}{s+2} + \frac{1}{s+1} - \frac{1}{s+3}\right) + \frac{1}{2} \frac{s+3}{s+2} P(E_{s+1}) \\ &= \frac{3}{8} - \frac{1}{4} \left(\frac{1}{s+2} + \frac{1}{s+3}\right) + \frac{1}{2} \frac{s+3}{s+2} P(E_{s+1}). \end{aligned} \tag{20}$$

Note that the coefficient in front of $P(E_{s+1})$ does *not* vanish in the limit that s tends to infinity. Indeed, in this limit we find formally

$$P(E_0) = \frac{3}{8} + \frac{1}{2} P(E_\infty),$$

where $P(E_\infty)$ is an indeterminate number in the interval $[0, 1]$. The infinite linear chain has in this case failed to produce a definite value for the probability. However, for the infinite loop we can set $P(E_{s+1}) = P(E_0)$ in Equation (20) and then we can solve the linear equation for $P(E_0)$ to obtain

$$\begin{aligned} P(E_0) &= \left[\frac{3}{8} - \frac{1}{4} \left(\frac{1}{s+2} + \frac{1}{s+3}\right)\right] / \left[1 - \frac{1}{2} \frac{s+3}{s+2}\right] \\ &= \frac{3}{4} - \frac{1}{4} \frac{1}{s+3}, \end{aligned}$$

which has the perfectly definite limit $\frac{3}{4}$. Thus, the infinite chain and the infinite ring only differ when γ_s tends to unity with sufficient rapidity. In the [Appendix](#) we formulate a necessary and sufficient condition under which this happens, thus delineating the entire class of cases in which the infinite chain fails to give a definite answer, but the infinite loop does so.

5 A Pluralistic Picture

In 1956 Sellars diagnosed the malaise of epistemology as an unpalatable either/or: “One seems forced to choose between the picture of an elephant which rests on a tortoise (What supports the tortoise?) and the picture of a great Hegelian serpent of knowledge with its tail in its mouth (Where does it begin?). Neither will do” ([3], p. 300). Sellars was presumably thinking of chains and loops, both involving entailment relations only, and then indeed neither is adequate. However, if support is interpreted probabilistically, then we are not confined to these two possibilities. For then a picture of justification emerges that is distinctly pluralistic. A target proposition, E_n , can be probabilistically justified by a finite or an infinite chain, or it can be justified by a finite or an infinite loop. In each of these four cases the conditional probabilities might be uniform or they might be nonuniform. These three parameters (finite versus infinite chain, finite versus infinite loop, uniform versus nonuniform) thus yield eight different varieties of probabilistic support.

The main result of our paper pertains to probabilistic support that is nonuniform and infinite. At first sight one might think that a nonuniform loop of infinite length cannot really be called a loop, since there is no end of the tail that the Hegelian

serpent can swallow. After all, is it not the case that a *Wiederkehr des Gleichen*, to quote another German philosopher, must require that the loop be finite? The loop may be long, indeed more than cosmologically long, but it seems that it may not be infinite, on pain of having no *Wiederkehr* at all. Moreover, even Poincaré, when he formulated his recurrence theorem, had to assume that the universe is finite in spatial extent and of finite energy: those are necessary conditions for a recurrence in finite time.

So it seems that a real loop differs from an infinite “loop.” However, from this it does not follow that, therefore, an infinite loop is in fact an infinite chain. Our investigation shows that such a conclusion would be unwarranted. It is true that an infinite *uniform* loop cannot be distinguished from an infinite *uniform* chain: both yield the same trivial result. It is also true that, usually, the infinite nonuniform loop produces the same value as does the infinite nonuniform chain. However, there are exceptional situations in which infinite nonuniform loops and infinite nonuniform chains yield different results. These exceptions consist in cases where the infinitely far away “end” of the chain can still exert some influence on the probability of the target proposition. As we have shown, an infinite loop has an even wider domain than does an infinite linear chain: an infinitely long serpent succeeds even when an infinite stack of tortoises fails.

Appendix

What is the general condition under which the infinite linear chain and the infinite loop fail to agree? Clearly if $\gamma_0\gamma_1\dots\gamma_s$ does not vanish in the limit of infinite s . Every γ_n lies in the open interval $(0, 1)$, the extreme values 0 and 1 being excluded by fiat. Since $\gamma_n = \exp[\log \gamma_n] = \exp[-|\log \gamma_n|]$, we have

$$\gamma_0\gamma_1\gamma_2\gamma_3\dots = \prod_{s=0}^{\infty} \gamma_s = \exp\left[-\sum_{s=0}^{\infty} |\log \gamma_s|\right].$$

So $\gamma_0\gamma_1\dots\gamma_s$ has a nonzero limit as s goes to infinity if and only if the sum

$$\sigma \equiv \sum_{s=0}^{\infty} |\log \gamma_s| \tag{21}$$

is convergent. Clearly, convergence can occur only if γ_s tends sufficiently quickly to 1. For example, if

$$\gamma_s \sim 1 - s^{-a}$$

for large s , then Equation (21) converges if $a > 1$, and in that case the coherentist loop (10) yields

$$P(E_0) = \frac{\beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \dots}{1 - e^{-\sigma}},$$

but the linear infinitist chain instead gives

$$P(E_0) = \beta_0 + \gamma_0\beta_1 + \gamma_0\gamma_1\beta_2 + \dots + e^{-\sigma} P(E_{\infty}).$$

Whereas the infinite loop gives a definite answer in terms of all the conditional probabilities, the linear chain is problematic in the infinite limit, for $P(E_{\infty})$ must be construed as an indeterminate number between 0 and 1.

Notes

1. For a more extensive discussion of how patterns of probabilistic dependence are relevant to an understanding of epistemic justification, see our earlier paper [1]. The present article may be regarded as a sequel to that work.
2. Equation (4) is the same as Equation (13) in [1].
3. If $\gamma_0\gamma_1 \dots \gamma_s = 1$, it follows that *all* the γ s are separately equal to one. But then all the α s are equal to one also, and all the β s are equal to zero, which is the condition of bi-implication. It is clear logically that the propositions could all be true, or all be false, for these two extreme possibilities are obviously consistent with the bi-implication. However, the lack of uniqueness goes further, for *any* probability between 0 and 1, if it is shared by all the propositions, is consistent with bi-implication. This is a direct consequence of the rule of total probability when $\alpha = 1$ and $\beta = 0$. So if the product of the γ s is unity, the indeterminacy is maximal, and if it is not unity, then the unconditional probabilities are determined uniquely.
4. As they must, these numbers satisfy

$$P(C) = \beta_0 + \gamma_0 P(P) \quad P(P) = \beta_1 + \gamma_1 P(S) \quad P(S) = \beta_2 + \gamma_2 P(C).$$

Incidentally, there is a good reason for considering a loop of at least three propositions. For in a “loop” of just two links, there are only three independent unconditional probabilities, for example $P(E_0)$, $P(E_1)$, and $P(E_0 \wedge E_1)$, whereas there are four conditional probabilities around the loop, $P(E_0|E_1)$, $P(E_0|\neg E_1)$, $P(E_1|E_0)$, and $P(E_1|\neg E_0)$. So there must be a linear relation between them, which means that all four may not be chosen independently. This difficulty does not arise for a loop of three links, for in this case there are seven independent unconditional probabilities and only six conditional probabilities around the loop, so the latter may be chosen arbitrarily. With more than three links on the loop there is even more freedom, so the conditional probabilities may again be chosen freely.

5. For a general value of m between 0 and s , we see by iteration of Equation (3) from $n = m$ to $n = s$ that $P(E_m) = \beta_m + \gamma_m\beta_{m+1} + \dots + \gamma_m \dots \gamma_{s-1}\beta_s + \gamma_m \dots \gamma_s P(E_{s+1})$. Much as in the case $m = 0$, we find $\gamma_m \dots \gamma_s = \frac{m+1}{s+2}$ and

$$\gamma_m \dots \gamma_{s-1}\beta_s = \frac{m+1}{2} \left(\frac{1}{s+1} - \frac{1}{s+3} \right).$$

With these expressions in hand, we work through steps entirely analogous to those given in (13), ending with formula (14).

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