Functors of Lindenbaum-Tarski, Schematic Interpretations, and Adjoint Cylinders between Sentential Logics

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In Memory of Professors K. Gödel (1906–1978) and S. Mac Lane (1909–2005)

Abstract   We prove, by using the concept of schematic interpretation, that the natural embedding from the category **ISL**, of intuitionistic sentential pretheories and i-congruence classes of morphisms, to the category **CSL**, of classical sentential pretheories and c-congruence classes of morphisms, has a left adjoint, which is related to the double negation interpretation of Gödel-Gentzen, and a right adjoint, which is related to the Law of Excluded Middle. Moreover, we prove that from the left to the right adjoint there is a pointwise epimorphic natural transformation and that since the two endofunctors at **CSL**, obtained by adequately composing the aforementioned functors, are naturally isomorphic to the identity functor for **CSL**, the string of adjunctions constitutes an adjoint cylinder. On the other hand, we show that the operators of Lindenbaum-Tarski of formation of algebras from pretheories can be extended to equivalences of categories from the category **CSL**, respectively, **ISL**, to the category **Bool**, of Boolean algebras, respectively, **Heyt**, of Heyting algebras. Finally, we prove that the functor of regularization from **Heyt** to **Bool** has, in addition to its well-known right adjoint (that is, the canonical embedding of **Bool** into **Heyt**) a left adjoint, that from the left to the right adjoint there is a pointwise epimorphic natural transformation, and, finally, that such a string of adjunctions constitutes an adjoint cylinder.

1 Introduction

Sentential logics, as any other logics, as is well known, can be investigated at least from two points of view, the syntactical or proof-theoretic and the semantical or model-theoretic. Both are, usually, strongly related through the Lindenbaum-Tarski operators, which associate to every theory in a sentential logic an algebraic construct of some definite species and this in such a way that the outcome is a representation of the algebraic constructs into the logical ones.

On the other hand, on some occasions, it happens that for a given couple of logical systems, as, for example, for the couple formed by (**CSL**, **Bool**) and (**ISL**, **Heyt**)—where **CSL** and **ISL** are the categories briefly described in the abstract and which will be formally defined in Section 2—it can be proved that there are adjunctions both between the proof-theoretic components and between the model-theoretic components of them, as is the case, for instance, whenever there are natural interpretations between the sets of formulas of the logical systems involved as well as naturally defined functors among the algebraic categories underlying them.

Our objective in this article will be to confirm the above by investigating a particular case, concretely that of classical and intuitionistic sentential logics. Thus, toward this goal, in Section 2, after defining the categories of classical and intuitionistic sentential pretheories, denoted, respectively, by $\mathbf{Pth}_c$ and $\mathbf{Pth}_i$, and having proved the existence of full and essentially surjective functors ℓt_c from $\mathbf{Pth}_c$ to **Bool**, the category of Boolean algebras and (Boolean) homomorphisms, and ℓt_i from $\mathbf{Pth}_i$ to **Heyt**, the category of Heyting algebras and (Heyting) homomorphisms, we define two congruences $\equiv_c$ on $\mathbf{Pth}_c$ and $\equiv_i$ on $\mathbf{Pth}_i$, and from them we obtain the *functors of Lindenbaum-Tarski* LT_c from $\mathbf{CSL} = \mathbf{Pth}_c/\equiv_c$, the category of classical sentential pretheories (and c-congruence classes of morphisms), to **Bool**, and LT_i from $\mathbf{ISL} = \mathbf{Pth}_i/\equiv_i$, the category of intuitionistic sentential pretheories (and i-congruence classes of morphisms), to **Heyt**. In this way the well-known operators of Lindenbaum-Tarski have been extended to functors; that is, they not only take as arguments pretheories giving as values corresponding algebraic constructs, but they also operate, in a natural way, on the morphisms between pretheories giving as values corresponding homomorphisms between the associated algebraic constructs. Moreover, it happens that both functors of Lindenbaum-Tarski are, as a matter of fact, equivalences of categories.

In Section 3, from a convenient full subcategory of the category $\mathbf{ALog}(\Sigma)$ of abstract logics and logical morphisms between abstract logics defined by Brown and Suszko in [2], p. 19, and from the compact entailment systems for classical and intuitionistic sentential logics, we define, from the standpoint of category theory, the concept of schematic interpretation between classical and intuitionistic sentential logics. In Section 4, the concept of schematic interpretation will be applied to prove that there are two adjunctions,

$$K \dashv J \colon \mathbf{CSL} \longrightarrow \mathbf{ISL} \quad \text{and} \quad J \dashv G \colon \mathbf{ISL} \longrightarrow \mathbf{CSL},$$

with the functor K, which embodies the category-theoretic content of the double negation interpretation of Gödel-Gentzen from classical to intuitionistic sentential logic, and the functor G, which is related to the Law of Excluded Middle, full embeddings from **CSL** to **ISL**, and the functor J injective on objects from **ISL** to **CSL**, together with a pointwise epimorphic natural transformation ξ from K to G. These facts allow us to assert that the string of adjunctions $K \dashv J \dashv G$ constitutes, in

particular, a special case of the concept of adjoint cylinder as defined by Lawvere in [7], p. 11. Finally, also in Section 4, we show that there is an adjunction,

$$T \dashv R \colon \mathbf{Bool} \longrightarrow \mathbf{Heyt},$$

with the functor $T \colon \mathbf{Bool} \longrightarrow \mathbf{Heyt}$ obtained from the syntactic functor K or by means of Freyd's Adjoint Functor Theorem, and R the regularization functor from **Heyt** to **Bool**, which, we recall, is a left adjoint of the canonical embedding I from **Bool** to **Heyt**, together with a pointwise epimorphic natural transformation ζ from T to I. Again, since T and I are, in addition, full embeddings, this situation allows us to assert that the string of adjunctions $T \dashv R \dashv I$ constitutes, in particular, a special case of the concept of adjoint cylinder.

To simplify matters, we agree that the logical signatures of classical and intuitionistic sentential logics are identical and denoted by Σ. Moreover, for a set (of variables) X, $\mathbf{T}_\Sigma(X)$ is the free Σ-algebra on X, and, for $\ell \in \{\,\mathrm{c},\mathrm{i}\,\}$, $\mathrm{Cn}_{\ell,X}$ is the consequence operator and $\vdash_{\ell,X}$ the consequence relation for the classical, respectively, intuitionistic, sentential logic relative to the set of variables X, and if $\Phi \subseteq \mathrm{T}_\Sigma(X)$, where $\mathrm{T}_\Sigma(X)$ is the underlying set of $\mathbf{T}_\Sigma(X)$, then $\equiv_{\ell,\Phi}$ is the congruence on $\mathbf{T}_\Sigma(X)$ defined, for every $\alpha, \beta \in \mathbf{T}_\Sigma(X)$, as follows:

$$\alpha \equiv_{\ell,\Phi} \beta \quad \text{iff} \quad \alpha \leftrightarrow \beta \in \mathrm{Cn}_{\ell,X}(\Phi).$$

Lastly, for a set A, we agree to denote by $\mathrm{Sub}(A)$ the set of all subsets of A and for a mapping $f \colon A \longrightarrow B$, by $f[\cdot]$ the operator of direct image formation, that is, the mapping from $\mathrm{Sub}(A)$ to $\mathrm{Sub}(B)$ which to a subset X of A assigns its direct image $f[X] \subseteq B$.

In all that follows we use standard concepts and constructions from category theory (see, for example, Mac Lane [9]), classical universal algebra (see, for example, Cohn [4]), and lattice theory (see, for example, Balbes and Dwinger [1] and Rasiowa and Sikorski [12]).

2   The Functors of Lindenbaum-Tarski between Categories of Sentential Pretheories and Algebraic Categories

Logicians are well aware of the fact that the operators of Lindenbaum-Tarski, explicitly introduced by Tarski in [14], p. 510, yield quotients of algebras of formulas of given sentential logics by congruences associated to subsets of the algebras of formulas under consideration. Thus, in particular, given a classical or intuitionistic pretheory (X, Φ), where X is a set of propositional variables and Φ a subset of $\mathrm{T}_\Sigma(X)$, not necessarily closed either under the consequence operator $\mathrm{Cn}_{\mathrm{c},X}$ or under $\mathrm{Cn}_{\mathrm{i},X}$, the values of the operators of Lindenbaum-Tarski ℓt_{c}, for classical sentential logic, and ℓt_{i}, for intuitionistic sentential logic, at (X, Φ) are the Boolean algebra $\ell t_{\mathrm{c}}(X, \Phi) = \mathbf{T}_\Sigma(X)/\equiv_{\mathrm{c},\Phi}$ and the Heyting algebra $\ell t_{\mathrm{i}}(X, \Phi) = \mathbf{T}_\Sigma(X)/\equiv_{\mathrm{i},\Phi}$, respectively. In this way an exact formulation of the connection between, on the one hand, the classical sentential pretheories and the Boolean algebras and, on the other hand, the intuitionistic sentential pretheories and the Heyting algebras, is obtained, but, we remark, only at the object level.

Our main aim in this section is to show that the operators of Lindenbaum-Tarski can also be defined in a *natural* way on suitable morphisms between classical, respectively, intuitionistic, pretheories and that they are compatible with definite congruences on the sets of morphisms between pretheories. This allows us to get, before taking quotients by the aforementioned congruences, a precise expression of the connection between the morphisms among classical, respectively, intuitionistic, pretheories and the corresponding homomorphisms among Boolean, respectively, Heyting algebras, extending in this way the operators of Lindenbaum-Tarski to functors. Moreover, by identifying the aforementioned morphisms between pretheories by means of the so-called ℓ-congruences, we get equivalences of categories from the category of classical sentential pretheories and c-congruence classes of morphisms to the category **Bool** and from the category of intuitionistic sentential pretheories and i-congruence classes of morphisms to the category **Heyt**.

To attain the aim mentioned above we begin by defining two compact entailment systems $\mathcal{E}_c$ and $\mathcal{E}_i$ for classical and intuitionistic propositional logics, respectively. From here we get, in a natural way, the categories **Pth**$_c$ and **Pth**$_i$ of classical and intuitionistic sentential pretheories, respectively. Then we prove that there are full and essentially surjective functors ℓt_c from **Pth**$_c$ to **Bool** and ℓt_i from **Pth**$_i$ to **Heyt**. Finally, by defining suitable congruences $\equiv_c$ and $\equiv_i$ on **Pth**$_c$ and **Pth**$_i$, respectively, we prove that there are equivalences of categories LT_c and LT_i from **Pth**$_c/\equiv_c$ to **Bool** and from **Pth**$_i/\equiv_i$ to **Heyt**, respectively.

For the common logical signature Σ of classical and intuitionistic sentential logics, we denote by $\mathbf{Kl}(\mathbb{T}_\Sigma)$ the Kleisli category for $\mathbb{T}_\Sigma = (T_\Sigma, \eta, \mu)$, the standard monad derived from the adjunction $\mathbf{T}_\Sigma \dashv \mathbf{G}_\Sigma$ between the category $\mathbf{Alg}(\Sigma)$, of Σ-algebras and homomorphisms, and the category **Set**, of sets and mappings, where T_Σ is the endofunctor $G_\Sigma \circ \mathbf{T}_\Sigma$. Then, for $\ell \in \{c, i\}$, the triple $\mathcal{E}_\ell = (\mathbf{Kl}(\mathbb{T}_\Sigma), G_{\mathbb{T}_\Sigma}, \vdash_\ell)$, where

1. $G_{\mathbb{T}_\Sigma}$ is the functor from $\mathbf{Kl}(\mathbb{T}_\Sigma)$ to **Set** which sends an object X in $\mathbf{Kl}(\mathbb{T}_\Sigma)$, that is, a set X, to $T_\Sigma(X)$, and a morphism f from X to Y in $\mathbf{Kl}(\mathbb{T}_\Sigma)$, that is, a mapping f from X to $T_\Sigma(Y)$, to the mapping $\mu_Y \circ \mathbf{T}_\Sigma(f)$ from $T_\Sigma(X)$ to $T_\Sigma(Y)$, and

2. $\vdash_\ell$ is a subfunctor of the functor $\mathrm{Sub}(T_\Sigma(\cdot)) \times T_\Sigma(\cdot)$ from $\mathbf{Kl}(\mathbb{T}_\Sigma)$ to **Set** which sends a set X to $\mathrm{Sub}(T_\Sigma(X)) \times T_\Sigma(X)$ and a morphism f from X to $T_\Sigma(Y)$ to the mapping $f^\sharp[\cdot] \times f^\sharp$ from $\mathrm{Sub}(T_\Sigma(X)) \times T_\Sigma(X)$ to $\mathrm{Sub}(T_\Sigma(Y)) \times T_\Sigma(Y)$, where $f^\sharp$ is the canonical extension of f to the free Σ-algebra on X,

is a *compact entailment system*; that is, $\vdash_\ell$ satisfies the following conditions:

1. for every $X \in \mathbf{Kl}(\mathbb{T}_\Sigma)$ and every $\varphi \in T_\Sigma(X)$, $\{\varphi\} \vdash_{\ell,X} \varphi$;
2. for every $X \in \mathbf{Kl}(\mathbb{T}_\Sigma)$, every $\Gamma, \Delta \subseteq T_\Sigma(X)$, and every $\varphi \in T_\Sigma(X)$, if $\Gamma \vdash_{\ell,X} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\ell,X} \varphi$;
3. for every $X \in \mathbf{Kl}(\mathbb{T}_\Sigma)$, every set I, every family $(\varphi_i)_{i \in I}$ in $T_\Sigma(X)$, every $\Gamma \subseteq T_\Sigma(X)$, and every $\psi \in T_\Sigma(X)$, if, for every $i \in I$, $\Gamma \vdash_{\ell,X} \varphi_i$, and $\Gamma \cup \{\varphi_i \mid i \in I\} \vdash_{\ell,X} \psi$, then $\Gamma \vdash_{\ell,X} \psi$;
4. for every morphism f from X to Y in $\mathbf{Kl}(\mathbb{T}_\Sigma)$, every $\Gamma \subseteq T_\Sigma(X)$, and every $\varphi \in T_\Sigma(X)$, if $\Gamma \vdash_{\ell,X} \varphi$, then $f^\sharp[\Gamma] \vdash_{\ell,Y} f^\sharp(\varphi)$;
5. for every $X \in \mathbf{Kl}(\mathbb{T}_\Sigma)$, every $\Gamma \subseteq T_\Sigma(X)$, and every $\varphi \in T_\Sigma(X)$, if we have that $\Gamma \vdash_{\ell,X} \varphi$, then there exists a finite subset Δ of Γ such that $\Delta \vdash_{\ell,X} \varphi$.

To the entailment systems $\mathcal{E}_c$ and $\mathcal{E}_i$ we associate, respectively, and in a *natural* way, certain categories of pretheories as stated in the following definition.

Definition 2.1   For the signature Σ we denote by **Pth**$_c$ the category which has as objects the *classical sentential pretheories*, that is, the pairs (X, Φ) with X a set and Φ a subset of $T_\Sigma(X)$, and as morphisms from (X, Φ) to (Y, Δ) those homomorphisms $f : \mathbf{T}_\Sigma(X) \longrightarrow \mathbf{T}_\Sigma(Y)$ which satisfy $f[Cn_{c,X}(\Phi)] \subseteq Cn_{c,Y}(\Delta)$, that is, such that, for every $\gamma \in T_\Sigma(X)$, if $\Phi \vdash_{c,X} \gamma$, then $\Delta \vdash_{c,Y} f(\gamma)$. The category **Pth**$_i$, of *intuitionistic sentential pretheories*, is similarly defined.

We point out that, for $\ell \in \{c, i\}$, in the category **Pth**$_\ell$ any pretheory (X, Φ) is isomorphic to the theory $(X, Cn_{\ell,X}(\Phi))$, where, we recall, a pretheory (X, Φ) is called a *theory* exactly if $\Phi = Cn_{\ell,X}(\Phi)$. Indeed we have that the category **Pth**$_\ell$ is *equivalent*, but not *isomorphic*, to the full subcategory **Th**$_\ell$ of **Pth**$_\ell$ determined by the theories. However, we maintain the distinction between pretheories and theories, because the concept of pretheory allows one to make finer distinctions that are important for proof-theoretic and computational purposes.

On the other hand, by composing the projection functor P_ℓ from **Pth**$_\ell$ to **Kl**$(\mathbb{T}_\Sigma)$, which sends a pretheory (X, Φ) to X and a morphism $f \in \text{Hom}_{\mathbf{Pth}_\ell}((X, \Phi), (Y, \Delta))$ to $f \circ \eta_X \in \text{Hom}_{\mathbf{Kl}(\mathbb{T}_\Sigma)}(X, Y)$, with the functor $G_{\mathbb{T}_\Sigma}$ from **Kl**$(\mathbb{T}_\Sigma)$ to **Set**, we obtain an extension of the functor $G_{\mathbb{T}_\Sigma}$ to a functor $G_{\ell,\mathbb{T}_\Sigma}$ from **Pth**$_\ell$ to **Set**. Moreover, we obtain an extension of the functor $\vdash_\ell$ from **Kl**$(\mathbb{T}_\Sigma)$ to **Set** to a functor, also denoted by $\vdash_\ell$, from **Pth**$_\ell$ to **Set** by defining, for every $(X, \Phi) \in \mathbf{Pth}_\ell$, every $\Delta \subseteq T_\Sigma(X)$, and every $\gamma \in T_\Sigma(X)$, the binary relation $\Delta \vdash_{\ell,(X,\Phi)} \gamma$ as follows:

$$\Delta \vdash_{\ell,(X,\Phi)} \gamma \quad \text{iff} \quad \Delta \cup \Phi \vdash_{\ell,X} \gamma.$$

Since it will be useful in what follows, we gather in the following proposition some characterizations of the concept of morphism between pretheories.

Proposition 2.2   *Let (X, Φ) and (Y, Δ) be two sentential pretheories, both intuitionistic or both classical, and f a homomorphism from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Sigma(Y)$. Then, for $\ell \in \{c, i\}$, the following conditions are equivalent:*

1. *the homomorphism f is a morphism from (X, Φ) to (Y, Δ);*
2. *the homomorphism f is such that $f[\Phi] \subseteq Cn_{\ell,Y}(\Delta)$;*
3. *for every $\alpha, \beta \in T_\Sigma(X)$, if we have that $\alpha \leftrightarrow \beta \in Cn_{\ell,X}(\Phi)$, then we also have that $f(\alpha) \leftrightarrow f(\beta) \in Cn_{\ell,Y}(\Delta)$.*

Following this we prove that there are full and essentially surjective, but not necessarily faithful, functors from **Pth**$_c$ to **Bool** and from **Pth**$_i$ to **Heyt**, rendering categorial the well-known Lindenbaum-Tarski representations of some algebraic constructs through logical systems.

Proposition 2.3   *There are full and essentially surjective functors ℓt_c from **Pth**$_c$ to **Bool** and ℓt_i from **Pth**$_i$ to **Heyt**.*

Proof   We restrict ourselves to the classical case, because the intuitionistic one is formally identical. We define the functor ℓt_c as follows.

1. On objects: If (X, Φ) is a classical sentential pretheory, then

$$\ell t_c(X, \Phi) = \mathbf{T}_\Sigma(X)/\equiv_{c,\Phi},$$

that is, the quotient Σ-algebra of $\mathbf{T}_\Sigma(X)$ by the congruence $\equiv_{c,\Phi}$ on it, which, as is well known, is a Boolean algebra.

2. On morphisms: If f is a morphism from (X, Φ) to (Y, Δ) in $\mathbf{Pth}_c$, then $\mathrm{Ker}(\mathrm{pr}_{\equiv_{c,\Phi}}) \subseteq \mathrm{Ker}(\mathrm{pr}_{\equiv_{c,\Delta}} \circ f)$, where $\mathrm{pr}_{\equiv_{c,\Phi}}$ is the canonical projection from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Sigma(X)/ \equiv_{c,\Phi}$ and $\mathrm{pr}_{\equiv_{c,\Delta}}$ the canonical projection from $\mathbf{T}_\Sigma(Y)$ to $\mathbf{T}_\Sigma(Y)/ \equiv_{c,\Delta}$. Hence there exists a unique homomorphism, denoted by $\ell t_c(f)$, from the Boolean algebra $\mathbf{T}_\Sigma(X)/ \equiv_{c,\Phi}$ to the Boolean algebra $\mathbf{T}_\Sigma(Y)/ \equiv_{c,\Delta}$ such that the homomorphisms $\mathrm{pr}_{\equiv_{c,\Delta}} \circ f$ and $\ell t_c(f) \circ \mathrm{pr}_{\equiv_{c,\Phi}}$ from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Sigma(Y)/ \equiv_{c,\Delta}$ are identical.

So defined ℓt_c is an essentially surjective functor from $\mathbf{Pth}_c$ to $\mathbf{Bool}$, that is, a functor such that, for every Boolean algebra $\mathbf{B}$, there exists a classical sentential pretheory (X, Φ) such that $\ell t_c(X, \Phi)$ is isomorphic to $\mathbf{B}$ (a proof of this assertion can be found, for example, in Monk [10], Theorem 9.60, p. 160).

Following this we prove that ℓt_c is full. Let g be a Boolean homomorphism from $\ell t_c(X, \Phi) = \mathbf{T}_\Sigma(X)/ \equiv_{c,\Phi}$ to $\ell t_c(Y, \Delta) = \mathbf{T}_\Sigma(Y)/ \equiv_{c,\Delta}$. Then, because the canonical projection $\mathrm{pr}_{\equiv_{c,\Delta}}$ is an epimorphism and the Σ-algebra $\mathbf{T}_\Sigma(X)$ is projective (since it is free), there exists a homomorphism f from $\mathbf{T}_\Sigma(X)$ into $\mathbf{T}_\Sigma(Y)$ such that the homomorphisms $\mathrm{pr}_{\equiv_{c,\Delta}} \circ f$ and $g \circ \mathrm{pr}_{\equiv_{c,\Phi}}$ from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Sigma(Y)/ \equiv_{c,\Delta}$ are identical. Moreover, f is a morphism from (X, Φ) to (Y, Δ). For suppose that $\alpha, \beta \in \mathrm{T}_\Sigma(X)$ are such that $\alpha \equiv_{c,\Phi} \beta$. Then $g([\alpha]_{\equiv_{c,\Phi}}) = g([\beta]_{\equiv_{c,\Phi}})$. Therefore, $[f(\alpha)]_{\equiv_{c,\Delta}} = [f(\beta)]_{\equiv_{c,\Delta}}$; that is, $f(\alpha) \equiv_{c,\Delta} f(\beta)$. Finally, it is obvious that $\ell t_c(f) = g$. Hence, as we wanted to prove, the functor ℓt_c is full. $\qquad\square$

As we have said above neither ℓt_c nor ℓt_i is necessarily faithful; however, we can get equivalences of categories from them by defining convenient congruences $\equiv_c$ and $\equiv_i$ on the categories $\mathbf{Pth}_c$ and $\mathbf{Pth}_i$, respectively.

Definition 2.4   We denote, for $\ell \in \{c, i\}$, by $\equiv_\ell$ the binary relation on the set of morphisms of $\mathbf{Pth}_\ell$ defined, for every $f, g \in \mathrm{Mor}(\mathbf{Pth}_\ell)$, as follows:

$$f \equiv_\ell g \;\; \text{iff} \;\; \begin{cases} d_0(f) = (X, \Phi) = d_0(g), \; \& \\ d_1(f) = (Y, \Delta) = d_1(g), \; \& \\ \mathrm{pr}_{\equiv_{\ell,\Delta}} \circ f = \mathrm{pr}_{\equiv_{\ell,\Delta}} \circ g, \end{cases}$$

where d_0 and d_1 are, among the structural mappings of $\mathbf{Pth}_\ell$, precisely those which assign to a morphism of $\mathbf{Pth}_\ell$ its domain and codomain, respectively. If f and g are in the relation $\equiv_\ell$, then we say that f and g are ℓ-congruent.

Proposition 2.5   *The relation $\equiv_\ell$, for $\ell \in \{c, i\}$, defined on the set of morphisms of* $\mathbf{Pth}_\ell$, *is a congruence on the category* $\mathbf{Pth}_\ell$.

Proof   It is easy to check that the relation $\equiv_\ell$ is an equivalence and right compatible with the composition of morphisms. In order to prove the left compatibility of the relation $\equiv_\ell$, let $f, g\colon (X, \Phi) \longrightarrow (Y, \Delta)$ be a coterminal pair of parallel morphisms in $\mathbf{Pth}_\ell$ such that $f \equiv_\ell g$ and let $h\colon (Y, \Delta) \longrightarrow (Z, \Theta)$ be another morphism in $\mathbf{Pth}_\ell$. Then $\mathrm{pr}_{\equiv_{\ell,\Delta}} \circ f = \mathrm{pr}_{\equiv_{\ell,\Delta}} \circ g$ and the homomorphisms $\mathrm{pr}_{\equiv_{\ell,\Theta}} \circ h$ and $\ell t_\ell(h) \circ \mathrm{pr}_{\equiv_{\ell,\Delta}}$ from $\mathbf{T}_\Sigma(Y)$ to $\mathbf{T}_\Sigma(Z)/ \equiv_{\ell,\Theta}$ are identical. Hence $\mathrm{pr}_{\equiv_{\ell,\Theta}} \circ (h \circ f) = \mathrm{pr}_{\equiv_{\ell,\Theta}} \circ (h \circ g)$; therefore, $h \circ f \equiv_\ell h \circ g$. $\qquad\square$

This proposition has as an immediate consequence the following corollary.

Corollary 2.6   *The quotient category* $\mathbf{CSL} = \mathbf{Pth}_c/ \equiv_c$, *called the category of* classical sentential pretheories *(and c-congruence classes of morphisms), is equivalent to* $\mathbf{Bool}$, *and we call the equivalence of categories* LT_c *from* $\mathbf{CSL}$ *to* $\mathbf{Bool}$ *the functor*

of Lindenbaum-Tarski for the classical sentential logic. *Analogously, the quotient category* **ISL** $=$ **Pth**$_i$/ $\equiv_i$*, called the category of* intuitionistic sentential pretheories *(and i-congruence classes of morphisms), is equivalent to* **Heyt***, and we call the equivalence of categories* LT$_i$ *from* **ISL** *to* **Heyt** *the functor of Lindenbaum-Tarski for the intuitionistic sentential logic.*

Remark 2.7   Since the operators of Lindenbaum-Tarski are really equivalences of categories, the quotient categories of the type **Pth**$/ \equiv$, which have an essentially logical character, can be investigated indirectly through the algebraic categories canonically equivalent to them. Thus, for example, since **Bool** and **Heyt** are bicomplete categories, we can assert that **CSL** and **ISL** also are bicomplete. On the other hand, since **Bool** and **Heyt** are Mal'cev varieties, we have, in a derived way, cohomology theories for **CSL** and **ISL**, as in Smith [13]; therefore, the extensions of one pretheory by another, as well as the obstructions to such extensions, can also be cohomologically investigated. Inasmuch as the indicated topics do not fall under the mainstream of this article, we leave the corresponding development for a future work.

3   Schematic Interpretations

Our objective in this section is to define, from the standpoint of category theory, a concept of transformation, which we call schematic interpretation, between sentential logics under which some, although not all, of the historical interpretations from classical to intuitionistic (or modal) sentential logics fall. To attain this aim, we begin by defining a full subcategory of the category **ALog**(Σ) of abstract logics and logical morphisms, defined by Brown and Suszko in [2], p. 19. But before that, and to make the article self-contained, we recall that **ALog**(Σ) has as objects the *abstract logics*, that is, the ordered pairs $(\mathbf{A}, \mathrm{Cn})$, where $\mathbf{A}$ is a Σ-algebra and Cn a closure operator on A, the underlying set of $\mathbf{A}$, and as morphisms from an abstract logic $(\mathbf{A}, \mathrm{Cn})$ to another $(\mathbf{A}', \mathrm{Cn}')$ the *logical* morphisms, that is, the homomorphisms f from $\mathbf{A}$ to $\mathbf{A}'$ such that, for every subset X of A, $f[\mathrm{Cn}(X)] \subseteq \mathrm{Cn}'(f[X])$.

Definition 3.1   We denote by **ALog**$_{s,a}(\Sigma)$ the category which has as objects precisely those abstract logics $(\mathbf{A}, \mathrm{Cn})$ which are *structural* and *algebraic*, that is, those for which the closure operator Cn on A satisfies the following two properties:

1. Cn is *structural*; that is, for every endomorphism f of $\mathbf{A}$ and every subset X of A, $f[\mathrm{Cn}(X)] \subseteq \mathrm{Cn}(f[X])$, and
2. Cn is *algebraic*; that is, for every subset X of A, $\mathrm{Cn}(X) = \bigcup_{K \in \mathrm{Sub}_f(X)} \mathrm{Cn}(K)$, where $\mathrm{Sub}_f(X)$ is the set of all finite subsets of X,

and as morphisms from a structural algebraic abstract logic $(\mathbf{A}, \mathrm{Cn})$ to another $(\mathbf{A}', \mathrm{Cn}')$ the logical morphisms.

From the definition of the category **ALog**$_{s,a}(\Sigma)$ it follows immediately that it is a full subcategory of the category **ALog**(Σ) of Brown and Suszko, and the reason why we work in this section in such a subcategory instead of working in **ALog**(Σ) rests ultimately upon the fact that classical and intuitionistic sentential logics, as is well known, have *structural* and *algebraic* consequence operators.

After having defined the category **ALog**$_{s,a}(\Sigma)$, we derive from the compact entailment system $\mathcal{E}_\ell = (\mathbf{Kl}(\mathbb{T}_\Sigma), G_{\mathbb{T}_\Sigma}, \vdash_\ell)$, for $\ell \in \{\mathrm{c, i}\}$, described, we recall, at the beginning of Section 2, the functor F$_\ell$, for $\ell \in \{\mathrm{c, i}\}$, from **Set** to **ALog**$_{s,a}(\Sigma)$

which assigns to a set X the abstract logic $(\mathbf{T}_\Sigma(X), \mathrm{Cn}_{\ell,X})$ and to a mapping f from X to Y its canonical extension $\mathbf{T}_\Sigma(f)$ from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Sigma(Y)$, which is a logical morphism from $(\mathbf{T}_\Sigma(X), \mathrm{Cn}_{\ell,X})$ to $(\mathbf{T}_\Sigma(Y), \mathrm{Cn}_{\ell,Y})$. Moreover, we notice that from $\mathbf{ALog}_{\mathrm{s,a}}(\Sigma)$ to $\mathbf{Alg}(\Sigma)$ we have the forgetful functor U_Σ which sends an object $(\mathbf{A}, \mathrm{Cn})$ of $\mathbf{ALog}_{\mathrm{s,a}}(\Sigma)$ to $\mathbf{A}$ and a logical morphism f from $(\mathbf{A}, \mathrm{Cn})$ to $(\mathbf{A}', \mathrm{Cn}')$ to the homomorphism f from $\mathbf{A}$ to $\mathbf{A}'$, and it happens that $U_\Sigma \circ F_\ell = \mathbf{T}_\Sigma$.

After these preliminaries, we define the concept of schematic interpretation from the pair (Σ, F_c) to the pair (Σ, F_i), which is related to the concepts of interpretation of Prawitz and Malmnäs in [11] and of Wójcicki in [15], p. 69. But before that, we choose, once and for all, an effectively enumerated set of variables $V = \{ v_n \mid n \in \mathbb{N} \}$ (isomorphic to $\mathbb{N}$) and make the following observations.

1. If d is a *derivor* from a signature Σ to another Σ', that is, a mapping from Σ to $\mathrm{T}_{\Sigma'}(V)$ such that, for every $n \in \mathbb{N}$ and every formal operation $\sigma \in \Sigma_n$, the term $d(\sigma) \in \mathrm{T}_{\Sigma'}(\downarrow v_n)$, where $\downarrow v_n = \{ v_0, \dots, v_{n-1} \} \subseteq V$, then the mapping d determines a functor d^*, from the category $\mathbf{Alg}(\Sigma')$ to the category $\mathbf{Alg}(\Sigma)$ that assigns to a Σ'-algebra $\mathbf{A}'$ the *derived* Σ-algebra $d^*(\mathbf{A}')$ whose underlying set is that of $\mathbf{A}'$ and whose structural operations are the term operations of the Σ'-algebra $\mathbf{A}'$ associated to the terms $d(\sigma)$, for every formal operation σ in Σ. In our case, since we have agreed that the signatures of classical and intuitionistic sentential logics are the same, we will speak of an *endoderivor*.

2. If $P \in \mathrm{T}_{\Sigma'}(\downarrow v_1)$ is a term in the variable $v_0 \in V$, then P induces a natural endotransformation $P^{\Sigma'}$ of the functor $G_{\Sigma'} \circ \mathbf{T}_{\Sigma'}$ that to a set X assigns the term operation $P^{\Sigma',X} : \mathrm{T}_{\Sigma'}(X) \longrightarrow \mathrm{T}_{\Sigma'}(X)$.

Definition 3.2   A *schematic interpretation* from (Σ, F_c) to (Σ, F_i) is an ordered triple (d, P, t), where d is an endoderivor of Σ, $P \in \mathrm{T}_\Sigma(\downarrow v_1)$, and t a natural transformation from F_c to $D \circ F_i$, where D is the endofunctor of $\mathbf{ALog}_{\mathrm{s,a}}(\Sigma)$ (uniquely determined by d) which sends an object $(\mathbf{A}, \mathrm{Cn})$ of $\mathbf{ALog}_{\mathrm{s,a}}(\Sigma)$ to $(d^*(\mathbf{A}), \mathrm{Cn})$ such that

$$U_\Sigma \circ D = d^* \circ U_\Sigma \text{ and } ((G_\Sigma \circ U_\Sigma) * t) \circ \eta^\Sigma = P^\Sigma \circ \eta^\Sigma,$$

where η^Σ is the unit of $\mathbf{T}_\Sigma \dashv G_\Sigma$ and $(G_\Sigma \circ U_\Sigma) * t$ is the horizontal composition of the natural transformation t and the natural transformation associated to the functor $G_\Sigma \circ U_\Sigma$.

Since the notion of strong subderived system, defined by Cleave in [3], p. 63, is used immediately below, we recall it next for the sake of completeness.

Definition 3.3   For an abstract logic $(\mathbf{A}', \mathrm{Cn}')$, with $\mathbf{A}'$ a Σ'-algebra, and a derivor d from Σ to Σ', an abstract logic $(\mathbf{A}, \mathrm{Cn})$, with $\mathbf{A}$ a Σ-algebra, is said to be a *strong subderived system* of $(\mathbf{A}', \mathrm{Cn}')$ if $\mathbf{A}$ is a subalgebra of $d^*(\mathbf{A}')$ and, for every subset X of A, $\mathrm{Cn}(X) = A \cap \mathrm{Cn}'(X)$.

Proposition 3.4   *The interpretation (d^G, P^G, t^G) from classical to intuitionistic sentential logic defined by Gödel in [6] (and slightly transformed) as follows,*

1. *d^G is the mapping from Σ to $\mathrm{T}_\Sigma(V)$ which assigns to $\neg$, $d^G(\neg) = \neg v_0$, to $\wedge$, $d^G(\wedge) = v_0 \wedge v_1$, to $\vee$, $d^G(\vee) = \neg(\neg v_0 \wedge \neg v_1)$, and, finally, to $\rightarrow$, $d^G(\rightarrow) = \neg(v_0 \wedge \neg v_1)$,*
2. *P^G is $\neg\neg v_0$,*

3. *for every set X, t_X^G, the component of t^G at X, is*

$$t_X^G(\gamma) = \begin{cases} \neg\neg x, & \text{if } \gamma = x(\in X); \\ \neg t_X^G(\varphi), & \text{if } \gamma = \neg\varphi; \\ t_X^G(\varphi) \wedge t_X^G(\psi), & \text{if } \gamma = \varphi \wedge \psi; \\ \neg(\neg t_X^G(\varphi) \wedge \neg t_X^G(\psi)), & \text{if } \gamma = \varphi \vee \psi; \\ \neg(t_X^G(\varphi) \wedge \neg t_X^G(\psi)), & \text{if } \gamma = \varphi \rightarrow \psi, \end{cases}$$

is schematic and such that, for each set X, each $\psi \in T_\Sigma(X)$, and each $\Phi \subseteq T_\Sigma(X)$, if $t_X^G(\psi) \in Cn_{i,X}(t_X^G[\Phi])$, then $\psi \in Cn_{c,X}(\Phi)$. Moreover, by [3], p. 103, for each set X, $(T_\Sigma(X), Cn_{c,X})$ is isomorphic to a strong subderived system of $(T_\Sigma(X), Cn_{i,X})$.

Remark 3.5   The schematic interpretations of Gödel, (d^G, P^G, t^G), and Gentzen, denoted by (d^{Gz}, P^{Gz}, t^{Gz}), this last defined by Gentzen in [5], p. 532, as follows,

1. d^{Gz} is the mapping from Σ to $T_\Sigma(V)$ which assigns to $\neg$, $d^{Gz}(\neg) = \neg v_0$, to $\wedge$, $d^{Gz}(\wedge) = v_0 \wedge v_1$, to $\vee$, $d^{Gz}(\vee) = \neg(\neg v_0 \wedge \neg v_1)$, and, finally, to $\rightarrow$, $d^{Gz}(\rightarrow) = v_0 \rightarrow v_1$,
2. P^{Gz} is $\neg\neg v_0$,
3. for every set X, t_X^{Gz}, the component of t^{Gz} at X, is

$$t_X^{Gz}(\gamma) = \begin{cases} \neg\neg x, & \text{if } \gamma = x(\in X); \\ \neg t_X^{Gz}(\varphi), & \text{if } \gamma = \neg\varphi; \\ t_X^{Gz}(\varphi) \wedge t_X^{Gz}(\psi), & \text{if } \gamma = \varphi \wedge \psi; \\ \neg(\neg t_X^{Gz}(\varphi) \wedge \neg t_X^{Gz}(\psi)), & \text{if } \gamma = \varphi \vee \psi; \\ t_X^{Gz}(\varphi) \rightarrow t_X^{Gz}(\psi), & \text{if } \gamma = \varphi \rightarrow \psi, \end{cases}$$

as is well known, are intuitionistically equivalent; that is, for every set X and every $\psi \in T_\Sigma(X)$, it happens that $\vdash_{i,X} t_X^G(\psi) \leftrightarrow t_X^{Gz}(\psi)$ (for a proof see, for example, Luckhardt [8], pp. 42–43).

If we agree, on the one hand, that P^+ is the endofunctor of **Set** which assigns to a set X its power set $P^+(X) = Sub(X)$ and to a mapping f from X to Y the mapping $P^+(f) = f[\cdot]$ from $Sub(X)$ to $Sub(Y)$, and, on the other hand, that, for every set X, $\{\cdot\}_{T_\Sigma(X)}$ is the mapping from $T_\Sigma(X)$ to $Sub(T_\Sigma(X))$ which to a $\psi \in T_\Sigma(X)$ assigns $\{\psi\}$, then, from the standpoint of category theory, the intuitionistic equivalence mentioned above can be explained by saying that the natural endotransformation $\varepsilon = (Cn_{i,X})_{X\in\mathbf{Set}}$ of $P^+ \circ T_\Sigma$ is such that $\varepsilon \circ \delta^G = \varepsilon \circ \delta^{Gz}$, where δ^G and δ^{Gz} are the natural transformations $(t_X^G[\cdot] \circ \{\cdot\}_{T_\Sigma(X)})_{X\in Ob(\mathbf{Set})}$ and $(t_X^{Gz}[\cdot] \circ \{\cdot\}_{T_\Sigma(X)})_{X\in Ob(\mathbf{Set})}$ from T_Σ to $P^+ \circ T_\Sigma$, respectively. Indeed, for every set X and every $\psi \in T_\Sigma(X)$, it happens that

$$(\varepsilon \circ \delta^G)_X(\psi) = Cn_{i,X}(\{t_X^G(\psi)\}) \text{ and } (\varepsilon \circ \delta^{Gz})_X(\psi) = Cn_{i,X}(\{t_X^{Gz}(\psi)\}).$$

Hence, $(\varepsilon \circ \delta^G)_X(\psi) = (\varepsilon \circ \delta^{Gz})_X(\psi)$; therefore, $\varepsilon \circ \delta^G = \varepsilon \circ \delta^{Gz}$.

4 Adjoint Cylinders between Classical and Intuitionistic Sentential Logics and between Boolean and Heyting Algebras

In this section we prove, first and foremost, that there are two adjunctions,

$$K \dashv J \colon \mathbf{CSL} \longrightarrow \mathbf{ISL} \quad \text{and} \quad J \dashv G \colon \mathbf{ISL} \longrightarrow \mathbf{CSL},$$

between the categories **CSL** and **ISL**, with the functor K, which embodies the category-theoretic content of the schematic interpretation of Gödel-Gentzen from classical to intuitionistic sentential logic, and the functor G, which is related to the Law of Excluded Middle, full embeddings from **CSL** to **ISL**, and the functor J injective on objects from **ISL** to **CSL**. Moreover, we prove that there exists a pointwise epimorphic natural transformation ξ from K to G.

Since not only are the three involved functors adjoint, but moreover the two composites at **CSL** are naturally isomorphic to the identity functor for **CSL**, because K and G are full embeddings, we have an example of a special case of the concept of *adjoint cylinder* as defined by Lawvere in [7], thus confirming Mac Lane's slogan ([9], Preface), "Adjoint functors arise everywhere," and what probably is more important in this case, Lawvere's slogan ([7], p. 11), "Adjoint cylinders (satisfying the additional conditions mentioned above) are mathematical models for many instances of the *Unity and Identity of Opposites*."

Following this, and connected with the equivalences of Lindenbaum-Tarski and the adjunctions mentioned above, we prove in a derived way that there exists an adjunction,

$$T \dashv R \colon \mathbf{Bool} \longrightarrow \mathbf{Heyt},$$

with R the regularization functor from **Heyt** to **Bool**, which we recall is a left adjoint of the canonical full embedding I from **Bool** to **Heyt**, and $T \colon \mathbf{Bool} \longrightarrow \mathbf{Heyt}$ the functor obtained as the composition $\mathrm{LT_i} \circ K \circ Q_\mathrm{c}$, where Q_c is an arbitrary but fixed inverse equivalence to the functor $\mathrm{LT_c}$, of Lindenbaum-Tarski for the classical sentential logic. However, before proving that there exists such an adjunction we state that the functors $R \circ \mathrm{LT_i}$ and $\mathrm{LT_c} \circ J$ from **ISL** to **Bool** are naturally isomorphic. This result has, on the one hand, a certain intrinsic value and, on the other hand, it is necessary to state one of the proofs of the aforementioned adjunction, precisely that one which proceeds by using natural isomorphisms between hom-sets. Moreover, we prove that there exists a pointwise epimorphic natural transformation ζ from T to I.

Proposition 4.1   *There are two adjunctions,*

$$K \dashv J \colon \mathbf{CSL} \longrightarrow \mathbf{ISL} \quad and \quad J \dashv G \colon \mathbf{ISL} \longrightarrow \mathbf{CSL},$$

*with K and G full embeddings from **CSL** to **ISL**, and J injective on objects from **ISL** to **CSL**, together with a pointwise epimorphic natural transformation ξ from K to G.*

Proof   We define the functor $J \colon \mathbf{ISL} \longrightarrow \mathbf{CSL}$ as follows.

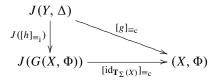1. On objects: If (X, Φ) is an intuitionistic sentential pretheory, then

$$J(X, \Phi) = (X, \Phi).$$

2. On morphisms: If $[f]_{\equiv_\mathrm{i}}$ is a morphism from (X, Φ) to (Y, Δ) in **ISL**, then
$$J([f]_{\equiv_\mathrm{i}}) = [f]_{\equiv_\mathrm{c}}.$$

For the functor J we restrict ourselves to prove that it is well defined on the morphisms; that is, if $[f]_{\equiv_i}$ is a morphism from (X, Φ) to (Y, Δ) in **ISL**, then $[f]_{\equiv_c}$ is a morphism from (X, Φ) to (Y, Δ) in **CSL** (that is, for every ψ in $\mathrm{T}_\Sigma(X)$, from $\Phi \vdash_{c,X} \psi$, it follows that $\Delta \vdash_{c,Y} f(\psi)$), since it is not particularly difficult to verify that J preserves identity morphisms and the composition of morphisms.

Let us suppose that $[f]_{\equiv_i}$ is a morphism from (X, Φ) to (Y, Δ) in **ISL** and that, for ψ in $\mathrm{T}_\Sigma(X)$, $\Phi \vdash_{c,X} \psi$. Then, since f is a homomorphism from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Sigma(Y)$, we have that $f[\Phi] \vdash_{c,Y} f(\psi)$, but it happens, on the one hand, that $f[\Phi] \subseteq \mathrm{Cn}_{i,Y}(\Delta)$, because by hypothesis $[f]_{\equiv_i}$ is a morphism from (X, Φ) to (Y, Δ) in **ISL**, and, on the other hand, that $\mathrm{Cn}_{i,Y}(\Delta) \subseteq \mathrm{Cn}_{c,Y}(\Delta)$, therefore $\Delta \vdash_{c,Y} f(\psi)$.

After having defined the functor $J \colon \textbf{ISL} \longrightarrow \textbf{CSL}$, which is neither faithful nor full, but which is obviously injective on the objects, we prove that J has a right adjoint G. To attain this goal, let (X, Φ) be a classical sentential pretheory. Then for the intuitionistic sentential pretheory $G(X, \Phi) = (X, \Phi \cup \mathrm{LEM}_X)$, where LEM_X is the *Law of Excluded Middle (relative to X)*, that is, the set $\{ \psi \vee \neg\psi \mid \psi \in \mathrm{T}_\Sigma(X) \}$, and for $[\mathrm{id}_{\mathbf{T}_\Sigma(X)}]_{\equiv_c}$ (which is a morphism from $J(G(X, \Phi))$ to (X, Φ), since, for every ψ in $\mathrm{T}_\Sigma(X)$, it happens that $\Phi \cup \mathrm{LEM}_X \vdash_{c,X} \psi$ if and only if $\Phi \vdash_{c,X} \psi$), we have that, for every intuitionistic sentential pretheory (Y, Δ) and every morphism $[g]_{\equiv_c}$ from $J(Y, \Delta)$ to (X, Φ), there exists a unique morphism $[h]_{\equiv_i}$ from (Y, Δ) to $G(X, \Phi)$ such that the following diagram commutes.

Indeed, put $[h]_{\equiv_i} = [g]_{\equiv_i}$ and let us check that $[g]_{\equiv_i}$ is a morphism from (Y, Δ) to $G(X, \Phi)$ in **ISL**; that is, for every ψ in $\mathrm{T}_\Sigma(Y)$, from $\Delta \vdash_{i,Y} \psi$, it follows that $\Phi \cup \mathrm{LEM}_X \vdash_{i,X} g(\psi)$.

Let ψ be an arbitrary element of $\mathrm{T}_\Sigma(Y)$ and let us suppose that $\Delta \vdash_{i,Y} \psi$. Then $\Delta \vdash_{c,Y} \psi$. Hence, keeping in mind that $[g]_{\equiv_c}$ is a morphism from $J(Y, \Delta)$ to (X, Φ) in **CSL**, $\Phi \vdash_{c,X} g(\psi)$. But we have, by Theorem 13.11, p. 410, stated by Rasiowa and Sikorski in [12], that $\Phi \vdash_{c,X} g(\psi)$ is equivalent to $\Phi \cup \mathrm{LEM}_X \vdash_{i,X} g(\psi)$; therefore, $\Phi \cup \mathrm{LEM}_X \vdash_{i,X} g(\psi)$.

It is easy to verify that $[h]_{\equiv_i} = [g]_{\equiv_i}$ is, in fact, the unique morphism from (Y, Δ) to $G(X, \Phi)$ such that $[\mathrm{id}_{\mathbf{T}_\Sigma(X)}]_{\equiv_c} \circ J([h]_{\equiv_i}) = [g]_{\equiv_c}$. On the other hand, the functor G is a full embedding from **CSL** to **ISL** because it has a left adjoint, J, and the counit of the adjunction $J \dashv G$ is, obviously, an isomorphism of functors.

Our next goal is to define a functor $K \colon \textbf{CSL} \longrightarrow \textbf{ISL}$ which has J as a right adjoint. Let K be defined as follows.

1. On objects: If (X, Φ) is a classical sentential pretheory, then

$$K(X, \Phi) = (X, t_X[\Phi]),$$

   t being the underlying natural transformation of an arbitrary, but fixed, schematic interpretation from classical to intuitionistic sentential logic.
2. On morphisms: If $[f]_{\equiv_c}$ is a morphism from (X, Φ) to (Y, Δ) in **CSL**, then $K([f]_{\equiv_c}) = [f]_{\equiv_i}$.
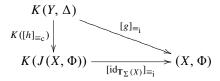
For the functor K, as was the case for J, we also restrict ourselves to prove that it is well defined on the morphisms; that is, if $[f]_{\equiv_c}$ is a morphism from (X, Φ) to (Y, Δ) in **CSL**, then $[f]_{\equiv_i}$ is a morphism from $(X, t_X[\Phi])$ to $(Y, t_Y[\Delta])$ in **ISL** (that is, for every ψ in $T_\Sigma(X)$, from $t_X[\Phi] \vdash_{i,X} \psi$, it follows that $t_Y[\Delta] \vdash_{i,Y} f(\psi)$), since it is not particularly difficult to verify that K preserves identity morphisms and the composition of morphisms.

Let us suppose that $[f]_{\equiv_c}$ is a morphism from (X, Φ) to (Y, Δ) in **CSL** and that, for ψ in $T_\Sigma(X)$, $t_X[\Phi] \vdash_{i,X} \psi$. Then, taking into account that f is a homomorphism from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Sigma(Y)$, we can affirm that $f[t_X[\Phi]] \vdash_{i,Y} f(\psi)$. However, what we want to show is that $t_Y[\Delta] \vdash_{i,Y} f(\psi)$. But to attain this goal it suffices, by transitivity, to prove, for every φ in Φ, $t_Y[\Delta] \vdash_{i,Y} f(t_X(\varphi))$; that is, $t_Y[\Delta] \vdash_{i,Y} f[t_X[\Phi]]$.

One way to state that $t_Y[\Delta] \vdash_{i,Y} f[t_X[\Phi]]$ is as follows. On the one hand, for each φ in Φ, it happens that $\Phi \vdash_{c,X} \varphi$; therefore, for each φ in Φ, since, by hypothesis, $[f]_{\equiv_c}$ is a morphism from (X, Φ) to (Y, Δ), it also happens that $\Delta \vdash_{c,Y} f(\varphi)$. But $t_Y \colon (\mathbf{T}_\Sigma(Y), \mathrm{Cn}_{c,Y}) \longrightarrow (d^*(\mathbf{T}_\Sigma(Y)), \mathrm{Cn}_{i,Y})$; hence $t_Y[\Delta] \vdash_{i,Y} t_Y(f(\varphi))$. On the other hand, from the fact that f is a homomorphism and taking into account that, for every φ in Φ, $\vdash_{i,X} t_X(\varphi) \leftrightarrow \neg\neg\varphi$, we conclude that $\vdash_{i,Y} f(t_X(\varphi)) \leftrightarrow \neg\neg f(\varphi)$. But also $\vdash_{i,Y} t_Y(f(\varphi)) \leftrightarrow \neg\neg f(\varphi)$; hence, $\vdash_{i,Y} f(t_X(\varphi)) \leftrightarrow t_Y(f(\varphi))$. Therefore, for every φ in Φ, $t_Y[\Delta] \vdash_{i,Y} f(t_X(\varphi))$ as we wanted to show. From this last result and by transitivity we can finally assert that $t_Y[\Delta] \vdash_{i,Y} f(\psi)$.

Before proving that K has J as a right adjoint, it seems to us appropriate to state otherwise that $t_Y[\Delta] \vdash_{i,Y} f[t_X[\Phi]]$. As above, from $\Phi \vdash_{c,X} \varphi$, for every φ in Φ, it follows that $\Delta \vdash_{c,Y} f[\Phi]$, because $[f]_{\equiv_c}$ is a morphism from (X, Φ) to (Y, Δ). Thus $\neg\neg\Delta \vdash_{i,Y} \neg\neg f[\Phi]$, where $\neg\neg\Delta$ is the set $\{\neg\neg\delta \mid \delta \in \Delta\}$ and $\neg\neg f[\Phi]$ is the set $\{\neg\neg f(\varphi) \mid \varphi \in \Phi\}$. But, bearing in mind that f is a homomorphism, we have that $\neg\neg f[\Phi] = f[\neg\neg\Phi]$, where $f[\neg\neg\Phi]$ is the set $\{f(\neg\neg\varphi) \mid \varphi \in \Phi\}$; therefore, since it happens that $\vdash_{i,Y} t_Y[\Delta] \leftrightarrow \neg\neg\Delta$, we conclude that $t_Y[\Delta] \vdash_{i,Y} f[\neg\neg\Phi]$. On the other hand, for every φ in Φ, it is the case that $\vdash_{i,X} t_X(\varphi) \leftrightarrow \neg\neg\varphi$; thus, for every φ in Φ, $\vdash_{i,Y} f(t_X(\varphi)) \leftrightarrow f(\neg\neg\varphi)$, or, what is equivalent, for every φ in Φ, $\vdash_{i,Y} f(t_X(\varphi)) \leftrightarrow \neg\neg f(\varphi)$. Hence, as we wanted to state, $t_Y[\Delta] \vdash_{i,Y} f[t_X[\Phi]]$.

Following this we prove that K has J as a right adjoint. Let (X, Φ) be an intuitionistic sentential pretheory. Then for the classical sentential pretheory $J(X, \Phi) = (X, \Phi)$ and for $[\mathrm{id}_{\mathbf{T}_\Sigma(X)}]_{\equiv_i}$, which is a morphism from $K(J(X, \Phi))$ to (X, Φ) (since, for every ψ in $T_\Sigma(X)$, from $t_X[\Phi] \vdash_{i,X} \psi$ it follows that $\Phi \vdash_{i,X} \psi$, because, for every φ in Φ, it happens that $\Phi \vdash_{i,X} t_X(\varphi)$) we have that, for every classical sentential pretheory (Y, Δ) and every morphism $[g]_{\equiv_i}$ from $K(Y, \Delta)$ to (X, Φ), there exists a unique morphism $[h]_{\equiv_c}$ from (Y, Δ) to $J(X, \Phi)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
K(Y, \Delta) & & \\
\Big\downarrow{\scriptstyle K([h]_{\equiv_c})} & \searrow{\scriptstyle [g]_{\equiv_i}} & \\
K(J(X, \Phi)) & \xrightarrow[{[\mathrm{id}_{\mathbf{T}_\Sigma(X)}]_{\equiv_i}}]{} & (X, \Phi)
\end{array}
$$

Indeed, put $[h]_{\equiv_c} = [g]_{\equiv_c}$ and let us check that $[g]_{\equiv_c}$ is a morphism from (Y, Δ) to $J(X, \Phi)$ in **CSL**; that is, for every ψ in $T_\Sigma(Y)$, if $\Delta \vdash_{c,Y} \psi$, then $\Phi \vdash_{c,X} g(\psi)$.

Let us suppose that, for ψ in $T_\Sigma(Y)$, $\Delta \vdash_{c,Y} \psi$. Then, since t_Y is a morphism from $(\mathbf{T}_\Sigma(Y), \mathrm{Cn}_{c,Y})$ to $(d^*(\mathbf{T}_\Sigma(Y)), \mathrm{Cn}_{i,Y})$, we have that $t_Y[\Delta] \vdash_{i,Y} t_Y(\psi)$, and

from here, and taking into account that $[g]_{\equiv_i}$ is a morphism from $K(Y, \Delta)$ to (X, Φ), we immediately obtain that $\Phi \vdash_{i,X} g(t_Y(\psi))$. Therefore, $\Phi \vdash_{c,X} g(t_Y(\psi))$. However, what we want to show is that $\Phi \vdash_{c,X} g(\psi)$. But to attain this goal it suffices, by transitivity, to show that, for every ψ in $T_\Sigma(Y)$, $\vdash_{c,X} g(\psi) \leftrightarrow g(t_Y(\psi))$.

One way to prove this last statement is to proceed as follows. It happens that, for ψ in $T_\Sigma(Y)$, $\vdash_{i,Y} t_Y(\psi) \leftrightarrow \neg\neg\psi$. Hence, bearing in mind that g is a homomorphism, it also happens that $\vdash_{i,Y} g(t_Y(\psi)) \leftrightarrow \neg\neg g(\psi)$; thus $\vdash_{c,X} g(t_Y(\psi)) \leftrightarrow \neg\neg g(\psi)$, but $\vdash_{c,X} g(\psi) \leftrightarrow \neg\neg g(\psi)$. Therefore, $\vdash_{c,X} g(\psi) \leftrightarrow g(t_Y(\psi))$ as we wanted to show. From this last result and by transitivity we can finally assert that $\Phi \vdash_{c,X} g(\psi)$.

It is easy to show that $[h]_{\equiv_c}$ is the unique morphism from (Y, Δ) to $J(X, \Phi)$ such that $[\mathrm{id}_{T_\Sigma(X)}]_{\equiv_i} \circ K([h]_{\equiv_c}) = [g]_{\equiv_i}$. On the other hand, the functor K is a full embedding from **CSL** to **ISL** because it has a right adjoint, J, and the unit of the adjunction $K \dashv J$ is, obviously, an isomorphism of functors (that is, $J \circ K$ and $\mathrm{Id}_{\mathbf{CSL}}$ are isomorphic). Let us point out that although the functor K has been defined by using an arbitrary but fixed schematic interpretation, this fact is, from the standpoint of category theory, inessential, because it will necessarily be naturally isomorphic to any other left adjoint of J.

Next let $\xi = (\xi_{(X,\Phi)})_{(X,\Phi)\in\mathrm{Ob}(\mathbf{CSL})}$ be the family defined, for every classical sentential pretheory (X, Φ), as $\xi_{(X,\Phi)} = [\mathrm{id}_{T_\Sigma(X)}]_{\equiv_i}$. Then $\xi_{(X,\Phi)}$ is a morphism from the intuitionistic sentential pretheory $K(X, \Phi) = (X, t_X[\Phi])$ to the intuitionistic sentential pretheory $G(X, \Phi) = (X, \Phi \cup \mathrm{LEM}_X)$. Let us suppose that, for ψ in $T_\Sigma(X)$, $t_X[\Phi] \vdash_{i,X} \psi$. Then $t_X[\Phi] \vdash_{c,X} \psi$; hence, by [12] (Theorem 13.11, p. 410) $t_X[\Phi] \cup \mathrm{LEM}_X \vdash_{i,X} \psi$. But $\Phi \vdash_{i,X} \neg\neg \Phi$ and $\neg\neg \Phi \vdash_{i,X} t_X[\Phi]$. Therefore, $\Phi \cup \mathrm{LEM}_X \vdash_{i,X} \psi$. Moreover, keeping in mind the definition of the congruence $\equiv_i$ (stated in Definition 2.4), it is evident that $\xi_{(X,\Phi)}$ is an epimorphism.

Finally, it is not difficult to show that ξ, as defined, is actually a natural transformation from K to G, and given that for every classical sentential pretheory (X, Φ) the component of ξ at (X, Φ) is an epimorphism, it is, in addition, pointwise epimorphic. Indeed, since $J \circ K$ is naturally isomorphic to $\mathrm{Id}_{\mathbf{CSL}}$, ξ is simply $\eta * K$, where η is the unit of the adjunction $J \dashv G$.   □

Since the functors G and K from **CSL** to **ISL** are full embeddings and both have the same functor J as a left and a right adjoint, respectively, we have obtained an instance of a special case of the concept of adjoint cylinder of Lawvere, and we could say that, from the syntactical, or proof-theoretic, point of view, the intellectual *reflection* of Gödel, and others, on the relationships between the classical and intuitionistic sentential logics, becomes category-theoretically *reflected* by the fact that **CSL** can be identified with a *full reflective* and a *full co-reflective* subcategory of **ISL**. Moreover, the category **ISL** is subordinated to **CSL** by virtue of the existence of the natural transformation ξ from the functor K to the functor G.

On the semantical, or model-theoretic, side, since the variety of Boolean algebras is a subvariety of the variety of Heyting algebras, we have the full embedding I from the category **Bool** to the category **Heyt**, which is, in addition, injective on the objects. On the other hand, we have the regularization functor R from the category **Heyt** to the category **Bool** which, we recall, is defined as follows.

1. On objects: If **H** is a Heyting algebra, then $R(\mathbf{H})$ is the Boolean algebra of the regular elements of **H**. Thus $R(\mathbf{H})$ has as elements those elements a of

H, the underlying set of $\mathbf{H}$, such that $a = \neg\neg a$, and as structural operations $\wedge_{R(\mathbf{H})}$, $\vee_{R(\mathbf{H})}$, and $\neg_{R(\mathbf{H})}$ those defined, for every $x, y \in R(H)$, respectively, as $x \wedge_{R(\mathbf{H})} y = x \wedge y$, $x \vee_{R(\mathbf{H})} y = \neg\neg(x \vee y)$, and $\neg_{R(\mathbf{H})}x = \neg x$.

2. On morphisms: If f is a Heyting homomorphism from $\mathbf{H}$ to $\mathbf{H}'$, then $R(f)$ is the bi-restriction of f to $R(\mathbf{H})$ and $R(\mathbf{H}')$.

The functor R is a left adjoint of I and such that, for every Heyting algebra $\mathbf{H}$, the value of r, the unit of the adjunction $R \dashv I$, at $\mathbf{H}$ (that is, the homomorphism $r_{\mathbf{H}}$ from $\mathbf{H}$ to $I(R(\mathbf{H}))$ which sends a to $\neg\neg a$, for every $a \in H$) is surjective and, considered as a functor between ordered sets, a left adjoint of the canonical embedding of $I(R(\mathbf{H}))$ into $\mathbf{H}$. Additionally, in this case, and mainly as a consequence of the existence of the functor K from $\mathbf{CSL}$ to $\mathbf{ISL}$, it happens that the reflector R has, in its turn, a left adjoint T.

But before proving it we need to establish that there exists a natural isomorphism between the functors $R \circ \mathrm{LT_i}$ and $\mathrm{LT_c} \circ J$ from $\mathbf{ISL}$ to $\mathbf{Bool}$. We notice that, as a matter of fact, this is exactly the categorial rendering of Theorem 13.10, p. 410, stated by Rasiowa and Sikorski in [12].

Lemma 4.2 *Let (Y, Δ) be an intuitionistic sentential pretheory. Then there is a natural isomorphism from the Boolean algebra $R(\mathrm{LT_i}(Y, \Delta)) = R(\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta})$ to the Boolean algebra $\mathrm{LT_c}(J(Y, \Delta)) = \mathbf{T}_\Sigma(Y)/\equiv_{c,\Delta}$. Therefore, for every classical sentential pretheory (X, Φ), since $(X, \Phi) \cong J(K(X, \Phi)) = (X, t_X[\Phi])$, there is a natural isomorphism from $R(\mathbf{T}_\Sigma(X)/\equiv_{i,t_X[\Phi]})$ to $\mathbf{T}_\Sigma(X)/\equiv_{c,\Phi}$.*

Proof From $\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta}$ to $I(R(\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta}))$ we have the canonical surjective homomorphism $r_{\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta}}$, which, to abbreviate, we agree to denote, simply, by $r_{(Y,\Delta)}$. On the other hand, from $\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta}$ to $I(\mathbf{T}_\Sigma(Y)/\equiv_{c,\Delta})$ there exists a canonical surjective homomorphism $\mathrm{pr}_{(Y,\Delta)}$ which sends $[\alpha]_{\equiv_{i,\Delta}}$ in $\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta}$ to $[\alpha]_{\equiv_{c,\Delta}}$ in $I(\mathbf{T}_\Sigma(Y)/\equiv_{c,\Delta})$. Hence, by the universal property of $R(\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta})$, there exists a unique homomorphism $\mathrm{pr}^\natural_{(Y,\Delta)}$ from $R(\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta})$ to $\mathbf{T}_\Sigma(Y)/\equiv_{c,\Delta}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta} & \xrightarrow{\;\; r_{(Y,\Delta)} \;\;} & I(R(\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta})) \\
& \searrow{\scriptstyle \mathrm{pr}_{(Y,\Delta)}} & \big\downarrow{\scriptstyle I(\mathrm{pr}^\natural_{(Y,\Delta)})} \\
& & I(\mathbf{T}_\Sigma(Y)/\equiv_{c,\Delta})
\end{array}
$$

Therefore, $\mathrm{pr}^\natural_{(Y,\Delta)}$ is a surjective homomorphism.

Following this we prove that the homomorphism $\mathrm{pr}^\natural_{(Y,\Delta)}$ is, in addition, injective. Let $[\alpha]_{\equiv_{i,\Delta}}$ and $[\beta]_{\equiv_{i,\Delta}}$ be elements of $R(\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta})$ such that $[\alpha]_{\equiv_{c,\Delta}} = [\beta]_{\equiv_{c,\Delta}}$. Then, by definition, we have that $\Delta \vdash_{c,Y} \alpha \leftrightarrow \beta$. Thus, $\neg\neg\Delta \vdash_{i,Y} \neg\neg(\alpha \leftrightarrow \beta)$, but $\vdash_{i,Y} \neg\neg(\alpha \leftrightarrow \beta) \to (\neg\neg\alpha \leftrightarrow \neg\neg\beta)$. Hence $\neg\neg\Delta \vdash_{i,Y} \neg\neg\alpha \leftrightarrow \neg\neg\beta$, and bearing in mind that $\Delta \vdash_{i,Y} \neg\neg\Delta$, we get that $\Delta \vdash_{i,Y} \neg\neg\alpha \leftrightarrow \neg\neg\beta$. On the other hand, since, by hypothesis, $[\alpha]_{\equiv_{i,\Delta}}$ and $[\beta]_{\equiv_{i,\Delta}}$ are regular elements of the Heyting algebra $\mathbf{T}_\Sigma(Y)/\equiv_{i,\Delta}$, we have that $\Delta \vdash_{i,Y} \alpha \leftrightarrow \neg\neg\alpha$ and $\Delta \vdash_{i,\Delta} \beta \leftrightarrow \neg\neg\beta$. From the just stated result and taking into account that $\Delta \vdash_{i,Y} \neg\neg\alpha \leftrightarrow \neg\neg\beta$, it follows that $\Delta \vdash_{i,\Delta} \alpha \leftrightarrow \beta$ or, what by definition is equivalent, that $[\alpha]_{\equiv_{i,\Delta}} = [\beta]_{\equiv_{i,\Delta}}$. Thus

$\text{pr}^\natural_{(Y,\Delta)}$ is an injective homomorphism. Consequently, $\text{pr}^\natural_{(Y,\Delta)}$ is an isomorphism from the Boolean algebra $R(\mathbf{T}_\Sigma(Y)/\equiv_{\text{i},\Delta})$ to the Boolean algebra $\mathbf{T}_\Sigma(Y)/\equiv_{\text{c},\Delta}$.

Since it is not difficult to verify that the isomorphisms $\text{pr}^\natural_{(Y,\Delta)}$ are the components of a natural transformation λ from $R \circ \text{LT}_\text{i}$ to $\text{LT}_\text{c} \circ J$ we leave this piece of work to the reader. Hence λ is a natural isomorphism from $R \circ \text{LT}_\text{i}$ to $\text{LT}_\text{c} \circ J$. $\square$

Corollary 4.3   *The regularization functor R from* **Heyt** *to* **Bool** *has a left adjoint T. Moreover, there exists a pointwise epimorphic natural transformation ζ from T to I.*

Proof   We begin by proving that the functor R from **Heyt** to **Bool** has a left adjoint T. Since LT_c is an equivalence of categories from **CSL** to **Bool**, let Q_c be an inverse equivalence to LT_c fixed once and for all (and similarly for LT_i). Let T be the functor from **Bool** to **Heyt** defined as the composite functor $\text{LT}_\text{i} \circ K \circ Q_\text{c}$, where K is the functor defined in Proposition 4.1. Thus, if we agree to represent the value of Q_c on a Boolean algebra **B** by the classical sentential pretheory (X, Φ), then we have that $T(\mathbf{B})$ is the Heyting algebra $\mathbf{T}_\Sigma(X)/\equiv_{\text{i},t_X[\Phi]}$. Therefore, the values of the object mapping of T have an explicit construction by generators and relations (for example, for a Boolean algebra **B**, $(X, \equiv_{\text{i},t_X[\Phi]})$ is a presentation of $T(\mathbf{B})$).

That the functor T is a left adjoint of R follows, taking into account Lemma 4.2 and after a long chain of natural isomorphisms, from the fact that, for every Boolean algebra **B** and every Heyting algebra **H**, the hom-sets $\text{Hom}_{\mathbf{Heyt}}(T(\mathbf{B}), \mathbf{H})$ and $\text{Hom}_{\mathbf{Bool}}(\mathbf{B}, R(\mathbf{H}))$ are naturally isomorphic.

On the other hand, the functor T is a full embedding from **Bool** to **Heyt** because it has a right adjoint, R, and the unit of the adjunction $T \dashv R$, by the second part of Lemma 4.2, is an isomorphism of functors; that is, $R \circ T$ and $\text{Id}_{\mathbf{Bool}}$ are isomorphic.

An alternative proof of the existence of a left adjoint for the regularization functor R, which rests upon Freyd's Adjoint Functor Theorem, is as follows. The category **Heyt** has small hom-sets and, since it is a Mal'cev variety, it is small-complete. The functor R preserves small limits because (1) R preserves small-products, since for a small set J and a J-indexed family $(\mathbf{H}_j)_{j \in J}$ of Heyting algebras, we have that $R(\prod_{j \in J} \mathbf{H}_j) = \prod_{j \in J} R(\mathbf{H}_j)$, and (2) R preserves equalizers, since for two homomorphisms $f, g: \mathbf{H} \longrightarrow \mathbf{H}'$, if we denote by $\mathbf{Eq}(f, g)$ the equalizer of f and g, then $R(\mathbf{Eq}(f, g)) = \mathbf{Eq}(R(f), R(g))$. Finally, R satisfies the solution set condition. In fact, given a Boolean algebra **B**, a Heyting algebra **H**, and a homomorphism $f: \mathbf{B} \longrightarrow R(\mathbf{H})$, the cardinal of $\mathbf{Sg}_\mathbf{H}(f[B])$, the Heyting subalgebra of **H** generated by $f[B]$, is bounded. Then taking one copy of each isomorphism class of such Heyting algebras $\mathbf{Sg}_\mathbf{H}(f[B])$ we obtain a small set of Heyting algebras, and the set of all homomorphisms $\mathbf{B} \longrightarrow R(\mathbf{Sg}_\mathbf{H}(f[B]))$ is a solution set for **B**. Therefore, the functor R has a left adjoint.

Finally, we prove that there exists a pointwise epimorphic natural transformation ζ from T to I. This follows, taking into account that we have the equivalences **CSL** $\simeq$ **Bool** and **ISL** $\simeq$ **Heyt**, from the fact, proved in Proposition 4.1, that ξ is a pointwise epimorphic natural transformation from K to G. Indeed, since $R \circ T$ is naturally isomorphic to $\text{Id}_{\mathbf{Bool}}$, the natural transformation ζ is $r * T$, the horizontal composite of r and T, where r is the unit of the adjunction $R \dashv I$.

An alternative proof of the existence of a pointwise epimorphic natural transformation ζ from T to I which, as distinguished from the proof just stated, is direct and, in addition, independent of the natural transformation ξ, is as follows. Let **B**

be an arbitrary, but fixed, Boolean algebra. Then, from the definition of T and bearing in mind the convention that has already been made about the value of Q_c at a Boolean algebra, we have that $T(\mathbf{B})$ is the Heyting algebra $\mathbf{T}_\Sigma(X)/\equiv_{i,t_X[\Phi]}$. On the other hand, from the fact that the functors $R \circ T$ and $\mathrm{Id}_{\mathbf{Bool}}$ are naturally isomorphic, it follows that $R(T(\mathbf{B}))$ is isomorphic to $\mathbf{T}_\Sigma(X)/\equiv_{c,\Phi}\cong \mathrm{Id}_{\mathbf{Bool}}(\mathbf{B})$ and that $I \circ R \circ T$ and I are also naturally isomorphic. Hence $I(\mathbf{B})$ is naturally isomorphic to $I(\mathbf{T}_\Sigma(X)/\equiv_{c,\Phi}) = \mathbf{T}_\Sigma(X)/\equiv_{c,\Phi}$. But it happens that the congruence $\equiv_{i,t_X[\Phi]}$ is included in the congruence $\equiv_{c,\Phi}$. In fact, for every $\alpha, \beta \in \mathrm{T}_\Sigma(X)$, from $t_X[\Phi] \vdash_{i,X} \alpha \leftrightarrow \beta$ it follows that $t_X[\Phi] \vdash_{c,X} \alpha \leftrightarrow \beta$ which, in its turn, is equivalent to $\Phi \vdash_{c,X} \alpha \leftrightarrow \beta$, because $\vdash_{c,X} t_X[\Phi] \leftrightarrow \Phi$. Hence there exists a unique homomorphism $\zeta_\mathbf{B}$ from $T(\mathbf{B}) = \mathbf{T}_\Sigma(X)/\equiv_{i,t_X[\Phi]}$ to $I(\mathbf{B}) \cong \mathbf{T}_\Sigma(X)/\equiv_{c,\Phi}$ such that $\mathrm{pr}_{\equiv_{c,\Phi}} = \zeta_\mathbf{B} \circ \mathrm{pr}_{\equiv_{i,t_X[\Phi]}}$. It is straightforward to prove that the homomorphisms $\zeta_\mathbf{B}$ are surjective and the components of a natural transformation ζ from T to I. Therefore, ζ is a pointwise epimorphic natural transformation. $\square$

Since the functors I and T from **Bool** to **Heyt** are full embeddings and both have the same functor R as a left and a right adjoint, respectively, we have that the string of adjunctions $T \dashv R \dashv I$ also constitutes a special case of the concept of adjoint cylinder of Lawvere, and we could say that, from the semantical, or model-theoretic, standpoint, the category **Bool** can be identified with a *full reflective* and a *full coreflective* subcategory of **Heyt**. Moreover, the category **Heyt** is subordinated to **Bool** by virtue of the existence of the natural transformation ζ from the functor T to the functor I.

For completeness, and owing to the information contained in the referee's report, it seems suitable to mention that there is one more functor C from **Heyt** to **Bool** in the aforementioned string of adjunctions: The functor assigning to a Heyting algebra the Boolean algebra of its complemented elements, which, we add, is a right adjoint of I.

Remark 4.4   For future research it would be useful and interesting (1) to determine the magnitude of the underlying set of the Heyting algebra $T(\mathbf{B})$ in function of that of the Boolean algebra $\mathbf{B}$; (2) to calculate, something which one can sensibly expect to achieve, for example, the values of T at the free Boolean algebras, or at the Boolean algebras of the form $\mathbf{Sub}(n) = (\mathrm{Sub}(n), \subseteq)$, for n in $\mathbb{N} - 1$, or at the finite-cofinite Boolean algebras on an infinite set; and (3) to complete the edges of the triangle with vertices adjoint cylinders of "logics" (classical and intuitionistic sentential pretheories), of "algebras" (Boolean and Heyting algebras), and of "geometries" (Stone and Priestley spaces) by explicitly showing the relation between algebra and geometry through the corresponding dualities. We notice that getting such a triangular correspondence would be suitable, in particular, to provide transference methods between the three involved fields.

Additionally, it would be also appropriate to investigate if there are further adjoint cylinders in between those that have been considered in this article (for example, by considering intermediate logics between classical and intuitionistic sentential logics, or intermediate algebraic constructs between Boolean and Heyting algebras).

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