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On Cofinal Submodels and Elementary Interstices

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Abstract We prove a number of results concerning the variety of first-order theories and isomorphism types of pairs of the form (N, M), where N is a countable recursively saturated model of Peano Arithmetic and M is its cofinal submodel. We identify two new isomorphism invariants for such pairs. In the strongest result we obtain continuum many theories of such pairs with the fixed greatest common initial segment of N and M and fixed lattice of interstructures K, such that $M \prec K \prec N$.

1 Introduction

Craig Smoryński wrote in [17]: *A relatively neglected aspect of the study of nonstandard models of arithmetic is the study of their cofinal extensions. These extensions certainly do not present themselves to the intuition as readily as do their more popular cousins the end extensions, but they are not exactly shrouded in mystery or unnatural objects of study either. They are equal partners with end extensions in the construction of general extensions of models; they offer both special advantages and disadvantages worthy of our interest; and, occasionally, they are useful in understanding the generally more simply behaved end extensions. Cofinal extensions deserve more attention than they have traditionally received.*

Smoryński's remark was justified in the 1980s, and its validity has not diminished since then. The study of elementary end extensions of models of PA is much more advanced than the study of cofinal ones. In this note we will heed Smoryński's advice and we will consider isomorphism types and the first-order theories of pairs of the form (N, M), where $N \models$ PA and M is an elementary cofinal submodel of N. We will review what is known about such pairs, we will prove some new results, and we will pose open problems. Much in this note is based on Smoryński's three papers [17], [18], and [19]. We are aiming at a more systematic study of some problems left open in those papers.

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The complete treatment of the general case of elementary pairs (N, M), even in the countable case, seems far out of reach. For example, there are many open basic questions concerning the interstructure lattices of elementary submodels $\mathrm{Lt}(N/M) = \{K : M \prec K \prec N\}$. We do not even know if there is a finite lattice which cannot be realized as an interstructure lattice with M cofinal in N.

In this paper, we will consider the case of countable recursively saturated models and their recursively saturated cofinal elementary submodels. By the well-known theorem of Gaifman the assumption of elementarity of the extension is redundant: if M is cofinal in N and both models are models of PA, then $M \prec N$.

All countable recursively saturated models of PA have continuum many nonisomorphic cofinal submodels. This can be proved in several different ways, providing different types of diversity. We give two proofs in Propositions 2.6 and 2.7, where we consider standard systems and sets of complete types realized in cofinal submodels. Once we know that there is a great variety of cofinal submodels of N, the next goal is to consider isomorphism types and first-order theories for pairs of models (N, M) for a fixed countable recursively saturated model N and a fixed isomorphism type of M. This is the main theme of the paper. Smoryński [18], [17] was the first to study such pairs, and, among other results, he proved that there are continuum many complete first-order theories of pairs (N, M), where $M \prec_{\mathsf{cof}} N$. He did this by showing that for each cut I of N that is closed under addition and multiplication, there is a cofinal submodel M such that $M \cong N$ and the greatest common initial segment of M and N is I. Since there are continuum many complete theories of such cuts I, the result follows. Our main results concern the case when M is isomorphic to N and the greatest common initial segment of N and M is fixed. We give two independent constructions yielding continuum many first-order theories of such pairs under different assumptions on the greatest common initial segment. Instead of greatest common initial segments, we use another isomorphism invariant: the base of M in N. To define the base of a submodel M of N, we first define the notions of an M-interstice and elementary M-interstice in analogy with interstices introduced by Bamber and Kotlarski [1]. The base of M is its least elementary interstice, if it exists. We give three independent constructions, each yielding continuum many theories of pairs (N, M). In each construction we use different assumptions on the greatest common initial segment of N and M, and each is based on a different isomorphism invariant. One construction uses bases, one uses a finitely generated cofinal extension, and one is based on Smoryńki's idea of *rational sequences of skies* from [19]. In preliminary sections we also prove a number of general results concerning bases and, in particular, bases of finitely generated submodels.

1.1 Notation Let N be a model of PA, and let $\mathcal{E}(N) = \{M : M \prec N\}$. Our convention is that $M \prec N$ allows for the possibility that $M = N$. We write $M \prec_{\mathsf{end}} N$ if $M \in \mathcal{E}(N)$ and M is an initial segment of N or, as we more often say, M is an elementary cut of N. The set $\{M : M \prec_{\mathsf{end}} N\}$ is denoted by $\mathcal{E}_{\mathsf{end}}(N)$. We write $M \prec_{\mathsf{cof}} N$ if $M \in \mathcal{E}(N)$ and M is cofinal in N; that is, for each $a \in N$ there is a $b \in M$ such that $a < b$. The set $\{M : M \prec_{\mathsf{cof}} N\}$ is denoted by $\mathcal{E}_{\mathsf{cof}}(N)$.

If $A \subseteq N$, then $\mathrm{Scl}^N(A)$ is the smallest model in $\mathcal{E}(N)$ which contains A. We call $\mathrm{Scl}^N(A)$ the *Skolem closure* of A in N. For $a \in N$, $\mathrm{Scl}(a)$ is $\mathrm{Scl}(\{a\})$. If the context is clear, we will omit the superscript N.

A *Skolem term* is an expression of the form $\min\{y : \exists y \varphi(x,y) \to \varphi(x,y)\}$, where $\varphi(x,y)$ is a formula of the language of PA. It is easy to see that for each $A \subseteq N$,

$$\mathrm{Scl}(A) = \{t(a) : a \in A \text{ and } t \text{ is a Skolem term}\}.$$

We will also consider Skolem closures in expansions of N that satisfy the induction schema. Such expansions are often referred to as *models of* PA*.

It is sometimes convenient to use arithmetic coding of finite sequences under which each element codes a sequence. We choose such a coding. For each a, $(a)_i$ denotes the ith term of the sequence coded by a. We also use Cantor's pairing function. For all a, b, $\langle a, b\rangle$ is Cantor's code of the pair (a, b).

$\mathrm{Def}(N)$ is the set of subsets of N which are definable with parameters; $\mathrm{Def}_0(N)$ is the set of subsets of N which are definable without parameters.

For $I \subseteq_{\mathrm{end}} N$, let $\mathrm{Cod}(N/I) = \{I \cap X : X \in \mathrm{Def}(N)\}$. As usual $\mathrm{SSy}(N) = \mathrm{Cod}(N/\mathbb{N})$, where $\mathbb{N}$ is the standard cut.

For $A \subseteq N$, $\sup(A) = \{x \in N : \exists y \in A\ N \models (x \leq y)\}$ and $\inf(A) = \{x \in N : \forall y \in A\ N \models (x \leq y)\}$.

For $a \in N$, the *gap* of a, denoted by $\mathrm{gap}^N(a)$, is the convex set $\sup(\mathrm{Scl}(a)) \setminus \{b \in N : \mathrm{Scl}(b) < a\}$. If $a > \sup(\mathrm{Scl}(0))$, then we say that $\mathrm{gap}(a)$ is *proper*. With few exceptions, we write $\mathrm{gap}(a)$ instead of $\mathrm{gap}^N(a)$.

Recall that a model $N \models$ PA is *short* if $\mathrm{Scl}(a)$ is cofinal in N for some $a \in N$. Otherwise N is *tall*. If $M \prec_{\mathrm{end}} N$ and there is an $a \in N \setminus M$ such that $M = \inf(\mathrm{gap}(a))$, then M is *coshort* in N.

Minimal types will be used several times in this paper; in particular, we will use the fact that they are *strongly indiscernible*, which means that if $\bar{a} = \langle a_0, \ldots, a_n\rangle$ and $\bar{b} = \langle b_0, \ldots, b_n\rangle$ are increasing tuples of elements realizing the same minimal type in a model N, then for every $c \in N$ such that $\sup(\mathrm{Scl}(c)) < \min\{a_0, b_0\}$, $\mathrm{tp}(\bar{a}, c) = \mathrm{tp}(\bar{b}, c)$.

A type is *rare* if it cannot be realized by two elements in the same gap. All minimal types are rare, and there are rare types that are not minimal (see Kossak and Schmerl [14, Chapter 3]).

For the rest of the paper let $\mathfrak{N}$ be a fixed countable recursively saturated model of PA. Let $\mathfrak{N}_0 = \sup(\mathrm{Scl}(0))$; with this notation, $\mathfrak{N} \equiv \mathbb{N}$ if and only if $\mathfrak{N}_0 = \mathbb{N}$.

For $A \subseteq \mathfrak{N}$, by $\mathrm{Tp}(A)$ we denote the set of complete types realized by elements of A in $\mathfrak{N}$, that is, $\mathrm{Tp}(A) = \{\mathrm{tp}^{\mathfrak{N}}(a) : a \in A\}$. Since $\mathfrak{N}$ is ω-homogenous, for all $a, b \in \mathfrak{N}$, either $\mathrm{Tp}(\mathrm{gap}(a)) = \mathrm{Tp}(\mathrm{gap}(b))$ or $\mathrm{Tp}(\mathrm{gap}(a)) \cap \mathrm{Tp}(\mathrm{gap}(b)) = \varnothing$.

2 Sources of Diversity

If every complete type realized in a countable model $M \models$ PA is also realized in $\mathfrak{N}$, then M can be elementarily embedded into $\mathfrak{N}$. This implies that $\mathcal{E}(\mathfrak{N})$ contains uncountably many pairwise nonisomorphic models and, in fact, that there are continuum many pairwise nonisomorphic models in $\mathcal{E}_{\mathrm{cof}}(\mathfrak{N})$. Full arguments are given in Propositions 2.6 and 2.7 below. Diversity among models in $\mathcal{E}_{\mathrm{end}}(\mathfrak{N})$ is more modest. If $M \in \mathcal{E}_{\mathrm{end}}(\mathfrak{N})$, then either M is tall, and in this case M is isomorphic to $\mathfrak{N}$, or M is short, and then M is one of the countably many elementary cuts of the form $\sup(\mathrm{Scl}(a))$. There are infinitely many pairwise nonisomorphic short elementary cuts. This was proved independently by Kotlarski and Smoryński [17], but it also follows from earlier results of Gaifman [3]. Here we will give a more direct argument with a stronger conclusion. We need two lemmas.

Lemma 2.1 *If $M \in \mathcal{E}(\mathfrak{N})$ is short, then $\omega \in \mathrm{Def}_0(\mathfrak{N}, M)$.*

Proof Let $M \in \mathcal{E}(\mathfrak{N})$ be short. Then, by recursive saturation, there is an $a \in \mathfrak{N}$ such that a codes an infinite increasing sequence and $\sup\{(a)_i : i < \omega\} = M$. This implies that $\omega \in \mathrm{Def}(\mathfrak{N}, M)$. To get the stronger conclusion, notice that if ω is definable in some expansion of a model of PA, then it is also definable in that expansion without parameters, for if a formula $\varphi(x, a)$ defines ω, for some a, then ω is also defined by the formula $\forall y[\psi(y) \longrightarrow \varphi(x, y)]$, where $\psi(y) = \varphi(0, y) \wedge \forall x[\varphi(x, y) \longrightarrow \varphi(x + 1, y)]$. □

Lemma 2.2 *Let $M \in \mathcal{E}(\mathfrak{N})$, and suppose that ω is definable in $(\mathfrak{N}, M)$. Then the satisfaction relation for M is definable without parameters in $(\mathfrak{N}, M)$.*

Proof Let $\Theta(x, S, M)$ be a formula expressing that S satisfies Tarski's inductive conditions for satisfaction in M for all formulas of the language of PA with no more than x unbounded quantifiers. For each arithmetic formula $\varphi(x)$ and each $a \in M$,

$$M \models \varphi(a) \text{ iff } (\mathfrak{N}, M) \models \exists x \in \omega \exists d[\Theta(x, d \cap M, M) \wedge \langle \varphi, a \rangle \in d]. \qquad \square$$

Theorem 2.3 *There are infinitely many different theories of pairs $(\mathfrak{N}, M)$, where $M \in \mathcal{E}_{\mathrm{end}}(\mathfrak{N})$ is short and the last gap of M realizes a minimal type.*

Proof In the proof of Schmerl [15, Theorem 4.6], two recursive sequences $\langle \varphi_i(v) : i < \omega \rangle$ and $\langle \theta_i(v) : i < \omega \rangle$ of formulas are defined with the properties that whenever T is a consistent completion of PA and $I \subseteq \omega$, then there is a unique complete 1-type $p_I(v) \supseteq T \cup \{\varphi_i(v) : i < \omega\} \cup \{\theta_i(v) : i \in I\} \cup \{\neg\theta_i(v) : i \in \omega \backslash I\}$. Moreover, each $p_I(v)$ is a minimal type, and whenever $I \neq J$, then $p_I(v)$, $p_J(v)$ are independent (i.e., they cannot be realized in the same gap of a model of T).

By recursive saturation, if $I \in \mathrm{SSy}(\mathfrak{N})$, then $p_I(v)$ is realized in $\mathfrak{N}$. Let a_I be a realization of $p_I(v)$ in $\mathfrak{N}$, and let $M_I = \sup(\mathrm{gap}(a_I))$. To finish the proof, notice that, by Lemma 2.2, $\mathrm{gap}(a_I) \in \mathrm{Def}_0(\mathfrak{N}, M_I)$. Since a_I is the only element of $\mathrm{gap}(a_I)$ realizing the recursive type $\langle \varphi_i(v) : i < \omega \rangle$, this implies that $a_I \in \mathrm{Def}_0(\mathfrak{N}, M_I)$. Finally, since $\omega \in \mathrm{Def}_0(\mathfrak{N}, M_I)$, we conclude that $I = \{i : \mathfrak{N} \models \theta_i(a_I)\} \in \mathrm{Def}_0(\mathfrak{N}, M_I)$. Furthermore, since the definitions of $\mathrm{gap}(a_I)$, a_I, and I are uniform, for $I, J \in \mathrm{SSy}(\mathfrak{N})$, if $I \neq J$, then $(\mathfrak{N}, M_I) \not\equiv (\mathfrak{N}, M_J)$. □

It is not hard to prove that for each $a > \mathfrak{N}_0$, either some element in $\mathrm{gap}(a)$ realizes a rare type, or each element in $\mathrm{gap}(a)$ realizes a *ubiquitous* type, meaning that for all $b, c \in \mathrm{gap}(a)$ there are $b', b'' \in \mathrm{gap}(a)$ such that $b' < c < b''$ and $\mathrm{tp}(b') = \mathrm{tp}(b'') = \mathrm{tp}(b)$ (see Kossak, Kotlarski, and Schmerl [12] for details). It is clear that if an element in $\mathrm{gap}(a)$ realizes a rare type, then no element in $\mathrm{gap}(a)$ realizes a ubiquitous type. It follows from the results in [12] that there are at least two $a, a' \in \mathfrak{N}$ such that $\mathrm{tp}(a)$ and $\mathrm{tp}(a')$ are ubiquitous and $\mathrm{Tp}(\mathrm{gap}(a)) \cap \mathrm{Tp}(\mathrm{gap}(a')) = \varnothing$. By a bit more elaborate construction, we have been able to prove the following counterpart of Theorem 2.3. We omit the proof.

Theorem 2.4 *There are infinitely many different theories of pairs $(\mathfrak{N}, M)$, where $M \in \mathcal{E}_{\mathrm{end}}(\mathfrak{N})$ is short and the last gap of M realizes a ubiquitous type.*

According to the terminology introduced in [12], a gap containing an element realizing a rare type is called *labeled*, and a gap without such elements is called *unlabeled*. It is shown in [12] that in a recursively saturated model each element of an unlabeled gap realizes a ubiquitous type. Thus, Theorems 2.3 and 2.4 show that in any recursively saturated model there are infinite sets L and U of labeled gaps and unlabeled gaps, such that for distinct gaps γ and γ' in either L or U, $\mathrm{Tp}(\gamma) \cap \mathrm{Tp}(\gamma') = \varnothing$. Moreover, both proofs show that there are infinitely many theories of pairs $(\mathfrak{N}, M)$, where $M \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ is short. The case of coshort M is different. It is shown in Kossak and Kotlarski [10] that for all coshort $M, M' \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$, $(\mathfrak{N}, M) \equiv (\mathfrak{N}, M')$. If γ and γ' are the least gaps in $\mathfrak{N} \setminus M$ and $\mathfrak{N} \setminus M'$, respectively, such that $\mathrm{Tp}(\gamma) \cap \mathrm{Tp}(\gamma') = \varnothing$, then $(\mathfrak{N}, M) \not\cong (\mathfrak{N}, M')$. Thus, Theorems 2.3 and 2.4 give us infinitely many elementarily equivalent and pairwise nonisomorphic pairs $(\mathfrak{N}, M)$, where $M \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ is coshort. The following interesting question remains open.

Question 2.5 *Let $M, M' \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ be short, and assume that $(\mathfrak{N}, M) \equiv (\mathfrak{N}, M')$. Is $(\mathfrak{N}, M)$ isomorphic to $(\mathfrak{N}, M')$?*

How do we know that there are continuum many nonisomorphic models which are elementarily embeddable into $\mathfrak{N}$? There are many arguments providing different kinds of diversity; we will present two in the next two propositions.

Recall that for $A \subseteq \mathfrak{N}$, $\mathrm{Tp}(A) = \{\mathrm{tp}^{\mathfrak{N}}(a) : a \in A\}$.

Proposition 2.6 *There is $\mathcal{I} \subseteq \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$ such that*

(1) $|\mathcal{I}| = 2^{\aleph_0}$;

(2) for all $M, M' \in \mathcal{I}$, $\mathrm{Tp}(M) = \mathrm{Tp}(M')$; moreover, we can request that $\mathrm{Tp}(M) = \mathrm{Tp}(\mathfrak{N})$, for all $M \in \mathcal{I}$;

(3) for all $M, M' \in \mathcal{I}$, if $M \neq M'$, then $M \not\cong M'$.

Proof We will present two constructions. The second will address the "moreover" part of (2). Let p and q be independent minimal types realized in $\mathfrak{N}$. For each $X \subseteq \omega$ let $M_X = \mathrm{Scl}(\{a_i^X : i < \omega\})$, where $\langle a_i^X : i < \omega\rangle$ is an increasing sequence that is cofinal in $\mathfrak{N}$ and such that for all $i < \omega$, if $i \in X$, then a_i^X realizes p, and otherwise it realizes q. Let $\mathcal{I} = \{M_X : X \subseteq \omega\}$. By the results of Gaifman [3] mentioned above, for $X, Y \subseteq \omega$, and for all $i < \omega$, $\mathrm{tp}(a_i^X) = \mathrm{tp}(a_i^Y)$ iff $\mathrm{Tp}(\mathrm{gap}(a_i^X)) = \mathrm{Tp}(\mathrm{gap}(a_i^Y))$. Since for each X, $M_X = \mathrm{Scl}(0) \cup \bigcup_{i<\omega} \mathrm{gap}^{M_X}(a_i^X)$, it follows that M_X is isomorphic to M_Y iff $X = Y$. Also, since every element of M_X is of the form $t(\bar{a}, \bar{b})$, where t is a Skolem term, $\bar{a}$ is an increasing sequence of elements realizing p, and $\bar{b}$ is an increasing sequence of elements realizing q, it is easy to see, by the strong indiscernibility of the types, that if both X and Y are infinite and coinfinite, then M_X and M_Y realize exactly the same types.

For the second construction, let p be a minimal type realized in $\mathfrak{N}$. For every countable linear ordering $(A, <)$, $\mathfrak{N}$ has a canonical A-extension $\mathfrak{N}_A$ generated over $\mathfrak{N}$ by a set that is order isomorphic to $(A, <)$ of elements realizing p over $\mathfrak{N}$ (see [14, Section 3.3] for details). By strong indiscernibility of p, for each $(A, <)$, $\mathrm{Tp}(\mathfrak{N}_A) = \mathrm{Tp}(\mathfrak{N})$, and hence $\mathfrak{N}_A$ elementarily embeds in $\mathfrak{N}$; moreover, if $(A, <)$ has no last element, then it is not hard to see that $\mathfrak{N}_A$ can be embedded as a cofinal submodel of $\mathfrak{N}$. Since for all $A, B \subseteq \omega$, $\mathfrak{N}_A \cong \mathfrak{N}_B$ if and only if $(A, <) \cong (B, <)$, the result follows. $\square$

We also have the other extreme. One can modify the construction from the proof of Proposition 2.6 as follows. Let $\{p_n : n < \omega\}$ be a collection of pairwise independent minimal types realized in $\mathfrak{N}$. For each $X \subseteq \omega$, let $M_X = \mathrm{Scl}(\{a_i^X : i < \omega\})$, where $\langle a_i^X : i < \omega\rangle$ is an increasing sequence that is cofinal in $\mathfrak{N}$ and such that for all $i < \omega$, if $i \in X$, then a_i^X realizes p_i, and if $i \notin X$, then $a_i^X = \langle a, a'\rangle$, where a and a' realize p_i and $a < a'$. It is easy to see that for all $X, Y \subseteq \omega$, we have $\mathrm{Tp}(M_X) = \mathrm{Tp}(M_Y)$, and if $X \neq Y$, then $M_X \ncong M_Y$. A different construction gives an even stronger result in the next proposition.

Proposition 2.7 *There is $\mathcal{J} \subseteq \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$ such that*

(1) $|\mathcal{J}| = 2^{\aleph_0}$;
(2) for all $M \in \mathcal{J}$, M is recursively saturated;
(3) for all $M, M' \in \mathcal{J}$, if $M \neq M'$, then $M \ncong M'$.

Proof By an unpublished result of Stephen Simpson, every Scott set has continuum many Scott subsets. Here is an outline of Simpson's argument: using [16, Theorem VIII.2.6] one can show that the set of Scott subsets of a given Scott set contains a subset that is densely ordered by inclusion. Since the union of a chain of Scott sets is a Scott set, the result follows.

By standard arguments, for every Scott set $\mathfrak{X} \subseteq \mathrm{SSy}(\mathfrak{N})$ such that $\mathrm{Th}(\mathfrak{N}) \in \mathfrak{X}$, there is a recursively saturated $M_{\mathfrak{X}} \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$ such that $\mathrm{SSy}(M_{\mathfrak{X}}) = \mathfrak{X}$. This concludes the proof, since for all recursively saturated models $M, M' \in \mathcal{E}(\mathfrak{N})$, $M \cong M'$ iff $\mathrm{SSy}(M) = \mathrm{SSy}(M')$. □

Since all tall elementary cuts of $\mathfrak{N}$ are recursively saturated, they are isomorphic to $\mathfrak{N}$. Still, there are continuum many nonisomorphic, and in fact even nonelementarily equivalent, structures of the form $(\mathfrak{N}, M)$, where $M \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ is tall. This can be shown in various ways (see [18] or Kossak [7]). As a corollary we have the following proposition.

Proposition 2.8 *Let $M \in \mathcal{E}(\mathfrak{N})$ be tall. Then there are continuum many complete theories of structures of the form $(\mathfrak{N}, K)$, where $K \in \mathcal{E}(\mathfrak{N})$ and $K \cong M$.*

Proof Let M be a tall model, and suppose that $K \in \mathcal{E}(\mathfrak{N})$ is isomorphic to M. Then $K' = \sup(K)$ is recursively saturated. By taking a recursively saturated elementary end extension of $\mathfrak{N}$ if necessary, we can assume that $K' < \mathfrak{N}$. By the results mentioned above, there are continuum many complete theories of structures of the form $(\mathfrak{N}, K'')$, where $K'' \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ and $K'' \cong K'$. Since K' is definable in $(\mathfrak{N}, K)$, the result follows. □

Proposition 2.8 cannot be improved by requiring that $K \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$. For example, it is not hard to show that if p is a minimal type realized in $\mathfrak{N}$, K and K' are generated by ω-sequences of elements realizing p, and they are both cofinal in $\mathfrak{N}$, then $(\mathfrak{N}, K) \cong (\mathfrak{N}, K')$.

2.1 The role of combinatorial properties In the known constructions of large families of pairwise nonisomorphic pairs $(\mathfrak{N}, M)$ with $M \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$, M is either not semiregular in $\mathfrak{N}$, or M is strong in $\mathfrak{N}$. Semiregularity and regularity of a cut $I \subseteq_{\mathsf{end}} M$ are defined by Kirby and Paris in terms of coded functions and partitions, and later were given the following characterizations. A cut I is *semiregular* iff $(I, A)_{A\in \mathrm{Cod}(\mathfrak{N}/I)}$ is a model of the Σ_1-induction schema $\mathrm{I}\Sigma_1$, and I is *regular* iff $(I, A)_{A\in \mathrm{Cod}(\mathfrak{N}/I)}$ is a model of the Σ_2-collection schema $\mathrm{B}\Sigma_2$.

Question 2.9 *Are there infinitely many pairwise nonisomorphic pairs* $(\mathfrak{N}, M)$ *such that* $M \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ *and* M *is semiregular but not regular in* $\mathfrak{N}$*?*

In the context of the known facts one would expect that Question 2.9 has a positive answer and that there should be continuum many nonisomorphic such pairs. At the moment we only know how to get two nonisomorphic pairs. Richard Kaye and Tin Lok Wong [5] defined a notion of genericity for elementary cuts and proved that

(1) all countable arithmetically saturated models have elementary generic cuts;
(2) if I and J are elementary generic cuts, then $(\mathfrak{N}, I) \cong (\mathfrak{N}, J)$;
(3) all elementary generic cuts are semiregular but not regular.

They also proved that if M is generic in $\mathfrak{N}$, then $(\mathfrak{N}, M)$ is not recursively saturated. By chronic resplendency, there are $M \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ such that M is semiregular but not regular in $\mathfrak{N}$, and $(\mathfrak{N}, M)$ is recursively saturated. It follows that M is not generic. This gives us an example of two nonisomorphic pairs $(\mathfrak{N}, M)$, one in which M is generic and one in which it is not.

3 Elementary Interstices and Bases

If $M \in \mathcal{E}(\mathfrak{N})$ is bounded in $\mathfrak{N}$, then $\sup(M)$ is a very useful isomorphism invariant for the pair $(\mathfrak{N}, M)$. For M which are cofinal in $\mathfrak{N}$ we have to look for something else. Smoryński proved that there are continuum many theories of pairs $(\mathfrak{N}, M)$ where M is cofinal in $\mathfrak{N}$ by considering the greatest common initial segment (GCIS) of M and $\mathfrak{N}$, denoted $\mathrm{GCIS}(\mathfrak{N}, M)$. Smoryński's proof is based on the fact that for any cut $I \subseteq_{\mathsf{end}} \mathfrak{N}$ that is closed under addition and multiplication, there is $M \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$ such that $\mathrm{GCIS}(\mathfrak{N}, M) = I$. Since $\mathrm{GCIS}(\mathfrak{N}, M)$ is definable in $(\mathfrak{N}, M)$, and there are continuum many complete theories of cuts that are closed under addition and multiplication, we get continuum many theories $\mathrm{Th}((\mathfrak{N}, M))$ where $M \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$. One of our goals is to analyze diversity in the collection of pairs $(\mathfrak{N}, M)$ with a fixed greatest common initial segment. Since GCIS is fixed, we need other isomorphism invariants. The idea is to look at the convex set separating $\mathrm{GCIS}(\mathfrak{N}, M)$ from the rest of M, if there is one.

Bamber and Kotlarski [1] introduced a useful notion of an interstice. For each $a \in \mathfrak{N} \setminus \mathrm{Scl}(0)$, the *interstice* of a, denoted $\Omega(a)$, is the largest convex subset of $\mathfrak{N}$ that contains a and is disjoint from $\mathrm{Scl}(0)$. We will adapt this definition as follows.

Definition 3.1 (M-interstices) For $M \in \mathcal{E}(\mathfrak{N})$ and $a \in \mathfrak{N} \setminus M$ the *M-interstice of a*, denoted $\Omega^M(a)$, is the largest convex subset of $\mathfrak{N}$ that contains a and is disjoint from M. Each M-interstice determines two cuts: $I_-^M(a) = \inf(\Omega^M(a))$ and $I_+^M(a) = \sup(\Omega^M(a))$. We say that an M-interstice $\Omega^M(a)$ is *elementary* if $I_-^M(a) \in \mathcal{E}(\mathfrak{N})$.

Notice that for all a, $\Omega^{\mathrm{Scl}(0)}(a) = \Omega(a)$. Instead of $\mathrm{Scl}(0)$-interstice, we will say (as usual) interstice.

In the definition above we are referring to $I_-^M(a)$, but we could have used $I_+^M(a)$ as well. This follows from the next proposition.

Proposition 3.2 *Suppose that* $M \in \mathcal{E}(\mathfrak{N})$ *and* $a \in \mathfrak{N}\setminus M$*. Then,* $I_-^M(a) \prec \mathfrak{N}$ *iff* $I_+^M(a) \prec \mathfrak{N}$.

Proof ($\Longrightarrow$) Suppose that $b \in I_+^M(a)$ and that $t(x)$ is a Skolem term. If $t(b) > I_+^M(a)$, then let $c \in M$ be such that $I_+^M(a) < c < b$. Let $d \in M$ be the least such that $t(d) > c$. Clearly, $d \leq b$, so that $d \in I_-^M(a)$. But then $t(d) > c$, contradicting that $I_-^M(a) \prec \mathfrak{N}$.

($\Longleftarrow$) Suppose that $b \in I_-^M(a)$ and that $t(x)$ is a Skolem term. Let $c \in M$ be such that $b < c < \Omega^M(a)$. Let $d = \max\{t(x) : x \leq c\}$. Then $d \in I_+^M(a)$ since $I_+^M(a) \prec \mathfrak{N}$ and also $d \in M$, so that $d \in I_-^M(a)$, so $t(b) \in I_-^M(a)$. □

It is not immediately obvious that every elementary submodel of $\mathfrak{N}$ has elementary interstices. To see that this indeed is the case, let us make the following observations. First of all, for each gap γ and each $a \in \gamma$, $\mathrm{Scl}(a) \cap \gamma$ is both upward and downward cofinal in γ. It follows that for each a, if $\mathrm{gap}(a) \subseteq \Omega^M(a)$, then $\Omega^M(a) = \bigcup\{\mathrm{gap}(b) : b \in \Omega^M(a)\}$, and, consequently, the M-interstice $\Omega^M(a)$ is elementary iff $\mathrm{gap}(a) \subseteq \Omega^M(a)$. Now the result will follow from the next proposition.

Proposition 3.3 *Let $M \in \mathcal{E}(\mathfrak{N})$ be such that for each $a \in \mathfrak{N}$, $M \cap \mathrm{gap}(a) \neq \varnothing$. Then $M = \mathfrak{N}$.*

Proof For each $a \in \mathfrak{N}$, there is a $b \in \mathfrak{N}$ such that $\mathrm{tp}(b)$ is rare, and $a \in \mathrm{Scl}(b)$. One way to get such a b is by realizing in $\mathfrak{N}$ a minimal type of the theory $\mathrm{Th}((\mathfrak{N}, a))$. By [14, Theorem 3.1.16], if a and b are as above, then $a \in \mathrm{Scl}(c)$ for each $c \in \mathrm{gap}(b)$. It follows that if $A \subseteq \mathfrak{N}$ meets every gap of $\mathfrak{N}$, then $\mathrm{Scl}(A) = \mathfrak{N}$, and the result follows. □

Corollary 3.4 *If $M \in \mathcal{E}(\mathfrak{N}) \setminus \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ is tall, then there is $a \in \mathfrak{N}$ such that $\mathrm{gap}(a) \cap M = \varnothing$, and, consequently, M has elementary interstices.*

Proof Since M is tall, $\sup(M)$ is recursively saturated, and the result follows directly from Proposition 3.3. □

The next two results are a short digression before we continue with the main theme of this section. They are related to the following question: which types can be realized in $\mathfrak{N} \setminus M$, for $M \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$? As observed in Kossak and Kotlarski [11], Corollary 3.4 can be strengthened if in addition M is recursively saturated. In this case one can show that there is a convex set $\beta \subseteq \mathfrak{N} \setminus M$ such that $\sup(\beta)$ is a tall elementary cut of $\mathfrak{N}$. This implies the following proposition.

Proposition 3.5 *Let $M \in \mathcal{E}(\mathfrak{N})$ be proper and recursively saturated. Then for each $a \in \mathfrak{N}$ such that $a > \mathfrak{N}_0$, there is a $b \in \mathfrak{N} \setminus M$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$.*

Corollary 3.6 *Let $M \in \mathcal{E}(\mathfrak{N})$ be proper and recursively saturated, and let p be an unbounded type realized in M. If each element of $\mathfrak{N}$ realizing p is in M, then $M = \mathfrak{N}$.*

Now we return to the main theme. In the rest of the paper we will be mostly interested in the situation when $I_-^M(a) = \mathrm{GCIS}(\mathfrak{N}, M)$. The question to consider is to what extent does $I_-^M(a)$ determine the properties of $I_+^M(a)$; that is, we would like to fix the former cut and vary the latter by varying M. In general, this seems to be a difficult problem, even in the case when $\mathrm{GCIS}(\mathfrak{N}, M) = \mathbb{N}$. The difficulty is that if $\mathfrak{N}$ has nonstandard definable elements, the structure of interstices inside $\mathfrak{N}_0$ is rather complex. For example, altering the question a bit one could ask for which interstices $\Omega(a)$ in $\mathfrak{N}_0$ are there M such that $\Omega(a)$ is the least interstice such that $\Omega(a) \cap (M \setminus \mathrm{Scl}(0))$ is nonempty. It follows from the results in Bigorajska [2] that in an arithmetically saturated model $\mathfrak{N}$ there are such interstices, but to analyze potential cuts of the form $I_+^M(a)$ in such interstices would require more precise tools. To avoid technical difficulties, here we will only consider the easier case of elementary M-interstices.

The set of M-interstices is naturally ordered by $<$. We refer to this ordering when we talk about the smallest elementary M-interstice in the definition below.

Definition 3.7 (Bases) Suppose that $M \prec \mathfrak{N}$ and $a \in \mathfrak{N}\setminus M$. If $\Omega^M(a)$ is the smallest elementary M-interstice, then the cut $\mathrm{B}(M) = I_+^M(a)$ is the *base* of M in $\mathfrak{N}$. The cut $\sup(\mathrm{B}(M) \cap M)$ is called the *subbase* of M and is denoted by $\mathrm{Sb}(M)$.

Notice that, by Proposition 3.2, if an $M \prec \mathfrak{N}$ has a base, then $B(M) \in \mathscr{E}_{\mathsf{end}}(\mathfrak{N})$. Moreover, we have the following proposition.

Proposition 3.8 *Assume that $M \in \mathscr{E}(\mathfrak{N})$ has a base. Then*

$$\mathrm{GCIS}(\mathfrak{N}, M) \subseteq \mathrm{Sb}(M) \prec_{\mathsf{end}} \mathrm{B}(M),$$

and there is an $a \in \mathfrak{N} \setminus M$ such that $(\mathrm{Sb}(M) \setminus \mathrm{GCIS}(\mathfrak{N}, M)) \subseteq \mathrm{gap}(a)$.

Proof The result follows from definitions and Corollary 3.4. Suppose that the last part of the conclusion fails. Then there is a gap that is a proper subset of $\mathrm{Sb}(M)$ and is disjoint from M, which is a contradiction. □

There are examples of $a > \mathrm{Scl}(0)$ such that $\mathrm{Scl}(a)$ has no base. Suppose that $\mathfrak{N} \equiv \mathbb{N}$ and that $\mathfrak{N}$ is not arithmetically saturated. Then, since $\mathbb{N}$ is not strong in $\mathfrak{N}$, there is an $a \in M$ such that the set $\{(a)_n : n < \omega \text{ and } (a)_n > \mathbb{N}\}$ is downward cofinal in $\mathfrak{N} \setminus \mathbb{N}$. We will generalize this remark in Theorem 4.3 below.

We are primarily interested in bases of models that are not finitely generated, but let us first examine the finitely generated case.

4 Bases of Finitely Generated Submodels

Let $M \in \mathscr{E}(\mathfrak{N})$ be finitely generated. If $a, b \in \mathfrak{N}$ are such that for all $n < \omega$, $a + n < b$, then, by recursive saturation, there is $c \notin M$ such that $a < c < b$. It follows that $\mathrm{GCIS}(\mathfrak{N}, M) = \mathbb{N}$. Moreover, we have the following proposition.

Proposition 4.1 *If a model $M \in \mathscr{E}(\mathfrak{N})$ is finitely generated and has a base, then there is an $a \in \mathfrak{N}$ such that $I_+^M(a) = \mathrm{B}(M)$ and $I_-^M(a) = \mathfrak{N}_0$.*

Proof Suppose that the conclusion fails. Then, by the observation preceding the proposition and by Corollary 3.4, there is no least elementary M-interstice. Hence, M has no base. This is a contradiction. □

Recall that a cut $M \in \mathscr{E}(\mathfrak{N})$ is *coshort* if $\mathfrak{N} \setminus M$ has a least gap. Since the gaps with ordering inherited from $\mathfrak{N}$ are densely ordered, coshort models are tall. Since $\mathfrak{N}$ has only countably many coshort elementary cuts, it has continuum many tall elementary cuts which are not coshort.

If $a > \mathfrak{N}_0$ and γ is the least proper gap of $\mathrm{Scl}(a)$, then $\mathrm{B}(\mathrm{Scl}(a)) = \inf(\gamma)$; hence $\mathrm{B}(\mathrm{Scl}(a))$ is coshort. One can ask whether every coshort elementary cut can be represented as a base of a finitely generated model. If $a > \mathrm{Scl}(0)$, $a < \mathrm{gap}(b)$,

$\operatorname{tp}(b)$ is rare, and $a \in \operatorname{Scl}(b)$, then $a \in \operatorname{Scl}(c)$ for each $c \in \operatorname{gap}(b)$. Then, it is easy to see that $\inf(\operatorname{gap}(b))$ cannot be a base of a finitely generated model. There is still a possibility that $\inf(\operatorname{gap}(b))$ is a base of a model that is not finitely generated. Corollary 5.7 below shows that there are proper gaps γ such that $\inf(\gamma)$ is not a base of any $M \in \mathcal{E}(\mathfrak{N})$ such that $M \cap \mathrm{B}(M) \prec_{\mathrm{end}} \mathfrak{N}$.

Proposition 4.2 *If $M \in \mathcal{E}(\mathfrak{N})$ is finitely generated and has a base, then $\mathrm{B}(M)$ is tall.*

Proof Since M has a base, we can pick a b such that $\mathfrak{N}_0 < b \in \mathrm{B}(M)$. Suppose that $M = \operatorname{Scl}(a)$, and let $\langle t_n(x) : n < \omega\rangle$ be a recursive enumeration of all Skolem terms in one variable. Consider the type

$$p(x,a,b) = \{t_n(b) < x < t_m(a) : m,n < \omega \text{ and } b < t_m(a)\}.$$

Since $b < \mathrm{B}(M)$, it is easy to see that $p(x,a,b)$ is finitely realizable in $\mathfrak{N}$, and since $p(x,a,b)$ is recursive in $\operatorname{tp}(a,b)$, it is realized in $\mathfrak{N}$. By Proposition 4.1, this finishes the proof. □

The next result tells us that if $\mathfrak{N}$ is arithmetically saturated, then finitely generated submodels of $\mathfrak{N}$ have bases. In fact, this property characterizes arithmetic saturation.

Theorem 4.3 *$\mathfrak{N}$ is arithmetically saturated if and only if every finitely generated $M \in \mathcal{E}(\mathfrak{N})$ has a base.*

Proof The proof of the $\Longrightarrow$ direction is the same as that of Proposition 4.2 once we notice that the type

$$\{t_n(0) < x < t_m(a) : m,n < \omega \text{ and } \mathfrak{N}_0 < t_m(a)\}$$

is arithmetic in $\operatorname{tp}(a)$.

If $\mathfrak{N}$ is not arithmetically saturated, then there is a $b \in \mathfrak{N}$ such that $\inf\{(b)_n : n < \omega \wedge (b)_n > \mathbb{N}\} = \mathbb{N}$. By recursive saturation there is $c \in \mathfrak{N}$ such that $\langle (c)_n : n < \omega\rangle$ is an increasing sequence cofinal in $\operatorname{Scl}(0)$. Let $a = \langle b, c\rangle$. We will show that $\inf(\operatorname{Scl}(a)\backslash\mathfrak{N}_0) = \mathfrak{N}_0$. Let $n_0 > \mathbb{N}$ be such that $\langle (c)_n : n < n_0\rangle$ is increasing. We can assume that for all $n < \omega$, $(b)_n < n_0$. Then for $X = \{(c)_{(b)_n} : n < \omega\}$, we have $X \subseteq \operatorname{Scl}(a)$ and $\inf\{x \in X : x > \operatorname{Scl}(0)\} = \mathfrak{N}_0$, which finishes the proof. □

Proposition 4.2 tells us what bases of finitely generated models in $\mathcal{E}(\mathfrak{N})$ cannot be. It is a bit harder to see what they can be. The partial answer is given in the next theorem. Quasi-selective types were defined in [12]. An element $a \in \mathfrak{N}$ realizes a *quasi-selective* type iff $\operatorname{gap}(a) \cap \operatorname{Scl}(a) = \operatorname{Scl}(a) \setminus \operatorname{Scl}(0)$.

Theorem 4.4 *Let $I \in \mathcal{E}_{\mathrm{end}}(\mathfrak{N})$ be coshort. Then the following are equivalent:*

(1) There is an $a \in \mathfrak{N}$ such that a realizes a quasi-selective type and $I = \inf(\operatorname{gap}(a))$.

(2) There is a finitely generated $M \prec \mathfrak{N}$ such that $I = \mathrm{B}(M)$.

Proof The theorem follows directly from definitions and the remark at the beginning of this section. □

As we noted earlier, if $a \in \mathfrak{N}$ is such that $\operatorname{Scl}(a)$ has a least proper gap γ, then $B(\operatorname{Scl}(a)) = \inf(\gamma)$. The case of finitely generated models with no least proper gap is more interesting. To begin, it is not obvious that there are such models. Let p be a minimal type realized in $\mathfrak{N}$, and let $\langle (a)_i : i < \omega \rangle$ be a coded sequence of elements realizing p such that for all $i < \omega$, $(a)_i > \mathrm{gap}(a)_{i+1}$. One can show that there is a minimal type q of $\mathrm{Th}((\mathfrak{N}, (a)_i)_{i<\omega})$ such that q is recursive in $\mathrm{tp}(a)$ (hence it is realized in $\mathfrak{N}$) and that for each $i < \omega$ there is a Skolem term $t_i(x)$ such that the formula $t_i(x) = (a)_i$ is in q. Lemma 2.1.10 of [14] is crucial for this construction. If b realizes q in $\mathfrak{N}$, then it follows from the general properties of minimal types that $B(\mathrm{Scl}(b)) = \inf\{(a)_i : i < \omega\}$, and hence $\mathrm{Scl}(b)$ has no least proper gap. (Recall that if c and d realize the same minimal type, then $c = d$ iff $\mathrm{gap}(c) = \mathrm{gap}(d)$.) The next result shows that if $\mathfrak{N}$ is arithmetically saturated, the construction described above captures all not-coshort bases of finitely generated models.

Theorem 4.5 *Suppose that $\mathfrak{N}$ is arithmetically saturated and that $I \in \mathscr{E}_{\mathsf{end}}(\mathfrak{N})$ is not coshort. Then the following are equivalent:*

(1) There is $a \in \mathfrak{N}$ such that $I = \inf\{(a)_n : n < \omega\}$.
(2) There is a finitely generated $M \prec \mathfrak{N}$ such that $I = \mathrm{B}(M)$.

Proof The proof of (1) $\Longrightarrow$ (2) follows from the construction preceding Theorem 4.5 once we notice that without loss of generality we can assume that all elements $(a)_n$, $n < \omega$, realize the same minimal type.

For the (2) $\Longrightarrow$ (1) implication, let a finitely generated M such that $I = \mathrm{B}(M)$ be given. Suppose that $M = \mathrm{Scl}(a)$, and let $\langle t_n(x) : n < \omega \rangle$ be a recursive enumeration of all Skolem terms in one variable with $t_0(x) = x$. Since $\mathfrak{N}$ is arithmetically saturated, there is $c \in \mathfrak{N}$ such that $\{(c)_n : n < \omega\} = \{x \in \mathrm{Scl}(a) : x > \mathfrak{N}_0\}$.

We define $f : \mathbb{N}^2 \to \mathfrak{N}$ as follows:

$$f(m,n) = \begin{cases} \min\{k : t_k(t_m(a)) \geq t_n(a)\} & \text{if there is such } k, \\ a & \text{otherwise.} \end{cases}$$

Since $\mathfrak{N}$ is arithmetically saturated, f is coded in $\mathfrak{N}$. Let $a_0 = t_0(a) = a$. If a_n is defined and is equal to $t_{k_n}(a)$, to define a_{n+1} we first let

$$k_{n+1} = \min\big\{k : t_k(a) < \min\big\{a_n, (c)_n\big\} \wedge f(k, k_n) = a\big\}.$$

Since I is not coshort, $\mathrm{Scl}(a)$ has no least proper gap; hence k_{n+1} is well defined. We let $a_{n+1} = t_{k_{n+1}}(a)$. By recursive saturation, the sequence $\langle a_n : n < \omega \rangle$ is coded in $\mathfrak{N}$ and $I = \inf\{a_n : n < \omega\}$. □

The next corollary indicates a limitation on the types of bases of finitely generated models, at least in the case of arithmetically saturated $\mathfrak{N}$.

Corollary 4.6 *Assume that $\mathfrak{N}$ is arithmetically saturated. Let $M, N \in \mathscr{E}(\mathfrak{N})$ be finitely generated with no least proper gaps. Then $(\mathfrak{N}, \mathrm{B}(M)) \cong (\mathfrak{N}, B(N))$.*

Proof Let $M, N \in \mathscr{E}(\mathfrak{N})$ be finitely generated with no least proper gap. By Theorem 4.5, there are $a, b \in \mathfrak{N}$ such that $\mathrm{B}(M) = \inf\{(a)_n : n < \omega\}$ and $\mathrm{B}(N) = \inf\{(b)_n : n < \omega\}$. Without loss of generality we can assume that all $(a)_n$ and $(b)_n$, $n < \omega$, realize the same minimal type, and then, by recursive saturation, we can further assume that $\mathrm{tp}(a) = \mathrm{tp}(b)$, and the result follows (see [14, Proposition 10.2.4] for full details). □

It is shown in Kossak [8] that for a countable recursively saturated model $\mathfrak{N}$ of PA, the following statements are equivalent:

(1) $\mathfrak{N}$ is arithmetically saturated.

(2) For any two tall models $M, K \in \mathcal{E}(\mathfrak{N})$, if there exist increasing coded sequences $\langle a_n : n < \omega\rangle$ and $\langle b_n : n < \omega\rangle$ such that $M = \sup\{a_n : n < \omega\}$ and $K = \sup\{b_n : n < \omega\}$, then $(\mathfrak{N}, M) \cong (\mathfrak{N}, K)$.

Attempts to use similar arguments to show that the condition in Corollary 4.6 characterizes arithmetic saturation have not so far been successful, so let us pose the following question.

Question 4.7 *Suppose that for all finitely generated $M, N \in \mathcal{E}(\mathfrak{N})$, if M and N have no least proper gaps, then $(\mathfrak{N}, \mathrm{B}(M)) \cong (\mathfrak{N}, B(N))$. Is $\mathfrak{N}$ arithmetically saturated?*

5 When is a Coshort Cut a Base?

We have seen that bases of finitely generated models are tall. It is not the case in general. If $I \in \mathcal{E}(\mathfrak{N}) \setminus \{\mathfrak{N}_0\}$ is not coshort, then I is a base of an $M \in \mathcal{E}(\mathfrak{N})$. For example, let p be a minimal type realized in $\mathfrak{N}$, and let M be the Skolem closure of the set of realizations of p in $\mathfrak{N} \setminus I$. Clearly, $\mathrm{B}(M) = I$. By a result of Smoryński [17], for every cut $I \subseteq_{\mathsf{end}} \mathfrak{N}$ that is closed under addition and multiplication, there is $M \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$ such that $\mathrm{GCIS}(\mathfrak{N}, M) = I$. Combining this with the argument given above, and the fact that cofinal extensions preserve recursive saturation, we get the following proposition.

Proposition 5.1 *Suppose that $I_1 \subseteq_{\mathsf{end}} I_2 \prec_{\mathsf{end}} I_3 \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$, I_1 is closed under addition and multiplication, $I_2 \setminus I_1$ is a subset of a gap, and I_3 is not coshort. Then there is $M \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$ such that $\mathrm{GCIS}(\mathfrak{N}, M) = I_1$, $\mathrm{Sb}(M) = I_2$, and $\mathrm{B}(M) = I_3$.*

Proof By Smoryński's theorem applied to I_2, there is $M' \prec_{\mathsf{cof}} I_2$ such that $\mathrm{GCIS}(M', N) = I_1$. Since $N \setminus I_3$ has no least gap, there is a set A of elements realizing a minimal type such that $\inf(A) = I_3$. The model $M = \mathrm{Scl}(M' \cup A)$ has the required properties. □

Thus, if we ask whether every elementary cut of $\mathfrak{N}$ is a base, the only case to consider is that of coshort cuts. We will start with a positive result.

Question 5.2 *For which coshort $I \in \mathcal{E}_{\mathsf{end}}(\mathfrak{N})$ are there $M \in \mathcal{E}(\mathfrak{N})$ such that $\mathrm{B}(M) = I$?*

In getting a partial answer to Question 5.2, the notion of a semiregular type will be useful.

Definition 5.3 We call a type $p(x)$ *semiregular* if it is unbounded, and if a realizes $p(x)$ in a model $M \models \mathsf{PA}$, then $\inf(\mathrm{gap}(a))$ is *semiregular in* $\mathrm{Scl}(a)$.

Theorem 5.4 *Let γ be a proper gap of $\mathfrak{N}$.*

(1) If $M \in \mathcal{E}(\mathfrak{N})$ is such that $M \cap \mathrm{B}(M) \prec_{\mathsf{end}} \mathfrak{N}$ and $\mathrm{B}(M) = \inf(\gamma)$, then each $a \in \gamma \cap M$ realizes a semiregular type.

(2) If $\mathfrak{N}$ is arithmetically saturated and $a \in \gamma$ realizes a semiregular type, then there is $M \in \mathcal{E}(\mathfrak{N})$ such that $a \in M$, $\mathrm{B}(M) = \inf(\gamma)$, and $M \cap \mathrm{B}(M) \prec_{\mathsf{end}} \mathfrak{N}$.

Proof Let us start by introducing some notation. Suppose that we have a gap γ and an elementary cut $I = \inf(\gamma)$ and that we are trying to determine if there is $M \prec \mathfrak{N}$ such that $\mathrm{B}(M) = I$. If there is such an M, then we can obtain it from some $a \in \gamma$ as follows. Define $M_0, M_1, M_2, \ldots$ and $N_0, N_1, N_2, \ldots$ recursively by $M_0 = \mathrm{Scl}(a)$, $N_i = \sup(M_i \cap I)$, and $M_{i+1} = \mathrm{Scl}(N_i \cup \{a\})$. Then, let $M = \bigcup_i M_i$. Observe that $M \cap I = \bigcup_i N_i$. We will refer to this M as $M[a]$ and also refer to each M_i as $M_i[a]$ and to each N_i as $N_i[a]$.[1] Thus, for any $b \in \mathfrak{N} \setminus \mathrm{Scl}(0)$, if there is $M \prec \mathfrak{N}$ such that $\mathrm{B}(M) = \inf(\mathrm{gap}(b))$, then there is $a \in \mathrm{gap}(b)$ such that $B(M[a]) = \inf(\mathrm{gap}(b))$. So it is of interest to know which $a \in \mathfrak{N} \setminus \mathrm{Scl}(0)$ are such that $B(M[a]) = \inf(\mathrm{gap}(a))$.

(1) Let $\gamma = \mathrm{gap}(a)$, $I = \inf(\gamma)$, and suppose that $B(M[a]) = I$ and $\mathrm{tp}(a)$ is not semiregular. Then the infimum of $\mathrm{gap}(a))$ in $\mathrm{Scl}(a)$ is not semiregular in $\mathrm{Scl}(a)$, and there are $b, c \in \mathrm{Scl}(a)$ such that $c \in \mathrm{gap}(a)$ and $\{(b)_x : x \in \mathrm{Scl}(a) \cap [0, c]\} \cap I$ is unbounded in I. Let $t(x)$ and $t'(x)$ be Skolem terms such that $\mathrm{Scl}(a) \models t(a) = b \wedge t'(a) = c$. Let $f_0(x), f_1(x), \ldots$ be a recursive sequence of all Skolem terms, and then let $t_i(x)$ be defined as the least y such that $\max\{f_j(y) : j \leq i\} > x$. Notice that $t_0(a), t_1(a), t_2(a), \ldots$ is a decreasing sequence downward cofinal in $\mathrm{gap}(a)$ and that for infinitely many $i < \omega$,

$$\mathrm{Scl}(a) \models \exists x \leq t'(a)\big[t_{i+1}(a) < \big(t(a)\big)_x < t_i(a)\big].$$

That sentence also holds in $\mathfrak{N}$. Then, it easily follows that $N_1[a] = I$, contradicting the fact that $B(M[a]) = I$.

(2) Again, let $\gamma = \mathrm{gap}(a)$ and $I = \inf(\gamma)$. Suppose that $\mathfrak{N}$ is arithmetically saturated and that $\mathrm{tp}(a)$ is semiregular. Our aim is to show that $M_1[a] = M[a]$ and $B(M[a]) = I$.

Let us start by showing that $I \setminus N_0[a] \neq \varnothing$. That is easy because by arithmetic saturation, there is a $c \in \mathfrak{N}$ such that for any Skolem term $t(x)$, $c < t(a)$ iff $t(a) \in \mathrm{gap}(a)$. (This is so even if $\mathrm{tp}(a)$ is not semiregular.)

Next, let us show that $N_1 = \sup(M_0 \cap I)$. If not, then there are $b \in N_0$ and a Skolem term $t(x, y)$ such that $N_0 < t(a, b) \in I$. Since $b \in N_0$, there is a Skolem term $t_0(x)$ such that $b < t_0(a) \in I$. Since $\mathrm{tp}(a)$ is semiregular, there are Skolem terms $t_1(x)$ and $t_2(x)$ such that $t_1(a) \in \gamma < t_2(a)$ and

$$\mathrm{Scl}(a) \models \forall y\big[y < t_0(a) \longrightarrow \big(t(a, y) < t_1(a) \vee t(a, y) > t_2(a)\big)\big].$$

Thus, $\mathfrak{N} \models t(a, b) < t_1(a) \vee t(a, b) > t_2(a)$, a contradiction. □

Corollary 5.5 *Suppose that $\mathfrak{N}$ is arithmetically saturated and γ is a proper gap of $\mathfrak{N}$. Then, γ contains an element realizing a semiregular type iff there is $M \in \mathcal{E}(\mathfrak{N})$ such that* $\mathrm{B}(M) = \inf(\gamma)$ *and* $M \cap \mathrm{B}(M) \prec_{\mathsf{end}} \mathfrak{N}$.

One can now ask whether every gap contains an element realizing a semiregular type. It turns out not to be the case. An example is provided by the next lemma.

Lemma 5.6 *There is $a \in \mathfrak{N}$ realizing a rare type such that $\langle (a)_{2i+1} : i < \omega \rangle$ is a decreasing sequence downward cofinal in* $\mathrm{gap}(a)$ *and $\langle (a)_{2i} : i < \omega \rangle$ is an enumeration of* $\mathrm{Scl}(0)$.

Proof To be definitive about the Skolem term $(x)_y$, we state our conventions. If $a \in \mathfrak{N}$, then we think of a as being a code for the sequence $\langle (a)_i : i \in \mathfrak{N} \rangle$ such that

- every coded sequence is eventually 0.
- every $\mathfrak{N}$-definable sequence that is eventually 0 is coded by unboundedly many codes.

To get $a \in \mathfrak{N}$, we will, as usual, construct a recursive sequence $\langle \varphi_n(x) : n < \omega \rangle$ of 1-ary formulas, where each $\varphi_n(x)$ defines the unbounded set $X_n \subseteq \mathfrak{N}$ such that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$, and then we pick any $a \in \bigcap_n X_n$. Notice that since $\mathfrak{N}$ is recursively saturated, $\bigcap_n X_n \neq \varnothing$.

We will make use of a recursive sequence $\langle f_n : n < \omega \rangle$ of all 0-definable functions $f : \mathfrak{N} \longrightarrow \mathfrak{N}$, and also a recursive, one-to-one enumeration $\langle c_n : n < \omega \rangle$ of $\mathrm{Scl}(0)$.

Each X_n will have the following properties:

(1) For each $y \in \mathfrak{N}$, there are unboundedly many $x \in X_n$ such that $(x)_i = (y)_i$ whenever $2n \leq i \in \mathfrak{N}$.
(2) If $i < n$ and $x \in X_n$, then $(x)_{2i} = c_i$.
(3) If $n > 0$, $x, y \in X_n$, $y < x$, and there is no $z \in X_n$ such that $y < z < x$, then $(x)_{2n-1} = y$.

Let $X_0 = \mathfrak{N}$. Next, suppose that we already have $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n$ satisfying (1)–(3). To obtain X_{n+1}, we inductively choose $x_0 < x_1 < x_2 < \cdots$ as follows. Let x_0 be the least $x \in X_n$ such that $(x)_{2n} = c_n$. Assuming that we have x_j, then let x_{j+1} be the least $x \in X_n$ such that $x > f_n(x_j)$, $(x)_{2n} = c_n$, $(x)_{2n+1} = x_j$, and $(x)_i = (j)_i$ for $i \in \mathfrak{N}$ such that $2n + 1 < i$. Then let $X_{n+1} = \{x_j : j \in \mathfrak{N}\}$.

One easily notices that this construction results in a decreasing recursive sequence $\langle X_n : n < \omega \rangle$ satisfying (1)–(3).

Pick any $a \in \bigcap_n X_n$. From (2), $(a)_{2n} = c_n$, so $\langle (a)_{2n} : n < \omega \rangle$ is an enumeration of $\mathrm{Scl}(0)$. It is clear from the construction that $\langle (a)_{2n+1} : n < \omega \rangle$ is a decreasing sequence. Moreover, it is downward cofinal in $\mathrm{gap}(a)$ since, for any zero-definable $f : \mathfrak{N} \longrightarrow \mathfrak{N}$, there is $n < \omega$ such that f_n dominates f and $f_n((a)_{2n-1}) < a$. Also, notice that there is no $x \in X_n$ such that $a < x < f_n(a)$, thereby showing that a realizes a rare type. □

Corollary 5.7 *There is a coshort elementary cut $I \in \mathcal{E}_{\mathrm{end}}(\mathfrak{N})$ such that whenever $M \in \mathcal{E}(\mathfrak{N})$, $M \cap \mathrm{B}(M) \prec_{\mathrm{end}} \mathfrak{N}$, and $\mathrm{B}(M)$ exists, then $\mathrm{B}(M) \neq I$.*

Proof Let $a \in \mathfrak{N}$ be as in Lemma 5.6, let $I = \inf(\mathrm{gap}(a))$, and let $M \in \mathcal{E}(\mathfrak{N})$ be such that $M \cap \mathrm{gap}(a) \neq \varnothing$. Since the type of a is rare, $a \in M$ (see [14, Lemma 3.1.15]). We will consider two cases.

Case 1. Suppose that $\mathrm{GCIS}(\mathfrak{N}, M) \subseteq \mathfrak{N}_0$. Then, by overspill, there is a nonstandard $i_0 \in M$ such that $\mathfrak{N}_0 < (a)_{i_0} \in I$. Hence $\mathrm{B}(M) \neq I$.

Case 2. Now assume that $\mathfrak{N}_0 < \mathrm{GCIS}(\mathfrak{N}, M) = J < I$. Let $d \in I$ be such that $J < d$. Again by overspill, there is $i_0 \in J$ such that $d < (a)_{i_0} \in I$, and the result follows. □

6 Elementary Pairs Sharing the Same Greatest Common Initial Segment

If $I = \mathrm{B}(M)$ for some $M \in \mathcal{E}(\mathfrak{N})$, we want to see what, if any, are the restrictions on the isomorphism type of M. Can M be recursively saturated? Can it be isomorphic to $\mathfrak{N}$?

If $I \in \mathcal{E}_{\mathrm{end}}(\mathfrak{N})$ is strong, then $\mathfrak{N}$ has a countable cofinal elementary extension $\mathfrak{N}'$ such that $I = \mathrm{GCIS}(\mathfrak{N}', \mathfrak{N})$ and $I_+^{\mathfrak{N}}(a) \prec_{\mathrm{end}} \mathfrak{N}'$ for any $a \in \mathfrak{N}'$ such that $I < a < (\mathfrak{N} \setminus I)$ (this is a result of Kirby [6]). Since $\mathfrak{N}$ is isomorphic to $\mathfrak{N}'$, this provides examples of bases of models in $\mathcal{E}_{\mathrm{cof}}(\mathfrak{N})$ that are isomorphic to $\mathfrak{N}$. Kirby [6] proved that $\mathfrak{N}'$ can be constructed in such a way that for any $a \in \mathfrak{N}'$ such that

$I < a < (\mathfrak{N} \setminus I)$, $I_+^{\mathfrak{N}}(a)$ is a strong elementary cut of $\mathfrak{N}'$, but beyond that there is not much more information.

The rest of the paper is devoted to theories and isomorphism types of pairs $(\mathfrak{N}, M)$, where $M \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$ and $\mathrm{GCIS}(\mathfrak{N}, M)$ is fixed. There are several constructions giving a variety of theories and isomorphism types of pairs, and a variety of greatest common initial segments. The strongest result is Theorem 7.6; nevertheless we will present other constructions first, as each might be of interest on its own, and each rests on explorations of different kinds of isomorphism invariants. We will start with a slight improvement on a theorem of Smoryński [18, Theorem 1.3].

Theorem 6.1 *There is an elementary cut J for which there are infinitely many theories* $\mathrm{Th}((\mathfrak{N}, M))$*, where $M \in \mathcal{E}_{\mathsf{cof}}(\mathfrak{N})$,* $\mathrm{GCIS}(\mathfrak{N}, M) = J$*, and $M \cong \mathfrak{N}$.*

Proof There is a recursive list $\langle \Phi_n : n < \omega \rangle$ of first-order properties of elementary cuts such that for all distinct n and m and all cuts $M \in \mathcal{E}(\mathfrak{N})$, $(\mathfrak{N}, M) \models (\Phi_n \to \neg\Phi_m)$. For example, Φ_n can be a sentence expressing that $(M, \mathrm{Cod}(\mathfrak{N}/M))$ satisfies all Σ_n-induction axioms with set parameters from $\mathrm{Cod}(\mathfrak{N}/M)$ plus the negation of a particular instance of the (Σ_{n+1})-collection axiom (see Kossak [9]). By chronic resplendency of $\mathfrak{N}$, we get $X, S \subseteq \mathfrak{N}$ such that $(\mathfrak{N}, X, S)$ is resplendent and for each $n < \omega$ the following conditions hold:

(1) $(X)_n \prec_{\mathsf{end}} (X)_{n+1} \prec_{\mathsf{end}} \mathfrak{N}$;
(2) $(\mathfrak{N}, (X)_n) \models \Phi_n$;
(3) S is a partial inductive satisfaction class such that for all $n < \omega$,

$$\big((X)_n, S \cap (X)_n\big) \prec (\mathfrak{N}, S).$$

Let p be a minimal type of $\mathrm{Th}((\mathfrak{N}, S))$ realized in $(\mathfrak{N}, S)$. Let P be the set of realizations of p in $(\mathfrak{N}, S)$. Let $J = \sup(\mathrm{Scl}^{(\mathfrak{N},S)}(0))$. Condition (3) implies that $J < (X)_0$. Finally, for each $n < \omega$ we define M_n to be $\mathrm{Scl}^{(\mathfrak{N},S)}\big(J \cup (P \setminus (X)_n)\big)$. Since p is strongly indiscernible, for all $n < \omega$, $\mathrm{GCIS}(\mathfrak{N}, M_n) = J$ and $\mathrm{B}(M_n) = (X)_n$. By condition (2), $\mathrm{Th}((\mathfrak{N}, M_m)) \neq \mathrm{Th}((\mathfrak{N}, M_n))$ for all $m \neq n$. $\square$

Notice that the cut J obtained in the proof of Theorem 6.1 is not semiregular. In the next result we will get a strong J and continuum many different theories of pairs $(\mathfrak{N}, M)$. We will need a series of lemmas. The first one applies to all cofinal extensions.

Lemma 6.2 *There is a sentence σ of $\mathcal{L}_{\mathsf{PA}}$ with an additional unary predicate such that for all $M, N \models \mathsf{PA}$, if $M \prec_{\mathsf{cof}} N$, then $(N, M) \models \sigma$ if and only if N is finitely generated over M.*

Proof Suppose that $M \prec_{\mathsf{cof}} N$ and $N = \mathrm{Scl}(M \cup \{a\})$ for some $a \in N$. Then every element of N is of the form $t(a, b)$ for some Skolem term t and some $b \in M$. Let $c \in M$ be such that $a < c$, and let $d \in M$ be a code of the sequence $\langle t(x, b) : x < c\rangle$. Then $t(a, b) = (d)_a$. It follows that $N = \{(d)_a : d \in M\}$. Hence the sentence $\sigma = \exists x \forall y \exists z (z \in M \wedge y = (z)_x)$ is as required. $\square$

Lemma 6.2 is specifically about cofinal extensions. Suppose that M is not cofinal in N and that N is finitely generated over M. Let (N', M') be a recursively saturated pair such that $(N, M) \equiv (N', M')$. It is easy to see that N' is not finitely generated over M'.

The next lemma is essentially due to Kirby and Paris, and it follows from standard results on strong cuts in models of PA (see [14, Theorem 7.3.4] for details).

Lemma 6.3 *For each $a \in \mathfrak{N}$ there is a strong elementary cut I such that $a \in I$. If $I \in \mathscr{E}_{\mathsf{end}}(\mathfrak{N})$ is strong, then there are $\mathfrak{N}'$ and $\mathfrak{N}''$ such that*

(1) $\mathfrak{N} \prec_{\mathsf{cof}} \mathfrak{N}'$ *and* $\mathfrak{N} \prec_{\mathsf{cof}} \mathfrak{N}''$*;*
(2) $I = \mathrm{GCIS}(\mathfrak{N}', \mathfrak{N}) = \mathrm{GCIS}(\mathfrak{N}'', \mathfrak{N})$ *and* $\mathrm{Cod}(\mathfrak{N}'/I) = \mathrm{Cod}(\mathfrak{N}''/I) = \mathrm{Cod}(\mathfrak{N}, /I)$*;*
(3) $B^{\mathfrak{N}'}(\mathfrak{N})$ *and* $B^{\mathfrak{N}''}(\mathfrak{N})$ *exist,* $\mathfrak{N}' = \mathrm{Scl}(\mathfrak{N} \cup B^{\mathfrak{N}'}(\mathfrak{N}))$*, and* $\mathfrak{N}'' = \mathrm{Scl}(\mathfrak{N} \cup B^{\mathfrak{N}''}(\mathfrak{N}))$*;*
(4) $\mathfrak{N}'$ *is finitely generated over* $\mathfrak{N}$*, and* $\mathfrak{N}''$ *is not finitely generated over* $\mathfrak{N}$*.*

Theorem 6.4 *For every strong elementary cut J there are continuum many theories* $\mathrm{Th}((\mathfrak{N}, M))$*, where* $M \in \mathscr{E}_{\mathsf{cof}}(\mathfrak{N})$*,* $\mathrm{GCIS}(\mathfrak{N}, M) = J$*, and* $M \cong \mathfrak{N}$*.*

Proof Let J be a strong elementary cut in $\mathfrak{N}$, and let $X \subseteq \omega$ be given. Using Lemma 6.3 we inductively define the sequence $\mathfrak{N} = M_0 \prec_{\mathsf{cof}} M_1 \prec_{\mathsf{cof}} M_2 \prec_{\mathsf{cof}} \cdots$ such that for all $n < \omega$,

(1) $B^{M_{n+1}}(M_n)$ exists and $B^{M_{n+1}}(M_n) < \mathrm{GCIS}(M_{n+2}, M_{n+1})$;
(2) $\mathrm{GCIS}(M_1, M_0) = J$ and $\mathrm{Cod}(M_0/J) = \mathrm{Cod}(M_1/J)$;
(3) M_{n+1} is finitely generated over M_n if and only if $n \in X$.

Let $K_X = K = \bigcup_{n<\omega} M_n$. We will consider the pair $(K, \mathfrak{N})$. Notice that for each n,

$$M_{n+1} = \mathrm{Scl}^K\big(M_n \cup B^K(M_n)\big) = \big\{(a)_b : a \in M_n \wedge b \in B^K(M_n)\big\}.$$

Since $B(M_0)$ is definable in $(K, \mathfrak{N})$, this allows us to inductively define each M_n in $(K, \mathfrak{N})$. Since K is countable and recursively saturated and $\mathrm{SSy}(\mathfrak{N}) = \mathrm{SSy}(K)$, $K \cong \mathfrak{N}$; moreover, by (2), there is an isomorphism that is identity on J (see [14, Theorem 8.5.2]). The result now follows from Lemma 6.2, since it implies that for $X \neq Y$, $\mathrm{Th}((K_X, \mathfrak{N})) \neq \mathrm{Th}((K_Y, \mathfrak{N}))$. □

The construction in the proof of the next theorem is a modification of one of the basic constructions from [19]. It yields pairs $(\mathfrak{N}, M)$ which are not recursively saturated and cuts J that are not semiregular. We obtain continuum many nonisomorphic pairs, but we have no control over their first-order theories.

Theorem 6.5 *There are* $J \prec_{\mathsf{end}} \mathfrak{N}$ *and continuum many nonisomorphic pairs* $(\mathfrak{N}, M)$ *such that* $M \in \mathscr{E}_{\mathsf{cof}}(\mathfrak{N})$*,* $\mathrm{GCIS}(\mathfrak{N}, M) = J$*, and* $M \cong \mathfrak{N}$*.*

Proof Let S be a partial inductive satisfaction class such that $(\mathfrak{N}, S)$ is recursively saturated, and let p be a minimal type of $\mathrm{Th}((\mathfrak{N}, S))$ realized in $(\mathfrak{N}, S)$. We will follow the proof of [19, Theorem 3.6]. Let $<_\eta$ be a recursive ordering of $\mathbb{N}$ of the order type of the rationals. By recursive saturation, there is a coded sequence $\langle a_r : r \in \mathbb{N}\rangle$ such that for all $r, s \in \mathbb{N}$, $\mathrm{gap}^{(\mathfrak{N},S)}(a_r) < a_s$ iff $r <_\eta s$. In addition, we require that all a_r realize p. Let $a \in \mathfrak{N}$ code $\langle a_r : r \in \mathbb{N}\rangle$.

Let C be a Dedekind cut in $(\mathbb{N}, <_\eta)$, and let $K[C] = \sup\{a_r : r <_\eta C\}$. It is easy to see that $K[C]$ is a tall elementary cut of $\mathfrak{N}$. Hence $K[C] \cong \mathfrak{N}$. One can show that C is uniformly defined in $(\mathfrak{N}, K[C], a)$ (see [19]). If $(\mathfrak{N}, K[C])$ and $(\mathfrak{N}, K[C'])$ are isomorphic, then $\mathrm{Def}_0(\mathfrak{N}, K[C], a) = \mathrm{Def}_0(\mathfrak{N}, K[C'], a')$, where a' is an automorphic image of a. Since there are only countably many parameters a, if there were only countably many isomorphism types of structures $(\mathfrak{N}, K[C])$, the union of all sets definable in all structures of the form $(\mathfrak{N}, K[C], a)$ would be countable, but we know that every cut C must be in this union.

As before, we define $J = \sup(\mathrm{Scl}^{(\mathfrak{N},S)}(0))$, and for a Dedekind cut C, $M[C] = \mathrm{Scl}^{(\mathfrak{N},S)}(J \cup \{a_s : C <_\eta a_s\})$. To finish the proof, notice that $M[C]$ is isomorphic to $\mathfrak{N}$ by an isomorphism fixing $K[C]$ pointwise. $\square$

7 Elementary Pairs and Lattices of Elementary Substructures

In this last section we will prove a general result on cofinal extensions of (arbitrary) countable models of PA, which will provide another source of diversity. It will allow us to improve Theorem 6.5 by replacing nonisomorphic pairs by pairs satisfying different first-order theories.

If I is a proper cut in a model M and $M \prec N$, then we say that N *fills* I if there is a $b \in N$ such that for all $a, c \in M$, if $a \in I < c$, then $a < b < c$.

Theorem 7.1 *Suppose that M is any countable model of* PA *and that $J \subseteq M$ is a proper cut closed under exponentiation. Then there is a set $\mathcal{C}$ of finitely generated, cofinal extensions of M such that*

(1) $|\mathcal{C}| = 2^{\aleph_0}$;

(2) for each $N \in \mathcal{C}$, $\mathrm{GCIS}(N, M) = J$, $\mathrm{Cod}(N/J) = \mathrm{Cod}(M/J)$, *and N does not fill J;*

(3) if $N_1, N_2 \in \mathcal{C}$ are distinct, then $\mathrm{Th}((N_1, M)) \neq \mathrm{Th}((N_2, M))$.

One possible approach to proving this theorem is via substructure lattices. Recall that if $N \models$ PA, then $\mathcal{E}(N)$ can be considered as a lattice, and when we do so we write $\mathrm{Lt}(N)$ instead of $\mathcal{E}(N)$ (see [14, Chapter 4]). If $M \prec N \models$ PA, then $\mathrm{Lt}(N/M)$ is the sublattice of $\mathrm{Lt}(N)$ consisting of those $K \in \mathrm{Lt}(N)$ such that $M \prec K$. We let $\mathrm{Lt}_0(N/M) = \{K \in \mathrm{Lt}(N/M) : K$ is a finitely generated extension of $M\}$. Notice that, in general, $\mathrm{Lt}_0(N/M)$ is not a lattice, but it is a $\vee$-subsemilattice and we will think of it as such.

For a model M, $M(a)$ will denote the elementary extension of M generated over M by a, that is, $M(a) = \mathrm{Scl}(M \cup \{a\})$.

Lemma 7.2 *Suppose that $M \prec_{\mathrm{cof}} N \models$ PA. Then,* $\mathrm{Lt}_0(N/M)$ *is interpretable in* (N, M).

Proof In (N, M), the relation $R = \{\langle x, y\rangle \in N : M(x) \prec M(y)\}$ is definable by the formula $\forall u \in M \exists v \in M[(u)_x = (v)_y]$. $\square$

Lemma 7.2 implies that if $M_1 \prec_{\mathrm{cof}} N_1$, $M_2 \prec_{\mathrm{cof}} N_2$ and $(N_1, M_1) \equiv (N_2, M_2)$, then $\mathrm{Lt}_0(N_1/M_1) \equiv \mathrm{Lt}_0(N_2/M_2)$. This suggests the following question.

Question 7.3 *Suppose that $M_1 \prec_{\mathrm{cof}} N_1$, $M_2 \prec_{\mathrm{cof}} N_2$, and $(N_1, M_1) \equiv (N_2, M_2)$. Is* $\mathrm{Lt}(N_1/M_1)$ *elementarily equivalent to* $\mathrm{Lt}(N_2/M_2)$*?*

By the method described in Kossak and Schmerl [13, Section 5], the following can be shown. If $X \subseteq \omega$ and $M \models$ PA is countable and nonstandard, then there is $N \succ_{\mathrm{cof}} M$ such that $\mathrm{Lt}_0(N/M) = \mathrm{Lt}(N/M) \setminus \{N\}$ and $\mathrm{Th}(\mathrm{Lt}(N/M))$ is Turing-reducible to X. Moreover, if $J \subseteq M$ is a proper cut closed under exponentiation, then, using Lemma 7.4 below, we can get N such that $\mathrm{GCIS}(N, M) = J$. Theorem 7.1 easily follows.

We give another approach to proving Theorem 7.1 in which all $N \in \mathcal{C}$ are such that $\mathrm{Lt}(N/M)$ is much simpler; in fact, $\mathrm{Lt}(N/M)$ is a 3-element chain **3**.

Lemma 7.4 *Suppose that M is a countable model of* PA, *$a \in M$, and $J \subseteq M$ is a proper cut closed under exponentiation. Then M has a minimal cofinal extension N such that* $\mathrm{GCIS}(N, M) = J$, $\mathrm{Cod}(N/J) = \mathrm{Cod}(M/J)$, *and N does not fill J.*

If, moreover, $M_0 \prec M = M_0(a)$, then N can have the additional property that whenever $M_0 \prec M_1 \prec N$ and $M_1 \neq N$, then $M_1 \prec M$.

Proof As is typical with proofs of lemmas like this, we will construct a decreasing sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of nonempty definable subsets of M that determine a type over M in the sense that whenever $Y \subseteq M$ is definable, then there is $n < \omega$ such that either $Y \subseteq X_n$ or $Y \cap X_n = \varnothing$. Then, we let N be the extension of M generated by an element realizing this type.

We will say that a subset $X \subseteq M$ is *large* if $X \subseteq X_0$, X is definable, and $M \models |X| = b$ for some $b > J$. To start the construction, we choose some $b \in M \setminus J$ and then let $X_0 = \{x \in M : x = \langle a, y \rangle$ for some $y < b\}$. Clearly, X_0 is large.

The construction of $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ will proceed in the usual way. Every large set X satisfies the following properties. These are just the properties that are needed to guarantee that the sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$, and then also N, has the desired properties:

(a) If $Y \subseteq X$ is definable, then either Y or $X \setminus Y$ is large.
(b) If $f : X \longrightarrow M$ is a definable function, then there is a large $Y \subseteq X$ such that either f is constant on Y or f is one-to-one on Y.
(c) If $J < b \in M$, then there is a large $Y \subseteq X$ such that $M \models |Y| \leq b$.
(d) For every definable function $f : X \longrightarrow M$, there is a large $Y \subseteq X$ such that either there is $b \in J$ such that $f(y) \leq b$ for all $y \in Y$ or else there is $b > J$ such that $f(y) \geq b$ for all $y \in Y$.
(e) If $f : X \longrightarrow M$ is a definable function such that for all $x \in X$ and all i, $(f(x))_i \in \{0, 1\}$, and there is $b > J$ such that $f(x) \geq b$ for all $x \in X$, then there are $d \in M$, $c > J$, and a large $Y \subseteq X$ such that for all $y \in Y$ and $i < c$, $(f(y))_i = (d)_i$.

We next give proofs of (a)–(e).

(a) The larger of the sets Y and $X \setminus Y$ is large.

(b) Let R be the range of f, and let $r = |R|$. If $r > J$, then let Y be the collection of the smallest elements of $f^{-1}(y)$, for $y \in R$. If $r \in J$, there is a $y \in R$ such that $|f^{-1}(y)| > J$, because otherwise we would have $|X| \leq r \times \max\{|f^{-1}(y)| : y \in R\} \in J$.

(c) Let $f : X \longrightarrow |X|$ be a definable bijection, and then let $Y = f^{-1}([0, b] \cap |X|)$.

(d) By (b), we can assume that f is one-to-one or constant on X. If f is constant on X, then just let $Y = X$. So, suppose that f is one-to-one. Define $h : M \longrightarrow M$ to be such that $h(x) = |f^{-1}([x, \infty)])|$. Clearly, for each $x \in J$, $h(x) > J$, so $x < h(x)$. By overspill, there is $c > J$ such that $h(c) > c$. Let $Y = f^{-1}([c, \infty))$.

(e) For each i, consider the function $h_i : X \longrightarrow 2^i$ that maps x onto $\langle (f(x))_j : j < i \rangle$. Since J is closed under exponentiation, by overspill, there is a $d > J$ such that $|X| > 2^d d$. Pick such a d, and then let $Y \subseteq X$ be large such that h_d is constant on Y.

Let us see why this works. From (a), we get that the sequence determines a type. Let N be an extension of M generated by an element $c \in N$ realizing that type. Notice that by the choice of X_0, there is a Skolem term $t(x)$ such that $t(c) = a$.

Since X_0 is bounded, N is a cofinal extension of M, and it is a minimal extension by (b).

Let $b \in M \setminus J$ be given. By (c), there is a large $Y \in M$ such that $N \models c \in Y$ and $|Y| \leq b$. Let $d \in M$ be such that $Y = \{(d)_i : i < |Y|\}$, and let $t(x) = \min\{i : x = (d)_i\}$. Then $t(c) \leq b$ and $t(c) \notin M$. This implies that $\mathrm{GCIS}(N, M) \subseteq J$.

Conditions (b) and (d) imply that $\mathrm{GCIS}(N, M) \supseteq J$. Thus, $\mathrm{GCIS}(N, M) = J$. By (d), N does not fill J.

We get from (e) that $\mathrm{Cod}(N/J) = \mathrm{Cod}(M/J)$.

Finally, we prove the additional minimality property. Suppose that $M_0 \prec M_1$ and that $d \in N \setminus M$. Because of the existence of $t(x)$, there is an M_0-definable $f : N \longrightarrow N$ such that $f(c) = d$. By (b) we can assume that f is one-to-one, so that $c = f^{-1}(d)$ and then $c \in M_1$. Again, by the existence of $t(x)$, we get that $M_1 = N$. □

We now prove Theorem 7.1 with additional conclusion:

(4) If $N \in \mathcal{C}$, then $\mathrm{Lt}(N/M) \cong \mathbf{3}$.

Proof of Theorem 7.1 Applying Lemma 7.4 (without the "moreover" part), let N_0 be a minimal cofinal extension of M such that $\mathrm{GCIS}(N_0, M) = J$ and $\mathrm{Cod}(N_0/J) = \mathrm{Cod}(M/J)$. Let $N_0 = M(a)$. It was shown first by Jensen and Ehrenfeucht [4] (and it also follows easily from Friedman's embedding theorem) that there are continuum many cuts $J_0 \subseteq N_0$ such that J_0 is closed under exponentiation, $J \subseteq J_0$, and for each of these continuum many cuts J_0, the theories $\mathrm{Th}((N_0, J_0))$ are distinct. Now, for each such J_0, apply Lemma 7.4 (now with the "moreover" part), and get a minimal cofinal extension N of N_0 such that $\mathrm{GCIS}(N, N_0) = J_0$ and such that whenever $M \prec M_1 \prec N$ and $M_1 \neq N$, then $M_1 \prec N_0$. Let $\mathcal{C}$ consist of one such N for each possible $\mathrm{Th}((N_0, J_0))$.

Now, consider an $N \in \mathcal{C}$, with corresponding J_0. Clearly, $M \prec N_0 \prec N$, and there is no $N' \neq N_0$ such that $M \prec N' \prec N$, and all inclusions are proper. Thus, by Lemma 7.2, N_0 is definable in (N, M) and therefore J_0 is also. Thus, $\mathrm{Th}((N, M))$ determines $\mathrm{Th}((N_0, J_0))$. Then, for distinct $N \in \mathcal{C}$, the theories $\mathrm{Th}((N, M))$ are distinct. □

Question 7.5 *Can the condition in (4) that* $\mathrm{Lt}(N/M) \cong \mathbf{3}$ *be replaced with* $\mathrm{Lt}(N/M) \cong \mathbf{2}$ *(i.e., N is a minimal extension of M)?*

Theorem 7.6 *Let $J \subseteq \mathfrak{N}$ be a proper cut closed under exponentiation. Then there is a set $\mathcal{D} \subseteq \mathcal{E}_{\mathrm{cof}}(\mathfrak{N})$ such that*

(1) $|\mathcal{D}| = 2^{\aleph_0}$;
(2) for each $M \in \mathcal{D}$, $\mathrm{GCIS}(\mathfrak{N}, M) = J$, $\mathrm{Cod}(M/J) = \mathrm{Cod}(\mathfrak{N}/J)$, $(M, J) \cong (\mathfrak{N}, J)$*, and for every $a \in \mathfrak{N}$, if $a > J$, then there is $b \in M$ such that $J < b < a$;*
(3) if $M_1, M_2 \in \mathcal{D}$ are distinct, then $\mathrm{Th}((\mathfrak{N}, M_1)) \neq \mathrm{Th}((\mathfrak{N}, M_2))$;
(4) if $M \in \mathcal{D}$, then $\mathrm{Lt}(\mathfrak{N}/M) \cong \mathbf{3}$.

Proof Let $\mathcal{D}'$ be a set of cofinal extensions of $\mathfrak{N}$ given by Theorem 7.1. Each $M \in \mathcal{D}'$ is recursively saturated, and since $\mathrm{Cod}(\mathfrak{N}/J) = \mathrm{Cod}(M/J)$, $M \cong \mathfrak{N}$. The only part now that requires an additional explanation is $(M, J) \cong (\mathfrak{N}, J)$. This follows from [14, Theorem 8.5.2] and two observations. In the theorem J is assumed to be elementary, but here since the identity on J is a partial isomorphism between M and $\mathfrak{N}$, the proof works. There is also an assumption that J is not an infimum of a coded ω-sequence either in M or in $\mathfrak{N}$. So, if J satisfies this, we are done. If not, suppose that $J = \inf\{(b)_i : i < \omega\}$ for some $b \in M$ coding a decreasing sequence. (We are still assuming $\mathfrak{N} \prec_{\mathsf{cof}} M$.) Then $b \mapsto b$ can be extended to an isomorphism $f : \mathfrak{N} \cong M$, and, clearly, $f(J) = J$. Now assume that $J = \inf\{(c)_i : i < \omega\}$ for some $c \in \mathfrak{N}$ coding a decreasing sequence. Then $\{\langle i, (c)_i\rangle : i > \omega\} \subseteq J$ is coded in $\mathfrak{N}$ and hence in M. Pick a code b of this set, and notice that there is a nonstandard $J' \subset J$ such that the identity on $J' \cup \{b\}$ extends to an isomorphism $f : \mathfrak{N} \cong M$. Again, $f(J) = J$. □

Note

1. In early papers on elementary cuts in recursively saturated models of PA, $\mathfrak{N}[a]$ denotes $\inf(\mathrm{gap}(a))$. This should not be confused with our notation here.

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