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PFA and Ideals on ω_2 Whose Associated Forcings Are Proper

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Abstract Given an ideal I, let $\mathbb{P}_I$ denote the forcing with I-positive sets. We consider models of forcing axioms $MA(\Gamma)$ which also have a normal ideal I with completeness ω_2 such that $\mathbb{P}_I \in \Gamma$. Using a bit more than a superhuge cardinal, we produce a model of PFA (proper forcing axiom) which has many ideals on ω_2 whose associated forcings are proper; a similar phenomenon is also observed in the standard model of $MA^{+\omega_1}(\sigma\text{-closed})$ obtained from a supercompact cardinal. Our model of PFA also exhibits weaker versions of ideal properties, which were shown by Foreman and Magidor to be inconsistent with PFA.

Along the way, we also show (1) the diagonal reflection principle for internally club sets ($DRP(IC_{\omega_1})$) introduced by the author in earlier work is equivalent to a natural weakening of "there is an ideal I such that $\mathbb{P}_I$ is proper"; and (2) for many natural classes Γ of posets, $MA^{+\omega_1}(\Gamma)$ is equivalent to an apparently stronger version which we call $MA^{+\,\mathrm{Diag}}(\Gamma)$.

1 Introduction

In [4], we introduced the *diagonal reflection principle (DRP)*—which is a highly simultaneous form of stationary set reflection—and proved that DRP follows from strong forcing axioms. DRP asserts that a certain naturally defined set of ω_1-sized structures—namely, those structures on which the nonstationary ideal condenses correctly—is *stationary* (see Section 3 below; such condensation notions originally appeared in Foreman [6]). Independently, Viale [13] proved similar results.

A motivation for this paper is the following observation: if F is a filter which concentrates on sets witnessing DRP, then by examining generic ultrapowers which use F (such ultrapowers have critical point ω_2), we see that $\mathbb{P}_{\breve{F}}$ exhibits properties which resemble *properness*. (This observation follows from Theorem 8 below.) So a natural way to strengthen DRP is to require that $\mathbb{P}_{\breve{F}}$ is actually proper.

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Most of this paper is devoted to models of strong forcing axioms $MA(\Gamma)$ which have an ideal I such that $\mathbb{P}_I \in \Gamma$; in particular, we obtain such ideals whose dual filter concentrates on sets witnessing DRP. In Section 4 we observe that, in the standard model of $MA^{+\omega_1}(\sigma\text{-closed})$ obtained from collapsing everything below a supercompact cardinal to ω_1, there are ideals whose duals concentrate on sets witnessing DRP and whose associated forcings have σ-closed dense subsets. It is natural to ask if a similar situation can arise under PFA (proper forcing axiom); we answer this question affirmatively. In particular, we prove the following theorem, which is stated more precisely as Theorem 12 in Section 5.

Theorem *Relative to a super-2-huge cardinal, it is consistent that PFA holds and that there are ideals I such that $\mathbb{P}_I$ is proper. Moreover, there are such ideals whose dual concentrates on sets witnessing DRP.*

This model has several other interesting features. In [9], Foreman and Magidor showed that if *either*

$$\begin{array}{l} \bullet\ (\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1) \text{ or} \\ \bullet \text{ there is a presaturated ideal on } \omega_2, \end{array} \tag{1}$$

then PFA fails. On the other hand, the model of PFA we produce in Section 5 shows that weaker versions of (1)—namely that $(\theta, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ holds for an inaccessible θ and that there is an ideal on ω_2 with closure properties resembling, but weaker than, presaturation—are (simultaneously) consistent with PFA.

Finally, it is well known that PFA implies that every proper poset $\mathbb{Q}$ completely embeds below some condition in $[H_\theta]^{\omega_1}/NS$ (for all sufficiently large θ); this is due to the existence of stationarily many $M \in \wp_{\omega_2}(H_\theta)$ such that $\omega_1 \subset M$ and there exists an $(M, \mathbb{Q})$-generic. Let $S^{H_\theta}_{\mathbb{Q}}$ denote this stationary set.[1] So, in particular, if PFA holds and I is an ideal whose associated forcing $\mathbb{P}_I$ is proper, then $\mathbb{P}_I$ completely embeds into another ideal forcing $\mathbb{P}_{I'}$ where $\breve{I}'$ concentrates on $S^{H_{\theta'}}_{\mathbb{P}_I}$. (Namely, I' is the nonstationary ideal restricted to $S^{H_{\theta'}}_{\mathbb{P}_I}$.) In general the nature of this complete embedding $\mathbb{P}_I \to \mathbb{P}_{I'}$ is mysterious; however, we produce a model of PFA with such complete embeddings of a simple form. (The proof appears in Section 6.)

Theorem *Relative to a super-3-huge cardinal, it is consistent that PFA holds and that there are ideals I, I' such that $\mathbb{P}_I, \mathbb{P}_{I'}$ are proper, I' projects to I in the Rudin–Keisler sense, and this projection is also a projection in the sense of forcing.*

The paper is organized as follows. Section 2 gives background material and notation, including a discussion "plus" versions of forcing axioms and Foreman's duality theorem (which is heavily used throughout the paper); Section 3 discusses DRP and characterizations in terms of ideals on ω_2; Section 4 is about models of $MA^{+\omega_1}(\sigma\text{-closed})$ with σ-closed ideal forcings; Section 5 is about models of PFA with proper ideal forcings; and Section 6 discusses "Ideal projections as forcing projections" in models of PFA.

2 Notation and Background

2.1 Ideals $\wp_\kappa(H_\theta)$ denotes the set of $M \prec H_\theta$ such that $|M| < \kappa$ and $M \cap \kappa \in \kappa$. If Z is a set, $NS \upharpoonright Z$ denotes the ideal $\{A \subset Z \mid A \text{ is nonstationary in } \bigcup Z\}$.[2] Throughout this paper, *ideal* always means a normal ideal. If I is an ideal, then $\breve{I}$ denotes the filter which is dual to I; similarly, if F is a filter, then $\breve{F}$ denotes its dual ideal. I^+ denotes the I-positive sets (i.e., if $I \subset \wp(Z)$, then I^+ is the collection of $S \in \wp(Z)$ such that $S \notin I$); if F is a filter, then F^+ means $\breve{F}^+$. If Δ is a class and F a filter, we say that F *concentrates on* Δ iff there is some $A \in F$ such that $A \subseteq \Delta$ (and an ideal I concentrates on Δ iff $\breve{I}$ concentrates on Δ). The *forcing associated with* I is $(I^+, \subset)$, which we will denote by $\mathbb{P}_I$; this is equivalent to forcing with $\wp(Z)/I - \{0\}$, where by definition $S =_I T$ iff the symmetric difference of S and T is in I.

$(\theta, \mu) \twoheadrightarrow (\theta', \mu')$ denotes the statement, "for every first order structure $\mathcal{A}$ on θ, there is a $Z \prec \mathcal{A}$ such that $|Z| = \theta'$ and $|Z \cap \mu| = \mu'$ (the classic *Chang's conjecture* is the special case $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$)."

We refer to a notion related to saturation of an ideal; this notion is analyzed in more detail in [3], though no results from that paper are used here.

Definition 1 Let H be a set (typically $\theta \subseteq H \subseteq H_\theta$ for some θ), let $Z \subset \wp(H)$, and let $I \subset \wp(Z)$ be an ideal. Suppose that I' is an ideal over some $Z' \subset H_{\theta'}$ for $\theta' \gg |H|$ such that I' projects to I; that is, $I = \{\{Z' \cap H \mid Z' \in A'\} \mid A' \in I'\}$. Let $\pi : (I')^+ \to I^+$ be the map $S \mapsto \{M \cap H \mid M \in S\}$.

- If π is a projection in the sense of forcing (i.e., pointwise preimages of maximal antichains are maximal), then we say that I, I' witnesses "ideal projections as forcing projections" and write $FP(I', I)$.
- If there is some I' such that $FP(I', I)$ holds, then we say that $FP(I)$ holds.
- If $FP(I', I)$ holds where I' is the conditional club filter relative to I at $H_{\theta'}$,[3] we say that $FP^{\text{Conditional Club}}(I)$ holds.

In [3] it is shown that $FP(I)$ implies precipitousness of I, and that $FP^{\text{Conditional Club}}(I)$ is equivalent to saturation of I. (The latter is essentially due to Foreman [7, Lemma 3.46].) In the present paper we produce a model of $FP(I)$ where I is not saturated (or even presaturated).

2.2 Forcing axioms By $MA(\Gamma)$ we always mean $MA_{\omega_1}(\Gamma)$; that is, for every $\mathbb{P} \in \Gamma$ and every ω_1-sized collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ there is a filter on $\mathbb{P}$ which meets every set in $\mathcal{D}$. For an ordinal α, $MA^{+\alpha}(\Gamma)$ means that for every $\mathbb{P} \in \Gamma$, every ω_1-sized collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, and every sequence $\langle \dot{S}_i \mid i < \alpha \rangle$ of $\mathbb{P}$-names such that $\Vdash_{\mathbb{P}}$ "$\dot{S}_i$ is stationary subset of ω_1" for every $i < \alpha$, there is a filter $F \subset \mathbb{P}$ which meets every D_i, and for every $i < \omega_1$: $(\dot{S}_i)_F := \{\beta : \text{there is some } q \in F \text{ such that } q \Vdash \check{\beta} \in \dot{S}_i\}$ is stationary.

The proof of Theorem 2.53 in Woodin [14] shows that $MA(\{\mathbb{P}\})$ is equivalent to stationarity of the following set:

$$S_{\mathbb{P}}^{H_\theta} := \big\{\text{the set of } M \prec (H_\theta, \in, \{\mathbb{P}\}) \text{ such that } \omega_1 \subset M \text{ and there exists an } (M, \mathbb{P})\text{-generic object}\big\} \tag{2}$$

for all (some) sufficiently large θ. The same argument shows that $MA^{+\alpha}(\{\mathbb{P}\})$ is equivalent to the statement: for every sequence $\langle \dot{S}_i \mid i < \alpha \rangle$ of $\mathbb{P}$-names for stationary subsets of ω_1, the set

$$S_{\dot{\vec{S}},\mathbb{P}}^{H_\theta} := \big\{\text{the set of } M \prec (H_\theta, \in, \{\mathbb{P}, \dot{\vec{S}}\}) \text{ such that } \omega_1 \subset M \text{and there exists a } g \text{ which is } (M, \mathbb{P})\text{-generic and } \dot{S}_g \text{ is stationaryfor every } i < \alpha\big\}, \tag{3}$$

is stationary for all (some) sufficiently large θ. Using the latter equivalence as motivation, let us define the following apparent strengthening of $MA^{+\omega_1}(\{\mathbb{P}\})$.

Definition 2

- For a poset $\mathbb{P}$ and a regular cardinal θ such that $\wp(\mathbb{P}) \in H_\theta$: $S_{\mathbb{P}}^{+\text{Diag},H_\theta}$ will denote the set of $M \prec (H_\theta, \in, \{\mathbb{P}\})$ such that $\omega_1 \subset M$ and there exists a g which is $(M, \mathbb{P})$-generic and such that $\dot{S}_g$ is stationary whenever $\dot{S} \in M$ is a $\mathbb{P}$-name for a stationary subset of ω_1.
- $MA^{+\text{Diag}}(\Gamma)$ denotes the statement that for all $\mathbb{P} \in \Gamma$ and for sufficiently large regular θ, $S_{\mathbb{P}}^{+\text{Diag},H_\theta}$ is stationary.

There are obvious generalizations of Definition 2 which require stationary evaluation of names of stationary subsets of $[\lambda]^\omega$ by g.[4] We note the following.

Lemma 3 *Suppose that Γ is a class of posets such that whenever $\mathbb{P} \in \Gamma$ then $\mathbb{P} * \text{Col}(\omega_1, \wp(\omega_1))^{V[\dot{G}]}) \in \Gamma$ (where $\dot{G}$ is the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic object). Then $MA^{+\omega_1}(\Gamma)$ is equivalent to $MA^{+\text{Diag}}(\Gamma)$.*

Proof To see that $MA^{+\text{Diag}}(\Gamma)$ implies $MA^{+\omega_1}(\Gamma)$, fix any $\mathbb{P} \in \Gamma$ and a regular θ such that $\wp(\mathbb{P}) \in H_\theta$. If $\langle \dot{S}_i \mid i < \omega_1 \rangle$ is a sequence of $\mathbb{P}$-names for stationary subsets of ω_1, then almost every $M \in S_{\mathbb{P}}^{+\text{Diag},H_\theta}$ has $\dot{\vec{S}}$ as an element; since $\omega_1 \subset M$, then $\dot{S}_i \in M$ for each i. Then if g is any $(M, \mathbb{P})$-generic which witnesses that $M \in S_{\mathbb{P}}^{+\text{Diag},H_\theta}$, then $(\dot{S}_i)_g$ is stationary for each $i < \omega_1$.

Now assume that $MA^{+\omega_1}(\Gamma)$, and let $\mathbb{P} \in \Gamma$; we want to show that $MA^{+\text{Diag}}(\mathbb{P})$ holds. Let $\dot{Q}$ be the $\mathbb{P}$-name for $\text{Col}(\omega_1, \wp(\omega_1))^{V[\dot{G}]})$, where $\dot{G}$ is the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic; so by assumption, $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$. Let $\langle \dot{T}_\alpha \mid \alpha < \omega_1 \rangle$ be a $(\mathbb{P} * \dot{\mathbb{Q}})$-name which is forced to enumerate $\{S \in V[G] \mid V[G] \models S \subseteq \omega_1 \text{ is stationary}\}$. Now for each $\alpha < \omega_1$, $\mathbb{P} * \dot{\mathbb{Q}}$ forces that $\dot{T}_\alpha$ is stationary (since $\dot{Q}_G$ is countably closed and thus stationary set preserving in $V[G]$). Then by $MA^{+\omega_1}(\mathbb{P} * \dot{\mathbb{Q}})$, $S^{H_\theta}_{\dot{\vec{T}}, \mathbb{P} * \dot{\mathbb{Q}}}$ is stationary. Let M be any element of $S^{H_\theta}_{\dot{\vec{T}}, \mathbb{P} * \dot{\mathbb{Q}}}$, let $\dot{S} \in M$ be a $\mathbb{P}$-name for a stationary subset of ω_1, and let $g * h$ be an $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic witnessing that $M \in S^{H_\theta}_{\dot{\vec{T}}, \mathbb{P} * \dot{\mathbb{Q}}}$. Then $\dot{S}_g = (\dot{T}_\alpha)_{g*h}$ for some $\alpha < \omega_1$, and the latter is stationary since $M \in S^{H_\theta}_{\dot{\vec{T}}, \mathbb{P} * \dot{\mathbb{Q}}}$. □

Corollary 4 *$PFA^{+\omega_1}$ is equivalent to $PFA^{+\text{Diag}}$, $MM^{+\omega_1}$ is equivalent to $MM^{+\text{Diag}}$, and $MA^{+\omega_1}(\sigma\text{-closed})$ is equivalent to $MA^{+\text{Diag}}(\sigma\text{-closed})$.*

Proof Let Γ be one of those three classes of posets (i.e., proper, stationary set preserving for subsets of ω_1, or σ-closed). Then for every $\mathbb{P} \in \Gamma$, $\mathbb{P} * \text{Col}(\omega_1, \wp(\omega_1))^{V[\dot{G}]}) \in \Gamma$. Apply Lemma 3. □

2.3 Diagonal reflection We recall the definition of DRP from [4].

Definition 5 Let Z be a class of ω_1-sized sets (e.g., Z could be the class of all ω-closed sets).

The *diagonal reflection principle at* θ relative to Z, abbreviated *DRP*(θ, Z), is the following statement:

There are stationarily many $M \in \wp_{\omega_2}(H_{(\theta^\omega)^+})$ such that

(1) $M \cap H_\theta \in Z$;
(2) whenever $R \in M$ is a stationary subset of $[\theta]^\omega$, then $R \cap [M \cap \theta]^\omega$ is stationary in $[M \cap \theta]^\omega$.

We say that DRP holds on Z iff $DRP(\theta, Z)$ holds for all regular $\theta \geq \omega_2$.

Definition 6 $wDRP(\theta, Z)$ is defined exactly like $DRP(\theta, Z)$ *except* that we replace clause 5 of the definition with the following:

- Whenever $R \in M$ is a projective stationary subset of $[\theta]^\omega$, then $R \cap [M \cap \theta]^\omega$ is stationary in $[M \cap \theta]^\omega$.

Independently, Viale [13] considered notions very similar to DRP and proved similar theorems to those in [4]. Adopting his terminology, for an ordinal λ of cofinality at least ω_2 and an $M \prec (H_\theta, \in, \{\lambda\} \ldots)$, let us say that M is $\lambda \cap \mathrm{cof}(\omega)$*-faithful* iff $R \cap \sup(M \cap \lambda)$ is stationary in $\sup(M \cap \lambda)$ for every $R \in M$ which is a stationary subset of $\lambda \cap \mathrm{cof}(\omega)$. Similarly, we will say that M is $[\lambda]^\omega$*-faithful* if the analogous statement holds for every $R \in M$ which is a stationary subset of $[\lambda]^\omega$.

For this paper, the most relevant classes Z in Definition 5 are the classes of *internally approachable* (IA_{ω_1}) and *internally club* (IC_{ω_1}) structures of size ω_1. IA_{ω_1} is the class of all M such that there is some $\in$-increasing and $\subset$-continuous chain $\langle N_\xi \mid \xi < \omega_1 \rangle$ of countable sets such that $M = \bigcup_{\xi < \omega_1} N_\xi$ and every proper initial segment of $\vec{N}$ is an element of M. IC_{ω_1} is the (possibly wider) class defined similarly, except that we only require that each N_ξ is an element of M (equivalently, $M \in IC_{\omega_1}$ iff $M \cap [M]^\omega$ contains a club).

2.4 Forcing quotients and Foreman's duality theorem Following [7], if $\mathbb{P}$ and $\mathbb{Q}$ are posets, then $i : \mathbb{P} \to \mathbb{Q}$ is a *regular embedding of* $\mathbb{P}$ *to* $\mathbb{Q}$ iff i preserves the order, preserves incompatibility, and pointwise maps maximal antichains in $\mathbb{P}$ to maximal antichains in $\mathbb{Q}$. If $i : \mathbb{P} \to \mathbb{Q}$ is regular and $G \subset \mathbb{P}$ is generic, then $\mathbb{Q}/i''G$ denotes the collection of $q \in \mathbb{Q}$ such that q is compatible with every member of $i''G$; the ordering on $\mathbb{Q}/i''G$ is just the ordering inherited from $\mathbb{Q}$. Regularity of i ensures that $\mathbb{Q}$ is forcing equivalent to $\mathbb{P} * \mathbb{Q}/i''\dot{G}$ (where $\dot{G}$ is the canonical $\mathbb{P}$-name for its generic object).

We will often use the following construction of precipitous filters and caution that it varies slightly from many of the constructions in [7] because the construction below yields a filter which does *not* extend the ground model's ultrafilter.[5] Suppose that $j : V \to_U N$ is an ultrapower embedding via some normal ultrafilter $U \in V$ which concentrates on $\{M \mid M \prec H_\lambda\}$ and has critical point κ (where $\kappa \leq \lambda$). Let $\mathbb{P} \in H_\lambda$ be a poset; note that $j \restriction \mathbb{P} : \mathbb{P} \to j(\mathbb{P})$ preserves order and incompatibility. Assume also that

$$j \restriction \mathbb{P} \text{ is a regular embedding from } \mathbb{P} \text{ to } j(\mathbb{P}). \tag{4}$$

This happens, for instance, whenever $\mathbb{P}$ has the κ-cc, which will always be the case in this paper.[6] So $j(\mathbb{P})$ is forcing equivalent to $\mathbb{P} * j(\mathbb{P})/j''\dot{G}$. Then whenever G is $(V, \mathbb{P})$-generic and H is $(V[G], j(\mathbb{P})/j''G)$-generic, in $V[G][H]$ the map j can be lifted to a $\hat{j}_{G*H} : V[G] \to N[G][H]$ via $\tau_G \mapsto j(\tau)_{H'}$, where H' is the $(V, \mathbb{P} * j(\mathbb{P})/j''\dot{G})$-generic obtained by transferring $G * H$ via the forcing equivalence of $\mathbb{P} * j(\mathbb{P})/j''\dot{G}$ with $j(\mathbb{P})$. This map will be well defined and elementary (see Cummings [5, Section 9] for details). Also, $\hat{j}_{G*H}(G)$ is equal to the generic for $(V, j(\mathbb{P}))$ obtained by transferring $G * H$ to a generic for $j(\mathbb{P})$ modulo the equivalence of $j(\mathbb{P})$ with $\mathbb{P} * j(\mathbb{P})/j''G$. This last fact can be used to show that $N \cap \hat{j}''_{G*H} H_\lambda[G] = j'' H_\lambda$. Now $\hat{j}''_{G*H} H_\lambda[G]$ is an element of $N[G][H]$, and

$$U_{G*H} := \left\{ A \in V[G] \;\middle|\; A \subset \wp(H_\lambda[G]) \text{ and } \hat{j}''_{G*H} H_\lambda[G] \in \hat{j}_{G*H}(A) \right\} \quad (5)$$

is a $V[G]$-normal ultrafilter.[7] We caution again that we are using $\hat{j}''_{G*H} H_\lambda[G]$, not $j'' H_\lambda$, to define the ultrafilter U_{G*H}, and as a result U_{G*H} will typically *not* extend U (e.g., if $\mathbb{P}$ turns κ into a successor cardinal, then U_{G*H} concentrates on models which believe that κ is a successor cardinal, while U concentrates on models which believe that κ is inaccessible). Still, using the definition of U_{G*H}, the fact that $N \cap \hat{j}''_{G*H} H_\lambda[G] = j'' H_\lambda$, and the assumption that j is an ultrapower embedding which maps $\mathbb{P}$ regularly into $j(\mathbb{P})$, then U_{G*H} will always concentrate on the set A of those $M' \prec H_\lambda[G]$ such that we have the following:

- $M' \cap V \in V$; let M denote $M' \cap V$.
- M is an element of the underlying set $\bigcup\bigcup U$ measured by U (e.g., if U is an ultrafilter on $\wp(S)$ then $M' \cap V \in S$).
- $V \models$ "$M \cap \mathbb{P}$ is a regular subposet of $\mathbb{P}$" (recall that we are assuming (4)).

Using these facts, it can be shown that $N[G][H] = \{\hat{j}_{G*H}(f)(\hat{j}''_{G*H} H_\lambda[G]) \mid f \in V[G] \cap {}^{A}V[G]\}$ (recall from above that $A \in V[G]$ will always have measure one in U_{G*H}), and it follows that $\hat{j}_{G*H}$ is the ultrapower map corresponding to $\text{ult}(V[G], U_{G*H})$.

In $V[G]$ define the following functions on A (here we again use the notation $M := M' \cap V$ whenever $M' \cap V \in V$):

(1) $Q(M') := \mathbb{P}/(G \cap M)$.
(2) $h(M') :=$ the generic for $\mathbb{P}/(G \cap M)$ obtained by G and the forcing equivalence between $\mathbb{P}$ and $(\mathbb{P} \cap M) * \mathbb{P}/(\dot{G} \cap \check{M})$.
(3) Given a $q \in j(\mathbb{P})/j''G$, note that $q \in V$ and that there is some $f_q \in V$ such that $q = [f_q]_U$ (here U was the ultrafilter in the ground model). Then define in $V[G]$ the function f'_q on A by $M' \mapsto f_q(M)$.

Then one can check:

$$\begin{aligned} &\text{For any } H \text{ which is } (V[G], j(\mathbb{P})/j''G)\text{-generic,} \\ &\quad (1) \quad [Q]_{U_{G*H}} = j(\mathbb{P})/j''G; \\ &\quad (2) \quad [h]_{U_{G*H}} = H; \\ &\quad (3) \quad \text{for every } q \in j(\mathbb{P})/j''G,\ [f'_q]_{U_{G*H}} = q. \end{aligned} \quad (6)$$

Definition 7 In $V[G]$, $F(j)$ denotes the collection of all $A \subset \wp(H_\lambda)$ such that $\Vdash_{j(\mathbb{P})/j''G} \check{A} \in \dot{U}_{G*\dot{H}}$.

$F(j)$ is a filter in $V[G]$ and inherits the completeness and normality properties from U (see [7, Section 3.2]). By (6) and a special case of Foreman's duality theorem—namely, [7, Proposition 7.13]—we have the following:

$$\text{In } V[G], \text{ the map } A \mapsto \|\check{A} \in \dot{U}_{G*\dot{H}}\|_{ro(j(\mathbb{P})/j''G)} \text{ is an isomorphism between } F(j)^+ \text{ and a dense subset of } ro(j(\mathbb{P})/j''G). \quad (7)$$

3 Equivalent Formulations of DRP and Effect of DRP on Generic Embeddings

In this section we show that the existence of ideals whose forcings are proper implies DRP (at least if the ideal concentrates on IC_{ω_1}), and, in turn, DRP always yields ideals whose associated forcings resemble proper forcings in a weak sense. Namely, DRP implies that for sufficiently small λ, stationary subsets of $[\lambda]^\omega$ are not destroyed in the generic ultrapower of V (though they may be destroyed in the generic extension!). In [4] we showed the following (see Section 2.3 for the definition of IC_{ω_1} and IA_{ω_1}):

- $MA^{+\omega_1}(\sigma\text{-closed})$ implies $DRP(\theta, IA_{\omega_1})$ for all regular $\theta \geq \omega_2$ (and MM implies $wDRP(\theta, IA_{\omega_1})$ for every regular $\theta \geq \omega_2$; see Definition 6).
- $DRP(\theta, IC_{\omega_1})$ implies that there is a stationary set of $M \in IC_{\omega_1} \cap \wp_{\omega_2}(H_{(\theta^\omega)^+})$ on which $NS \restriction [H_\theta]^\omega$ *condenses correctly*; that is, if $\sigma : H_M \to M$ is the inverse of the transitive collapse of M and $\bar{H} := \sigma^{-1}(H_\theta)$, then $(NS \restriction [\bar{H}]^\omega)^{H_M}$ coheres with $(NS \restriction [\bar{H}]^\omega)^V$.

There are several equivalent formulations of $DRP(\theta, IC_{\omega_1})$; note that formulations (C) and (D) below look like weak versions of saying $\mathbb{P}_I$ is proper.

Theorem 8 *For regular $\theta \geq \omega_2$, let Z^θ denote the collection of $M \in \wp_{\omega_2}(H_{(\theta^\omega)^+})$ such that $M \cap H_\theta \in IC_{\omega_1}$. The following are equivalent:*

(A) $DRP(\theta, Z^\theta)$;
(B) there are stationarily many $M \in Z^\theta$ such that $NS \restriction [H_\theta]^\omega$ condenses correctly via M;
(C) there is a normal ideal I whose dual contains the club filter over Z^θ such that $\Vdash_{\mathbb{P}_I}$ "every stationary $R \subset [H_\theta]^\omega$ from the ground model remains stationary in $\text{ult}(V, \dot{G})$*";*
(D) there is a stationary $S \subset Z^\theta$ such that $S \Vdash_{NS}$ "every stationary $R \subset [H_\theta]^\omega$ from the ground model remains stationary in $\text{ult}(V, \dot{G})$*."*

Proof Note that if $M \cap H_\theta \in IC_{\omega_1}$, then $M \cap [M \cap H_\theta]^\omega$ contains a club.[8] This implies that, letting $\sigma_M : H_M \to M$ be the inverse of the Mostowski collapse map and $\bar{H} := \sigma_M^{-1}(H_\theta)$, if $S \in M$ is a stationary subset of $[H_\theta]^\omega$, then $S \cap [M \cap H_\theta]^\omega$ is stationary if and only if V believes that $\sigma_M^{-1}(S)$ is stationary in $[\bar{H}]^\omega$ (see the proof of Theorem 3.6 in [4] for more details). This gives the equivalence of (A) with (B).

For the remainder of the proof, we use the standard fact[9] that if U is a normal V-ultrafilter on some $Z \subset \wp_{\omega_2}(H_\Gamma)$ (e.g., if U is generic for the forcing with a normal ideal) and $j : V \to_U \text{ult}(V, U)$ is the ultrapower, then

$$j \restriction H_\Gamma \in \text{ult}(V, U) \text{ and is represented by } [M \mapsto \sigma_M]_U, \text{ where } \sigma_M \text{ is the inverse of the Mostowski collapse of } M. \tag{8}$$

To see that (A) implies (D), let S be the stationary set which witnesses $DRP(\theta, Z^\theta)$, let G be generic for $(V, \wp(Z^\theta)/(NS \restriction Z^\theta))$ with $S \in G$, and let $j : V \to_G \text{ult}(V, G)$ be the generic ultrapower. Let $R \in V$ be a stationary subset of $[H_\theta]^\omega$; note that, by the definition of Z^θ, there are G-many models with $[H_\theta]^\omega$ and R as elements. By (8), together with Los's theorem and the fact that (A) implies (B), $\text{ult}(V, G)$ believes that $NS \restriction [j(H_\theta)]^\omega$ condenses correctly via j^*, where $j^* := j \restriction H_{(\theta^\omega)^+}$. In particular, $\text{ult}(V, G)$ believes that $(j^*)^{-1}(j(R)) = R$ is stationary.

(D) clearly implies (C).

Finally, we prove that (C) implies (A). Let G be generic for $(V, \mathbb{P}_I)$, and let $j : V \to_G \text{ult}(V, G)$ be the generic embedding. Then for every stationary $R \subset [H_\theta]^\omega$, since R remains stationary in $\text{ult}(V, G)$ and since $\text{ult}(V, G) \models$

"$j''H_{(\theta^\omega)^+} \cap j(H_\theta)$ is internally club," we have that $\mathrm{ult}(V, G) \models$ "$j(R)$ reflects to $j''H_\theta$." So by Los's theorem there are G-many structures M such that every $R \in M$ that is a stationary subset of $[H_\theta]^\omega$ reflects to $M \cap H_\theta$. Since the dual of I extends the club filter, this collection of M is stationary and witnesses $DRP(\theta, Z^\theta)$. □

Corollary 9 *Suppose that I is a normal ideal which concentrates on $IC_{\omega_1} \cap \wp_{\omega_2}(H_{(\theta^\omega)^+})$ such that $\mathbb{P}_I$ is proper. Then $DRP(\theta, IC_{\omega_1})$ holds.*

Proof This is an immediate corollary of part C of Theorem 8, the definition of properness, and the fact that stationarity is downward absolute from $V[G]$ to $\mathrm{ult}(V, G)$. □

We also state, without proof, a similar characterization for $wDRP(\mathrm{Unif}_{\omega_1})$ (here Unif_{ω_1} denotes the class of ω_1-sized M such that $cf(\sup(M \cap \lambda)) = \omega_1$ for every $\lambda \in M$ of uncountable cofinality). The proof is very similar to the proof of Theorem 8, except that one would instead use the fact that for $M \in \mathrm{Unif}_{\omega_1}$, if $S \in M$ is a stationary subset of $\theta \cap \mathrm{cof}(\omega)$, then S reflects to $\sup(M \cap \theta)$ if and only if V believes that $\sigma_M^{-1}(S)$ is stationary in $\sigma_M^{-1}(\theta)$.

Theorem 10 *For regular $\theta \geq \omega_2$, let Z^θ denote the collection of $M \in \wp_{\omega_2}(H_{(\theta^\omega)^+})$ such that $M \cap H_\theta \in \mathrm{Unif}_{\omega_1}$. The following are equivalent:*

- *(A) $wDRP(\theta, Z^\theta)$;*
- *(B) there are stationarily many $M \in Z^\theta$ such that $NS \restriction \theta \cap \mathrm{cof}(\omega)$ condenses correctly via M;*
- *(C) there is a normal ideal I whose dual contains the club filter over Z^θ such that $\Vdash_{\mathbb{P}_I}$ "every stationary $R \subset \theta \cap \mathrm{cof}(\omega)$ from the ground model remains stationary in $\mathrm{ult}(V, \dot{G})$";*
- *(D) there is a stationary $S \subset Z^\theta$ such that $S \Vdash_{NS}$ "every stationary $R \subset \theta \cap \mathrm{cof}(\omega)$ from the ground model remains stationary in $\mathrm{ult}(V, \dot{G})$."*

4 Models of $DRP(\theta)$ With Ideal Forcings That Have a σ-Closed Dense Subforcing

In Foreman, Magidor, and Shelah [10] it was noted that if κ is supercompact, then $MA^{+\omega_1}(\sigma\text{-closed})$ holds in $V^{\mathrm{Col}(\omega_1,<\kappa)}$.[10] In that model, a special case of Foreman's duality theorem (see [8, Proposition 7.13]) implies that for every σ-closed poset $\mathbb{Q}$ and every λ, there is an ideal which concentrates on $S_{\mathbb{Q}} \cap \wp_{\omega_2}(H_\lambda)$ and such that the forcing with the ideal is equivalent to a σ-closed forcing. We give a rough sketch of the argument; more details and similar arguments appear in Section 5, where we obtain analogous results for PFA (but starting from much larger cardinals than a supercompact).

Let κ be supercompact, let $\mathbb{P} := \mathrm{Col}(\omega_1, < \kappa)$, and let G be $(V, \mathbb{P})$-generic. Let $\mathbb{Q}$ be a σ-closed forcing in $V[G]$, and let $\dot{\mathbb{Q}}$ be a name for it. Inside $V[G]$, pick a λ sufficiently large such that $\mathrm{Col}(\omega_1, \lambda)$ is forcing equivalent to $\mathbb{Q} \times \mathrm{Col}(\omega_1, \lambda)$ (see [5, Section 14]). Note that V and $V[G]$ compute $\mathrm{Col}(\omega_1, \lambda)$ the same way since they have the same ω sequences. Let $j : V \to M$ be a H_λ-supercompact ultrapower of V (i.e., the ultrapower map by some normal fine measure on $\wp_\kappa(H_\lambda)$). Since $\mathbb{P}$ has the κ-cc in V, then $j \restriction \mathbb{P} : \mathbb{P} \to j(\mathbb{P})$ is a regular embedding, and so the construction in Section 2.4 is applicable; inside $V[G]$ let $F(j)$ be as in Definition 7.

Since $\mathrm{Col}(\omega_1, \lambda)$ is forcing equivalent to $\mathbb{Q} \times \mathrm{Col}(\omega_1, \lambda)$ and $M[G]$ is closed under λ sequences from $V[G]$, then $M[G]$ sees the forcing equivalence. So

$M[G][H]$ always has a $(H_\lambda^{V[G]} = H_\lambda^{M[G]}, \mathbb{Q})$-generic; call it g. Then g generic clearly transfers via $\hat{j} \restriction H_\lambda^{V[G]}$ to a $(\hat{j}'' H_\lambda^{V[G]}, \hat{j}(\mathbb{Q}))$-generic $\tilde{g}$, and, moreover, $\hat{j} \restriction H_\lambda^{V[G]}$ is an element of $M[G][H]$. Thus $M[G][H]$ always believes that there is a $(\hat{j}'' H_\lambda^{V[G]}, \hat{j}(\mathbb{Q}))$-generic, and so by definition of $F(j)$ we have $V[G] \models S_\mathbb{Q} \in F(j)$.

Furthermore, suppose that $\dot{S} \in \hat{j}'' H_\lambda^{V[G]}$ is a $\hat{j}(\mathbb{Q})$-name for a stationary subset of ω_1; say $\dot{S} = \hat{j}(\dot{\bar{S}})$. Then $H_\lambda[G][g] = M_\lambda[G][g] \models$ "$\dot{\bar{S}}_g$ is stationary." Since λ is sufficiently large with respect to $\mathbb{Q}$, $M[G][g] \models$ "$\dot{S}_g$ is stationary" and thus so does $M[G][H]$, since $M[G][H]$ is a stationary set-preserving forcing extension of $M[G][g]$. From this it follows that $MA^{+\,\mathrm{Diag}}(\sigma\text{-closed})$ holds in $V[G]$.

By (7), forcing with $F(j)$ is equivalent (in $V[G]$) to forcing with $\mathrm{Col}(\omega_1, < j(\kappa))/G$. In particular, the forcing associated with $F(j)$ has a σ-closed dense subset. If the arbitrary $\mathbb{Q}$ we chose at the beginning was, say, $\mathrm{Col}(\omega_1, H_\theta)$ for some regular θ then $F(j)$ concentrates on the set $S_\mathbb{Q}^{+\omega_1}$, which consists of sets which witness $DRP(\theta)$ structures.

Finally, we observe that this model satisfies "ideal projections as forcing projections" (see Section 2.1) for a pair of ideals whose forcings are σ-closed. Suppose that $\kappa < \lambda \ll \lambda'$ and that U is a normal measure on $\wp_\kappa(\lambda')$ and $j_U : V \to M_U$, the ultrapower map. Let $\mathrm{proj}(U)$ be the projection of U to $\wp_\kappa(\lambda)$ and $j_{\mathrm{proj}(U)} : V \to M_{\mathrm{proj}(U)}$, the ultrapower map. In $V[G]$ consider the filters $F(U)$ and $F(\mathrm{proj}(U))$ (see Definition 7). We need to show that $FP(F(U), F(\mathrm{proj}(U)))$ holds in $V[G]$. The simple but key observation is that $\mathrm{Col}^{M_{\mathrm{proj}(U)}}(\omega_1, < j_{\mathrm{proj}(U)}(\kappa))$ is a complete subforcing *in the sense of* $V[G]$ of $\mathrm{Col}^{M_U}(\omega_1, < j_U(\kappa))$, since all the listed models are closed under ω-sequences (so they all compute the same relevant Levy collapse posets), and $\mathrm{Col}(\omega_1, < j_{\mathrm{proj}(U)}(\kappa))$ is a regular subforcing of $\mathrm{Col}(\omega_1, < j_U(\kappa))$. This ensures that the canonical map $(F')^+ \to F^+$ given by $A \mapsto \{Z \cap \lambda \mid Z \in A\}$ is a forcing projection. We omit the details here, but a similar argument is given in Section 6. The point is that the ideal projection from $(F(U))^+ \to (F(\mathrm{proj}(U)))^+$ is the same as the composition of the following sequence of *forcing* projections. (It is important that the map numbered 4 in this list is a forcing projection in the sense of $V[G]$.) Here $\mathbb{P} := \mathrm{Col}(\omega_1, < \kappa)$ and $\mathbb{P}'', \mathbb{P}'$ are $j_U(\mathbb{P})$ and $j_{\mathrm{proj}(U)}(\mathbb{P})$, respectively:

(1) the dense embedding from $(F(U))^+ \to ro(\mathbb{P}''/G)$ from (7), given by $A \mapsto \| \check{A} \in \tilde{U}^{\dot{H}''} \|_{ro(\mathbb{P}''/G)}$;
(2) the forcing projection from $ro(\mathbb{P}''/G) \to ro(\mathbb{P}'/G)$ obtained from the forcing projection $\mathbb{P}''/G \to \mathbb{P}'/G$ (i.e., obtained via the map which restricts conditions in $\mathbb{P}''$ to their support on $j_{\mathrm{proj}(U)}(\kappa)$; recall that this is indeed a forcing projection in the sense of $V[G]$ by the remarks above);
(3) the isomorphism from (a dense subset of) $ro(\mathbb{P}'/G) \to (F(\mathrm{proj}(U)))^+$ obtained from (7) (i.e., the inverse of the map $B \mapsto \| \check{B} \in \widetilde{\mathrm{proj}(U)}^{\dot{H}'} \|_{ro(\mathbb{P}'/G)}$.

5 PFA and Proper Ideals on ω_2

In this section we construct a model of PFA which has many ideals I such that $\mathbb{P}_I$ is proper. Since there are known models of PFA where stationary reflection for subsets of $\omega_2 \cap \mathrm{cof}(\omega)$ fails (see Beaudoin [1]), and since the existence of an ideal

I with completeness ω_2 whose forcing is proper implies such stationary reflection,[11] we know that PFA does not imply the existence of ideals on ω_2 whose forcing is proper. In Section 3, however, we showed that DRP (which follows from $PFA^{+\omega_1}$) implies that there are ideals whose associated forcings satisfy a very weak form of properness. In this section we show that $PFA^{+\,\mathrm{Diag}}$ is consistent with the existence of many ideals[12] whose associated forcings are proper.

Our model of PFA also exhibits weaker versions of properties which are known to be inconsistent with PFA. Foreman and Magidor [9] proved that, under PFA, there is no presaturated ideal on ω_2 and that $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ fails (and that these facts are preserved under mild forcing extensions of PFA). In particular, there is no ideal on ω_2 that has some well-founded generic ultrapower $j_G : V \to_G \mathrm{ult}(V, G)$ where $\mathrm{ult}(V, G)$ is closed under sequences of length $< j_G(\omega_2)$ from $V[G]$. The following theorem shows that PFA *is* consistent with the existence of generic ultrapowers that are closed under sequences of length $j_G(\omega_2)$ *from the ground model*.

The theorem uses a *superhuge* cardinal.

Definition 11 Let $n \in \omega$, and let κ be a cardinal.

- κ is *n-huge* iff there is an elementary $j : V \to M$ where M is transitive, $\mathrm{crit}(j) = \kappa$, and M is closed under $j^n(\kappa)$ sequences. κ is *huge* iff it is 1-huge.
- κ is *super-n-huge* iff for every $\gamma > \kappa$ there is an n-huge embedding with critical point κ such that $j(\kappa) > \gamma$. κ is *superhuge* iff κ is super-1-huge.

See Kanamori [12] for more information about notions related to hugeness. In particular, κ is n-huge iff there is a κ-complete normal ultrafilter U over some $\wp(\lambda)$ and cardinals $\kappa = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda$ such that $\{x \subset \lambda \mid \mathrm{otp}(x \cap \lambda_{i+1}) = \lambda_i\} \in U$ for each $i < n$.

Theorem 12 *Suppose that there is a super-2-huge cardinal. Then there is a proper forcing extension which satisfies $PFA^{+\,\mathrm{Diag}}$ and has the following property.*

Let $\mathbb{Q}$ be any proper poset. Then for a proper class of inaccessible λ there is an ideal $I_{\lambda,\mathbb{Q}}$ such that

- *$\breve{I}_{\lambda,\mathbb{Q}}$ concentrates on $S^{\mathrm{Diag}}_{\mathbb{Q}} \cap \{M \subset H_\lambda \mid \mathrm{otp}(M \cap \lambda) = \omega_2$ and $M \cap \omega_2 \in \omega_2\}$ (i.e., $\breve{I}_{\lambda,\mathbb{Q}}$ concentrates on sets in $S^{\mathrm{Diag}}_{\mathbb{Q}}$ which witness $(\lambda, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$);*
- *the forcing associated with $I_{\lambda,\mathbb{Q}}$ is proper;*
- *whenever $j_G : V \to_G \mathrm{ult}(V, G)$ is a generic ultrapower by $I_{\lambda,\mathbb{Q}}$, then $\mathrm{ult}(V, G)$ is closed under $\lambda = j_G(\omega_2)$-sequences from V (though not necessarily from $V[G]$).*

Proof Let κ be a super-2-huge cardinal. We will do the standard construction, but for that we need a Laver function for huge embeddings.[13]

Fact 13 Let $n \in \omega$, and let $n \geq 1$. If κ is super-$(n+1)$-huge, then there is a Laver function $\mathrm{Lav} : \kappa \to V_\kappa$ for n-huge embeddings; that is, for every x and every λ there is an n-huge embedding j such that $cr(j) = \kappa$, $j(\kappa) \geq \lambda$, and $j(\mathrm{Lav})(\kappa) = x$.

This is well known, but we include a sketch of the proof, since there is a technical issue here that does not arise when constructing a Laver function for supercompact embeddings.[14] I also thank the anonymous referee for pointing out an error in the original version.

Proof Mimic the proof of Theorem 20.21 in [11]. Let us say that i is an $(\alpha, \geq \lambda)$-*n-huge* embedding iff i is n-huge, $cr(i) = \alpha$, and $i(\alpha) \geq \lambda$; similarly, we say that i is (α, λ)-*n-huge* if i is n-huge with critical point α and $i(\alpha) = \lambda$. For any α and any partial function $g : \alpha \to V_\alpha$, set the following:

$\lambda_{\alpha,g}$:≃ the least ordinal λ such that for some $x \in V_\lambda$, there is *no* $(\alpha, \geq \lambda)$-n-huge embedding i such that $i(g)(\alpha) = x$ (if such a λ exists). (9)

Recursively, define a partial function $f^* : \kappa \to V_\kappa$ by the following:

Assuming $f^* \restriction \alpha$ has been defined,

(1) if $\lambda_{\alpha, f^* \restriction \alpha}$ is not defined, then $f^*(\alpha)$ is not defined either;
(2) if $\lambda_{\alpha, f^* \restriction \alpha}$ is defined but none of the x witnessing this fact are in V_κ, then again $f^*(\alpha)$ is not defined; (10)
(3) otherwise $f^*(\alpha)$ is defined to be some $x \in V_\kappa$ which witnesses that $\lambda_{\alpha, f^* \restriction \alpha}$ is defined.

Note that if $f^*(\alpha)$ is defined, say $f^*(\alpha) = x_\alpha$, then $x_\alpha \in V_\kappa$ and there is *no* $(\alpha, \geq \lambda_{\alpha, f^* \restriction \alpha})$-$n$-huge embedding i such that $i(f^* \restriction \alpha)(\alpha) = x_\alpha$.

Suppose for a contradiction that κ is super-$(n+1)$-huge and that there is no Laver function from $\kappa \to V_\kappa$ for n-huge embeddings. So $\lambda_{\kappa,f}$ is defined for every partial $f : \kappa \to V_\kappa$, and, in particular,

$$\lambda_{\kappa, f^*} \text{ is defined (in } V). \tag{11}$$

By super-$(n+1)$-hugeness of κ, there is some $(n+1)$-huge embedding $j : V \to M$ such that $cr(j) = \kappa$ and $j(\kappa) > \lambda_{\kappa, f^*}$.

Claim 14 *λ^M_{κ, f^*} and $j(f^*)(\kappa)$ are defined, and $\lambda^M_{\kappa, f^*} \leq \lambda^V_{\kappa, f^*}$.*

Proof By (11) and the definition of λ_{κ, f^*}, there is some $x^* \in V_{\lambda_{\kappa, f^*}}$ which witnesses (in V) that λ_{κ, f^*} is defined. Since $\lambda_{\kappa, f^*} < j(\kappa)$ and M is closed under $j(\kappa)$ sequences, then this x^* is an element of $V^M_{j(\kappa)}$.

Set $\lambda^* := \lambda^V_{\kappa, f^*}$. Note that since f^* maps into V_κ, then $j(f^*) \restriction \kappa = f^*$. We will show that $M \models$ "There is no $(\kappa, \geq \lambda^*)$-n-huge embedding i such that $i(f^*)(\kappa) = x^*$." If we manage to prove this, then since $x^* \in V_{\lambda^*} = V^M_{\lambda^*}$ and $\lambda^* < j(\kappa)$ (by choice of j), then $j(f^*)(\kappa)$ will indeed be defined (since then x^* is an element of $V^M_{j(\kappa)}$ and thus we would be in the last clause, rather than the middle clause, in the definition (10) of $j(f^*)$ inside M). Moreover, this will also show that $\lambda^M_{\kappa, f^*} \leq \lambda^*$ (by minimality in the definition of the $\lambda_{(-,-)}$ function as defined in M). Note that x^* need not be equal to $j(f^*)(\kappa)$, but this is not necessary, as we only are trying to show that $j(f^*)(\kappa)$ is defined.

So suppose for a contradiction that $M \models$ "There *is* some $(\kappa, \geq \lambda^*)$-n-huge embedding i such that $i(f^*)(\kappa) = x^*$"; the quoted statement is Σ_2 in parameters κ, λ^*, f^*, x^*. Since κ is super-$(n+1)$-huge, then, in particular, κ is supercompact in V and $j(\kappa)$ is supercompact in M. Then $V^M_{j(\kappa)} \prec_{\Sigma_2} M$ (see Kanamori [12, Proposition 22.3]). So $V^M_{j(\kappa)} = V_{j(\kappa)} \models$ "There is some $(\kappa, \geq \lambda^*)$-n-huge embedding i such that $i(f^*)(\kappa) = x^*$." Then there is some $\eta \in [\lambda^*, j(\kappa))$ and a normal measure $U \in V_{j(\kappa)}$ such that $U \subset \wp\wp(\eta)$ codes this n-huge embedding; but since $j(\kappa)$ is inaccessible in V and $\eta < j(\kappa)$, then U also codes an n-huge embedding from the point of view of V.[15] This contradicts the fact that (in V) there is no $(\kappa, \geq \lambda^*)$-n-huge embedding i such that $i(f^*)(\kappa) = x^*$. □

So by Claim 14 we may set $\bar{x} := j(f^*)(\kappa)$ and $\bar{\lambda} := \lambda^M_{\kappa, f^*}$. We reach a contradiction by showing that $M \models$ "There is some $(\kappa, \geq \bar{\lambda})$-$n$-huge embedding i such that $i(f^*)(\kappa) = \bar{x}$." This is the first point where we use the $(n+1)$-hugeness of j (up to now we have only used 1-hugeness of j). Let U be the n-huge ultrafilter derived from j; that is, set $\kappa_n := j^n(\kappa)$ and $U := \{A \subset \wp(\kappa_n) \mid j''\kappa_n \in j(A)\}$. By standard arguments, if $i_U : V \to N$ is the ultrapower, then $i_U(f^*)(\kappa) = j(f^*)(\kappa) = \bar{x}$ and $i_U(\kappa) = j(\kappa) > \lambda_{\kappa, f^*} \geq \bar{\lambda}$. Since M is closed under $j^{n+1}(\kappa)$-sequences from V, then $U \in M$ and $i_U \restriction M$ is the ultrapower map as computed in M; so, in particular, $M \models$ "$i_U(f^*)(\kappa) = \bar{x}$ and $i_U(f^*)(\kappa) > \bar{\lambda}$," which is a contradiction. □

We note that the assumptions of Fact 13 can be weakened a bit: in the proof, we only needed the embedding $j : V \to M$ to have the property that the n-huge ultrafilter derived from j is an element of M. For this it would suffice to assume that κ is "super-$(n+1)$-almost-huge" (or in fact much less, though more than super-n-hugeness seems to be required for the argument).

Let $\mathbb{P}$ be the standard Baumgartner forcing to produce a model of PFA, but using the Laver function from Fact 13. So $\mathbb{P}$ is a countable support iteration of length κ where for $\alpha < \kappa$, the αth component is the $(\mathbb{P} \restriction \alpha)$th evaluation of Lav$(\alpha)$, *if* $\mathbb{P} \restriction \alpha$ forces Lav(α) to be a proper forcing (see [11] for details). Let G be $(V, \mathbb{P})$-generic. Let $\mathbb{Q} \in V[G]$ be a proper poset, let λ be a (regular) cardinal, let $\dot{Q} \in V$ be a name for $\mathbb{Q}$, and let $j : V \to N$ be a huge embedding with critical point κ such that $j(\kappa) > \lambda$ and $\dot{Q} = j(\mathrm{Lav})(\kappa)$.

In $V[G]$ let F_j be as in Definition 7. We need to show the following:

(1) forcing with $(F_j)^+$ is a proper forcing;
(2) forcing with $(F_j)^+$ yields a generic ultrapower which is closed under $\pi(\omega_2)$ sequences from the ground model, where π is the generic ultrapower embedding;
(3) F_j concentrates on $S^{\mathrm{Diag}}_{\mathbb{Q}}$.

Now N believes that $j(\mathbb{P})$ is a countable support iteration of proper forcings, where each forcing in the iteration has size $< j(\kappa)$. Since N is closed under $< j(\kappa)$ sequences, then for every $\xi < j(\kappa)$, N is correct whenever it believes that $j(\mathbb{P}) \restriction \xi$ forces $j(\mathrm{Lav})(\xi)$ to be proper; also, N correctly computes the countable support iteration of $j(\mathrm{Lav})$.[16] Thus $V \models$ "$j(\mathbb{P})$ is a countable support iteration of proper forcings and is thus proper." Then we use the following fact.[17]

Fact 15 If $\mathbb{R}$ is a countable support iteration of proper forcings, then for every $\alpha < lh(\mathbb{R})$: $\mathbb{R}_\alpha \Vdash$ "$\mathbb{R}/\dot{G}_\alpha$ is proper."

Then with $j(\mathbb{P})$ playing the role of the $\mathbb{R}$ from Fact 15, we have the following in $V[G]$:

$$j(\mathbb{P})/(j''G) = j(\mathbb{P})/G \text{ is proper.} \tag{12}$$

Then (7) and (12) imply that $\mathbb{P}_{F_j}$ is proper in $V[G]$.

We want to see that $S^{\mathrm{Diag}}_{\mathbb{Q}} \in F_j$. (This part only requires the supercompactness of κ and is just the standard argument due to Baumgartner.) Let H be any $(V[G], j(\mathbb{P})/G)$-generic. $V[G]$ and $N[G]$ have sufficient agreement so that $N[G] \models$ "$\mathbb{Q}$ is a proper forcing." So by the definition of Baumgartner's forcing, the κth component of H is generic for $(N[G], \mathbb{Q})$ and, equivalently, generic for $(V[G], \mathbb{Q})$. Let g denote this component. Now $\hat{j}^* := \hat{j}_{G*H} \restriction H_\lambda^{V[G]} \in N[G][H]$; this is standard (it is the inverse of the collapsing map obtained from $j''H_\lambda^V$ and $j''G$). Thus $N[G][H]$ sees that g can be transferred to a $(\hat{j}''_{G*H}H_\lambda[G], \hat{j}_{G*H}(\mathbb{Q}))$-generic; call this generic $\tilde{g}$. In addition, if $\dot{S} = \hat{j}_{G*H}(\dot{\bar{S}})$ is any element of $\hat{j}''_{G*H}H_\lambda[G]$ which is a $\hat{j}_{G*H}(\mathbb{Q})$-name for a stationary subset of ω_1, then $\dot{\bar{S}}_g$ is stationary in $H_\lambda^V[G][g] = H_\lambda^N[G][g]$. Since λ is sufficiently large with respect to $\mathbb{Q}$ this implies that $\dot{\bar{S}}_g$ is stationary in $N[G][g]$, and since $N[G][H]$ is a stationary set-preserving forcing extension of $N[G][g]$ (by Fact 15), $\dot{\bar{S}}_g$ is still stationary in $N[G][H]$.

Thus we have shown that, for arbitrary H, the model $\hat{j}''_{G*H}H_\lambda[G]$ is an element of $\hat{j}_{G*H}(S_{\mathbb{Q}}^{\text{Diag}})$; so $S_{\mathbb{Q}}^{\text{Diag}} \in F_j$ by the definition of F_j. Also note that since $j''\lambda \in N$ and has order type $\lambda = j(\kappa) = \aleph_2^{N[G][H]}$, then $N[G][H] \models \text{otp}(\hat{j}''_{G*H}H_\lambda[G] \cap ORD) = \aleph_2$ and $\hat{j}''_{G*H}H_\lambda[G] \cap \aleph_2 \in \aleph_2$. So F_j concentrates on sets which witness $(\lambda, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$.

The closure of the generic ultrapower is proved as follows. We say that $f : \eta \to ORD$ is a *function in* $V[G]$, where $\eta \leq i_K(\omega_2)$ and K is any generic for forcing with F_j^+ over $V[G]$, and that i_K is the generic ultrapower map. By (7) there is some H such that $i_K = \hat{j}_{G*H}$ and $\hat{j}_{G*H} : V[G] \to N[G][H]$ is the generic ultrapower. Say that $f = \dot{f}_G$, and for each $\xi < \eta$ let A_ξ be a maximal antichain of conditions in $\mathbb{P}$ which decide the value of $\dot{f}(\check{\xi})$; note each $A_\xi \in V_\kappa \subset N$, so by the hugeness of j we have that $\langle A_\xi \mid \xi < \eta \rangle$ is an element of N. Thus f is an element of $N[G]$. □

6 PFA Plus "Ideal Projections as Forcing Projections"

Finally, we prove that if we start with a super-3-huge cardinal, then the ideals in Theorem 12 can instantiate very special forms of "ideal projections as forcing projections." This is a property which falls somewhere between precipitousness and saturation; moreover, certain instances of "ideal projections as forcing projections" are *equivalent* to saturation of the projection ideal (see [3]). In particular, we obtain a model of PFA where $FP(I', I)$ for some I' whose dual concentrates on $[\Omega(I)]^\omega$-faithful models;[18] this property is stronger than the properness of $\mathbb{P}_I$.

Suppose that κ is super-3-huge (see Definition 11). By Fact 13, there is a Laver function Lav for 2-huge embeddings; that is, for every x and every λ there is a $\nu \geq \lambda$ and a 2-huge (κ, ν)-embedding i such that $i(\text{Lav})(\kappa) = x$. Now use this Laver function in the Baumgartner forcing $\mathbb{P}$ for PFA, as in the proof of Theorem 12. Now consider some normal ultrafilter U on $\{X \subset \kappa'' \mid \text{otp}(X) = \kappa' \text{ and } \text{otp}(X \cap \kappa') = \kappa\}$; note that U gives rise to a 2-huge embedding. Let $j_U : V \to M_U$; note that $j_U(\kappa) = \kappa'$ and that M_U is closed under $j_U^2(\kappa) = \kappa''$ sequences.

Let proj(U) be the projection of U to a (κ, κ')-huge ultrafilter,[19] and let $j_{\text{proj}(U)} : V \to_{\text{proj}(U)} M_{\text{proj}(U)}$. Then j_U factors as $k \circ j_{\text{proj}(U)}$ such that $k \restriction \kappa' + 1 = id$. Let $\mathbb{P}' := j_{\text{proj}(U)}(\mathbb{P})$. Note that

$$j_U(\mathbb{P}) = j_{\text{proj}(U)}(\mathbb{P}) = \mathbb{P}'. \tag{13}$$

Let G be $(V, \mathbb{P})$-generic. In $V[G]$ let $\dot{U}''$ denote the $\mathbb{P}'/G$-name for the $V[G]$-ultrafilter on $[\kappa'']^{\kappa'}$ derived from $(\hat{j}_U)_H$ (see Section 2.4), and let $\dot{U}''$ denote the

$\mathbb{P}'/G$-name for the $V[G]$-ultrafilter on $[\kappa']^\kappa$ derived from $(\hat{j}_{\text{proj}(U)})_H$ (where H is $(V[G], \mathbb{P}'/G)$-generic). Let $F'' := F(U)$ and $F' := F(\text{proj}(U))$ be as in Definition 7.

Consider the map on $(F'')^+$ defined by $A \mapsto \text{proj}(A, \kappa') := \{Z \cap \kappa' \mid Z \in A\}$. We want to see that this is the ideal projection of F'' onto F' and that this is also a projection in the sense of forcing. So assume that $\mathcal{A}$ is a maximal antichain in $(F')^+$ and that $A \in (F'')^+$. We need to show that there is some $B \in \mathcal{A}$ such that A has F''-positive intersection with $\text{Lift}(B) := \{Z \in [\kappa'']^{\kappa'} \mid Z \cap \kappa' \in B\}$.

Let H be $(V[G], \mathbb{P}'/G)$-generic such that $A \in \dot{U}''_H$ (such a generic exists since $A \in (F'')^+$, by the definition of F''). Then (note that k fixes $\mathbb{P}'$) k can be extended to $\tilde{k} : M_{\text{proj}(U)}[G][H] \to M_U[G][H]$ and $\hat{j}_U^{G*H} = \tilde{k} \circ \hat{j}^{G*H}_{\text{proj}(U)}$, and the map $\hat{j}^{G*H}_{\text{proj}(U)}$ is (from the point of view of $V[G][H]$) the ultrapower of $V[G]$ by the projection of $\dot{U}''_H$ to $[\kappa']^\kappa$. By (7), $(F')^+$ is forcing equivalent (in $V[G]$) to $\mathbb{P}'$ and so this ultrapower is a generic ultrapower of $V[G]$ by $(F')^+$; so, in particular, the ultrafilter $\text{proj}(\dot{U}''_H)$ meets $\mathcal{A}$; say that B is the element of $\mathcal{A}$ which is in $\text{proj}(\dot{U}''_H)$. Let $B'' \in \dot{U}''_H$ be such that $B = \text{proj}(B'')$. Then $B'' \cap A$ is F''-positive. This completes the proof that $FP(F'', F')$ holds.

Finally, by the same argument as in the proof of Theorem 12, the almost-hugeness of the embeddings and the fact that $\mathbb{P}$ is a countable support iteration of proper forcings ensures that $\mathbb{P}'$ is proper from the point of view of $V[G]$. Moreover, we have that $(F'')^+$ and $(F')^+$ are both isomorphic to a dense subset of $\mathbb{P}'$. In particular, forcing with $(F'')^+$ is proper and so F'' must concentrate on $[\Omega(F')]^\omega$-faithful (in fact $[ORD]^\omega$-faithful) models.

Notes

1. This is a general fact about forcing axioms, not just PFA; see Section 2.

2. For example, if $Z = [H_\theta]^\omega$, then $A \in NS \restriction Z$ iff $A \subset [H_\theta]^\omega$ and there is some $F : [H_\theta]^{<\omega} \to H_\theta$ such that no element of A is closed under F.

3. This is defined in [7].

4. If $\mathbb{P}$ satisfies certain requirements, this often follows from $MA^{+\text{Diag}}(\mathbb{P})$ for sufficiently small λ; for example, if $\mathbb{P}$ is σ-closed and $\Vdash_{\mathbb{P}}$ "$|H_\theta^V| = \omega_1$," then $MA^{+\omega_1}(\mathbb{P})$ implies this generalized version for all small λ.

5. The reason for this is that we want the filter to concentrate on elementary substructures of the generic extension and thus cannot hope that a measure one set of such structures has any intersection at all with the ground model.

6. If $A \subset \mathbb{P}$ is a maximal antichain, then $M \models$ "$j(A)$ is a maximal antichain in $j(\mathbb{P})$" and this clearly then holds in V as well. Since $|A| < \kappa = cr(j)$, $j(A) = j[A]$; so $j[A]$ is maximal in $j(\mathbb{P})$.

7. Whenever $j : M \to N$ is an embedding (not necessarily definable in M), $d \in M$, and $j''d \in N$, then the ultrafilter $\{A \in M \mid j''d \in j(A)\}$ will always be normal with respect to M.

8. Thus the name; in fact, this characterizes IC_{ω_1}.

9. This is essentially Claim 2.26 of [7], though that claim is specifically about generic ultrapowers by normal ideals. Roughly, normality of U with respect to functions from V ensures that the $\in_U$-extension of $[id]_U$ is equal to $j_U'' H_\Gamma$. This, in turn, implies that the transitive collapse of $[id]_U$ *as computed in* $\text{ult}(V, U)$ is in the well-founded part of $\text{ult}(V, U)$. Assuming that the well-founded part of $\text{ult}(V, U)$ has been transitivized, this transitive collapse is equal to H_Γ and the inverse of the collapsing map is $j \restriction H_\Gamma$.

10. Here $\text{Col}(\mu, \theta)$ denotes the usual Levy collapse to add a surjection from μ onto θ as in Jech [11, (15.18), p. 238]. $\text{Col}(\mu, < \theta)$ denotes the version which adds surjections from μ to η for every $\eta < \theta$, as in [11, (15.19), p. 238].

11. This is well known. If an ideal forcing (where the ideal has, say, completeness ω_2) is proper and G is generic for the forcing, then for any $S \subset \omega_2 \cap \text{cof}(\omega)$ in V, S remains stationary in $V[G]$ and thus $S = j(S) \cap \omega_2$ is stationary in $\text{ult}(V, G)$, where $j : V \to_G \text{ult}(V, G)$. So by elementarity, V believes that every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ reflects.

12. In the sense that for every proper $\mathbb{Q}$ there are proper class many ideal forcings whose ideals concentrate on $S_{\mathbb{Q}}$.

13. Corazza [2] considered Laver functions for super-almost-huge embeddings, starting from a super-1-huge cardinal. An argument similar to the one below using only a Laver function for almost huge cardinals would still give proper ideals, but we want our ideals to concentrate on models which witness $(\theta, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$; this is why we instead use a Laver function for huge embeddings.

14. This technical issue occurs in the proof of Claim 14. The issue is that if U is a (κ, λ)-supercompact ultrafilter and $\mu < \lambda$, then the projection of U to $\wp_\kappa(\mu)$ is a (κ, μ)-supercompact ultrafilter. This need not be the case for ultrafilters witnessing hugeness.

15. And $V \models$ "i_U is $(\kappa, \geq \eta)$-n-huge and $i_U(f^*)(\kappa) = x^*$."

16. Note that these facts only use the *almost* hugeness of j.

17. The author is not aware of a reference for this fact, but it is apparently widely known. One way to prove it is to use the fact that $\mathbb{P}_\alpha$ has the ω-covering property to show that $V^{\mathbb{P}_\alpha}$ believes the following: "The tail forcing $\mathbb{P}/G_\alpha$ is forcing equivalent to my countable support iteration of the same iterands."

18. Here $\Omega(I)$ is some regular cardinal sufficiently large so that the properness, precipitousness, and so on of I^+ is correctly decided by $H_{\Omega(I)}$. Clearly, if there is an inaccessible $\mu > \text{trcl}(I)$, then $\Omega(I)$ can be taken to be strictly less than the least such μ; the precise value of $\Omega(I)$ is not important in the present application.

19. That is, $\text{proj}(U) = \{\{X \cap \kappa' \mid X \in A\} \mid A \in U\}$.

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