Notre Dame Journal of Formal Logic
Volume 53, Number 3, 2012

Transplendent Models: Expansions Omitting a Type

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Abstract We expand the notion of resplendency to theories of the kind $T + p\uparrow$, where T is a first-order theory and $p\uparrow$ expresses that the type p is omitted; both T and p are in languages extending the base language. We investigate two different formulations and prove necessary and sufficient conditions for countable recursively saturated models of PA.

1 Introduction

In the late seventies the notions of *recursive saturation* and *resplendency* were introduced by Barwise and Schlipf [1] and, independently, Ressayre [11] as a useful saturation notion weaker than full saturation with plenty of models for all recursive theories, and in all cardinalities, and many if not all of the pleasant properties of full saturation. Recursive saturation is particularly helpful in the context of models of arithmetic, but it has other applications too. For a long time it seemed that there were no other useful and significantly different variations on the idea of resplendency. This seemed in part due to the fact that recursive saturation and resplendency are closely allied with those recursive sets of formulas that are a consequence of Σ^1_1-sentences. This logic has very nice properties, and there do not seem to be many analogous logics with similar properties. Then in a paper on the automorphism group of recursively saturated models of PA (Peano Arithmetic; see Kaye, Kossak, and Kotlarski [7]) the notion of *arithmetical saturation* was discovered, and its elegant equivalent (for countable recursively saturated models of PA) that a model is arithmetically saturated if and only if there is an automorphism moving all nondefinable elements. In fact, Körner in [8] proved that every arithmetically saturated countable model of any first-order theory in a countable language has an automorphism moving every nonalgebraic element. This was discovered though there was no formulation of resplendency equivalent to arithmetical saturation in the case of countable models.

Received August 30, 2010; accepted August 27, 2011
2010 Mathematics Subject Classification: Primary 03C62; Secondary 03C30
Keywords: models of arithmetic, resplendent models, standard cut, satisfaction classes
 10.1215/00294527-1716739

If we examine the logical structure of the statement, "there is an automorphism moving all nondefinable elements," we see that this is a property stating that there is an expansion adding a function g to the model satisfying certain first-order properties (that it is an automorphism of the underlying structure) and omitting a type (realized by some x that is nondefinable and fixed by g). This naturally suggests the investigation of "extended Σ^1_1-sentences" of the form $\exists \bar{X}\ (T + p\uparrow)$, stating that there is an expansion satisfying a first-order theory T and omitting a type p, and analogies with recursive saturation and resplendency.

A second example of the same logical structure, again in the context of models of arithmetic, is that of the theory of an initial segment K closed under taking successors:

$$\forall x, y\big(x < y \wedge K(y) \to K(x)\big),$$
$$\forall x\big(K(x) \to K(x+1)\big),$$

where $K(x)$ is a new predicate symbol and the type $p(x)$ omitted is the one saying that x is a nonstandard element in the initial segment K:

$$\big\{K(x) \wedge x > n \;\big|\; n \in \mathbb{N}\big\}.$$

(So such K will be the standard cut $\mathbb{N}$.) On its own, this will always be satisfied in some expansion, but modifications of this example, as we will see, are even more powerful than arithmetic saturation. In fact, arithmetic saturation can be expressed in this form by adding to T the first-order statement saying that the initial segment K is strong (see Section 4). This is so since arithmetic saturation is equivalent to that $\mathbb{N}$ being strong under the assumption of recursive saturation.

This paper takes these new ideas and explores them in a general model-theoretic context. Although the bulk of the paper is model theoretic, we will necessarily touch on aspects of proof theory for the fragment of infinitary logic in which one can say that a type is omitted, computability, and descriptive set theory. The main notion is that of transplendent models (previously called "transcendent," but this—we have been told—could potentially be confused with Morley's notion of transcendental theory) which is the version of resplendent for such extended Σ^1_1-sentences. There are many interesting questions left open in this work. Some of the results in this paper can be found in Engström's doctoral thesis [4].

2 Preliminaries

In this paper we will only consider recursive first-order languages $\mathcal{L}$ and recursive language extensions, so if we have theories $T \supseteq T_0$, in languages $\mathcal{L} \supseteq \mathcal{L}_0$, respectively, then we shall tacitly assume that both languages are recursive and the set $\mathcal{L} \setminus \mathcal{L}_0$ of new symbols in the larger language together with their arities is also recursive. Thus, $\mathcal{L}$ is what is usually called a recursive extension of $\mathcal{L}_0$.

Similarly, all models and all cardinal numbers will be tacitly assumed to be infinite.

Types p are sets of formulas whose free variables are among some finite tuple of variables $\bar{x}$. When we want to indicate the variables we denote a type by $p(\bar{x})$. We make no a priori assumptions on completeness or consistency of types. The $\mathcal{L}_{\omega_1\omega}$-sentence

$$\forall \bar{x} \bigvee_{\psi(\bar{x}) \in p(\bar{x})} \neg\psi(\bar{x})$$

will be denoted by $p\uparrow$, where the universal quantifier binds all free variables in p; clearly $M \models p\uparrow$ iff M omits p.

Let us recall the definition of resplendency.

Definition 2.1 Let M be any structure for a language $\mathcal{L}_0$. We say that M is *resplendent* if for all finite or recursive theories T in a language $\mathcal{L}$ extending $\mathcal{L}_0 \cup \{\bar{a}\}$ for some finite tuple $\bar{a} \in M$ such that $T + \mathrm{Th}(M, \bar{a})$ is consistent, then there is an expansion M^+ of M such that $M^+ \models T$.

The existence of (countable) resplendent models and the Robinson joint consistency theorem are tightly connected (see, e.g., Kaye [6, Chapter 15]); by using the existence of resplendent models there is a short proof of the joint consistency theorem for recursive theories. If we instead consider our extended Σ^1_1-sentences of the form $\exists \bar{X}$ $(T + p\uparrow)$ the analogous version of the joint consistency theorem is false for simple reasons. It is possible that $\mathrm{Th}(M) + T$ and $\mathrm{Th}(M) + p\uparrow$ are both semantically consistent (i.e., have models), and yet T implies that some $\mathcal{L}_0$-type is realized, but $p\uparrow$ implies that it must be omitted. For an example of this take M to be the standard model of PA, let T be the theory $\{c > n \mid n \in \mathbb{N}\}$, where c is a new constant, and let $p(x) = \{x > n \mid n \in \mathbb{N}\}$.

To rescue the situation, we restrict our notion of consistency to only those extended Σ^1_1-sentences that say nothing about omitting types over the base language; that is, they are true even when we move to a more saturated model of $\mathrm{Th}(M)$. Definition 2.4 states this more formally.

Definition 2.2 A set $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$, is called a *Scott set* if it is a Boolean algebra closed under relative recursion and such that if τ is an infinite binary tree coded in $\mathcal{X}$ (using some fixed coding), then there is an infinite path through τ coded in $\mathcal{X}$.

Definition 2.3 If $\mathcal{X}$ is a Scott set, a model M is said to be

- *$\mathcal{X}$-saturated* if for every complete type $p(\bar{x}, \bar{a})$ over M the type is realized in M iff it is coded in $\mathcal{X}$,
- *weakly $\mathcal{X}$-saturated* if it is recursively saturated and $\mathcal{Y}$-saturated for some $\mathcal{Y} \supseteq \mathcal{X}$.

Definition 2.4 Let $T_0 \subseteq T$ be theories in languages $\mathcal{L}_0 \subseteq \mathcal{L}$, and let $p(\bar{x})$ be a type in the language $\mathcal{L}$ of T.

(i) $T + p\uparrow$ is *$\mathcal{X}$-consistent* over T_0 if there are a model $N \models T_0$ which is weakly $\mathcal{X}$-saturated and an expansion of N satisfying $T + p\uparrow$. Furthermore, $T + p\uparrow$ is *fully consistent* over T_0 if $T + p\uparrow$ is *$\mathcal{P}(\mathbb{N})$-consistent* over T_0.

(ii) Given a model M we say that $T + p\uparrow$ (which may include finitely many parameters $\bar{a}$ from M) is *fully consistent* over M if it is fully consistent over $\mathrm{Th}(M, \bar{a})$. In other words, $T + p\uparrow$ is fully consistent over M iff there are an ω-saturated model N of $\mathrm{Th}(M)$ and an expansion of N satisfying $T + p\uparrow$.

In many cases, a model M has a distinguished Scott set, the *standard system* $\mathrm{SSy}(M)$ of the model. Such cases include models of set theory, arithmetic, and also recursively saturated models of rich theories (see Kaye [6]). In other cases, although there may not be a unique or distinguished Scott set, there may be some other appropriate Scott set. We recall the definition of a rich theory.

Definition 2.5 A theory T in a recursive language is *rich* if there is a recursive sequence of formulas $\varphi_k(x)$, $k \geq 0$, such that for any disjoint finite sets $X, Y \subset \mathbb{N}$,

$$T \vdash \exists x \Big(\bigwedge_{k \in X} \varphi_k(x) \wedge \bigwedge_{k \in Y} \neg\varphi_k(x) \Big).$$

Definition 2.6 If M is a recursively saturated model of a rich theory, then the *standard system of M* is defined by

$$\mathrm{SSy}(M) = \big\{ A \subseteq \mathbb{N} \;\big|\; \forall k\ (k \in A \leftrightarrow M \models \varphi_k(a)) \text{ for some } k \in M \big\}.$$

Definition 2.7 Let M be any $\mathcal{L}_0$-structure, and let $\mathcal{X}$ be a Scott set. We say that M is *$\mathcal{X}$-transplendent* if for all $T, p(\bar{x}) \in \mathcal{X}$ in some language $\mathcal{L} \supseteq \mathcal{L}_0 \cup \{\bar{a}\}$ (where $\bar{a} \in M$ is finite) such that $T + p\uparrow$ is fully consistent over M there is an expansion M^+ of $(M, \bar{a})$ such that $M^+ \models T + p\uparrow$ and $\mathrm{Th}(M^+, \bar{a}) + p\uparrow$ is fully consistent over M.

If we remove the condition that $\mathrm{Th}(M^+, \bar{a}) + p\uparrow$ is fully consistent over M we get a similar notion; however, it is not known to us if this gives us the same notion or something weaker. For the proof of Theorem 3.8 to go through we need to define transplendence as above.

Proposition 2.8 *If M is a countable model satisfying the definition of $\mathcal{X}$-transplendence for $\mathcal{L}$ not involving parameters from M (so $\mathcal{L} \supseteq \mathcal{L}_0$), then M is $\mathcal{X}$-transplendent.*

Proof Assume that M is a countable parameter-free $\mathcal{X}$-transplendent model. From the definition it follows that M is parameter-free resplendent and so is resplendent, since by a satisfaction class argument any countable parameter-free resplendent model is resplendent. Therefore, any such model is homogeneous.

By replacing the parameters $\bar{a}$ in $T + p\uparrow$ by new constant symbols and adding the complete $\mathcal{L}_0$-type of $\bar{a}$ to T we can find an expansion of M satisfying this new theory. Let $\bar{b} \in M$ be the interpretation of the constant symbols in the expansion, and let g be an automorphism of M taking $\bar{b}$ to $\bar{a}$. The image under g of the expansion will satisfy $T + p\uparrow$. □

Definition 2.9 We say that a model M is *transplendent* if it is $\mathcal{X}$-transplendent for some Scott set $\mathcal{X}$.

Note that any $\mathcal{X}$-transplendent structure is weakly $\mathcal{X}$-saturated. Thus, in the case where M is transplendent and has a well-defined standard system, M is $\mathcal{X}$-transplendent iff $\mathcal{X} \subseteq \mathrm{SSy}(M)$.

3 Existence of Transplendent Models

Our first remark is that transplendent models exist. We will give below a characterization of the transplendent models among the countable recursively saturated models of first-order arithmetic in terms of closure properties of the standard system.

We start off by finding a sufficient condition on the standard system for the existence of expansions omitting a specific type. This result will then be used to prove that there are many countable transplendent models of any rich theory.

Let M be a countable recursively saturated $\mathcal{L}_0$-model of a rich theory, and let $T, p(\bar{x}) \in \mathrm{SSy}(M)$ be a theory and a type in a language $\mathcal{L} \supseteq \mathcal{L}_0(\bar{a})$, where $\bar{a}$ are finitely many parameters from M. Suppose also that $T + p\uparrow$ is fully consistent over M.

We will prove that under certain conditions on $\mathrm{SSy}(M)$ there exists an expansion of M satisfying $T + p\uparrow$. The proof is a Henkin construction, but let us formulate it in the language of model-theoretic forcing.

Definition 3.1 Given a countable $\mathcal{L}_0$-model M,

- a *notion of forcing in* $\mathcal{L}$ consists of a set of (forcing) conditions which are sets of sentences in the language $\mathcal{L}(M)$ consistent with $\mathrm{Th}(M, a)_{a \in M}$;
- a *forcing property* is a property of conditions;
- a forcing property P is *dense* if for every condition S there is a condition $S' \supseteq S$ satisfying P;
- a *filter* F is a set of conditions such that if $S' \subseteq S$ are conditions and $S \in F$, then $S' \in F$, and for any $S_1, S_2 \in F$ there is a condition $S \supseteq S_1 \cup S_2$ in F;
- a filter F *meets a property* P if there is $S \in F$ satisfying P;
- the condition S satisfies the *witness property* for $\varphi(x)$, denoted $W_{\varphi(x)}$, if either $\neg\exists x \varphi(x) \in S$ or there is $a \in M$ such that $\varphi(a) \in S$;
- the condition S satisfies the *completeness property* for φ, denoted C_φ, if either $\varphi \in S$ or $\neg\varphi \in S$.

Theorem 3.2

(a) *Given a notion of forcing for M and countably many dense properties there is a filter meeting all given properties.*

(b) *Furthermore, if the filter meets the witness and the completeness properties for every formula, then there is an expansion of M satisfying all conditions in the filter.*

Proof

(a) By using the denseness we can choose a sequence $S_0 \subseteq S_1 \subseteq \cdots$ of conditions such that S_i satisfies the ith property. Let F be the set of conditions S such that there is $S_i \supseteq S$.

(b) It is easy to see that the $\mathcal{L}_0$-reduct of the canonical model of the union of F is isomorphic to M. □

Returning to the existence of transplendent models we let the forcing conditions be finite sets S of sentences in the language $\mathcal{L}(M)$ such that $T + S + p\uparrow$ is fully consistent over M.

For $m, \bar{b} \in M$ define the following (countably many) properties of forcing conditions S:

- P_m: $m = m \in S$;
- $P_{\bar{b}}$: for some $\psi(\bar{x}) \in p(\bar{x})$ we have $\neg\psi(\bar{b}) \in S$.

Lemma 3.3

(1) *P_m, C_φ, and $P_{\bar{b}}$ are all dense.*

(2) *Also, given the extra condition on $\mathrm{SSy}(M)$ that for any formula $\psi(c)$ in the language $\mathcal{L}(\bar{b}, c)$ (where c is a new constant symbol and $\bar{b} \in M$ are parameters) such that $T + \psi(c) + p\uparrow$ is fully consistent over M, there is a complete theory $S_c \in \mathrm{SSy}(M)$ in the language $\mathcal{L}(\bar{b}, c)$ such that $\psi(c) \in S_c$ and $T + S_c + p\uparrow$ is fully consistent over M; then $W_{\varphi(x)}$ is dense for $\varphi(x)$ in $\mathcal{L}(\bar{b})$.*

Proof P_m: Given a condition S with parameters $\bar{b}$ we have that $T + S + p{\uparrow}$ is fully consistent over $\mathrm{Th}(M, \bar{a}, \bar{b})$, meaning that there is an ω-saturated model N of $\mathrm{Th}(M, \bar{a}, \bar{b})$ with an expansion N^+ satisfying $T + S + p{\uparrow}$. Choose some element $n \in N$ that realizes the type $\mathrm{tp}^M(m/\bar{a}, \bar{n})$. Then (N, n) is an ω-saturated model of $\mathrm{Th}(M, \bar{a}, \bar{b}, m)$ with an expansion (N^+, n) satisfying $T + S + p{\uparrow}$. Thus $T + S + p{\uparrow}$ is fully consistent over $\mathrm{Th}(M, \bar{a}, \bar{b}, m)$, and so $T + S + m = m + p{\uparrow}$ is also, proving that $S + m = m$ is a condition.

C_φ: We may assume that all the parameters of φ already occur in S. Thus either $T + S + \varphi + p{\uparrow}$ or $T + S + \neg\varphi + p{\uparrow}$ is fully consistent over M, and either $S + \varphi$ or $S + \neg\varphi$ is a condition.

$P_{\bar{b}}$: We may again assume that all the parameters $\bar{b}$ already occur in S. Since $T + S + p{\uparrow}$ is fully consistent over M we have that $T + S + \neg\psi(\bar{b}) + p{\uparrow}$ is fully consistent over M for some $\psi(\bar{x}) \in p(\bar{x})$.

$W_{\varphi(x)}$: As above we may assume that all the parameters of $\varphi(x)$ occur in S and that either $\exists x\varphi(x)$ or $\neg\exists x\varphi(x)$ is in S. We need to prove that if $\exists x\varphi(x) \in S$, then $S + \varphi(m)$ is a condition for some $m \in M$.

It should be obvious that $T + S + \varphi(c) + p{\uparrow}$ is fully consistent over M, where c is a new constant symbol. By the assumption on $\mathrm{SSy}(M)$ there is a complete theory S_c including $S + \varphi(c)$ such that $T + S_c + p{\uparrow}$ is fully consistent over M.

Let $\bar{b}$ be all parameters occurring in S. We have $\varphi(d) \in S_c$ for some parameter $d \in \bar{b}$; in this case $S + \varphi(d)$ is a condition. Otherwise $c \neq d \in S_c$ for every $d \in \bar{b}$. Let $q(x) = \{\psi(x) \mid \psi(c) \in S_c, \psi \in \mathcal{L}(\bar{b})\}$ be the restriction of S_c to the language $\mathcal{L}(\bar{b})$. It is clear that $q(x)$ is a coded type over M and so is realized by, say, $m \in M$. Clearly $m \neq d$ for any parameter d in the language of S_c, so if $S_c[m/c]$ is S_c with the constant replaced by the parameter m, then $T + S_c[m/c] + p{\uparrow}$ is fully consistent over $\mathrm{Th}(M, \bar{b})$, and since $\mathrm{Th}(M, \bar{b}, m) = q(m) \subseteq S_c[m/c]$ it should be clear that $T + S_c[m/c] + p{\uparrow}$ is fully consistent over $\mathrm{Th}(M, \bar{b}, m)$ and thus over M; that is, $S + \varphi(m)$ is a condition. □

Given that for any forcing condition S there is a completion $S_c \in \mathrm{SSy}(M)$ of S such that $T + S_c + p{\uparrow}$ is fully consistent over M, let F be a filter meeting all countably many dense properties $W_{\varphi(x)}, C_\varphi, P_a, P_{\bar{b}}$, and let M^+ be the canonical model of the union of the filter. It is easy to see that $M^+ \models T + p{\uparrow}$ and that the $\mathcal{L}_0(\bar{a})$-reduct of M^+ is (isomorphic to) $(M, \bar{a})$. Thus there is an expansion of $(M, \bar{a})$ satisfying $T + p{\uparrow}$.

Definition 3.4 A Scott set $\mathcal{X}$ is *closed* if for any $T_0, T, p \in \mathcal{X}$ such that $T + p{\uparrow}$ is fully consistent over T_0 there is a completion $T_c \in \mathcal{X}$ of T such that $T_c + p{\uparrow}$ is fully consistent over T_0.

Combining the results above with this definition we get the following.

Theorem 3.5 *If M is a countable recursively saturated model of a rich theory such that* $\mathrm{SSy}(M)$ *is closed, then M is transplendent.*

Proof Given T and p as in the definition of transplendency, start by replacing T with a complete $T' \in \mathrm{SSy}(M)$ such that $T' + p{\uparrow}$ is fully consistent over M. Then do the construction of M^+ above. We know that $T' = \mathrm{Th}(M^+, \bar{a})$, and so $\mathrm{Th}(M^+, \bar{a})$ is fully consistent over M. □

These models do indeed exist, as the following easy proposition shows.

Proposition 3.6 *Any infinite set $\mathcal{X}_0 \subseteq \mathcal{P}(\mathbb{N})$ can be extended to a closed Scott set $\mathcal{X} \supseteq \mathcal{X}_0$ of the same cardinality as $\mathcal{X}_0$.*

Proof Let $F(T_0, T, p)$ be a (consistent) completion of T such that $F(T_0, T, p) + p\!\uparrow$ is fully consistent over T_0 if $T + p\!\uparrow$ is. Let $\mathcal{X}$ be the closure of $\mathcal{X}_0$ under the operation F. □

Corollary 3.7 *Any countable model of a rich theory has an elementary extension which is transplendent.*

Proof Use Proposition 3.6 and the fact that if $\mathcal{X}$ is countable and extends $\mathrm{SSy}(M)$ for some countable model M, then there is a countable recursively saturated elementary extension M' of M with $\mathrm{SSy}(M) = \mathcal{X}$. □

We are now ready to characterize, in terms of their standard systems, the recursively saturated countable models of PA that are transplendent.

Theorem 3.8 *Let $M \models$ PA be a countable recursively saturated model; then M is transplendent iff $\mathrm{SSy}(M)$ is closed.*

Proof One direction is just Theorem 3.5. For the other suppose that $T, p(\bar{x}) \in \mathrm{SSy}(M)$ are such that $T + p\!\uparrow$ is fully consistent over M, and let T' be the extension of T with every instance of the scheme

$$\varphi \leftrightarrow (c)_{\ulcorner \varphi \urcorner} \neq 0,$$

where φ is a sentence in the language of T and $p(\bar{x})$, and c is a new constant symbol. It is not hard to see that $T' + p\!\uparrow$ is fully consistent over M, and so there is an expansion M^+ of M satisfying $T' + p\!\uparrow$.

Then we have that $\mathrm{Th}(M^+) + p\!\uparrow$ is fully consistent over M, and clearly the same is true for $\mathrm{Th}(M') + p\!\uparrow$, where M' is the reduct of M^+ forgetting about the constant c. Furthermore, the (interpretation of the) constant c will code the theory of M', so we have $\mathrm{Th}(M') \in \mathrm{SSy}(M')$. Since $\mathrm{SSy}(M') = \mathrm{SSy}(M)$ there is a completion T^c of T such that $T^c \in \mathrm{SSy}(M)$ and $T^c + p\!\uparrow$ is fully consistent over M, namely, $T^c = \mathrm{Th}(M')$. Thus $\mathrm{SSy}(M)$ is closed. □

4 The Standard Predicate

As mentioned in the introduction, we have two key examples for applying the idea of transplendence. One of them is the theory $T_{K=\mathbb{N}}$: $\{K(n) \mid n \in \mathbb{N}\} + p\!\uparrow$, where $p(x)$ is $\{K(x) \land x > n \mid n \in \mathbb{N}\}$, considered over models of arithmetic.

Working in a model of arithmetic the only predicate satisfying $T_{K=\mathbb{N}}$ is the standard cut. On its own, this is not very interesting as all models of arithmetic have such an expansion, but we can add other first-order properties to $T_{K=\mathbb{N}}$ to get more interesting expansions. One example is the property that K is strong, which is first order:

$$\forall c \exists d \big(\neg K(d) \land \forall x \big(K(x) \to (K((c)_x) \leftrightarrow (c)_x > d)\big)\big).$$

We now take a look at some notions from the theory of second-order arithmetic. We will use $v_0, v_1, \ldots$ as first-order variables, $V_0, V_1, \ldots$ as second-order variables, $x, y, z, \ldots$ as metavariables ranging over first-order variables, and $X, Y, Z, \ldots$ over second-order variables. Any set $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ can be regarded as a second-order model of arithmetic by letting the first-order part be the standard model of first-order arithmetic and the domain of the second-order quantifiers be $\mathcal{X}$.

Definition 4.1 If $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{P}(\mathbb{N})$, then

- $\mathcal{X} \prec_{\Sigma^1_k} \mathcal{Y}$ if for all Σ^1_k-formulas $\Psi(\bar{X})$ and all $\bar{A} \in \mathcal{X}$, $\mathcal{X} \models \Psi(\bar{A})$ iff $\mathcal{Y} \models \Psi(\bar{A})$;
- $\mathcal{X} \prec \mathcal{Y}$ if $\mathcal{X} \prec_{\Sigma^1_k} \mathcal{Y}$ holds for all $k \in \mathbb{N}$;
- $\mathcal{X}$ is a β_ω*-model* if $\mathcal{X} \prec \mathcal{P}(\mathbb{N})$.

Given a term t, let t' be like t except that all first-order variables v_i are replaced by v_{2i}. (This substitution is made simultaneously for all variables.) Define the K-translate, Θ^K, of any second-order arithmetic formula Θ so that the following hold (for simplicity we assume the logical symbols are only the symbols $\vee$, $\neg$, and $\exists$):

$$\begin{aligned}
(t = r)^K &\text{ is } t' = r', \\
(V_i = V_j)^K &\text{ is } \big(\forall v_0 (v_0 \in V_i \leftrightarrow v_0 \in V_j)\big)^K, \\
(t \in V_i)^K &\text{ is } (v_{2i+1})_{t'} \neq 0, \\
(\Psi_1 \vee \Psi_2)^K &\text{ is } \Psi_1^K \vee \Psi_2^K, \\
(\neg\Psi)^K &\text{ is } \neg\Psi^K, \\
(\exists v_i \Psi)^K &\text{ is } \exists v_{2i} \big(K(v_{2i}) \wedge \Psi^K\big), \qquad \text{and} \\
(\exists V_i \Psi)^K &\text{ is } \exists v_{2i+1} \Psi^K,
\end{aligned}$$

where t and r are terms. Please remember that $(x)_y = z$ is a first-order formula in $\mathcal{L}_A$ saying that the yth element coded by x is z. Observe that if $v_{i_0}, \ldots, v_{i_k}, V_{j_0}, \ldots, V_{j_l}$ are the free variables of Θ, then $v_{2i_0}, \ldots, v_{2i_k}$ and $v_{2j_0+1}, \ldots, v_{2j_l+1}$ are the free variables of Θ^K. We will assume that the free variables are listed in this order.

Definition 4.2 If $a \in M \models \mathrm{PA}$, then $\mathrm{set}_M(a) = \{n \in \mathbb{N} \mid (a)_n \neq 0\}$.

Lemma 4.3 *For any $M \models \mathrm{PA}$, any second-order arithmetic formula $\Theta(\bar{x}, \bar{X})$, and any $\bar{n} \in \mathbb{N}$, $\bar{a} \in M$ we have*

$$\begin{aligned}
&(M, \mathbb{N}) \models \Theta^K(\bar{n}, \bar{a}) \quad \textit{iff} \\
&\mathrm{SSy}(M) \models \Theta\big(\bar{n}, \mathrm{set}_M(a_0), \ldots, \mathrm{set}_M(a_{k-1})\big).
\end{aligned}$$

Proof The proof is by induction on the construction of Θ. First assume Θ to be atomic. There are three cases.

- Θ is $t = r$ for some terms t and r. Clearly $M \models t(\bar{n}) = r(\bar{n})$ iff $\mathbb{N} \models t'(\bar{n}) = r'(\bar{n})$.
- Θ is $t \in V_i$, and we have $(M, \mathbb{N}) \models (t \in V_i)^K(\bar{n}, d)$ iff $(M, \mathbb{N}) \models K(t(\bar{n})) \wedge (d)_{t(\bar{n})} \neq 0$ iff $t(\bar{n}) \in \mathrm{set}_M(d)$.
- Θ is $X = Y$. This case reduces to the other cases.

If Θ is not atomic, it is composite; there are three cases here as well.

- Θ is $\neg\Psi$ or $\Psi_1 \vee \Psi_2$. This is obvious from the definition (since the K-translate and $\neg/\vee$ commute).
- Θ is $\exists v_i \Psi(v_i, \bar{y}, \bar{X})$; then $(M, \mathbb{N}) \models \exists v_{2i}(K(v_{2i}) \wedge \Psi^K)(\bar{n}, \bar{d})$ iff there is $n \in \mathbb{N}$ such that $(M, \mathbb{N}) \models \Psi^K(\bar{n}, n, \bar{d})$ iff there is $n \in \mathbb{N}$ such that $\mathrm{SSy}(M) \models \Psi(\bar{n}, n, \bar{D})$ iff $\mathrm{SSy}(M) \models \exists v_i \Psi(\bar{n}, v_i, \bar{D})$, where $\bar{D}$ are the sets coded by the elements $\bar{d}$.

- Θ is $\exists V_i \Psi(\bar{x}, X, \bar{Y})$. We have

$$(M, \mathbb{N}) \models \exists v_{2i+1} \Psi^K(\bar{n}, \bar{d})$$

iff there is $e \in M$ such that

$$(M, \mathbb{N}) \models \Psi^K(\bar{n}, e, \bar{d})$$

iff there is $E \in \mathrm{SSy}(M)$ such that

$$\mathrm{SSy}(M) \models \Psi(\bar{n}, E, \bar{D})$$

iff

$$\mathrm{SSy}(M) \models \exists V_i \Psi(\bar{n}, \bar{D}).$$

By induction the lemma holds for any second-order arithmetic formula Θ. □

Theorem 4.4 (Engström [4, Theorem 2.17]) *If $M \models \mathrm{PA}$ is transplendent, then $\mathrm{SSy}(M)$ is a β_ω-model.*

Proof Let $\Psi(\bar{A})$, where $\bar{A} \in \mathrm{SSy}(M)$, be a second-order sentence true in $\mathcal{P}(\mathbb{N})$. Let $a_i \in M$ code A_i. By taking N to be an ω-saturated model of $\mathrm{Th}(M, \bar{a})$ we have, by using the fact that $\mathrm{SSy}(N) = \mathcal{P}(\mathbb{N})$ and Lemma 4.3, that $\mathcal{P}(\mathbb{N}) \models \Psi(\bar{A})$ iff $(N, \mathbb{N}) \models \Psi^K(\bar{a})$. Thus, by the choice of Ψ, we have $(N, \mathbb{N}) \models \Psi^K(\bar{a})$. Therefore $T_{K=\mathbb{N}} + \Psi^K(\bar{a})$ is fully consistent over M, and so by the transplendence of M there is an expansion of M satisfying $T_{K=\mathbb{N}} + \Psi^K(\bar{a})$. There could only be one such expansion, and so we have

$$(M, \mathbb{N}) \models \Psi^K(\bar{a}).$$

By using the lemma once again we see that

$$\mathrm{SSy}(M) \models \Psi(\bar{A}),$$

and thus $\mathrm{SSy}(M)$ is a β_ω-model. □

The set of all arithmetically definable subsets of $\mathbb{N}$ is an arithmetically closed Scott set. However, this set is not a β_ω-model since it does not satisfy Π^1_1-comprehension. Thus, there are countable arithmetically closed Scott sets which are not β_ω-models, giving us the following.

Corollary 4.5 *There are countable arithmetically saturated models of* PA *that are not transplendent.*

This leaves us with an open question which is not answered in this paper: Is there a resplendent-like property which is equivalent to arithmetic saturation?

In fact, we can say more about the standard system of a transplendent model. Given $A \subseteq \mathbb{N}$ let the second-order theory of A be

$$\mathrm{Th}^2(A) = \{\Psi(X) \mid \mathcal{P}(\mathbb{N}) \models \Psi(A)\}.$$

Theorem 4.6 (Engström [4]) *If $M \models \mathrm{PA}$ is transplendent and $A \in \mathrm{SSy}(M)$, then $\mathrm{Th}^2(A) \in \mathrm{SSy}(M)$.*

Proof Assume that $A \in \mathrm{SSy}(M)$ is coded by $a \in M$. Let $T + p\!\uparrow$ be

$$T_{K=\mathbb{N}} + \{(c)_{\ulcorner\Theta(X)\urcorner} \neq 0 \leftrightarrow \Theta^K(a) \mid \Theta(X) \text{ second-order formula}\}.$$

If N is an ω-saturated model of $\mathrm{Th}(M)$ and $b \in N$ codes the set $\mathrm{Th}^2(A)$, then

$$(N, \mathbb{N}, b) \models T + p\!\uparrow,$$

since

$$\mathcal{P}(\mathbb{N}) \models \Theta(A) \quad \text{iff} \quad (N, \mathbb{N}, a) \models \Theta^K(a)$$

for all second-order $\Theta(X)$. By the transplendence of M there is $d \in M$ such that

$$(M, \mathbb{N}, d) \models T + p{\uparrow}.$$

Thus, d codes the theory of the second-order model $(\mathrm{SSy}(M), A)$ which is elementary equivalent to $(\mathcal{P}(\mathbb{N}), A)$ since $\mathrm{SSy}(M)$ is a β_ω-model. □

Under certain set-theoretic assumptions, Theorem 4.6 generalizes Theorem 4.4 since then, by some well-known basis theorems (see Hinman [5, Corollaries V.2.7, V.3.6]), the set $\Delta^{1;A}_{\infty}$ is a basis for itself for every $A \subseteq \mathbb{N}$.[1] Thus we have the following.

Proposition 4.7 *If $V = L$ or PD (projective determinacy) holds, then any Scott set closed under the operation $A \mapsto \mathrm{Th}^2(A)$ is a β_ω-model.*

To us it seems to be a difficult question to identify in terms of recursion theory or descriptive set theory necessary and sufficient conditions on a Scott set to be closed, in the sense of Definition 3.4. The following question is open.

Question 4.8 Is a β_ω-model which is closed under the operation

$$A \mapsto \mathrm{Th}^2(A)$$

closed in the sense of Definition 3.4?

5 Subtransplendence

Resplendency is strictly stronger than recursive saturation, which a recursively saturated ω_1-like model of PA shows.[2] However, it is easy to find a resplendency-like property which is equivalent to recursive saturation.

Definition 5.1 Let M be any $\mathcal{L}_0$-structure, and let $\mathcal{X}$ be a Scott set. We say that M is *$\mathcal{X}$-subresplendent* if for all $T \in \mathcal{X}$ in a language $\mathcal{L} \supseteq \mathcal{L}_0 \cup \{\bar{a}\}$ (where $\bar{a} \in M$ is finite) such that $T + \mathrm{Th}(M, \bar{a})$ is consistent, there are an elementary submodel N of M such that $\bar{a} \in N$ and an expansion N^+ of N satisfying T. A model is *subresplendent* if it is $\mathcal{X}$-subresplendent for some Scott set $\mathcal{X}$.

The following theorem is easily proved by an argument similar to the argument proving that recursive saturation implies resplendency for countable models. It could be worth noting that with this construction the elementary submodels N will all be countable.

Theorem 5.2 *A model is subresplendent iff it is recursively saturated.*

Thus, for countable models subresplendency and resplendency coincide. In the case where we also omit a type the situation is quite different; the notion of subtransplendence will be strictly weaker than transplendence even for countable models.

Definition 5.3 Let M be any $\mathcal{L}_0$-structure, and let $\mathcal{X}$ be a Scott set. We say that M is *$\mathcal{X}$-subtransplendent* if for all $T, p(\bar{x}) \in \mathcal{X}$ in some language $\mathcal{L} \supseteq \mathcal{L}_0 \cup \{\bar{a}\}$ (where $\bar{a} \in M$ is finite) such that there is a model of $T + p{\uparrow} + \mathrm{Th}(M, \bar{a})$, there are an elementary submodel $\bar{a} \in N$ of M and an expansion N^+ of N such that $N^+ \models T + p{\uparrow}$. We say that a model M is *subtransplendent* if it is $\mathcal{X}$-subtransplendent for some Scott set $\mathcal{X}$.

Observe that in this definition we only demand that $T + p{\uparrow} + \mathrm{Th}(M, \bar{a})$ be consistent, not that it be fully consistent.

In the case of rich theories, we characterize those recursively saturated models (in any cardinality) which are subtransplendent in terms of their standard system.

Definition 5.4 A Scott set $\mathcal{X}$ is a *β-model* if $\mathcal{X} \prec_{\Sigma_1} \mathcal{P}(\mathbb{N})$. We say that M is *β-saturated* if there is a β-model $\mathcal{X}$ such that M is $\mathcal{X}$-saturated.

The following theorem is proved by a construction not very different from the one in the case of transplendent models. However, there is a difference in that the properties need not be monotonic and that there may be uncountably many properties that need to be satisfied.

Theorem 5.5 *Every β-saturated model is subtransplendent.*

Proof Let M be a β-saturated model, and let $\mathcal{X}$ be a β-model such that M is $\mathcal{X}$-saturated. Let $\mathcal{L}$, $\bar{a}$, T, and $p(\bar{x})$ be as in the definition of subtransplendence.

The forcing conditions $S \in \mathcal{X}$ we use in the argument are complete theories in languages $\mathcal{L}(\bar{b})$, where $\bar{b} \in M$ is finite such that $\mathrm{Th}(M, \bar{a}, \bar{b}) + S + p{\uparrow}$ is consistent. Such forcing conditions exist, as the following lemma shows.

Lemma 5.6 *If $S_0 \in \mathcal{X}$ is a set of $\mathcal{L}(\bar{b}, c)$-formulas, where $\bar{b} \in M$ is finite and c is a (new) constant symbol, such that $S_0 + \mathrm{Th}(M, \bar{b}) + p{\uparrow}$ is consistent, then there is a completion $S \in \mathcal{X}$ of S_0 consistent with $\mathrm{Th}(M, \bar{b}) + p{\uparrow}$.*

Proof For a complete theory T to say that $T + p{\uparrow}$ is consistent is equivalent to saying that $p(\bar{x})$ is not isolated in T. Therefore, letting $\theta(X, S_0, p)$ be the first-order formula expressing that X is a completion of S_0 such that $p(x)$ is not isolated in X, it is easy to see that $\mathcal{P}(\mathbb{N}) \models \exists X \Theta(X, S_0, p)$ and so, since $\mathcal{X}$ is a β-model, $\mathcal{X} \models \exists X \Theta(X, S_0, p)$. □

The following properties of forcing conditions S will be used.

- $P_{\varphi(x,\bar{b})}$: If $\exists x \varphi(x, \bar{b}) \in S$, then there is $m \in M$ such that $\varphi(m, \bar{b}) \in S$.

Here $\bar{b} \in M$, and $\varphi(x, \bar{y})$ is an $\mathcal{L}$-formula.

Observe that these properties are not monotonic; that is, it might happen that $S \subseteq S'$ and S satisfies a property $P_{\varphi(x,\bar{b})}$ but S' does not. However, they are dense in the sense that if S is a condition and $P_{\varphi(x,\bar{b})}$ is a property, then there is a condition $S' \supseteq S$ satisfying the property. To see this let S be a condition, and assume that $\exists x \varphi(x, \bar{b}) \in S$. By the lemma there is a completion S' of $S + \varphi(c, \bar{b})$ coded in $\mathcal{X}$. Let $q(x)$ be the restriction of $S'[x/c]$ to the language $\mathcal{L}_0(\bar{a}, \bar{b})$. It should be clear that $q(x)$ is a coded type over M and so is realized, say, by $m \in M$. It is easy to see that $S'[m/c]$ is a condition since $\mathrm{Th}(M, \bar{a}, \bar{b}, m) \subseteq S'[m/c]$. Thus S' is a condition including S which satisfies $P_{\varphi(x,\bar{b})}$.

To construct a complete theory meeting all properties we enumerate all $\mathcal{L}$-formulas as $\varphi_k(x, \bar{y})$ in such a way that every formula occurs an infinite number of times in the enumeration. Start with some forcing condition S_0, and build a countable chain of conditions $S_k \subseteq S_{k+1}$. Let $\bar{b}_k$, $k \leq n$, be a finite enumeration of all sequences of parameters occurring in S_0. Find S_1 satisfying $P_{\varphi_0(x,\bar{b}_0)}$, S_2 satisfying $P_{\varphi_0(x,\bar{b}_1)}$, and so on. When S_{n+1} is found start over with a new enumeration of all finite sequences of parameters occurring in S_{n+1}: $\bar{b}_k$, $k \leq n'$, and start satisfying properties $P_{\varphi_1(x,\bar{b}_0)}$, $P_{\varphi_1(x,\bar{b}_1)}$, and so on.

Let S_∞ be the complete theory $\bigcup_{k\geq 0} S_k$. We claim that S_∞ satisfies every (potentially uncountably many) properties. Let $P_{\varphi(x,\bar{b})}$ be a property; we may assume that all parameters in the sequence $\bar{b}$ occur in S_∞ and that $\exists x\varphi(x,\bar{b}) \in S_\infty$. There are k and n such that $\exists x\varphi(x,\bar{b}) \in S_k$ and $\varphi(x,\bar{y})$ is $\varphi_n(x,\bar{y})$. Thus there is a k' such that $\varphi(m,\bar{b}) \in S_{k'}$ for some $m \in M$, and therefore $\varphi(m,\bar{b}) \in S_\infty$.

Let N^+ be the canonical model of S_∞. It is straightforward to check that N^+ satisfies $T + p\!\uparrow$ and that the $\mathcal{L}$-reduct of N^+ is elementary embedded in $(M,\bar{a})$. □

It could be worth noting that the constructed elementary submodel N^+ is countable.

Corollary 5.7 *If $M \models$ PA is transplendent, then it is subtransplendent.*

We have a converse to Theorem 5.5.

Theorem 5.8 *If $M \models$ PA is subtransplendent, then it is β-saturated.*

Proof Let $\theta(X,\bar{A})$ be an arithmetic first-order formula with set parameters $\bar{A}$ from $\mathrm{SSy}(M)$, such that $\mathcal{P}(\mathbb{N}) \models \exists X\theta(X,\bar{A})$. We will find $B \in \mathrm{SSy}(M)$ such that $\mathcal{P}(\mathbb{N}) \models \theta(B,\bar{A})$.

Let $T + p\uparrow$ be $\exists x\Theta^K(x,\bar{a}) + T_{K=\mathbb{N}}$, where $\bar{a}$ codes $\bar{A}$ in M. To see that $\mathrm{Th}(M,\bar{a}) + T + p\uparrow$ is consistent, take a model N of $\mathrm{Th}(M,\bar{a})$ such that N is β-saturated; then $(N,\mathbb{N}) \models \mathrm{Th}(M,\bar{a}) + T + p\!\uparrow$.

By the assumption that M is subtranscendent there are an elementary submodel $\bar{a} \in N$ and an expansion N^+ of N such that $N^+ \models T + p\!\uparrow$.

Thus, if $b \in N^+$ is such that $N^+ \models \Theta^K(b,\bar{a})$, then $\mathcal{P}(\mathbb{N}) \models \Theta(\mathrm{set}_{N^+}(b),\bar{A})$, and since N is elementary embedded in M the set $B = \mathrm{set}_{N^+}(b)$ is in $\mathrm{SSy}(M)$. This completes the proof. □

Corollary 5.9 *A model of PA is subtransplendent iff it is β-saturated.*

From these results we get a characterization of β-models in terms of closure under completions of theories.

Corollary 5.10 *A Scott set $\mathcal{X}$ is a β-model iff for every T, $p(\bar{x}) \in \mathcal{X}$ such that $T + p\uparrow$ is consistent there is a completion $T^c \in \mathcal{X}$ of T such that $T^c + p\uparrow$ is consistent.*

Proof Assume that $\mathcal{X}$ is a β-model and that T, $p(\bar{x}) \in \mathcal{X}$ are such that $T + p\!\uparrow$ is consistent. Let $\theta(X,T,p)$ be a first-order arithmetic formula expressing

$$X \text{ is a complete theory} \wedge p(\bar{x}) \text{ is not isolated in } X \wedge T \subseteq X.$$

Since there is $X \in \mathcal{P}(\mathbb{N})$ satisfying $\theta(X,T,p)$ and $\mathcal{X}$ is a β-model, there is $T^c \in \mathcal{X}$ such that $\mathcal{P}(\mathbb{N}) \models \theta(T^c,T,p)$. By the omitting types theorem $T^c + p\!\uparrow$ is consistent.

For the other direction let $\mathcal{X}$ be such, and let M be an $\mathcal{X}$-saturated model of PA. The proof of Theorem 5.5 goes through since it uses only that $\mathcal{X}$ is closed under such completions and no other properties of β-models; thus M is subtransplendent. Theorem 5.8 then says that $\mathrm{SSy}(M) = \mathcal{X}$ is a β-model. □

In the proof of Corollary 5.9 it is easy to observe that it is enough to assume that $p(x)$ is the type

$$\{K(x) \wedge x > n \mid n \in \mathbb{N}\}.$$

Let us formulate this as a corollary.

Corollary 5.11 *A Scott set $\mathcal{X}$ is a β-model iff for every $T \in \mathcal{X}$ in the language $\mathcal{L}_A(K, c)$ such that $T + T_{K=\mathbb{N}}$ is consistent there is a completion T^c of T such that $T^c \in \mathcal{X}$ and $T^c + T_{K=\mathbb{N}}$ is consistent.*

6 The Standard Predicate, Revisited

Let $\mathcal{L}_A^{\mathbb{N}}$ be the language of arithmetic with an extra predicate K whose intended interpretation is the standard predicate $\mathbb{N}$ and with all Skolem functions in the language of $\mathcal{L}_A$ added. The set of all $\mathcal{L}_A^{\mathbb{N}}$-formulas whose only quantifiers are of the form $\exists x \in K$ or $\forall x \in K$ are denoted $\Delta_0^{\mathbb{N}}$. $\Sigma_1^{\mathbb{N}}$ is the set of formulas of the form $\exists \bar{x} \varphi(\bar{x}, \bar{y})$ where φ is $\Delta_0^{\mathbb{N}}$. It should be noted that if $\Theta(\bar{X})$ is a Δ_0^1- (or Σ_1^1-) formula in second-order arithmetic, then $\Theta^K(\bar{x})$, as defined above, is a $\Delta_0^{\mathbb{N}}$- (or $\Sigma_1^{\mathbb{N}}$-) formula.

Please observe that if $M \prec {}^*M$, then $(M, \mathbb{N}) \prec_{\Delta_0^{\mathbb{N}}} ({}^*M, \mathbb{N})$.

In [12] Stuart Smith proved the following two theorems which might be helpful for the reader to bear in mind.

Theorem 6.1 ([12, Theorem 0.3]) *Let M be a countable model of PA and $A \subseteq M$ which is not parametrically definable in M; then there is $M \prec N$ such that no $B \subseteq N$ satisfies $(M, A) \not\prec (N, B)$.*

Theorem 6.2 ([12, Theorem 3.13]) *If $M \prec_{\text{end}} N$, then $(M, \mathbb{N}) \prec (N, \mathbb{N})$.*

The first theorem implies that there is no countable nonstandard model M such that for any $M \prec N$ there is $A \subseteq N$ satisfying $(M, \mathbb{N}) \prec (N, A)$.

Definition 6.3 M is $\Sigma_1^{\mathbb{N}}$*-closed* if whenever $M \prec {}^*M$, then $(M, \mathbb{N}) \prec_{\Sigma_1^{\mathbb{N}}} ({}^*M, \mathbb{N})$.

Proposition 6.4 *If M is $\Sigma_1^{\mathbb{N}}$-closed, then* $\mathrm{SSy}(M)$ *is a β-model.*

Proof Let $\bar{A} \in \mathrm{SSy}(M)$ be coded by $\bar{a} \in M$, and let $\mathcal{P}(\mathbb{N}) \models \Theta(\bar{A})$, where $\Theta(\bar{A})$ is a Σ_1^1-sentence. Let $M \prec {}^*M$ be such that $\mathrm{SSy}({}^*M) \models \Theta(\bar{A})$. By Lemma 4.3 $({}^*M, \mathbb{N}) \models \Theta^K(\bar{a})$, and so $(M, \mathbb{N}) \models \Theta^K(\bar{a})$ since $\Theta^K(\bar{a})$ is $\Sigma_1^{\mathbb{N}}$. Again by using the same lemma we get that $\mathrm{SSy}(M) \models \Theta(\bar{A})$. □

Proposition 6.5 *If M is subtransplendent, then M is $\Sigma_1^{\mathbb{N}}$-closed.*

Proof Assume $M \prec {}^*M$ and $({}^*M, \mathbb{N}) \models \varphi(\bar{a})$, where $\varphi(\bar{x})$ is $\Sigma_1^{\mathbb{N}}$ and $\bar{a} \in M$. The theory $\mathrm{Th}(M, \bar{a}) + \varphi(\bar{a}) + T_{K=\mathbb{N}}$ is consistent, and so, since M is subtransplendent, there is an elementary submodel $\bar{a} \in N \prec M$ such that $(N, \mathbb{N}) \models \varphi(\bar{a})$. Since $(N, \mathbb{N}) \prec_{\Delta_0^{\mathbb{N}}} (M, \mathbb{N})$ this is also true in $(M, \mathbb{N})$. □

Combining these two propositions with Theorem 5.5 we get the following.

Corollary 6.6 *Let M be a model of* PA*; then following are equivalent.*

1. *M is subtransplendent.*
2. *M is recursively saturated and $\Sigma_1^{\mathbb{N}}$-closed.*
3. *M is β-saturated.*

Next, we will try to apply these ideas to transplendence. We need a notion stronger than $\Sigma_1^{\mathbb{N}}$-closed. The first naive try might be: If $M \prec {}^*M$, then $(M, \mathbb{N}) \prec ({}^*M, \mathbb{N})$, but this is a too strong requirement, which Smith's result above shows. Therefore, we use the following definition.

Definition 6.7 M is $\mathbb{N}$-*correct* if whenever $M \prec {}^*M$ and ${}^*M$ is ω-saturated, then $(M,\mathbb{N}) \prec ({}^*M,\mathbb{N})$.

This makes sense since we have the following.

Proposition 6.8 *If* $M \equiv N$, $\mathrm{SSy}(M) = \mathrm{SSy}(N)$ *and both are recursively saturated, then* $(M,\mathbb{N}) \equiv (N,\mathbb{N})$.

Proof $I = \{(\bar{a},\bar{b}) \mid (M,\bar{a}) \equiv (N,\bar{b})\}$ is a back-and-forth system between $(M,\mathbb{N})$ and $(N,\mathbb{N})$. □

Proposition 6.9 *Any transplendent model of PA is* $\mathbb{N}$*-correct.*

Proof Consider given ω-saturated ${}^*M$ such that $M \prec {}^*M$ and ${}^*M \models \varphi(\bar{a})$, $\bar{a} \in M$, and φ is an $\mathcal{L}_A(K)$-formula. Clearly $\mathrm{Th}(M,\bar{a}) + \varphi(\bar{a}) + T_{K=\mathbb{N}}$ is fully consistent over M. Thus by the transplendence of M there is an expansion of M satisfying this theory, that is, $(M,\mathbb{N}) \models \varphi(\bar{a})$. This proves that $(M,\mathbb{N}) \prec ({}^*M,\mathbb{N})$. □

We do not know if $\mathbb{N}$-correctness, which really is a form of transplendency for a single fixed type being omitted and finite theories in the language $\mathcal{L}_A(K)$ (with parameters), by itself is enough to prove transplendency.

Question 6.10 Are all $\mathbb{N}$-correct recursively saturated countable models of PA transplendent?

7 Satisfaction Classes: An Application

Recently, in [2], Enayat and Visser gave a new proof of the conservativity of PA + "there exists a full satisfaction class" over PA. We will give a very short outline of that proof below.

Theorem 7.1 (Kotlarski, Krajewski, and Lachlan [10]) PA+ *"S is a full satisfaction class" is conservative over* PA.

Proof Let $M \models$ PA; we will build a chain of elementary extensions of M such that the limit of this chain has a full satisfaction class. Let $M_0 = M$, let $\mathcal{L}_0 = \mathcal{L}_A(M_0)$, and let M_{i+1} be a model of

$$T_i = \mathrm{Th}_{\mathcal{L}_i}(M_i,a)_{a\in M_i} + \{\bar{S}_\varphi \mid \varphi \in M_i\}$$

in the language

$$\mathcal{L}_i \cup \{S_\varphi | \varphi \in M_i \models \varphi \text{ is an } \mathcal{L}_A\text{-formula}\},$$

where the S_φs are unary predicates and $\bar{S}_\varphi$ is the Tarski condition for φ; for example,

$$\bar{S}_{\psi\vee\psi'} \text{ is } \forall x\big(S_{\psi\vee\psi'}(x) \leftrightarrow S_\psi(x) \vee S_{\psi'}(x)\big)$$

and

$$\bar{S}_{\exists v_i\varphi} \text{ is } \forall x\big(S_{\exists v_i\varphi}(x) \leftrightarrow \exists y S_\varphi(x[y/i])\big).$$

$\mathcal{L}_{i+1}$ is then

$$\mathcal{L}_A(M_{i+1}) \cup \{S_\varphi | \varphi \in M_{i+1} \models \varphi \text{ is an } \mathcal{L}_A\text{-formula}\}.$$

By compactness the theory T_i is consistent, so such an M_{i+1} exists. Let $M' = \bigcup M_i$, and define

$$S = \big\{\langle\varphi,a\rangle \bigm| M' \models S_\varphi(a)\big\}.$$

It can be checked that S is a satisfaction class for the $\mathcal{L}_A$-reduct of M'. □

Ali Enayat observed that this proof allows us to construct models of PA with a satisfaction class S satisfying $\epsilon_i \in S$ iff $i \in \mathbb{N}$, where ϵ_0 if $0 = 0$ and ϵ_{i+1} is $\epsilon_i \vee \epsilon_i$. By observing that we in fact can construct ω-saturated such models, we see that over any countable model of PA the theory

$$\text{“}S \text{ is a full satisfaction class”} + \forall x\big(S(\epsilon_x) \leftrightarrow K(x)\big) + T_{K=\mathbb{N}}$$

is fully consistent over M. Thus we have the following.

Corollary 7.2 *Any transplendent model of* PA *has a full satisfaction class S such that $\mathbb{N}$ is definable in (M, S).*

This idea can be somewhat extended if S_0 is a set of pairs of formulas (in the sense of a model M) and elements $a \in M$ such that S_0 is definable in $(M, \mathbb{N})$ and the set of finite approximations of S_0 is consistent.[3] Then there is a full satisfaction class S on M extending S_0.

Notes

1. $\Delta_k^{1,A}$ denotes both the collection of sets of natural numbers definable in $\mathcal{P}(\mathbb{N})$ by a Δ_k^1-formula $\theta(x, A)$, and the collection of subsets of $\mathcal{P}(\mathbb{N})$ definable in $\mathcal{P}(\mathbb{N})$ by a Δ_k^1-formula $\theta(X, \bar{A})$. The set $\Delta_\infty^{1,A}$ is the union of all $\Delta_k^{1,A}$. If Δ is a collection of sets of natural numbers and Γ is a collection of subsets of $\mathcal{P}(\mathbb{N})$, then Δ is a *basis for* Γ if for any $\gamma \in \Gamma$ we have $\gamma \cap \Delta \neq \emptyset$.

2. A nice exposition of ω_1-like models of PA can be found in Kossak and Schmerl [9, Chapter 10].

3. See [10] or Engström [3] for more on finite approximations.

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