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DE RHAM COHOMOLOGY OF LOCAL COHOMOLOGY MODULES: THE GRADED CASE

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Abstract. Let K be a field of characteristic zero, and let $R = K[X_1, \ldots, X_n]$. Let $A_n(K) = K\langle X_1, \ldots, X_n, \partial_1, \ldots, \partial_n\rangle$ be the nth Weyl algebra over K. We consider the case when R and $A_n(K)$ are graded by giving $\deg X_i = \omega_i$ and $\deg \partial_i = -\omega_i$ for $i = 1, \ldots, n$ (here ω_i are positive integers). Set $\omega = \sum_{k=1}^{n} \omega_k$. Let I be a graded ideal in R. By a result due to Lyubeznik the local cohomology modules $H^i_I(R)$ are holonomic $(A_n(K))$-modules for each $i \geq 0$. In this article we prove that the de Rham cohomology modules $H^*(\partial; H^*_I(R))$ are concentrated in degree $-\omega$; that is, $H^*(\partial; H^*_I(R))_j = 0$ for $j \neq -\omega$. As an application when $A = R/(f)$ is an isolated singularity, we relate $H^{n-1}(\partial; H^1_{(f)}(R))$ to $H^{n-1}(\partial(f); A)$, the $(n-1)$th Koszul cohomology of A with respect to $\partial_1(f), \ldots, \partial_n(f)$.

Let K be a field of characteristic zero, and let $R = K[X_1, \ldots, X_n]$. We consider R graded with $\deg X_i = \omega_i$ for $i = 1, \ldots, n$; here ω_i are positive integers. Set $\mathfrak{m} = (X_1, \ldots, X_n)$. Let I be a graded ideal in R. The local cohomology modules $H^*_I(R)$ are clearly graded R-modules. Let $A_n(K) = K\langle X_1, \ldots, X_n, \partial_1, \ldots, \partial_n\rangle$ be the nth Weyl algebra over K. By a result due to Lyubeznik (see [3, Section 2.2.d]), the local cohomology modules $H^i_I(R)$ are holonomic $(A_n(K))$-modules for each $i \geq 0$. We can consider $A_n(K)$ graded by giving $\deg \partial_i = -\omega_i$ for $i = 1, \ldots, n$.

Let N be a graded left $(A_n(K))$-module. Now $\partial = \partial_1, \ldots, \partial_n$ are pairwise commuting K-linear maps, so we can consider the de Rham complex $K(\partial; N)$. Notice that the de Rham cohomology modules $H^*(\partial; N)$ are in general only graded K-vector spaces. They are finite-dimensional if N is holonomic (see [1, Chapter 1, Theorem 6.1]). In particular, $H^*(\partial; H^*_I(R))$ are finite-dimensional graded K-vector spaces.

Our first result is as follows.

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THEOREM 1. *Let I be a graded ideal in R. Set $\omega = \sum_{i=1}^{n} \omega_i$. Then the de Rham cohomology modules $H^*(\partial_1, \dots, \partial_n; H^*_I(R))$ are concentrated in degree $-\omega$; that is,*

$$H^*\big(\partial_1, \dots, \partial_n; H^*_I(R)\big)_j = 0, \quad \text{for } j \neq -\omega.$$

We give an application of Theorem 1. Let f be a homogeneous polynomial in R, with $A = R/(f)$ an isolated singularity; that is, A_P is regular for all homogeneous prime ideals $P \neq \mathfrak{m}$. Let $H^i(\partial(f); A)$ be the ith Koszul cohomology of A with respect to $\partial_1(f), \dots, \partial_n(f)$. We show the following.

THEOREM 2 (with hypotheses as above). *There exists a filtration $\mathcal{F} = \{\mathcal{F}_\nu\}_{\nu \geq 0}$ consisting of K-subspaces of $H^{n-1}(\partial; H^1_{(f)}(R))$ with $\mathcal{F}_\nu = H^{n-1}(\partial; H^1_{(f)}(R))$ for $\nu \gg 0$, $\mathcal{F}_\nu \supseteq F_{\nu-1}$, and $\mathcal{F}_0 = 0$ and injective K-linear maps*

$$\eta_\nu \colon \frac{\mathcal{F}_\nu}{\mathcal{F}_{\nu-1}} \to H^{n-1}\big(\partial(f); A\big)_{(\nu+1)\deg f - \omega}.$$

The techniques used in this theorem are generalized in [6] to show that $H^i(\partial; H^1_{(f)}(R)) = 0$ for $1 < i < n-1$ and $H^1(\partial; H^1_{(f)}(R)) \cong K$. There is no software to compute de Rham cohomology of an $(A_n(K))$-module M. As an application of Theorem 2, we prove the following.

EXAMPLE 0.1. Let $R = K[X_1, \dots, X_n]$, and let $f = X_1^2 + X_2^2 + \cdots + X_{n-1}^2 + X_n^m$ with $m \geq 2$. Then

(1) if m is odd, then $H^{n-1}(\partial; H^1_{(f)}(R)) = 0$;
(2) if m is even, then
  (a) if n is odd, then $H^{n-1}(\partial; H^1_{(f)}(R)) = 0$, and
  (b) if n is even, then $\dim_K H^{n-1}(\partial; H^1_{(f)}(R)) \leq 1$.

We now describe in brief the contents of this article. In Section 1 we discuss a few preliminaries that we need. In Section 2 we introduce the concept of generalized Eulerian modules. In Section 3 we give a proof of Theorem 1. In Section 4 we give an outline of proof of Theorem 2. In Section 5 we prove Theorem 2. In Section 6 we give a proof of Example 0.1.

§1. Preliminaries

In this section we discuss a few preliminary results that we need.

REMARK 1.1. Although all the results are stated for de Rham cohomology of an $(A_n(K))$-module M, we will actually work with de Rham homology.

Note that $H_i(\partial, M) = H^{n-i}(\partial, M)$ for any $(A_n(K))$-module M. Let $S = K[\partial_1, \ldots, \partial_n]$. Consider it as a subring of $A_n(K)$. Then note that $H_i(\partial, M)$ is the ith Koszul homology module of M with respect to ∂.

1.2. Let M be a holonomic $(A_n(K))$-module. Then for the case where $i = 0, 1$, the de Rham homology modules $H_i(\partial_n, M)$ are holonomic $(A_{n-1}(K))$-modules (see [1, Theorem 6.2]).

The following result is well known (see [2, Corollary 1.6.13]).

LEMMA 1.3. *Let $\partial = \partial_r, \partial_{r+1}, \ldots, \partial_n$, and let $\partial' = \partial_{r+1}, \ldots, \partial_n$. Let M be a left $(A_n(K))$-module. For each $i \geq 0$ there exists an exact sequence*

$$0 \to H_0\big(\partial_r; H_i(\partial'; M)\big) \to H_i(\partial; M) \to H_1\big(\partial_r; H_{i-1}(\partial'; M)\big) \to 0.$$

§2. Generalized Eulerian modules

Consider the Eulerian operator

$$\mathcal{E}_n = \omega_1 X_1 \partial_1 + \omega_2 X_2 \partial_2 + \cdots + \omega_n X_n \partial_n.$$

If $r \in R$ is homogeneous, then recall that $\mathcal{E}_n r = (\deg r) \cdot r$. Note that degree of $\mathcal{E}_n$ is zero.

Let M be a graded $(A_n(K))$-module. If m is homogeneous, we set $|m| = \deg m$. We say that M is *Eulerian* $(A_n(K))$-module if $\mathcal{E}_n m = |m| \cdot m$ for each homogeneous $m \in M$. This notion was discovered by Ma and Zhang (see their excellent paper [4]). They prove that local cohomology modules $H^*_I(R)$ are Eulerian $(A_n(K))$-modules (see [4, Theorem 5.3]). In fact, they prove this when R is standard graded. The same proof can be adapted to prove the general case.

It can easily be seen that if M is an Eulerian $(A_n(K))$-module, then so are each graded submodule and graded quotient of M. However, extensions of Eulerian modules need not be Eulerian (see [4, Remark 3.6]). To rectify this, we introduce the following notion. A graded $(A_n(K))$-module M is said to be *generalized Eulerian* if for a homogeneous element m of M there exists a positive integer a (here a may depend on m) such that

$$\big(\mathcal{E}_n - |m|\big)^a m = 0.$$

We now prove that the class of generalized Eulerian modules is closed under extensions.

PROPOSITION 2.1. *Let* $0 \to M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \to 0$ *be a short exact sequence of graded* $(A_n(K))$*-modules. Then the following are equivalent:*

(1) M_2 *is generalized Eulerian,*
(2) M_1 *and* M_3 *are generalized Eulerian.*

Proof. The assertion (1) $\implies$ (2) is clear. We prove (2) $\implies$ (1). Let $m \in M_2$ be homogeneous. Because M_3 is generalized Eulerian, we have

$$\big(\mathcal{E}_n - |m|\big)^b \alpha_2(m) = 0 \quad \text{for some } b \geq 1.$$

Set $v_2 = (\mathcal{E}_n - |m|)^b m \in M_2$. Because α_2 is $(A_n(K))$-linear, we get $\alpha_2(v_2) = 0$. So $v_2 = \alpha_1(v_1)$ for some $v_1 \in M_1$. Note that $\deg v_1 = \deg v_2 = |m|$. Because M_1 is generalized Eulerian, we have

$$\big(\mathcal{E}_n - |m|\big)^a v_1 = 0 \quad \text{for some } a \geq 1.$$

Because α_1 is $(A_n(K))$-linear, we get $(\mathcal{E}_n - |m|)^a v_2 = 0$. It follows that

$$\big(\mathcal{E}_n - |m|\big)^{a+b} m = 0.$$ ∎

If M is a graded $(A_n(K))$-module, then for $l \in \mathbb{Z}$ the module $M(l)$ denotes the shift of M by l; that is, $M(l)_n = M_{n+l}$ for all $n \in \mathbb{Z}$. The following result was proved for Eulerian $(A_n(K))$-modules in [4, Remark 2.5].

PROPOSITION 2.2. *Let* M *be a nonzero generalized Eulerian* $(A_n(K))$*-module. Then for* $l \neq 0$*, the module* $M(l)$ *is not a generalized Eulerian* $(A_n(K))$*-module.*

Proof. Suppose that $M(l)$ is a generalized Eulerian $(A_n(K))$-module for some $l \neq 0$. Let $m \in M$ be homogeneous of degree r and nonzero. Because M is generalized Eulerian $(A_n(K))$-module, we have

$$(\mathcal{E}_n - r)^a m = 0 \quad \text{for some } a \geq 1.$$

We may assume that $(\mathcal{E}_n - r)^{a-1} m \neq 0$. Now $m \in M(l)_{r-l}$. Because $M(l)$ is generalized Eulerian, we get

$$(\mathcal{E}_n - r + l)^b m = 0 \quad \text{for some } b \geq 1.$$

Notice that

$$0 = (\mathcal{E}_n - r + l)^b m = \Big(l^b + \sum_{i=1}^{b} \binom{b}{i} l^{b-i} (\mathcal{E}_n - r)^i\Big) m.$$

Multiply the term on the left by $(\mathcal{E}_n - r)^{a-1}$. We obtain

$$l^b(\mathcal{E}_n - r)^{a-1}m = 0.$$

Because $l \neq 0$, we get $(\mathcal{E}_n - r)^{a-1}m = 0$, a contradiction. ∎

§3. Proof of Theorem 1

In this section we prove Theorem 1. Notice that $H^i_I(R)$ are Eulerian $(A_n(K))$-modules for all $i \geq 0$. Hence, Theorem 1 follows from the following more general result.

Theorem 3.1. *Let M be a generalized Eulerian $(A_n(K))$-module. Then $H_i(\partial; M)$ is concentrated in degree $-\omega = -\sum_{k=1}^n \omega_k$.*

Before proving Theorem 3.1, we need to prove a few preliminary results.

Proposition 3.2. *Let M be a generalized Eulerian $(A_n(K))$-module. Then for $i = 0, 1$, the $(A_{n-1}(K))$-modules $H_i(\partial_n; M)(-\omega_n)$ are generalized Eulerian.*

Proof. Clearly, $H_i(\partial_n; M)(-\omega_n)$ are $(A_{n-1}(K))$-modules for $i = 0, 1$. We have an exact sequence of $(A_{n-1}(K))$-modules

$$0 \to H_1(\partial_n; M) \to M(\omega_n) \xrightarrow{\partial_n} M \to H_0(\partial_n; M) \to 0.$$

Note that $H_1(\partial_n; M)(-\omega_n) \subset M$. Let $\xi \in H_1(\partial_n; M)(-\omega_n)$ be homogeneous. As M is generalized Eulerian, we have

$$\big(\mathcal{E}_n - |\xi|\big)^a \xi = 0 \quad \text{for some } a \geq 1.$$

Notice that $\mathcal{E}_n = \mathcal{E}_{n-1} + \omega_n X_n \partial_n$. Also note that $X_n\partial_n$ commutes with $\mathcal{E}_{n-1}$. Thus,

$$0 = \big(\mathcal{E}_{n-1} - |\xi| + \omega_n X_n \partial_n\big)^a \xi = \Big(\big(\mathcal{E}_{n-1} - |\xi|\big)^a + (*)X_n\partial_n\Big)\xi.$$

Because $\partial_n \xi = 0$, we get $(\mathcal{E}_{n-1} - |\xi|)^a \xi = 0$. It follows that $H_1(\partial_n; M)(-\omega_n)$ is a generalized Eulerian $(A_{n-1}(K))$-module.

Let $\xi \in H_0(\partial_n; M)(-\omega_n)$ be homogeneous of degree r. Then $\xi = \alpha + \partial_n M$, where $\alpha \in M_{r-\omega_n}$. Because M is generalized Eulerian, we get

$$(\mathcal{E}_n - r + \omega_n)^a \alpha = 0 \quad \text{for some } a \geq 1.$$

Notice that $\mathcal{E}_n = \mathcal{E}_{n-1} + \omega_n X_n \partial_n = \mathcal{E}_{n-1} + \omega_n \partial_n X_n - \omega_n$, so $\mathcal{E}_n - r + \omega_n = \mathcal{E}_{n-1} - r + \omega_n \partial_n X_n$. Notice that $\partial_n X_n$ commutes with $\mathcal{E}_{n-1}$. Thus,

$$0 = (\mathcal{E}_{n-1} - r + \omega_n \partial_n X_n)^a \alpha = (\mathcal{E}_{n-1} - r)^a \alpha + \partial_n \cdot * \alpha.$$

Going mod $\partial_n M$, we get

$$(\mathcal{E}_{n-1} - r)^a \xi = 0.$$

It follows that $H_0(\partial_n; M)(-\omega_n)$ is a generalized Eulerian $(A_{n-1}(K))$-module. ∎

Remark 3.3. If M is Eulerian, then the same proof shows that $H_i(\partial_n; M)(-\omega_n)$ is an Eulerian $(A_{n-1}(K))$-module for $i = 0, 1$. However, as the proof of the following theorem shows, we can prove only that $H_1(\partial_{n-1}, \partial_n; M)(-\omega_{n-1} - \omega_n)$ is a generalized Eulerian $(A_{n-1}(K))$-module.

Proposition 3.4. *Let M be a generalized Eulerian $(A_n(K))$-module. Let $\partial = \partial_i, \partial_{i+1}, \ldots, \partial_n$; here $i \geq 2$. Then for each $j \geq 0$ the de Rham homology module*

$$H_j(\partial; M)\Big(-\sum_{k=i}^{n} \omega_k\Big)$$

is a generalized Eulerian $(A_{i-1}(K))$-module.

Proof. We prove this result by descending induction on i. For $i = n$, the result holds by Proposition 3.2. Set $\partial' = \partial_{i+1}, \ldots, \partial_n$. By induction hypothesis $H_j(\partial'; M)(-\sum_{k=i+1}^{n} \omega_k)$ is generalized Eulerian $(A_i(K))$-module. By Proposition 3.2 again, for $l = 0, 1$ and for each $j \geq 0$,

$$H_l\Big(\partial_i; H_j(\partial'; M)\Big(-\sum_{k=i+1}^{n} \omega_k\Big)\Big)(-\omega_i) = H_l\big(\partial_i; H_j(\partial'; M)\big)\Big(-\sum_{k=i}^{n} \omega_k\Big)$$

is generalized Eulerian. By Lemma 1.3 we have the exact sequence

$$0 \to H_0\big(\partial_i; H_j(\partial'; M)\big) \to H_j(\partial; M) \to H_1\big(\partial_i; H_{j-1}(\partial'; M)\big) \to 0.$$

The modules at the left and right end become generalized Eulerian after shifting by $-\sum_{k=i}^{n} \omega_k$. By Proposition 2.1 it follows that for each $j \geq 0$ the de Rham homology module

$$H_j(\partial; M)\Big(-\sum_{k=i}^{n} \omega_k\Big)$$

is a generalized Eulerian $(A_{i-1}(K))$-module. ∎

We now consider the case when $n=1$.

Proposition 3.5. *Let M be a generalized Eulerian $(A_1(K))$-module. Then for $l=0,1$ the modules $H_l(\partial_1;M)$ are concentrated in degree $-\omega_1$.*

Proof. We have an exact sequence of K-vector spaces

$$0 \to H_1(\partial_1;M) \to M(\omega_1) \xrightarrow{\partial_1} M \to H_0(\partial_1;M) \to 0.$$

Let $\xi \in H_1(\partial_1;M)(-\omega_1)$ be homogeneous and nonzero. Because $\xi \in M$, we have

$$\big(\omega_1 X_1 \partial_1 - |\xi|\big)^a \xi = 0 \quad \text{for some } a \geq 1.$$

Notice that $(\omega_1 X_1 \partial_1 - |\xi|)^a = (*)\partial_1 + (-1)^a|\xi|^a$. Because $\partial_1 \xi = 0$, we get $(-1)^a|\xi|^a\xi = 0$. Because $\xi \neq 0$, we get $|\xi| = 0$. It follows that $H_1(\partial_1;M)$ is concentrated in degree $-\omega_1$.

Let $\xi \in H_0(\partial_1, M)$ be nonzero and homogeneous of degree r. Let $\xi = \alpha + \partial_1 M$, where $\alpha \in M_r$. Because M is generalized Eulerian, we get

$$(\omega_1 X_1 \partial_1 - r)^a \alpha = 0 \quad \text{for some } a \geq 1.$$

Notice that $\omega_1 X_1 \partial_1 = \omega_1 \partial_1 X_1 - \omega_1$, so we have

$$0 = \big(\omega_1 \partial_1 X_1 - (r+\omega_1)\big)^a \alpha = \big(\partial_1 * + (-1)^a (r+\omega_1)^a\big)\alpha.$$

In $M/\partial_1 M$, we have $(-1)^a(r+\omega_1)^a\xi = 0$. Because $\xi \neq 0$, we get $r = -\omega_1$. It follows that $H_0(\partial_1;M)$ is concentrated in degree $-\omega_1$. ∎

We now give the following.

Proof of Theorem 3.1. Set $\partial' = \partial_2, \ldots, \partial_n$. By Proposition 3.4, $N_j = H_j(\partial';M)(-\sum_{k=2}^n \omega_k)$ is a generalized Eulerian $(A_1(K))$-module, for each $j \geq 0$. We use exact sequence in Lemma 1.3 and shift it by $-\sum_{k=2}^n \omega_k$ to obtain an exact sequence

$$0 \to H_0(\partial_1, N_j) \to H_j(\partial;M)\Big(-\sum_{k=2}^n \omega_k\Big) \to H_1(\partial_1, N_{j-1}) \to 0$$

for each $j \geq 0$. By Proposition 3.5, the modules on the left and right of the above exact sequence are concentrated in degree $-\omega_1$. It follows that for each $j \geq 0$ the K-vector space $H_j(\partial;M)$ is concentrated in degree $-\omega = -\sum_{k=1}^n \omega_k$. ∎

§4. Outline of proof of Theorem 2

The proof of Theorem 2 is a bit long and has a lot of technical details. For the convenience of the reader, we give an outline of the proof.

4.1. By [5, Lemma 2.7], we have $H_1(\partial, R_f) \cong H_1(\partial, H^1_{(f)}(R))$. Thus, it is sufficient to work with $H_1(\partial, R_f)$ in order to prove Theorem 2. We consider elements of R_f^m as column vectors. For $x \in R_f^m$, we write $x = (x_1, \ldots, x_m)'$; here $'$ indicates "transpose".

4.2. Let $\xi \in R_f^m \setminus R^m$. The element $(a_1/f^i, a_2/f^i, \ldots, a_m/f^i)'$, with $a_j \in R$ for all j, is said to be a *normal form* of ξ if

(1) $\xi = (a_1/f^i, a_2/f^i, \ldots, a_m/f^i)'$,
(2) f does not divide a_j for some j, and
(3) $i \geq 1$.

It can easily be shown that the normal form of ξ exists and is unique (see Proposition 5.1). Let $(a_1/f^i, a_2/f^i, \ldots, a_m/f^i)'$ be the normal form of ξ. Set $L(\xi) = i$. Notice that $L(\xi) \geq 1$.

4.3. Construction of a function $\theta\colon Z_1(\partial, R_f) \setminus R^n \to H_1(\partial(f); A)$
Let $\xi \in Z_1(\partial, R_f) \setminus R^n$. Let $(a_1/f^i, a_2/f^i, \ldots, a_n/f^i)'$ be the normal form of ξ. Thus, we have $\sum_{j=1}^n \partial/\partial X_j(a_j/f^i) = 0$, so we have

$$\frac{1}{f^i}\Big(\sum_{j=1}^n \frac{\partial a_j}{\partial X_j}\Big) - \frac{i}{f^{i+1}}\Big(\sum_{j=1}^n a_j \frac{\partial f}{\partial X_j}\Big) = 0.$$

It follows that

$$f \text{ divides } \sum_{j=1}^n a_j \frac{\partial f}{\partial X_j}.$$

Thus, $(\overline{a_1}, \ldots, \overline{a_n})' \in Z_1(\partial(f); A)$. We set

$$\theta(\xi) = \big[(\overline{a_1}, \ldots, \overline{a_n})'\big] \in H_1\big(\partial(f); A\big).$$

Remark 4.4. It can be shown that if $\xi \in Z_1(\partial, R_f)_{-\omega}$ is nonzero, then $\xi \notin R^n$ (see Section 5.2). If $L(\xi) = i$, then by Section 5.3 we have

$$\theta(\xi) \in H_1\big(\partial(f); A\big)_{(i+1)\deg f - \omega}.$$

The next result uses the fact that A is an isolated singularity.

PROPOSITION 4.5. *If $\xi \in B_1(\partial, R_f)_{-\omega}$ is nonzero, then $\theta(\xi) = 0$.*

4.6. Let $\xi \in R_f^m$. We define $L(f)$ as follows.

Case 1: $\xi \in R_f^m \setminus R^m$. Let $(a_1/f^i, a_2/f^i, \dots, a_m/f^i)'$ be the normal form of ξ. Set $L(\xi) = i$. Notice that $L(\xi) \geq 1$ in this case.

Case 2: $\xi \in R^m \setminus \{0\}$. Set $L(\xi) = 0$.

Case 3: $\xi = 0$. Set $L(\xi) = -\infty$.

The following properties of the function L can be easily verified.

PROPOSITION 4.7 (with hypotheses as above). *Let $\xi, \xi_1, \xi_2 \in R_f^m$, and let $\alpha, \alpha_1, \alpha_2 \in K$. Then we have the following:*

(1) *if $L(\xi_1) < L(\xi_2)$, then $L(\xi_1 + \xi_2) = L(\xi_2)$;*
(2) *if $L(\xi_1) = L(\xi_2)$, then $L(\xi_1 + \xi_2) \leq L(\xi_2)$;*
(3) *$L(\xi_1 + \xi_2) \leq \max\{L(\xi_1), L(\xi_2)\}$;*
(4) *if $\alpha \in K^*$, then $L(\alpha\xi) = L(\xi)$;*
(5) *$L(\alpha\xi) \leq L(\xi)$ for all $\alpha \in K$;*
(6) *$L(\alpha_1\xi_1 + \alpha_2\xi_2) \leq \max\{L(\xi_1), L(\xi_2)\}$;*
(7) *let $\xi_1, \dots, \xi_r \in R_f^m$, and let $\alpha_1, \dots, \alpha_r \in K$. Then*

$$L\Big(\sum_{j=1}^{r} \alpha_j \xi_j\Big) \leq \max\big\{L(\xi_1), L(\xi_2), \dots, L(\xi_r)\big\}.$$

4.8. We now use the fact that $H_1(\partial, R_f)$ is concentrated in degree $-\omega = -\sum_{k=1}^{n} \omega_k$. Thus,

$$H_1(\partial, R_f) = H_1(\partial, R_f)_{-\omega} = \frac{Z_1(\partial, R_f)_{-\omega}}{B_1(\partial, R_f)_{-\omega}}.$$

Let $x \in H_1(\partial, R_f)$ be nonzero. Define

$$L(x) = \min\big\{L(\xi) \;\big|\; x = [\xi], \text{ where } \xi \in Z_1(\partial, R_f)_{-\omega}\big\}.$$

It can be shown that $L(x) \geq 1$. If $x = 0$, then set

$$L(0) = -\infty.$$

We now define a function

$$\widetilde{\theta}\colon H_1(\partial, R_f) \to H_1\big(\partial(f); A\big)$$
$$x \mapsto \begin{cases} \theta(\xi) & \text{if } x \neq 0, x = [\xi], \text{ and } L(x) = L(\xi), \\ 0 & \text{if } x = 0. \end{cases}$$

It can be shown that $\widetilde{\theta}(x)$ is independent of choice of ξ (see Proposition 5.6). Also note that if $L(x) = i$, then $\widetilde{\theta}(x) \in H_1(\partial(f); A)_{(i+1)\deg f - \omega}$.

4.9. We now construct a filtration $\mathcal{F} = \{\mathcal{F}_\nu\}_{\nu \geq 0}$ of $H_1(\partial, R_f)$. Set

$$\mathcal{F}_\nu = \big\{x \in H_1(\partial, R_f) \bigm| L(x) \leq \nu\big\}.$$

In Section 5, we prove the following.

Proposition 4.10. *We have the following:*

(1) *$\mathcal{F}_\nu$ is a K subspace of $H_1(\partial, R_f)$,*
(2) *$\mathcal{F}_\nu \supseteq \mathcal{F}_{\nu-1}$ for all $\nu \geq 1$,*
(3) *$\mathcal{F}_\nu = H_1(\partial, R_f)$ for all $\nu \gg 0$,*
(4) *$\mathcal{F}_0 = 0$.*

Let $\mathcal{G} = \bigoplus_{\nu \geq 1} \mathcal{F}_\nu / \mathcal{F}_{\nu-1}$. For $\nu \geq 1$, we define

$$\eta_\nu\colon \frac{\mathcal{F}_\nu}{\mathcal{F}_{\nu-1}} \to H_1\big(\partial(f); A\big)_{(\nu+1)\deg f - \omega},$$
$$\xi \mapsto \begin{cases} 0 & \text{if } \xi = 0, \\ \widetilde{\theta}(x) & \text{if } \xi = x + \mathcal{F}_{\nu-1} \text{ is nonzero.} \end{cases}$$

It can be shown that $\eta_\nu(\xi)$ is independent of choice of x (see Proposition 5.10). Finally we prove the following result.

Theorem 4.11 (with notation as above). *For all $\nu \geq 1$,*

(1) *η_ν is K-linear, and*
(2) *η_ν is injective.*

§5. Proof of Theorem 2

In this section we give a proof of Theorem 2 with all details. The reader is advised to read the preceding section before reading this section.

We first prove the following.

Proposition 5.1. *Let $\xi \in R_f^m \setminus R^m$. Then a normal form of ξ exists and is unique.*

Proof. Existence: Let $\xi \in R_f^m \setminus R^m$. Let $\xi = (b_1/f^{i_1}, b_2/f^{i_2}, \dots, b_m/f^{i_m})'$ with $f \nmid b_j$ if $b_j \neq 0$. Note that $i_j \leq 0$ is possible. Let

$$i_r = \max\{i_j \mid i_j \geq 1 \text{ and } b_j \neq 0\}.$$

Notice that $i_r \geq 1$. Then

$$\xi = \Big(\frac{b_1 f^{i_r - i_1}}{f^{i_r}}, \frac{b_2 f^{i_r - i_2}}{f^{i_r}}, \dots, \frac{b_m f^{i_r - i_m}}{f^{i_r}}\Big)'.$$

Note that $f \nmid b_r$. Thus, the expression above is a normal form of ξ.

Uniqueness: Let $(a_1/f^i, \dots, a_m/f^i)'$ and $(b_1/f^r, \dots, b_m/f^r)'$ be two normal forms of ξ. We first assert that $i < r$ is not possible, for if this holds, then because $a_j/f^i = b_j/f^r$, we get $b_j = a_j f^{r-i}$, so $f \mid b_j$ for all j, a contradiction.

A similar argument shows that $i > r$ is not possible, so $i = r$. Thus, $a_j = b_j$ for all j. Thus, the normal form of ξ is unique. ∎

5.2. Let $\xi \in Z_1(\partial, R_f)_{-\omega}$ be nonzero. Let $\xi = (\xi_1, \dots, \xi_n)'$. Note that

$$\xi \in \big(R_f(\omega_1) \oplus R_f(\omega_2) \oplus \cdots \oplus R_f(\omega_n)\big)_{-\omega}.$$

It follows that

$$\xi_j \in (R_f)_{-\sum_{k \neq j} \omega_k}.$$

It follows that $\xi \in R_f^n \setminus R^n$.

5.3. Let $(a_1/f^i, \dots, a_n/f^i)'$ be the normal form of ξ. Then

$$\deg a_j = i \deg f - \sum_{k \neq j} \omega_k.$$

In particular, going mod f, we get

$$\overline{a_j} \in A(-\deg f + \omega_j)_{(i+1)\deg f - \omega}.$$

Notice that $\deg \partial f/\partial X_j = \deg f - \omega_j$. It follows that

$$(\overline{a_1}, \dots, \overline{a_n})' \in Z_1\big(\partial(f); A\big)_{(i+1)\deg f - \omega}.$$

Thus, $\theta(\xi) \in H_1(\partial(f); A)_{(i+1)\deg f - \omega}$.

5.4. Let $\mathbb{K} = \mathbb{K}(\partial; R_f)$ be the de Rham complex on R_f written homologically, so

$$\mathbb{K} = \cdots \to \mathbb{K}_3 \xrightarrow{\phi_3} \mathbb{K}_2 \xrightarrow{\phi_2} \mathbb{K}_1 \xrightarrow{\phi_1} \mathbb{K}_0 \to 0.$$

Here $\mathbb{K}_0 = R_f$, $\mathbb{K}_1 = \bigoplus_{k=1}^n R_f(\omega_k)$,

$$\mathbb{K}_2 = \bigoplus_{1\le i<j\le n} R_f(\omega_i + \omega_j), \qquad \text{and} \qquad \mathbb{K}_3 = \bigoplus_{1\le i<j<l\le n} R_f(\omega_i + \omega_j + \omega_l).$$

Let $\mathbb{K}' = \mathbb{K}(\partial(f); A)$ be the Koszul complex on A with respect to $\partial f/\partial X_1, \ldots, \partial f/\partial X_n$. Thus,

$$\mathbb{K}' = \cdots \to \mathbb{K}'_3 \xrightarrow{\psi_3} \mathbb{K}'_2 \xrightarrow{\psi_2} \mathbb{K}'_1 \xrightarrow{\psi_1} \mathbb{K}'_0 \to 0.$$

Here $\mathbb{K}'_0 = A$, $\mathbb{K}'_1 = \bigoplus_{k=1}^n A(-\deg f + \omega_k)$,

$$\mathbb{K}'_2 = \bigoplus_{1\le i<j\le n} A(-2\deg f + \omega_i + \omega_j), \qquad \text{and}$$

$$\mathbb{K}'_3 = \bigoplus_{1\le i<j<l\le n} A(-3\deg f + \omega_i + \omega_j + \omega_l).$$

We now prove Proposition 4.5.

Proof of Proposition 4.5. Let $u \in B_1(\partial; R_f)_{-\omega}$ be nonzero. Let $\xi \in (\mathbb{K}_2)_{-\omega}$ be homogeneous, with $\phi_2(\xi) = u$. Let $\xi = (\xi_{ij} \mid 1 \le i < j \le n)'$. Notice that

$$\xi_{ij} \in R_f(\omega_i + \omega_j)_{-\omega} = (R_f)_{-\sum_{k\neq i,j}\omega_k}.$$

It follows that $\xi \in R_f^{\binom{n}{2}} \setminus R^{\binom{n}{2}}$. Set

$$c = \min\bigl\{ j \bigm| j = L(\xi) \text{ where } \phi_2(\xi) = u \text{ and } \xi \in (\mathbb{K}_2)_{-\omega} \text{ is homogeneous}\bigr\}.$$

Notice that $c \ge 1$. Let $\xi \in (\mathbb{K}_2)_{-\omega}$ be such that $L(\xi) = c$ and $\phi_2(\xi) = u$. Let $(b_{ij}/f^c \mid 1 \le i < j \le n)'$ be the normal form of ξ. Let $u = (u_1, \ldots, u_n)'$. Then for $l = 1, \ldots, n$,

$$u_l = \sum_{i<l} \frac{\partial}{\partial X_i}\Bigl(\frac{b_{il}}{f^c}\Bigr) - \sum_{j>l} \frac{\partial}{\partial X_j}\Bigl(\frac{b_{lj}}{f^c}\Bigr).$$

So

$$u_l = \frac{f}{f^{c+1}}\Bigl(\sum_{i<l}\frac{\partial(b_{il})}{\partial X_i} - \sum_{j>l}\frac{\partial(b_{lj})}{\partial X_j}\Bigr) + \frac{c}{f^{c+1}}\Bigl(-\sum_{i<l} b_{il}\frac{\partial f}{\partial X_i} + \sum_{j>l} b_{lj}\frac{\partial f}{\partial X_j}\Bigr).$$

Set

$$v_l = c\Big(-\sum_{i<l} b_{il}\frac{\partial f}{\partial X_i} + \sum_{j>l} b_{lj}\frac{\partial f}{\partial X_j}\Big).$$

Therefore,

$$u_l = \frac{f * + v_l}{f^{c+1}}.$$

CLAIM. *$f \nmid v_l$ for some l. First assume the claim. Then $((f * + v_1)/f^{c+1}, \ldots, (f * + v_n)/f^{c+1})'$ is the normal form of u. Thus,*

$$\theta(u) = \big[(\overline{v_1}, \ldots, \overline{v_n})'\big] = \big[\psi_2(-c\overline{b})\big] = 0.$$

We now prove our claim. Suppose, if possible, that $f \mid v_l$ for all l. Then

$$\psi_2(-c\overline{b}) = (\overline{v_1}, \ldots, \overline{v_l})' = 0,$$

so $-cb \in Z_2(\partial(f); A)$. Because $H_2(\partial(f); A) = 0$, we get $-cb \in B_2(\partial(f); A)$. Thus, $-c\overline{b} = \psi_3(\overline{\gamma})$. Here

$$\gamma = (\gamma_{ijl} \mid 1 \leq i < j < l \leq n)'.$$

Thus,

$$-cb_{ij} = \sum_{k<i<j} \gamma_{kij}\frac{\partial f}{\partial X_k} - \sum_{i<k<j} \gamma_{ikj}\frac{\partial f}{\partial X_k} + \sum_{i<j<k} \gamma_{ijk}\frac{\partial f}{\partial X_k} + \alpha_{ij} f. \tag{5.4.1}$$

We need to compute the degree of γ_{ijl}. Note that $\xi \in (\mathbb{K}_2)_{-\omega}$, so

$$\frac{b_{ij}}{f^c} \in \big(R_f(\omega_i + \omega_j)\big)_{-\omega}.$$

It follows that

$$\deg b_{ij} = c \deg f - \omega + \omega_i + \omega_j. \tag{5.4.2}$$

It can be easily checked that

$$\overline{b} \in (\mathbb{K}_2')_{(c+2)\deg f - \omega},$$

so

$$\gamma \in (\mathbb{K}_3')_{(c+2)\deg f - \omega}.$$

It follows that

$$\deg \gamma_{ijl} = (c-1)\deg f - \omega + \omega_i + \omega_j + \omega_l. \tag{5.4.3}$$

We first consider the case when $c = 1$. Then by (5.4.1), we have $\alpha_{ij} = 0$. Also,

$$\deg \gamma_{ijl} = -\omega + \omega_i + \omega_j + \omega_l < 0 \quad \text{if } n > 3,$$

so if $n > 3$, we get $\gamma_{ijl} = 0$. Thus, $b = 0$, so $\xi = 0$, a contradiction.

We now consider the case when $n = 3$. Note that $\gamma = \gamma_{123}$ is a constant. Thus,

$$b = \left(\gamma\frac{\partial f}{\partial X_3}, -\gamma\frac{\partial f}{\partial X_2}, \gamma\frac{\partial f}{\partial X_3}\right)'.$$

A direct computation yields $u = 0$, a contradiction.

We now consider the case when $c \geq 2$. Notice that by (5.4.1), we have

$$\frac{-cb_{ij}}{f^c} = \frac{1}{f^c}\sum_{k<i<j}\gamma_{kij}\frac{\partial f}{\partial X_k} - \frac{1}{f^c}\sum_{i<k<j}\gamma_{ikj}\frac{\partial f}{\partial X_k} + \frac{1}{f^c}\sum_{i<j<k}\gamma_{ijk}\frac{\partial f}{\partial X_k} + \frac{\alpha_{ij}}{f^{c-1}}.$$

Notice that

$$\frac{\gamma_{kij}\partial f/\partial X_k}{f^c} = \frac{\partial}{\partial X_k}\left(\frac{\gamma_{kij}/(1-c)}{f^{c-1}}\right) - \frac{*}{f^{c-1}}.$$

Put

$$\widetilde{\gamma_*} = \frac{1}{c(c-1)}\gamma_*.$$

Thus, we obtain

$$\frac{b_{ij}}{f^c} = \sum_{k<i<j}\frac{\partial}{\partial X_k}\left(\frac{\widetilde{\gamma_{kij}}}{f^{c-1}}\right) - \sum_{i<k<j}\frac{\partial}{\partial X_k}\left(\frac{\widetilde{\gamma_{ikj}}}{f^{c-1}}\right) + \sum_{i<j<k}\frac{\partial}{\partial X_k}\left(\frac{\widetilde{\gamma_{ijk}}}{f^{c-1}}\right) + \frac{\widetilde{b_{ij}}}{f^{c-1}}.$$

Set

$$\delta = \left(\frac{\widetilde{\gamma_{ijl}}}{f^{c-1}} \;\middle|\; 1 \leq i < j < l \leq n\right) \qquad \text{and} \qquad \widetilde{\xi} = \left(\frac{\widetilde{b_{ij}}}{f^{c-1}} \;\middle|\; 1 \leq i < j \leq n\right).$$

Then

$$\xi = \phi_3(\delta) + \widetilde{\xi},$$

so we have $u = \phi_2(\xi) = \phi_2(\widetilde{\xi})$. This contradicts choice of c. ∎

5.5. By Theorem 3.1 we have

$$H_1(\partial;R_f)=H_1(\partial;R_f)_{-\omega}=\frac{Z_1(\partial;R_f)_{-\omega}}{B_1(\partial;R_f)_{-\omega}}.$$

Let $x\in H_1(\partial;R_f)$ be nonzero. Define

$$L(x)=\min\bigl\{L(\xi)\bigm| x=[\xi],\ \text{where } \xi\in Z_1(\partial,R_f)_{-\omega}\bigr\}.$$

Let $\xi=(\xi_1,\dots,\xi_n)'\in Z_1(\partial,R_f)_{-\omega}$ be such that $x=[\xi]$, so $\xi\in(\mathbb{K}_1)_{-\omega}$. Thus, $\xi_i\in R_f(+\omega_i)_{-\omega}$, so if $\xi\neq 0$, then $\xi\in R_f^n\setminus R^n$. It follows that $L(\xi)\geq 1$. Thus, $L(x)\geq 1$.

We now define a function

$$\widetilde{\theta}\colon H_1(\partial,R_f)\to H_1\bigl(\partial(f);A\bigr),$$
$$x\mapsto\begin{cases}\theta(\xi) & \text{if } x\neq 0, x=[\xi],\ \text{and } L(x)=L(\xi),\\ 0 & \text{if } x=0.\end{cases}$$

Proposition 5.6 (with hypotheses as above). *The element $\widetilde{\theta(x)}$ is independent of the choice of ξ.*

Proof. Suppose that $x=[\xi_1]=[\xi_2]$ is nonzero and that $L(x)=L(\xi_1)=L(\xi_2)=i$. Let $(a_1/f^i,\dots,a_n/f^i)'$ be the normal form of ξ_1, and let $(b_1/f^i,\dots,b_n/f^i)'$ be the normal form of ξ_2. It follows that $\xi_1=\xi_2+\delta$, where $\delta\in B_1(\partial;R_f)_{-\omega}$. By Proposition 4.7(1), we get $j=L(\delta)\leq i$. Let $(c_1/f^j,\dots,c_n/f^j)'$ be the normal form of δ. We consider two cases.

Case 1: $j<i$. Then note that $a_k=b_k+f^{i-j}c_k$ for $k=1,\dots,n$. It follows that

$$\theta(\xi_1)=\bigl[(\overline{a_1},\dots,\overline{a_n})\bigr]=\bigl[(\overline{b_1},\dots,\overline{b_n})\bigr]=\theta(\xi_2).$$

Case 2: $j=i$. Then note that $a_k=b_k+c_k$ for $k=1,\dots,n$. It follows that

$$\theta(\xi_1)=\theta(\xi_2)+\theta(\delta).$$

However, by Proposition 4.5, $\theta(\delta)=0$, so $\theta(\xi_1)=\theta(\xi_2)$. Thus, $\widetilde{\theta(x)}$ is independent of choice of ξ. ∎

5.7. We now construct a filtration $\mathcal{F} = \{\mathcal{F}_\nu\}_{\nu \geq 0}$ of $H_1(\partial, R_f)$. Set

$$\mathcal{F}_\nu = \big\{x \in H_1(\partial, R_f) \,\big|\, L(x) \leq \nu\big\}.$$

We prove the following proposition.

Proposition 5.8. *We have the following:*

(1) $\mathcal{F}_\nu$ *is a* K *subspace of* $H_1(\partial; R_f)$*,*
(2) $\mathcal{F}_\nu \supseteq \mathcal{F}_{\nu-1}$ *for all* $\nu \geq 1$*,*
(3) $\mathcal{F}_\nu = H_1(\partial; R_f)$ *for all* $\nu \gg 0$*,*
(4) $\mathcal{F}_0 = 0$*.*

Proof. (1) Let $x \in \mathcal{F}_\nu$, and let $\alpha \in K$. Then by Proposition 4.7,

$$L(\alpha x) \leq L(x) \leq \nu,$$

so $\alpha x \in \mathcal{F}_\nu$.

Let $x, x' \in \mathcal{F}_\nu$ be nonzero. Let $\xi, \xi' \in Z_1(\partial; R_f)$ be such that $x = [\xi], x' = [\xi']$ and $L(x) = L(\xi)$, $L(x') = L(\xi')$. Then $x + x' = [\xi + \xi']$. It follows that

$$L(x + x') \leq L(\xi + \xi') \leq \max\big\{L(\xi), L(\xi')\big\} \leq \nu.$$

Note that the second inequality follows from Proposition 4.7. Thus, $x + x' \in \mathcal{F}_\nu$.

(2) This is clear.

(3) Let $\mathcal{B} = \{x_1, \ldots, x_m\}$ be a K-basis of $H_1(\partial; R_f) = H_1(\partial; R_f)_{-\omega}$. Let

$$c = \max\big\{L(x_i) \,\big|\, i = 1, \ldots, m\big\}.$$

We claim that

$$\mathcal{F}_\nu = H_1(\partial; R_f) \quad \text{for all } \nu \geq c.$$

Fix $\nu \geq c$. Let $\xi_i \in Z_1(\partial; R_f)_{-\omega}$ be such that $x_i = [\xi_i]$ and $L(x_i) = L(\xi_i)$ for $i = 1, \ldots, m$.

Let $u \in H_1(\partial; R_f)$. Say that $u = \sum_{i=1}^m \alpha_i x_i$ for some $\alpha_1, \ldots, \alpha_m \in K$. Then $u = [\sum_{i=1}^m \alpha_i \xi_i]$. It follows that

$$L(u) \leq L\Big(\sum_{i=1}^m \alpha_i x_i\Big) \leq \max\big\{L(\xi_i) \,\big|\, i = 1, \ldots, m\big\} = c \leq \nu.$$

Here the second inequality follows from Proposition 4.7, so $u \in \mathcal{F}_\nu$. Thus, $\mathcal{F}_\nu = H_1(\partial; R_f)$.

(4) If $x \in H_1(\partial; R_f)$ is nonzero, then $L(x) \geq 1$. It follows that $\mathcal{F}_0 = 0$. ∎

5.9. Let $\mathcal{G} = \bigoplus_{\nu \geq 1} \mathcal{F}_\nu / \mathcal{F}_{\nu-1}$. For $\nu \geq 1$ we define

$$\eta_\nu \colon \frac{\mathcal{F}_\nu}{\mathcal{F}_{\nu-1}} \to H_1\big(\partial(f); A\big)_{(\nu+1)\deg f - \omega},$$
$$u \mapsto \begin{cases} 0 & \text{if } u = 0, \\ \widetilde{\theta}(x) & \text{if } u = x + \mathcal{F}_{\nu-1} \text{ is nonzero.} \end{cases}$$

PROPOSITION 5.10 (with hypotheses as above). *The element $\eta_\nu(u)$ is independent of choice of x.*

Proof. Suppose that $u = x + \mathcal{F}_{\nu-1} = x' + \mathcal{F}_{\nu-1}$ is nonzero. Then $x = x' + y$, where $y \in \mathcal{F}_{\nu-1}$. Because $u \neq 0$, we have $x, x' \in \mathcal{F}_\nu \setminus \mathcal{F}_{\nu-1}$, so $L(x) = L(x') = \nu$. Say that $x = [\xi]$, $x' = [\xi']$ and that $y = [\delta]$, where $\xi, \xi', \delta \in Z_1(\partial; R_f)$ with $L(\xi) = L(\xi') = \nu$ and $L(\delta) = L(y) = k \leq \nu - 1$. Thus, we have $\xi = \xi' + \delta + \alpha$, where $\alpha \in B_1(\partial; R_f)_{-\omega}$. Let $L(\alpha) = r$. Note that $r \leq \nu$.

Let $(a_1/f^\nu, \ldots, a_n/f^\nu)'$, $(a'_1/f^\nu, \ldots, a'_n/f^\nu)'$, $(b_1/f^k, \ldots, b_n/f^k)'$, and $(c_1/f^r, \ldots, c_n/f^r)\prime$ be normal forms of ξ, ξ', δ, and α, respectively. Thus, we have

$$a_j = a'_j + f^{\nu-k} b_j + f^{\nu-r} c_j \quad \text{for } j = 1, \ldots, n.$$

Case 1: $r < \nu$. In this case we have $\overline{a_j} = \overline{a'_j}$ in A for each $j = 1, \ldots, n$, so $\theta(\xi) = \theta(\xi')$. Thus, $\widetilde{\theta}(x) = \widetilde{\theta}(x')$.

Case 2: $r = \nu$. In this case notice that $\overline{a_j} = \overline{a'_j} + \overline{c_j}$ in A for each $j = 1, \ldots, n$, so $\theta(\xi) = \theta(\xi') + \theta(\alpha)$. However, $\theta(\alpha) = 0$ as $\alpha \in B_1(\partial; R_f)_{-\omega}$ (see Proposition 4.5). Thus, $\widetilde{\theta}(x) = \widetilde{\theta}(x')$. ∎

Note that neither θ nor $\widetilde{\theta}$ is linear. However, we prove the following.

PROPOSITION 5.11 (with notation as above). *For all $\nu \geq 1$, η_ν is K-linear.*

Proof. Let $u, u' \in \mathcal{F}_\nu / \mathcal{F}_{\nu-1}$. We first show that $\eta_\nu(\alpha u) = \alpha \eta_\nu(u)$ for all $\alpha \in K$. We have nothing to show if $\alpha = 0$ or if $u = 0$, so assume that $\alpha \neq 0$ and that $u \neq 0$. Say that $u = x + \mathcal{F}_{\nu-1}$. Then $\alpha u = \alpha x + \mathcal{F}_{\nu-1}$. Because $\widetilde{\theta}(\alpha x) = \alpha \widetilde{\theta}(x)$, we get the result.

Next we show that $\eta_\nu(u + u') = \eta_\nu(u) + \eta_\nu(u')$. We have nothing to show if u or u' is zero. Next we consider the case when $u + u' = 0$. Then $u = -u'$, so $\eta_\nu(u) = -\eta_\nu(u')$. Thus, in this case

$$\eta_\nu(u + u') = 0 = \eta_\nu(u) + \eta_\nu(u').$$

Now consider the case when u, u' are nonzero and $u+u'$ is nonzero. Say that $u = x + \mathcal{F}_{\nu-1}$ and that $u' = x' + \mathcal{F}_{\nu-1}$. Note that because $u + u'$ is nonzero, $x + x' \in \mathcal{F}_\nu \setminus \mathcal{F}_{\nu-1}$. Let $x = [\xi]$, and let $x' = [\xi']$, where $\xi, \xi' \in Z_1(\partial; R_f)_{-\omega}$ and $L(\xi) = L(\xi') = \nu$. Then $x + x' = [\xi + \xi']$. Note that $L(\xi + \xi') \leq \nu$ by Proposition 4.7. But $L(x+x') = \nu$, so $L(\xi+\xi') = \nu$. Let $(a_1/f^\nu, \dots, a_n/f^\nu)'$, $(a_1'/f^\nu, \dots, a_n'/f^\nu)'$ be normal forms of ξ and ξ', respectively. Note that $((a_1 + a_1')/f^\nu, \dots, (a_n + a_n')/f^\nu)'$ is the normal form of $\xi + \xi'$. It follows that $\theta(\xi + \xi') = \theta(\xi) + \theta(\xi')$. Thus, $\widetilde{\theta}(x + x') = \widetilde{\theta}(x) + \widetilde{\theta}(x')$. Therefore,

$$\eta_\nu(u + u') = \eta_\nu(u) + \eta_\nu(u'). \qquad \square$$

Finally we have the main result of this section.

Proof of Theorem 2. Let $\nu \geq 1$. By Proposition 5.11, we know that η_ν is a linear map of K-vector spaces. We now prove that η_ν is injective.

Suppose, if possible, that η_ν is not injective. Then there exists nonzero $u \in \mathcal{F}_\nu/\mathcal{F}_{\nu-1}$ with $\eta_\nu(u) = 0$. Say that $u = x + \mathcal{F}_{\nu-1}$. Also, let $x = [\xi]$, where $\xi \in Z_1(\partial; R_f)_{-\omega}$ and $L(\xi) = L(x) = \nu$. Let $(a_1/f^\nu, \dots, a_n/f^\nu)'$ be the normal form of ξ. Thus, we have

$$0 = \eta_\nu(u) = \widetilde{\theta}(x) = \theta(\xi) = \big[(\overline{a_1}, \dots, \overline{a_n})'\big].$$

It follows that $(\overline{a_1}, \dots, \overline{a_n})' = \psi_2(\overline{b})$, where $\overline{b} = (\overline{b_{ij}} \mid 1 \leq i < j \leq n)'$. It follows that, for $l = 1, \dots, n$,

$$\overline{a_l} = \sum_{i<l} \overline{b_{il}} \frac{\partial f}{\partial X_i} - \sum_{l>j} \overline{b_{lj}} \frac{\partial f}{\partial X_j}.$$

Then it follows that for $l = 1, \dots, n$ we have the following equation in R:

$$a_l = \sum_{i<l} b_{il} \frac{\partial f}{\partial X_i} - \sum_{l>j} b_{lj} \frac{\partial f}{\partial X_j} + d_l f, \tag{5.11.1}$$

for some $d_l \in R$. Note that (5.11.1) is of homogeneous elements in R. Thus, we have the following:

$$\frac{a_l}{f^\nu} = \frac{\sum_{i<l} b_{il} \frac{\partial f}{\partial X_i}}{f^\nu} - \frac{\sum_{l>j} b_{lj} \frac{\partial f}{\partial X_j}}{f^\nu} + \frac{d_l}{f^{\nu-1}}. \tag{5.11.2}$$

We consider two cases.

Case 1: $\nu \geq 2$. Set $\widetilde{b_{ij}} = -b_{il}/(c-1)$. Then note that

$$\frac{b_{il}\frac{\partial f}{\partial X_i}}{f^\nu} = \frac{\partial}{\partial X_i}\left(\frac{\widetilde{b_{il}}}{f^{\nu-1}}\right) - \frac{*}{f^{\nu-1}}.$$

By (5.11.2) we have, for $l = 1, \ldots, n$,

$$\frac{a_l}{f^\nu} = \sum_{i<l} \frac{\partial}{\partial X_i}\left(\frac{\widetilde{b_{il}}}{f^{\nu-1}}\right) - \sum_{l<j} \frac{\partial}{\partial X_j}\left(\frac{\widetilde{b_{lj}}}{f^{\nu-1}}\right) + \frac{c_l}{f^{\nu-1}}.$$

Put $\xi' = (c_1/f^{\nu-1}, \ldots, c_n/f^{\nu-1})'$, and put $\delta = (\widetilde{b_{ij}}/f^{\nu-1} \mid 1 \leq i < j \leq n)$. Then we have

$$\xi = \phi_2(\delta) + \xi',$$

so we have $x = [\xi] = [\xi']$. This yields $L(x) \leq L(\xi') \leq \nu - 1$. This is a contradiction.

Case 2: $\nu = 1$. Note that $\xi \in (\mathbb{K}_1)_{-\omega}$. Thus, for $l = 1, \ldots, n$ we have

$$\frac{a_l}{f} \in \big(R_f(\omega_l)\big)_{-\omega}.$$

It follows that

$$\deg a_l = \deg f - \sum_{k \neq l} \omega_k.$$

Also note that $\deg \partial f/\partial X_i = \deg f - \omega_i$. By comparing degrees in (5.11.1) we get $a_l = 0$ for all l. Thus, $\xi = 0$, so $x = 0$. Therefore, $u = 0$, a contradiction. ∎

§6. Example 0.1

Let $R = K[X_1, \ldots, X_n]$, and let $f = X_1^2 + \cdots + X_{n-1}^2 + X_n^m$, with $m \geq 2$. Set $A = R/(f)$. In this section we compute $H_1(\partial; H^1_{(f)}(R))$.

6.1. We give $\omega_i = \deg X_i = m$ for $i = 1, \ldots, n-1$, and we give $\omega_n = \deg X_n = 2$. Note that f is a homogeneous polynomial in R of degree $2m$. Also note that $\omega = \sum_{k=1}^n \omega_k = (n-1)m + 2$.

6.2. First note that the Jacobian ideal J of f is primary to the unique graded maximal ideal of R. It follows that A is an isolated singularity. Note that $J = (X_1, \ldots, X_{n-1}, X_n^{m-1})$. Let $H_i(J; A)$ be the ith Koszul homology of A with respect to J.

PROPOSITION 6.3. *The Hilbert series, $P(t)$, of $H_1(J; A)$ is*

$$P(t) = \sum_{k=0}^{m-2} t^{2m+2k}.$$

Proof. It is easily verified that $X_1, \ldots, X_{n-1}$ is an A-regular sequence. Set

$$B = A/(X_1, \ldots, X_{n-1})A = \frac{K[X_n]}{(X_n^m)} = K \oplus KX_n \oplus X_n^2 \oplus \cdots \oplus KX_n^{m-1}.$$

Note that we have an exact sequence

$$0 \to H_1(J; A) \to B\big(-2(m-1)\big) \xrightarrow{X_n^{m-1}} B.$$

It follows that $H_1(J; A) = X_n B(-2(m-1))$. The result follows. ∎

6.4. By Theorem 2 there exists a filtration $\mathcal{F} = \{\mathcal{F}_\nu\}_{\nu \geq 0}$ consisting of K-subspaces of $H_1(\partial; H^1_{(f)}(R))$ with $\mathcal{F}_\nu = H^{n-1}(\partial; H^1_{(f)}(R))$ for $\nu \gg 0$, $\mathcal{F}_\nu \supseteq F_{\nu-1}$, and $\mathcal{F}_0 = 0$ and injective K-linear maps

$$\eta_\nu \colon \frac{\mathcal{F}_\nu}{\mathcal{F}_{\nu-1}} \longrightarrow H_1\big(\partial(f); A\big)_{(\nu+1)\deg f - \omega}.$$

Notice that

$$(\nu + 1)\deg f - \omega = (\nu + 1)2m - (n - 1)m - 2 = (2\nu - n + 3)m - 2.$$

If $\eta_\nu \neq 0$, then by Proposition 6.3 it follows that

$$(2\nu - n + 3)m - 2 = 2m + 2j \quad \text{for some } j = 0, \ldots, m - 2.$$

Thus, we obtain

$$2\nu m = (n - 1)m + 2(j + 1). \tag{6.4.1}$$

It follows that m divides $2(j+1)$. Because $2(j+1) \leq 2m-2$, it follows that $2(j+1) = m$. Thus, m is even.

6.5. Say that $m = 2r$. Then by (6.4.1) we have

$$2\nu r = (n-1)r + r,$$

so $\nu = n/2$. It follows that n is even. Furthermore, note that $\eta_\nu = 0$ for $\nu \neq n/2$ and that if $\nu = n/2$ then by (6.3) $\dim \mathcal{F}_{n/2}/\mathcal{F}_{n/2-1} \leq 1$. It follows that in this case $\dim H_1(\partial; H^1_{(f)}(R)) \leq 1$.

6.6. In conclusion we have the following:

(1) if m is odd, then $H^{n-1}(\partial; H^1_{(f)}(R)) = 0$;
(2) if m is even, then
  (a) if n is odd then $H^{n-1}(\partial; H^1_{(f)}(R)) = 0$,
  (b) if n is even then $\dim_K H^{n-1}(\partial; H^1_{(f)}(R)) \leq 1$.

This proves Example 0.1.

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