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ERRATUM: LINEAR PROJECTIONS AND SUCCESSIVE MINIMA

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§1. Erratum

The proof of Proposition 1 and Theorem 2 in [3] is incorrect. Indeed, Sections 2.5 and 2.7 in [3] contain a vicious circle: the definition of the filtration V_i, $1 \leq i \leq n$, in Section 2.5 of that article depends on the choice of the integers n_i, when the definition of the integers n_i in Section 2.7 depends on the choice of the filtration (V_i). Thus, only Theorem 1 and Corollary 1 in [3] are proved. In the following we will prove another result instead of [3, Proposition 1].

§2. An inequality

2.1. Let K be a number field, let O_K be its ring of algebraic integers, and let $S = \mathrm{Spec}(O_K)$ be the associated scheme. Consider a Hermitian vector bundle (E, h) over S. Define the ith successive minima μ_i of (E, h) as in [3, Section 2.1]. Let $X_K \subset \mathbb{P}(E_K^\vee)$ be a smooth, geometrically irreducible curve of genus g and degree d. We assume that $X_K \subset \mathbb{P}(E_K^\vee)$ is defined by a complete linear series on X_K and that $d \geq 2g+1$. The rank of E is thus $N = d+1-g$. Let $h(X_K)$ be the Faltings height of X_K (see [3, Section 2.2]).

For any positive integer $i \leq N$, we define the integer f_i by the formulas

$$f_i = i - 1 \quad \text{if } i - 1 \leq d - 2g,$$
$$f_i = i - 1 + \alpha \quad \text{if } i - 1 = d - 2g + \alpha, 0 \leq \alpha \leq g.$$

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Fix two natural integers s and t and suppose that $2 \leq s < t \leq N-2$. When $2 \leq i \leq s$, we let

$$A_i = \frac{f_i^2}{(i-1)f_i - \sum_{j=2}^{i-1} f_j},$$

and, when $t \leq i \leq N$,

$$A_i = \frac{f_i^2}{((i-t+s)f_i - (f_1 + f_2 + \cdots + f_s + f_t + \cdots + f_{i-1}))}$$

(with the convention that $f_t + \cdots + f_{t-1} = 0$). Consider

$$A(s,t) = \max_{2 \leq i \leq s \text{ or } t \leq i \leq N} A_i.$$

THEOREM 1. *There exists a constant $c(d)$ such that the following inequality holds:*

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + \big(2d - A(s,t)(N-t+s+1)\big)\mu_1 + A(s,t)\Big(\sum_{\alpha=1}^{N+1-t} \mu_\alpha + \sum_{\alpha=N+1-s}^{N} \mu_\alpha\Big) + c(d) \geq 0.$$

2.2. To prove Theorem 1, we start by the following variant of Corollary 1 in [1].

PROPOSITION 1. *Fix an increasing sequence of integers $0 = e_1 \leq e_2 \leq \cdots \leq e_N$ and a decreasing sequence of numbers $r_1 \geq r_2 \geq \cdots \geq r_N$. Assume that $e_s = e_{s+1} = \cdots = e_{t-1}$ and that $e_{i-1} < e_i$ when $i \leq s$ or $i \geq t$. Let*

$$S = \min_{1 = i_0 < \cdots < i_\ell = N} \sum_{j=0}^{\ell-1} (r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}).$$

Then

$$S \leq B(s,t)\Big(\sum_{j=1}^{s}(r_j - r_N) + \sum_{j=t}^{N}(r_j - r_N)\Big),$$

where

$$B(s,t) = \max_{2 \leq i \leq s \text{ or } t \leq i \leq N} B_i,$$

and B_i is defined by the same formula as A_i, each f_j being replaced by e_j.

Proof. We can assume that $r_N = 0$. As in [1, proof of Theorem 1], we may first assume that $S = 1$ and seek to minimize $\sum_{j=1}^{s} r_j + \sum_{j=t}^{N} r_j$. If we graph the points (e_j, r_j), $S/2$ is the area under the Newton polygon they determine in the first quadrant. Moving the points not lying on the polygon down onto it only reduces $\sum_{j=1}^{s} r_j + \sum_{j=t}^{N} r_j$, so we may assume that all the points actually lie on the polygon. In particular, we assume that the point $(e_j, r_j) = (e_s, r_j)$ lies on this polygon when $s \le j \le t-1$. For such r_i's we have

$$S = \sum_{i=1}^{N-1} (r_i - r_{i+1})(e_i + e_{i+1}).$$

Let $\sigma_i = r_{i-1} - r_i$, $i = 2, \dots, N$. The condition that the points (e_i, r_i) lie on their Newton polygon and that the r_i decrease becomes, in terms of the σ_i,

$$\frac{\sigma_2}{e_2 - e_1} \ge \frac{\sigma_3}{e_3 - e_2} \ge \cdots \ge \frac{\sigma_s}{e_s - e_{s-1}} \ge \frac{\sigma_t}{e_t - e_{t-1}} \ge \cdots \ge 0. \tag{1}$$

Furthermore

$$\sigma_{s+1} = \cdots = \sigma_{t-1} = 0.$$

Next, we impose the constraint $\sum_{j=1}^{s} r_j + \sum_{j=t}^{N} r_j = 1$, that is,

$$\sum_{j=2}^{s} (j-1)\sigma_j + \sum_{j=t}^{N} (j-t+s)\sigma_j = 1 \tag{2}$$

(recall that $r_N = 0$). In the subspace of the points $\sigma = (\sigma_2, \dots, \sigma_s, \sigma_t, \dots, \sigma_N)$ defined by (2), the inequalities (1) define a simplex. The linear function

$$S = \sum_{2 \le j \le s} \sigma_j (e_{j-1} + e_j) + \sum_{t \le j \le N} \sigma_j (e_{j-1} + e_j)$$

must achieve its maximum on this simplex at one of the vertices, that is, a point where, for some i and α, we have

$$\alpha = \frac{\sigma_2}{e_2 - e_1} = \cdots = \frac{\sigma_i}{e_i - e_{i-1}} > \frac{\sigma_{i+1}}{e_{i+1} - e_i} = \cdots = 0.$$

We get

$$\sigma_j = \begin{cases} \alpha(e_j - e_{j-1}) & \text{if } j \le i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, using (2), we get, if $i \leq s$,

$$\alpha = \Big((i-1)e_i - \sum_{j=2}^{i-1} e_j\Big)^{-1},$$

and, when $i \geq t$,

$$\alpha = \big((i-t+s)e_i - e_1 - e_2 - \cdots - e_s - e_t - \cdots - e_{i-1}\big)^{-1}.$$

Since

$$S = \alpha \sum_{j=2}^{i} (e_j^2 - e_{j-1}^2) = \alpha e_i^2,$$

Proposition 1 follows. ∎

2.3. We come back to the situation of Theorem 1. For every complex embedding $\sigma : K \to \mathbb{C}$, the metric h defines a scalar product h_σ on $E \otimes_{O_K} \mathbb{C}$. If $v \in E$, we let

$$\|v\| = \max_\sigma \sqrt{h_\sigma(v,v)}.$$

Choose N elements $x_1, \dots, x_N$ in E, linearly independent over K and such that

$$\log \|x_i\| = \mu_{N-i+1}, \quad 1 \leq i \leq N.$$

Let $y_1, \dots, y_N \in E_K^\vee$ be the dual basis of $x_1, \dots, x_N$. Let $A(d)$ be the constant appearing in [3, Theorem 1]. From [3, Corollary 1], we deduce the following.

Lemma 1. *Assume that $1 \leq s \leq t \leq N-2$. We may choose integers n_i, $s+1 \leq i \leq t-1$, such that the following holds.*

(i) *For all i, $|n_i| \leq A(d) + d$.*

(ii) *Let $w_i = y_i$ if $1 \leq i \leq s$ or $t \leq i \leq N$, and let $w_i = y_i + n_i y_{i+1}$ if $s+1 \leq i \leq t-1$. Let $\langle w_1, \dots, w_i \rangle \subset E_K^\vee$ be the subspace spanned by $w_1, \dots, w_i$, and*

$$W_i = E_K^\vee / \langle w_1, \dots, w_i \rangle$$

$(W_0 = E_K^\vee)$. Then, when $s+1 \leq i \leq t-1$, the linear projection from $\mathbb{P}(W_{i-1})$ to $\mathbb{P}(W_i)$ does not change the degree of the image of X_K.

2.4. Let $(v_i) \in E_K^N$ be the dual basis of (w_i). We have

$$v_i = x_i \quad \text{when } i \leq s+1 \text{ or } i \geq t+1$$

and

$$v_i = x_i - n_{i-1}x_{i-1} + n_{i-1}n_{i-2}x_{i-2} - \cdots \pm n_{i-1} \cdots n_{s+1}x_{s+1}$$

when $s+2 \leq i \leq t$.

From these formulas it follows that there exists a positive constant $c_1(d)$ such that

$$\log \|v_i\| \leq r_i = \begin{cases} \mu_{N+1-i} + c_1(d) & \text{if } i \leq s \text{ or } i \geq t+1, \\ \mu_{N-s} + c_1(d) & \text{if } s+1 \leq i \leq t. \end{cases}$$

Let d_i be the degree of the image of X_K in $\mathbb{P}(W_i)$, and let $e_i = d - d_i$. By Lemma 1, we have

$$e_s = e_{s+1} = \cdots = e_{t-1}.$$

Therefore we can argue as in [2, Theorem 1] and [3, pp. 50–53] to deduce Theorem 1 from Proposition 1.

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