Nagoya Math. J. **214** (2014), 1–52
DOI 10.1215/00277630-2402198

DIFFERENTIAL OPERATORS ON QUANTIZED FLAG MANIFOLDS AT ROOTS OF UNITY, II

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To Etsuro Date on his 60th birthday

Abstract. We formulate a Beilinson–Bernstein-type derived equivalence for a quantized enveloping algebra at a root of 1 as a conjecture. It says that there exists a derived equivalence between the category of modules over a quantized enveloping algebra at a root of 1 with fixed regular Harish-Chandra central character and the category of certain twisted D-modules on the corresponding quantized flag manifold. We show that the proof is reduced to a statement about the (derived) global sections of the ring of differential operators on the quantized flag manifold. We also give a reformulation of the conjecture in terms of the (derived) induction functor.

§0. Introduction

0.1.

Let G be a connected, simply connected simple algebraic group over $\mathbb{C}$, and let H be a maximal torus of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Let Q and Λ be the root lattice and the weight lattice, respectively. Let h_G be the Coxeter number of G. We fix an odd integer $\ell > h_G$, which is prime to the order of Λ/Q and prime to 3 if $\mathfrak{g}$ is of type G_2, F_4, E_6, E_7, E_8, and we consider the De Concini–Kac-type quantized enveloping algebra U_ζ at $q=\zeta=\exp(2\pi\sqrt{-1}/\ell)$.

In [20], we started the investigation of the corresponding quantized flag manifold $\mathcal{B}_\zeta$, which is a noncommutative scheme, and the category of D-modules on it. In view of a general philosophy saying that quantized objects at roots of 1 resemble ordinary objects in positive characteristics, it

Received April 21, 2012. Revised January 28, 2013. Accepted May 14, 2013.
First published online January 7, 2014.
2010 Mathematics Subject Classification. Primary 20G05; Secondary 17B37.
The author was partially supported by Grants-in-Aid for Scientific Research, Challenging Exploratory Research award 21654005 and by Grants-in-Aid for Scientific Research (C) award 24540026 from the Japan Society for the Promotion of Science.

is natural to pursue an analogue of the theory of D-modules on the ordinary flag manifolds in positive characteristics due to Bezrukavnikov, Mirković, and Rumynin [6]. Along this line, we have established in [20] certain Azumaya properties of the ring of differential operators on the quantized flag manifold. The aim of the present article is to investigate an analogue of another main point of [6] about the Beilinson–Bernstein-type derived equivalence.

0.2.

We denote by $\mathcal{D}_{\mathcal{B}_\zeta,1}$ the sheaf of rings of differential operators on the quantized flag manifold $\mathcal{B}_\zeta$. More generally, for each $t \in H$ we have its twisted analogue denoted by $\mathcal{D}_{\mathcal{B}_\zeta,t}$. It is obtained as the specialization $\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[H]} \mathbb{C}$ of the universally twisted sheaf $\mathcal{D}_{\mathcal{B}_\zeta}$ with respect to the ring homomorphism $\mathbb{C}[H] \to \mathbb{C}$ corresponding to $t \in H$.

Let $\mathcal{B}$ be the ordinary flag manifold for G. Then we have a Frobenius morphism $\mathrm{Fr} : \mathcal{B}_\zeta \to \mathcal{B}$, which is a finite morphism from a noncommutative scheme to an ordinary scheme. Taking the direct images, we obtain sheaves $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}, \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$ $(t \in H)$ of rings on $\mathcal{B}$ (in the ordinary sense). Denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ the category of coherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. Let $Z_{\mathrm{Har}}(U_\zeta)$ be the Harish-Chandra center of U_ζ, and let $\mathbb{C}_t$ be the corresponding 1-dimensional $Z_{\mathrm{Har}}(U_\zeta)$-module. Denote by $\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)$ the category of finitely generated $U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t$-modules. Then we have a functor

$$R\Gamma(\mathcal{B}, \bullet) : D^b\big(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})\big) \to D^b\big(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)\big) \tag{0.1}$$

between derived categories. It is natural in view of [6] to conjecture that (0.1) gives an equivalence if t is regular. By imitating the argument of [6], we can show that this is true if we have

$$R\Gamma(\mathcal{B}, \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}) \cong U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda]. \tag{0.2}$$

However, we do not know how to prove (0.2) at present; hence, we can only state it as a conjecture. We have also a stronger conjecture,

$$R\Gamma\big(\mathcal{B}, \mathrm{Fr}_*(\mathcal{D}_{\mathcal{B}_\zeta})_f\big) \cong U_{\zeta,f} \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda], \tag{0.3}$$

which is the analogue of (0.2) regarding the adjoint finite parts $(\mathcal{D}_{\mathcal{B}_\zeta})_f$, $U_{\zeta,f}$ of $\mathcal{D}_{\mathcal{B}_\zeta}$, U_ζ, respectively. We will give a reformulation of (0.3) in terms of the induction functor (see Conjecture 5.2 below). It turns out that (0.3) is equivalent to some assertions in Backelin and Kremnizer [2], [3] stated to be true under certain conditions on ℓ (see Remark 5.4 below).

It is also an interesting problem to find a formulation which works even in the case when the parameter $t \in H$ is singular. In the case of Lie algebras in positive characteristics, Bezrukavnikov, Mirković, and Rumynin in [5] have succeeded in giving a more general framework, which works even for singular parameters, using partial flag manifolds (quotients of G by parabolic subgroups). In their case, the parameter space is $\mathfrak{h}^*$, and one can associate for each $h \in \mathfrak{h}^*$ a parabolic subgroup whose Levi subgroup is the centralizer of h; however, in our case the centralizer of $t \in H$ is not necessarily a Levi subgroup of a parabolic subgroup, and hence the method in [5] cannot be directly applied to our case.

0.3.

This article has the following organization. In Section 1, we recall basic facts on quantized enveloping algebras at roots of 1 and the corresponding quantized flag manifolds. In Section 2, we investigate properties of the category of D-modules. In particular, we show that (0.2) implies (0.1) for regular t and that (0.3) implies (0.2). In Sections 3 and 4, we recall some known results on the representations of quantized enveloping algebras and the induction functor, respectively. Finally, in Section 5 we give a reformulation of (0.3) in terms of the induction functor.

§1. Quantized flag manifold

1.1. Quantized enveloping algebras

1.1.1. Let G be a connected simply connected simple algebraic group over the complex number field $\mathbb{C}$. We fix Borel subgroups B^+ and B^- such that $H = B^+ \cap B^-$ is a maximal torus of G. Set $N^+ = [B^+, B^+]$, and set $N^- = [B^-, B^-]$. We denote the Lie algebras of G, B^+, B^-, H, N^+, N^- by $\mathfrak{g}$, $\mathfrak{b}^+$, $\mathfrak{b}^-$, $\mathfrak{h}$, $\mathfrak{n}^+$, $\mathfrak{n}^-$, respectively. Let $\Delta \subset \mathfrak{h}^*$ be the root system of $(\mathfrak{g}, \mathfrak{h})$. We denote by $\Lambda \subset \mathfrak{h}^*$ and $Q \subset \mathfrak{h}^*$ the weight lattice and the root lattice, respectively. For $\lambda \in \Lambda$ we denote by θ_λ the corresponding character of H. The coordinate algebra $\mathbb{C}[H]$ of H is naturally identified with the group algebra $\mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e(\lambda)$ via the correspondence $\theta_\lambda \leftrightarrow e(\lambda)$ for $\lambda \in \Lambda$. We take a system of positive roots Δ^+ such that $\mathfrak{b}^+$ is the sum of weight spaces with weights in $\Delta^+ \cup \{0\}$. Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots, and let $\{\varpi_i\}_{i \in I}$ be the corresponding set of fundamental weights. We denote by Λ^+ the set of dominant integral weights. We set $Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geqq 0} \alpha_i$. Let $W \subset$

$GL(\mathfrak{h}^*)$ be the Weyl group. For $i \in I$ we denote by $s_i \in W$ the corresponding simple reflection. We take a W-invariant symmetric bilinear form

$$(\, , \,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$$

such that $(\alpha, \alpha) = 2$ for short roots α. For $\alpha \in \Delta$ we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. For $i \in I$ we fix $\bar{e}_i \in \mathfrak{g}_{\alpha_i}$, $\bar{f}_i \in \mathfrak{g}_{-\alpha_i}$ such that $[\bar{e}_i, \bar{f}_i] = \alpha_i^\vee$ under the identification $\mathfrak{h} = \mathfrak{h}^*$ induced by $(\, , \,)$.

1.1.2. For $n \in \mathbb{Z}_{\geqq 0}$ we set

$$[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}],$$
$$[n]_t! = [n]_t [n-1]_t \cdots [2]_t [1]_t \in \mathbb{Z}[t, t^{-1}].$$

We denote by $U_{\mathbb{F}}$ the quantized enveloping algebra over $\mathbb{F} = \mathbb{Q}(q^{1/|\Lambda/Q|})$ associated to $\mathfrak{g}$. Namely, $U_{\mathbb{F}}$ is the associative algebra over $\mathbb{F}$ generated by elements

$$k_\lambda \quad (\lambda \in \Lambda), \qquad e_i, f_i \quad (i \in I)$$

satisfying the relations

$$k_0 = 1, \qquad k_\lambda k_\mu = k_{\lambda + \mu} \quad (\lambda, \mu \in \Lambda),$$
$$k_\lambda e_i k_\lambda^{-1} = q^{(\lambda, \alpha_i)} e_i \quad (\lambda \in \Lambda, i \in I),$$
$$k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda, \alpha_i)} f_i \quad (\lambda \in \Lambda, i \in I),$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I),$$
$$\sum_{n=0}^{1-a_{ij}} (-1)^n e_i^{(1-a_{ij}-n)} e_j e_i^{(n)} = 0 \quad (i, j \in I, i \neq j),$$
$$\sum_{n=0}^{1-a_{ij}} (-1)^n f_i^{(1-a_{ij}-n)} f_j f_i^{(n)} = 0 \quad (i, j \in I, i \neq j),$$

where $q_i = q^{(\alpha_i, \alpha_i)/2}, k_i = k_{\alpha_i}, a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ for $i, j \in I$, and

$$e_i^{(n)} = e_i^n / [n]_{q_i}!, \qquad f_i^{(n)} = f_i^n / [n]_{q_i}!$$

for $i\in I$ and $n\in\mathbb{Z}_{\geqq 0}$. We will use the Hopf algebra structure of $U_{\mathbb{F}}$ given by

$$\begin{aligned}
&\Delta(k_\lambda)=k_\lambda\otimes k_\lambda \quad (\lambda\in\Lambda),\\
&\Delta(e_i)=e_i\otimes 1+k_i\otimes e_i, \qquad \Delta(f_i)=f_i\otimes k_i^{-1}+1\otimes f_i \quad (i\in I),\\
&\varepsilon(k_\lambda)=1, \qquad \varepsilon(e_i)=\varepsilon(f_i)=0 \quad (\lambda\in\Lambda, i\in I),\\
&S(k_\lambda)=k_\lambda^{-1}, \qquad S(e_i)=-k_i^{-1}e_i, \qquad S(f_i)=-f_ik_i \quad (\lambda\in\Lambda, i\in I).
\end{aligned}$$

Define subalgebras $U_{\mathbb{F}}^0$, $U_{\mathbb{F}}^+$, $U_{\mathbb{F}}^-$, $U_{\mathbb{F}}^{\geqq 0}$, $U_{\mathbb{F}}^{\leqq 0}$ of $U_{\mathbb{F}}$ by

$$\begin{gathered}
U_{\mathbb{F}}^0=\langle k_\lambda \mid \lambda\in\Lambda\rangle, \qquad U_{\mathbb{F}}^+=\langle e_i \mid i\in I\rangle, \qquad U_{\mathbb{F}}^-=\langle f_i \mid i\in I\rangle,\\
U_{\mathbb{F}}^{\geqq 0}=\langle k_\lambda, e_i \mid \lambda\in\Lambda, i\in I\rangle, \qquad U_{\mathbb{F}}^{\leqq 0}=\langle k_\lambda, f_i \mid \lambda\in\Lambda, i\in I\rangle.
\end{gathered}$$

The multiplication of $U_{\mathbb{F}}$ induces isomorphisms

$$U_{\mathbb{F}}\cong U_{\mathbb{F}}^-\otimes U_{\mathbb{F}}^0\otimes U_{\mathbb{F}}^+\cong U_{\mathbb{F}}^+\otimes U_{\mathbb{F}}^0\otimes U_{\mathbb{F}}^-, \tag{1.1}$$

$$U_{\mathbb{F}}^{\geqq 0}\cong U_{\mathbb{F}}^0\otimes U_{\mathbb{F}}^+\cong U_{\mathbb{F}}^+\otimes U_{\mathbb{F}}^0, \tag{1.2}$$

$$U_{\mathbb{F}}^{\leqq 0}\cong U_{\mathbb{F}}^0\otimes U_{\mathbb{F}}^-\cong U_{\mathbb{F}}^-\otimes U_{\mathbb{F}}^0, \tag{1.3}$$

of $\mathbb{F}$-modules. The fact (1.1) is called the *triangular decomposition* of $U_{\mathbb{F}}$. For $\gamma\in Q$ we set

$$U_{\mathbb{F},\gamma}^{\pm}=\left\{u\in U_{\mathbb{F}}^{\pm} \;\middle|\; k_\mu u k_{-\mu}=q^{(\gamma,\mu)}u \ (\mu\in\Lambda)\right\}.$$

Then we have

$$U_{\mathbb{F}}^{\pm}=\bigoplus_{\gamma\in Q^+}U_{\mathbb{F},\pm\gamma}^{\pm}.$$

For $i\in I$ we denote by T_i the automorphism of the algebra $U_{\mathbb{F}}$ given by

$$\begin{aligned}
&T_i(k_\mu)=k_{s_i\mu}(\mu\in\Lambda),\\
&T_i(e_j)=\begin{cases}\sum_{k=0}^{-a_{ij}}(-1)^kq_i^{-k}e_i^{(-a_{ij}-k)}e_je_i^{(k)} & (j\in I, j\neq i),\\ -f_ik_i & (j=i),\end{cases}\\
&T_i(f_j)=\begin{cases}\sum_{k=0}^{-a_{ij}}(-1)^kq_i^{k}f_i^{(k)}f_jf_i^{(-a_{ij}-k)} & (j\in I, j\neq i),\\ -k_i^{-1}e_i & (j=i)\end{cases}
\end{aligned}$$

(see [15]). Let w_0 be the longest element of W. We fix a reduced expression

$$w_0 = s_{i_1} \cdots s_{i_N}$$

of w_0, and we set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad (1 \leqq k \leqq N).$$

Then we have $\Delta^+ = \{\beta_k \mid 1 \leqq k \leqq N\}$. For $1 \leqq k \leqq N$ we set

$$e_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(e_{i_k}), \qquad f_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f_{i_k}).$$

Then $\{e_{\beta_N}^{m_N} \cdots e_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$ (resp., $\{f_{\beta_N}^{m_N} \cdots f_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$) is an $\mathbb{F}$-basis of $U_{\mathbb{F}}^+$ (resp., $U_{\mathbb{F}}^-$), called the *PBW (Poincaré–Birkhoff–Witt) basis* (see [14]). We have $e_{\alpha_i} = e_i$ and $f_{\alpha_i} = f_i$ for any $i \in I$.

Denote by

$$\tau : U_{\mathbb{F}}^{\geqq 0} \times U_{\mathbb{F}}^{\leqq 0} \to \mathbb{F} \tag{1.4}$$

the Drinfeld pairing. It is characterized as the unique bilinear form satisfying

$$\begin{aligned}
\tau(x, y_1 y_2) &= (\tau \otimes \tau)\big(\Delta(x), y_1 \otimes y_2\big) \quad (x \in U_{\mathbb{F}}^{\geqq 0}, y_1, y_2 \in U_{\mathbb{F}}^{\leqq 0}),\\
\tau(x_1 x_2, y) &= (\tau \otimes \tau)\big(x_2 \otimes x_1, \Delta(y)\big) \quad (x_1, x_2 \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}),\\
\tau(k_\lambda, k_\mu) &= q^{-(\lambda,\mu)} \quad (\lambda, \mu \in \Lambda),\\
\tau(k_\lambda, f_i) &= \tau(e_i, k_\lambda) = 0 \quad (\lambda \in \Lambda, i \in I),\\
\tau(e_i, f_j) &= \delta_{ij}/(q_i^{-1} - q_i) \quad (i, j \in I)
\end{aligned}$$

(see [15], [18]). It also satisfies the following.

Lemma 1.1 ([15, Section 1.2], [18, Proposition 2.1.1]). *We have the following:*

(i) $\tau(S(x), S(y)) = \tau(x, y)$ *for* $x \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}$;

(ii) *for* $x \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}$ *we have*

$$yx = \sum_{(x)_2,(y)_2} \tau\big(x_{(0)}, S(y_{(0)})\big)\tau(x_{(2)}, y_{(2)}) x_{(1)} y_{(1)},$$

$$xy = \sum_{(x)_2,(y)_2} \tau(x_{(0)}, y_{(0)})\tau\big(x_{(2)}, S(y_{(2)})\big) y_{(1)} x_{(1)};$$

(iii) $\tau(xk_\lambda, yk_\mu) = q^{-(\lambda,\mu)}\tau(x,y)$ *for* $\lambda,\mu\in\Lambda, x\in U_{\mathbb{F}}^+, y\in U_{\mathbb{F}}^-$;
(iv) $\tau(U^+_{\mathbb{F},\beta}, U^-_{\mathbb{F},-\gamma}) = \{0\}$ *for* $\beta,\gamma\in Q^+$ *with* $\beta\neq\gamma$;
(v) *for any* $\beta\in Q^+$*, the restriction* $\tau|_{U^+_{\mathbb{F},\beta}\times U^-_{\mathbb{F},-\beta}}$ *is nondegenerate.*

We define an algebra homomorphism

$$\mathrm{ad}: U_{\mathbb{F}} \to \mathrm{End}_{\mathbb{F}}(U_{\mathbb{F}})$$

by

$$\mathrm{ad}(u)(v) = \sum_{(u)} u_{(0)} v(Su_{(1)}) \quad (u,v\in U_{\mathbb{F}}).$$

1.1.3. We fix an integer $\ell>1$ satisfying
(a) ℓ is odd;
(b) ℓ is prime to 3 if G is of type G_2, F_4, E_6, E_7, E_8;
(c) ℓ is prime to $|\Lambda/Q|$;
and a primitive ℓth root $\zeta'\in\mathbb{C}$ of 1. Define a subring $\mathbb{A}$ of $\mathbb{F}$ by

$$\mathbb{A} = \left\{ f(q^{1/|\Lambda/Q|}) \;\middle|\; f(x)\in\mathbb{Q}(x), f \text{ is regular at } x=\zeta' \right\}.$$

We set $\zeta = (\zeta')^{|\Lambda/Q|}$. We note that ζ is also a primitive ℓth root of 1 by condition (c).

We denote by $U^L_{\mathbb{A}}$, $U_{\mathbb{A}}$ the $\mathbb{A}$-forms of $U_{\mathbb{F}}$ called the *Lusztig form* and the *De Concini–Kac form*, respectively. Namely, we have

$$U^L_{\mathbb{A}} = \langle e_i^{(m)}, f_i^{(m)}, k_\lambda \mid i\in I, m\in\mathbb{Z}_{\geqq 0}, \lambda\in\Lambda\rangle_{\mathbb{A}\text{-alg}} \subset U_{\mathbb{F}},$$
$$U_{\mathbb{A}} = \langle e_i, f_i, k_\lambda \mid i\in I, \lambda\in\Lambda\rangle_{\mathbb{A}\text{-alg}} \subset U_{\mathbb{F}}.$$

We have obviously $U_{\mathbb{A}}\subset U^L_{\mathbb{A}}$. The Hopf algebra structure of $U_{\mathbb{F}}$ induces Hopf algebra structures over $\mathbb{A}$ of $U^L_{\mathbb{A}}$ and $U_{\mathbb{A}}$. We set

$$U^{L,\flat}_{\mathbb{A}} = U^L_{\mathbb{A}}\cap U^\flat_{\mathbb{F}}, \qquad U^\flat_{\mathbb{A}} = U_{\mathbb{A}}\cap U^\flat_{\mathbb{F}} \quad (\flat = +,-,0,\geqq 0,\leqq 0),$$
$$U^{L,\pm}_{\mathbb{A},\pm\gamma} = U^L_{\mathbb{A}}\cap U^\pm_{\mathbb{F},\pm\gamma}, \qquad U^\pm_{\mathbb{A},\pm\gamma} = U_{\mathbb{A}}\cap U^\pm_{\mathbb{F},\pm\gamma} \quad (\gamma\in Q^+).$$

Then we have triangular decompositions

$$U^L_{\mathbb{A}} \cong U^{L,-}_{\mathbb{A}}\otimes_{\mathbb{A}} U^{L,0}_{\mathbb{A}}\otimes_{\mathbb{A}} U^{L,+}_{\mathbb{A}},$$
$$U_{\mathbb{A}} \cong U^-_{\mathbb{A}}\otimes_{\mathbb{A}} U^0_{\mathbb{A}}\otimes_{\mathbb{A}} U^+_{\mathbb{A}}.$$

Moreover, we have

$$U_{\mathbb{A}}^{L,\pm}=\bigoplus_{\gamma\in Q^+}U_{\mathbb{A},\pm\gamma}^{L,\pm},\qquad U_{\mathbb{A}}^{\pm}=\bigoplus_{\gamma\in Q^+}U_{\mathbb{A},\pm\gamma}^{\pm}.$$

The Drinfeld pairing (1.4) induces

$$(1.5)\qquad {}^L\tau_{\mathbb{A}}:U_{\mathbb{A}}^{L,\geqq 0}\times U_{\mathbb{A}}^{\leqq 0}\to\mathbb{A},\qquad \tau_{\mathbb{A}}^L:U_{\mathbb{A}}^{\geqq 0}\times U_{\mathbb{A}}^{L,\leqq 0}\to\mathbb{A}.$$

Lemma 1.2. *We have* $\mathrm{ad}(U_{\mathbb{A}}^L)(U_{\mathbb{A}})\subset U_{\mathbb{A}}$.

Proof. It is sufficient to show that

$$(1.6)\qquad \mathrm{ad}(k_\lambda)(U_{\mathbb{A}})\subset U_{\mathbb{A}}\quad(\lambda\in\Lambda),$$

$$(1.7)\qquad \mathrm{ad}(e_i^{(n)})(U_{\mathbb{A}})\subset U_{\mathbb{A}}\quad(i\in I,n\in\mathbb{Z}_{\geqq 0}),$$

$$(1.8)\qquad \mathrm{ad}(f_i^{(n)})(U_{\mathbb{A}})\subset U_{\mathbb{A}}\quad(i\in I,n\in\mathbb{Z}_{\geqq 0}).$$

The proof of (1.6) is easy and omitted. By the formulas

$$\mathrm{ad}(x)(uv)=\sum_{(x)}\mathrm{ad}(x_{(0)})(u)\mathrm{ad}(x_{(1)})(v)\quad(x\in U_{\mathbb{A}}^L,u,v\in U_{\mathbb{A}}),$$

$$\Delta(e_i^{(n)})=\sum_{r=0}^n q_i^{r(n-r)}e_i^{(n-r)}k_i^r\otimes e_i^{(r)}\quad(i\in I,n\geqq 0),$$

$$\Delta(f_i^{(n)})=\sum_{r=0}^n q_i^{-r(n-r)}f_i^{(r)}\otimes k_i^{-r}f_i^{(n-r)}\quad(i\in I,n\geqq 0),$$

we have only to show that

$$(1.9)\qquad \mathrm{ad}(e_i^{(n)})(u)\in U_{\mathbb{A}}\quad(i\in I,n\in\mathbb{Z}_{\geqq 0},u=k_\lambda,e_j,f_jk_j),$$

$$(1.10)\qquad \mathrm{ad}(f_i^{(n)})(u)\in U_{\mathbb{A}}\quad(i\in I,n\in\mathbb{Z}_{\geqq 0},u=k_\lambda,e_j,f_j).$$

For $\lambda\in\Lambda$, $i,j\in I$ with $i\neq j$ and $n\in\mathbb{Z}_{>0}$, we have

$$\mathrm{ad}(e_i^{(n)})(k_\lambda)=\frac{(-1)^nq_i^{n(n-1)}}{[n]_{q_i}!}\Big(\prod_{j=0}^{n-1}(q_i^{(\lambda,\alpha_i^\vee)}-q_i^{-2j})\Big)e_i^nk_\lambda,$$

$$\mathrm{ad}(e_i^{(n)})(e_i)=q_i^{-n(n+1)/2}(q_i-q_i^{-1})^ne_i^{n+1},$$

$$\mathrm{ad}(e_i^{(n)})(e_j)=\begin{cases}\sum_{r=0}^{n}(-1)^r q_i^{r(n-1+a_{ij})}e_i^{(n-r)}e_je_i^{(r)} & (n<1-a_{ij}),\\ 0 & (n\geqq 1-a_{ij}),\end{cases}$$

$$\mathrm{ad}(e_i^{(n)})(f_ik_i)=\begin{cases}(k_i^2-1)/(q_i-q_i^{-1}) & (n=1),\\ (-1)^{n-1}q_i^{(n-1)(n+2)/2}(q_i-q_i^{-1})^{n-2}e_i^{n-1}k_i^2 & (n>1),\end{cases}$$

$$\mathrm{ad}(e_i^{(n)})(f_jk_j)=0,$$

and hence (1.9) holds. (Note that $[r]_{q_i}!$ is invertible in $\mathbb{A}$ for $r\leqq -a_{ij}$.) The proof of (1.10) is similar and omitted. ∎

1.1.4. Let us consider the specialization

$$\mathbb{A}\to\mathbb{C}\quad (q^{1/|\Lambda/Q|}\mapsto\zeta').$$

Note that q is mapped to $\zeta=(\zeta')^{|\Lambda/Q|}\in\mathbb{C}$, which is also a primitive ℓth root of 1. We set

$$U_\zeta^L=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^L,\qquad U_\zeta=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}},$$
$$U_\zeta^{L,\flat}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^{L,\flat},\qquad U_\zeta^{\flat}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^{\flat}\quad(\flat=+,-,0,\geqq 0,\leqq 0),$$
$$U_{\zeta,\pm\gamma}^{L,\pm}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A},\pm\gamma}^{L,\pm},\qquad U_{\zeta,\pm\gamma}^{\pm}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A},\pm\gamma}^{\pm}\quad(\gamma\in Q^+).$$

Then U_ζ^L and U_ζ are Hopf algebras over $\mathbb{C}$, and we have triangular decompositions

$$U_\zeta^L\cong U_\zeta^{L,-}\otimes U_\zeta^{L,0}\otimes U_\zeta^{L,+},$$
$$U_\zeta\cong U_\zeta^{-}\otimes U_\zeta^{0}\otimes U_\zeta^{+}.$$

Moreover, we have

$$U_\zeta^{L,\pm}=\bigoplus_{\gamma\in Q^+}U_{\zeta,\pm\gamma}^{L,\pm},\qquad U_\zeta^{\pm}=\bigoplus_{\gamma\in Q^+}U_{\zeta,\pm\gamma}^{\pm}.$$

By De Concini and Kac [8, Proposition 1.7], we have the following.

LEMMA 1.3.

(i) *The set* $\{e_{\beta_N}^{m_N}\cdots e_{\beta_1}^{m_1}\mid m_1,\dots,m_N\geqq 0\}$ *(resp.,* $\{f_{\beta_N}^{m_N}\cdots f_{\beta_1}^{m_1}\mid m_1,\dots,m_N\geqq 0\}$*) forms a* $\mathbb{C}$*-basis of* U_ζ^+ *(resp.,* U_ζ^-*).*

(ii) *The set* $\{k_\lambda\mid\lambda\in\Lambda\}$ *forms a* $\mathbb{C}$*-basis of* U_ζ^0*.*

The Drinfeld pairings (1.5) induce

$$ {}^{L}\tau_\zeta : U_\zeta^{L,\geqq 0} \times U_\zeta^{\leqq 0} \to \mathbb{C}, \qquad \tau_\zeta^{L} : U_\zeta^{\geqq 0} \times U_\zeta^{L,\leqq 0} \to \mathbb{C}. \tag{1.11} $$

Moreover, we have the following (see [20, Lemma 1.5]).

PROPOSITION 1.4. *For any $\gamma \in Q^+$, the restrictions of ${}^{L}\tau_\zeta$ and τ_ζ^{L} to*

$$ U_{\zeta,\gamma}^{L,+} \times U_{\zeta,-\gamma}^{-} \to \mathbb{C}, \qquad U_{\zeta,\gamma}^{-} \times U_{\zeta,-\gamma}^{L,-} \to \mathbb{C}, $$

respectively, are nondegenerate.

By Lemma 1.2 we have an algebra homomorphism

$$ \mathrm{ad} : U_\zeta^{L} \to \mathrm{End}_{\mathbb{C}}(U_\zeta). $$

In general, for a Lie algebra $\mathfrak{s}$ we denote its enveloping algebra by $U(\mathfrak{s})$. We denote by

$$ \pi : U_\zeta^{L} \to U(\mathfrak{g}) \tag{1.12} $$

Lusztig's Frobenius homomorphism (see [14]). Namely, π is the $\mathbb{C}$-algebra homomorphism given by

$$ \pi(e_i^{(m)}) = \begin{cases} \bar{e}_i^{(m/\ell)} & (\ell | m) \\ 0 & (\ell \nmid m), \end{cases} \qquad \pi(f_i^{(m)}) = \begin{cases} \bar{f}_i^{(m/\ell)} & (\ell | m) \\ 0 & (\ell \nmid m), \end{cases} \qquad \pi(k_\lambda) = 1 $$

for $i \in I$, $m \in \mathbb{Z}_{\geqq 0}$, $\lambda \in \Lambda$. Here, $\bar{e}_i^{(n)} = \bar{e}_i^{n}/n!$, $\bar{f}_i^{(n)} = \bar{f}_i^{n}/n!$ for $i \in I$ and $n \in \mathbb{Z}_{\geqq 0}$. Then π is a homomorphism of Hopf algebras.

We recall the description of the center $Z(U_\zeta)$ of the algebra U_ζ due to De Concini and Kac [8, Section 3] and De Concini and Procesi [9, Section 21]. Denote by $Z(U_{\mathbb{F}})$ the center of $U_{\mathbb{F}}$, and define a subalgebra $Z_{\mathrm{Har}}(U_\zeta)$ of $Z(U_\zeta)$ by

$$ Z_{\mathrm{Har}}(U_\zeta) = \mathrm{Im}\big(Z(U_{\mathbb{F}}) \cap U_{\mathbb{A}} \to U_\zeta\big). $$

We define a shifted action of W on the group algebra $\mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e(\lambda)$ of Λ by

$$ w \circ e(\lambda) = \zeta^{(w\lambda - \lambda, \rho)} e(w\lambda) \quad (w \in W, \lambda \in \Lambda). \tag{1.13} $$

Let

$$ \iota : Z_{\mathrm{Har}}(U_\zeta) \to \mathbb{C}[\Lambda] \tag{1.14} $$

be the composite of

$$Z_{\mathrm{Har}}(U_\zeta) \hookrightarrow U_\zeta \cong U_\zeta^- \otimes U_\zeta^0 \otimes U_\zeta^+ \xrightarrow{\varepsilon\otimes 1\otimes\varepsilon} U_\zeta^0 \cong \mathbb{C}[\Lambda],$$

where $U_\zeta^0 \cong \mathbb{C}[\Lambda]$ is given by $k_\lambda \leftrightarrow e(\lambda)$. Then by [8, Lemma 3.9], ι is an injective algebra homomorphism with image

$$\mathbb{C}[2\Lambda]^{W\circ} = \{ f \in \mathbb{C}[2\Lambda] \mid w \circ f = f \ (\forall w \in W) \}.$$

In particular, we have an isomorphism

$$Z_{\mathrm{Har}}(U_\zeta) \simeq \mathbb{C}[2\Lambda]^{W\circ} \tag{1.15}$$

of $\mathbb{C}$-algebras. By [8, Section 3.1] the elements

$$e_\beta^\ell, \qquad f_\beta^\ell, \qquad k_{\ell\lambda} \quad (\beta \in \Delta^+, \lambda \in \Lambda)$$

are central in U_ζ. Let $Z_{\mathrm{Fr}}(U_\zeta)$ be the subalgebra of U_ζ generated by them. It is a Hopf subalgebra of U_ζ. Define an algebraic subgroup K of $B^+ \times B^-$ by

$$K = \{ (gh, g'h^{-1}) \mid h \in H, g \in N^+, g' \in N^- \}.$$

By [9, Section 19.1] we have an isomorphism

$$Z_{\mathrm{Fr}}(U_\zeta) \cong \mathbb{C}[K] \tag{1.16}$$

of Hopf algebras (see also [10, Theorem 7.4]). We refer the reader to [20, Section 1.5] for the explicit description of the isomorphism (1.16). By [9], $Z(U_\zeta)$ is generated by $Z_{\mathrm{Fr}}(U_\zeta)$ and $Z_{\mathrm{Har}}(U_\zeta)$. Moreover, we have an isomorphism

$$Z(U_\zeta) \cong Z_{\mathrm{Har}}(U_\zeta) \otimes_{Z_{\mathrm{Har}}(U_\zeta)\cap Z_{\mathrm{Fr}}(U_\zeta)} Z_{\mathrm{Fr}}(U_\zeta) \quad (z_1 z_2 \leftrightarrow z_1 \otimes z_2)$$

of algebras.

1.2. Sheaves on quantized flag manifolds

1.2.1. We denote by $C_{\mathbb{F}}$ the subspace of $U_{\mathbb{F}}^* = \mathrm{Hom}_{\mathbb{F}}(U_{\mathbb{F}}, \mathbb{F})$ spanned by the matrix coefficients of finite-dimensional $U_{\mathbb{F}}$-modules of type 1 in the sense of Lusztig, and we denote by

$$\langle\ ,\ \rangle : C_{\mathbb{F}} \times U_{\mathbb{F}} \to \mathbb{F} \tag{1.17}$$

the canonical pairing. Then $C_{\mathbb{F}}$ is endowed with a Hopf algebra structure dual to $U_{\mathbb{F}}$ via (1.17). We have a $U_{\mathbb{F}}$-bimodule structure of $C_{\mathbb{F}}$ given by

$$\langle u_1 \cdot \varphi \cdot u_2, u \rangle = \langle \varphi, u_2 u u_1 \rangle \quad (\varphi \in C_{\mathbb{F}}, u, u_1, u_2 \in U_{\mathbb{F}}).$$

Define a Λ-graded ring $A_{\mathbb{F}} = \bigoplus_{\lambda \in \Lambda^+} A_{\mathbb{F}}(\lambda)$ by

$$A_{\mathbb{F}} = \left\{ \varphi \in C_{\mathbb{F}} \,\middle|\, \varphi \cdot f_i = 0 \ (i \in I) \right\},$$
$$A_{\mathbb{F}}(\lambda) = \left\{ \varphi \in A_{\mathbb{F}} \,\middle|\, \varphi \cdot k_\mu = q^{(\mu,\lambda)} \varphi \ (\mu \in \Lambda) \right\}.$$

Note that $A_{\mathbb{F}}$ is a left $U_{\mathbb{F}}$-submodule of $C_{\mathbb{F}}$. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$, we set

$$A_{\mathbb{F}}(\lambda)_\xi = \left\{ \varphi \in A_{\mathbb{F}}(\lambda) \,\middle|\, k_\mu \cdot \varphi = q^{(\xi,\mu)} \varphi \right\}.$$

Then we have

$$A_{\mathbb{F}}(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_{\mathbb{F}}(\lambda)_\xi.$$

We define $\mathbb{A}$-forms $C_{\mathbb{A}}$, $A_{\mathbb{A}}$, $A_{\mathbb{A}}(\lambda)$ $(\lambda \in \Lambda^+)$ of $C_{\mathbb{F}}$, $A_{\mathbb{F}}$, $A_{\mathbb{F}}(\lambda)$, respectively, by

$$C_{\mathbb{A}} = \left\{ \varphi \in C_{\mathbb{F}} \,\middle|\, \langle \varphi, U^L_{\mathbb{A}} \rangle \subset \mathbb{A} \right\}, \qquad A_{\mathbb{A}} = A_{\mathbb{F}} \cap C_{\mathbb{A}}, \qquad A_{\mathbb{A}}(\lambda) = A_{\mathbb{F}}(\lambda) \cap C_{\mathbb{A}}.$$

Then $C_{\mathbb{A}}$ is a Hopf algebra over $\mathbb{A}$, and $A_{\mathbb{A}}$ is its $\mathbb{A}$-subalgebra. Moreover, $C_{\mathbb{A}}$ is a $U^L_{\mathbb{A}}$-bimodule, and $A_{\mathbb{A}}$ is its left $U^L_{\mathbb{A}}$-submodule. We also set $A_{\mathbb{A}}(\lambda)_\xi = A_{\mathbb{F}}(\lambda)_\xi \cap A_{\mathbb{A}}$ for $\lambda \in \Lambda^+$, $\xi \in \Lambda$.

We set

$$C_\zeta = \mathbb{C} \otimes_{\mathbb{A}} C_{\mathbb{A}}, \qquad A_\zeta = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}, \qquad A_\zeta(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\lambda) \quad (\lambda \in \Lambda^+).$$

Then C_ζ is a Hopf algebra over $\mathbb{C}$. Moreover, the $U_{\mathbb{F}}$-bimodule structure of $C_{\mathbb{F}}$ induces a U^L_ζ-bimodule structure of C_ζ. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$, we set $A_\zeta(\lambda)_\xi = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\lambda)_\xi$. Then we have

$$A_\zeta(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_\zeta(\lambda)_\xi.$$

We have a natural pairing

$$\langle\ ,\ \rangle : C_\zeta \times U^L_\zeta \to \mathbb{C} \tag{1.18}$$

induced by (1.17).

1.2.2. For a ring (resp., Λ-graded ring) $\mathcal{R}$ we denote by $\mathrm{Mod}(\mathcal{R})$ (resp., $\mathrm{Mod}_\Lambda(\mathcal{R})$) the category of $\mathcal{R}$-modules (resp., Λ-graded left $\mathcal{R}$-modules). Assume that we are given a homomorphism $\jmath : A \to B$ of Λ-graded rings satisfying

$$\jmath\big(A(\lambda)\big)B(\mu) = B(\mu)\jmath\big(A(\lambda)\big) \quad (\lambda, \mu \in \Lambda). \tag{1.19}$$

For $M \in \mathrm{Mod}_\Lambda(B)$, let $\mathrm{Tor}(M)$ be the subset of M consisting of $m \in M$ such that there exists $\lambda \in \Lambda^+$ satisfying $\jmath(A(\lambda+\mu))m = \{0\}$ for any $\mu \in \Lambda^+$. Then $\mathrm{Tor}(M)$ is a subobject of M in $\mathrm{Mod}_\Lambda(B)$ by (1.19). We denote by $\mathrm{Tor}_{\Lambda^+}(A,B)$ the full subcategory of $\mathrm{Mod}_\Lambda(B)$ consisting of $M \in \mathrm{Mod}_\Lambda(B)$ such that $\mathrm{Tor}(M) = M$. Note that $\mathrm{Tor}_{\Lambda^+}(A,B)$ is closed under taking subquotients and extensions in $\mathrm{Mod}_\Lambda(B)$. Let $\Sigma(A,B)$ denote the collection of morphisms f of $\mathrm{Mod}_\Lambda(B)$ such that its kernel $\mathrm{Ker}(f)$ and its cokernel $\mathrm{Coker}(f)$ belong to $\mathrm{Tor}_{\Lambda^+}(A,B)$. Then we define an abelian category $\mathcal{C}(A,B) = \mathrm{Mod}_\Lambda(B)/\mathrm{Tor}_{\Lambda^+}(A,B)$ as the localization

$$\mathcal{C}(A,B) = \Sigma(A,B)^{-1}\mathrm{Mod}_\Lambda(B)$$

of $\mathrm{Mod}_\Lambda(B)$ with respect to the multiplicative system $\Sigma(A,B)$ (see, e.g., [16] for the notion of localization of categories). We denote by

$$\omega(A,B)^* : \mathrm{Mod}_\Lambda(B) \to \mathcal{C}(A,B) \tag{1.20}$$

the canonical exact functor. It admits a right adjoint

$$\omega(A,B)_* : \mathcal{C}(A,B) \to \mathrm{Mod}_\Lambda(B), \tag{1.21}$$

which is left exact. It is known that $\omega(A,B)^* \circ \omega(A,B)_* \cong \mathrm{Id}$. By taking the degree 0 part of (1.21), we obtain a left exact functor

$$\Gamma_{(A,B)} : \mathcal{C}(A,B) \to \mathrm{Mod}\big(B(0)\big). \tag{1.22}$$

The abelian category $\mathcal{C}(A,B)$ has enough injectives, and we have the right derived functors

$$R^i\Gamma_{(A,B)} : \mathcal{C}(A,B) \to \mathrm{Mod}\big(B(0)\big) \quad (i \in \mathbb{Z}) \tag{1.23}$$

of (1.22).

We apply the above arguments to the case $A = B = A_\zeta$. Then $\mathrm{Tor}(M)$ for $M \in \mathrm{Mod}_\Lambda(A_\zeta)$ consists of $m \in M$ such that there exists $\lambda \in \Lambda^+$ satisfying $A_\zeta(\lambda)m = \{0\}$ (see [20, Lemma 3.4]). We set

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) = \mathcal{C}(A_\zeta, A_\zeta). \tag{1.24}$$

In this case, the natural functors (1.20), (1.21), (1.22) are simply denoted as

$$\omega^*: \mathrm{Mod}_\Lambda(A_\zeta) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}), \tag{1.25}$$

$$\omega_*: \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}_\Lambda(A_\zeta), \tag{1.26}$$

$$\Gamma: \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C}). \tag{1.27}$$

REMARK 1.5. In the terminology of noncommutative algebraic geometry, $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ is the category of *quasicoherent sheaves* on the quantized flag manifold $\mathcal{B}_\zeta$, which is a noncommutative projective scheme. The notations $\mathcal{B}_\zeta$, $\mathcal{O}_{\mathcal{B}_\zeta}$ have only symbolical meaning.

1.2.3. Using Lusztig's Frobenius homomorphism (1.12), we will relate the quantized flag manifold $\mathcal{B}_\zeta$ with the ordinary flag manifold $\mathcal{B} = B^-\backslash G$. Taking the dual Hopf algebras in (1.12), we obtain an injective homomorphism $\mathbb{C}[G] \to C_\zeta$ of Hopf algebras. Moreover, its image is contained in the center of C_ζ (see [14]). We will regard $\mathbb{C}[G]$ as a central Hopf subalgebra of C_ζ in the following. Setting

$$A_1 = \{\varphi \in \mathbb{C}[G] \mid \varphi(ng) = \varphi(g)\ (n \in N^-, g \in G)\},$$
$$A_1(\lambda) = \{\varphi \in A_1 \mid \varphi(tg) = \theta_\lambda(t)\varphi(g)\ (t \in H, g \in G)\} \quad (\lambda \in \Lambda^+),$$

we have a Λ-graded algebra

$$A_1 = \bigoplus_{\lambda \in \Lambda^+} A_1(\lambda).$$

We have a left G-module structure of A_1 given by

$$(x\varphi)(g) = \varphi(gx) \quad (\varphi \in A_1, x, g \in G).$$

In particular, A_1 is a $U(\mathfrak{g})$-module. Moreover, for each $\lambda \in \Lambda^+$, $A_1(\lambda)$ is a $U(\mathfrak{g})$-submodule of A_1 which is an irreducible highest-weight module with highest-weight λ. Regarding $\mathbb{C}[G]$ as a subalgebra of C_ζ, we have

$$A_1 = A_\zeta \cap \mathbb{C}[G], \qquad A_1(\lambda) = A_\zeta(\ell\lambda) \cap \mathbb{C}[G].$$

Since the Λ-graded algebra A_1 is the homogeneous coordinate algebra of the projective variety $\mathcal{B} = B^-\backslash G$, we have an identification

$$\mathrm{Mod}(\mathcal{O}_\mathcal{B}) = \mathcal{C}(A_1, A_1) \tag{1.28}$$

of abelian categories, where $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$ denotes the category of quasicoherent $\mathcal{O}_{\mathcal{B}}$-modules on the ordinary flag manifold $\mathcal{B}$. We set

$$\omega_{\mathcal{B}*} = \omega(A_1, A_1)_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}_\Lambda(A_1). \tag{1.29}$$

For $\lambda \in \Lambda$, we denote by $\mathcal{O}_{\mathcal{B}}(\lambda) \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$ the invertible G-equivariant $\mathcal{O}_{\mathcal{B}}$-module corresponding to λ. Then under identification (1.28), we have

$$\omega_{\mathcal{B}*} M = \bigoplus_{\lambda \in \Lambda} \Gamma\big(\mathcal{B}, M \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}}(\lambda)\big) \quad \big(M \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})\big),$$

where $\Gamma(\mathcal{B},) : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathbb{C}$ is the global section functor for the algebraic variety $\mathcal{B}$. In particular, the functor $\Gamma_{(A_1,A_1)} : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}(\mathbb{C})$ is identified with $\Gamma(\mathcal{B},)$.

For a Λ-graded $\mathbb{C}$-algebra B, we define a new Λ-graded $\mathbb{C}$-algebra $B^{(\ell)}$ by

$$B^{(\ell)}(\lambda) = B(\ell\lambda) \quad (\lambda \in \Lambda).$$

Let

$$(\,)^{(\ell)} : \mathrm{Mod}_\Lambda(B) \to \mathrm{Mod}_\Lambda(B^{(\ell)}) \tag{1.30}$$

be the exact functor given by

$$M^{(\ell)}(\lambda) = M(\ell\lambda) \quad (\lambda \in \Lambda)$$

for $M \in \mathrm{Mod}_\Lambda(B)$.

We have the following results (see [20, Lemma 3.9]).

LEMMA 1.6. *Let B be a Λ-graded $\mathbb{C}$-algebra. Assume that we are given a homomorphism $\jmath : A_\zeta \to B$ of Λ-graded $\mathbb{C}$-algebras. We denote by $\jmath' : A_1 \to B^{(\ell)}$ the induced homomorphism of Λ-graded $\mathbb{C}$-algebras. Assume that*

$$\jmath\big(A_\zeta(\lambda)\big)B(\mu) = B(\mu)\jmath\big(A_\zeta(\lambda)\big) \quad (\lambda, \mu \in \Lambda),$$

$$\jmath'\big(A_1(\lambda)\big)B^{(\ell)}(\mu) = B^{(\ell)}(\mu)\jmath'\big(A_1(\lambda)\big) \quad (\lambda, \mu \in \Lambda).$$

Then the exact functor

$$(\,)^{(\ell)} : \mathrm{Mod}_\Lambda(B) \to \mathrm{Mod}_\Lambda(B^{(\ell)})$$

induces an equivalence

$$\mathrm{Fr}_* : \mathcal{C}(A_\zeta, B) \to \mathcal{C}(A_1, B^{(\ell)}) \tag{1.31}$$

of abelian categories. Moreover, we have

$$\omega(A_1, B^{(\ell)})_* \circ \mathrm{Fr}_* = (\,)^{(\ell)} \circ \omega(A_\zeta, B)_*. \tag{1.32}$$

Lemma 1.7. *Let F be a Λ-graded $\mathbb{C}$-algebra, and let $A_1 \to F$ be a homomorphism of Λ-graded $\mathbb{C}$-algebras. Assume that $\mathrm{Im}(A_1 \to F)$ is central in F. Regard F as an object of $\mathrm{Mod}_\Lambda(A_1)$, and consider $\omega_{\mathcal{B}}^* F \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$. Then the multiplication of F induces an $\mathcal{O}_{\mathcal{B}}$-algebra structure of $\omega_{\mathcal{B}}^* F$, and we have an identification*

$$\mathcal{C}(A_1, F) = \mathrm{Mod}(\omega_{\mathcal{B}}^* F) \tag{1.33}$$

of abelian categories, where $\mathrm{Mod}(\omega_{\mathcal{B}}^* F)$ denotes the category of quasicoherent $\omega_{\mathcal{B}}^* F$-modules. Moreover, under identification (1.33) we have

$$\Gamma_{(A_1,F)}(M) = \Gamma(\mathcal{B}, M) \in \mathrm{Mod}\big(F(0)\big) \quad \big(M \in \mathrm{Mod}(\omega_{\mathcal{B}}^* F)\big).$$

We define an $\mathcal{O}_{\mathcal{B}}$-algebra $\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta}$ by

$$\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta} = \omega_{\mathcal{B}}^* (A_\zeta^{(\ell)}).$$

We denote by $\mathrm{Mod}(\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta})$ the category of quasicoherent $\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta}$-modules. By Lemmas 1.6 and 1.7, we have the following.

Lemma 1.8. *We have an equivalence*

$$\mathrm{Fr}_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta})$$

of abelian categories. Moreover, for $M \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ we have

$$R^i \Gamma(M) \simeq R^i \Gamma\big(\mathcal{B}, \mathrm{Fr}_*(M)\big),$$

where $\Gamma(\mathcal{B},) : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}(\mathbb{C})$ on the right-hand side is the global section functor for $\mathcal{B}$.

§2. The category of D-modules

2.1. Ring of differential operators

2.1.1. We define a subalgebra $D_{\mathbb{F}}$ of $\mathrm{End}_{\mathbb{F}}(A_{\mathbb{F}})$ by

$$D_{\mathbb{F}} = \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{F}}, u \in U_{\mathbb{F}}, \lambda \in \Lambda \rangle,$$

where

$$\ell_\varphi(\psi) = \varphi\psi, \qquad r_\varphi(\psi) = \psi\varphi, \qquad \partial_u(\psi) = u \cdot \psi, \qquad \sigma_\lambda(\psi) = q^{(\lambda,\mu)}\psi$$

for $\psi \in A_{\mathbb{F}}(\mu)$. In fact, we have

$$D_{\mathbb{F}} = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{F}}, u \in U_{\mathbb{F}}, \lambda \in \Lambda \rangle$$

by [20, Lemma 4.1].

We have a natural grading

$$D_{\mathbb{F}} = \bigoplus_{\lambda \in \Lambda^+} D_{\mathbb{F}}(\lambda),$$
$$D_{\mathbb{F}}(\lambda) = \left\{ \Phi \in D_{\mathbb{F}} \,\middle|\, \Phi\big(A_{\mathbb{F}}(\mu)\big) \subset A_{\mathbb{F}}(\lambda + \mu) \ (\mu \in \Lambda) \right\} \quad (\lambda \in \Lambda)$$

of $D_{\mathbb{F}}$. It is easily checked that

$$\partial_u \ell_\varphi = \sum_{(u)} \ell_{u_{(0)} \cdot \varphi} \partial_{u_{(1)}} \quad (u \in U_{\mathbb{F}}, \varphi \in A_{\mathbb{F}}),$$
$$\partial_u \sigma_\lambda = \sigma_\lambda \partial_u \quad (u \in U_{\mathbb{F}}, \lambda \in \Lambda),$$
$$\sigma_\lambda \ell_\varphi = q^{(\lambda, \mu)} \ell_\varphi \sigma_\lambda \quad \big(\lambda \in \Lambda, \varphi \in A_{\mathbb{F}}(\mu)\big).$$

Set

$$E_{\mathbb{F}} = A_{\mathbb{F}} \otimes U_{\mathbb{F}} \otimes \mathbb{F}[\Lambda].$$

We have a natural $\mathbb{F}$-algebra structure of $E_{\mathbb{F}}$ such that $A_{\mathbb{F}} \otimes 1 \otimes 1$, $1 \otimes U_{\mathbb{F}} \otimes 1$, $1 \otimes 1 \otimes \mathbb{F}[\Lambda]$ are subalgebras of $E_{\mathbb{F}}$ naturally isomorphic to $A_{\mathbb{F}}$, $U_{\mathbb{F}}$, $\mathbb{F}[\Lambda]$, respectively, and such that we have the relations

$$u\varphi = \sum_{(u)} (u_{(0)} \cdot \varphi) u_{(1)} \quad (u \in U_{\mathbb{F}}, \varphi \in A_{\mathbb{F}}),$$
$$ue(\lambda) = e(\lambda)u \quad (u \in U_{\mathbb{F}}, \lambda \in \Lambda),$$
$$e(\lambda)\varphi = q^{(\lambda, \mu)} \varphi e(\lambda) \quad \big(\lambda \in \Lambda, \varphi \in A_{\mathbb{F}}(\mu)\big)$$

in $E_{\mathbb{F}}$. Here, we identify $A_{\mathbb{F}} \otimes 1 \otimes 1$, $1 \otimes U_{\mathbb{F}} \otimes 1$, $1 \otimes 1 \otimes \mathbb{F}[\Lambda]$ with $A_{\mathbb{F}}$, $U_{\mathbb{F}}$, $\mathbb{F}[\Lambda]$, respectively. Then we have a surjective algebra homomorphism

$$E_{\mathbb{F}} \to D_{\mathbb{F}} \tag{2.1}$$

sending $\varphi \in A_{\mathbb{F}}$, $u \in U_{\mathbb{F}}$, $e(\lambda) \in \mathbb{F}[\Lambda]$ $(\lambda \in \Lambda)$ to ℓ_φ, ∂_u, σ_λ, respectively. Moreover, $E_{\mathbb{F}}$ has an obvious Λ-grading so that (2.1) preserves the Λ-grading.

2.1.2. Set

$$D_{\mathbb{A}} = \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{A}}, u \in U_{\mathbb{A}}, \lambda \in \Lambda \rangle_{\mathbb{A}\text{-alg}} \subset D_{\mathbb{F}},$$
$$E_{\mathbb{A}} = A_{\mathbb{A}} \otimes U_{\mathbb{A}} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{A}}.$$

They are Λ-graded $\mathbb{A}$-subalgebras of $D_{\mathbb{F}}$ and $E_{\mathbb{F}}$, respectively. Again, we have

$$D_{\mathbb{A}} = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{A}}, u \in U_{\mathbb{A}}, \lambda \in \Lambda \rangle_{\mathbb{A}\text{-alg}}$$

by [20]. In particular, we have a surjective homomorphism

$$E_{\mathbb{A}} \to D_{\mathbb{A}}$$

of Λ-graded algebras. Note that there is a canonical embedding

$$D_{\mathbb{A}} \to \mathrm{End}_{\mathbb{A}}(A_{\mathbb{A}}).$$

2.1.3. We set

$$D_\zeta = \mathbb{C} \otimes_{\mathbb{A}} D_{\mathbb{A}}, \qquad E_\zeta = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A}} = A_\zeta \otimes U_\zeta \otimes \mathbb{C}[\Lambda].$$

D_ζ is a Λ-graded $\mathbb{C}$-algebra generated by elements of the form

$$\ell_\varphi, \qquad \partial_u, \qquad \sigma_\lambda \quad (\varphi \in A_\zeta, u \in U_\zeta, \lambda \in \Lambda).$$

We have a surjective homomorphism

$$E_\zeta \to D_\zeta$$

of Λ-graded $\mathbb{C}$-algebras.

Lemma 2.1. *Let $z \in Z_{\mathrm{Har}}(U_\zeta)$, and write $\iota(z) = \sum_{\lambda \in \Lambda} c_\lambda k_{2\lambda}$ $(c_\lambda \in \mathbb{C})$. Then we have*

$$\partial_z = \sum_{\lambda \in \Lambda} c_\lambda \sigma_{2\lambda}.$$

Proof. This follows from the corresponding statement over $\mathbb{F}$, which is given in [19, Section 5.1]. □

Remark 2.2. The natural algebra homomorphism $D_\zeta \to \mathrm{End}_{\mathbb{C}}(A_\zeta)$ is not injective.

2.1.4. Define an $\mathcal{O}_{\mathcal{B}}$-algebra $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$ by

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta} = \omega_{\mathcal{B}}^* D_\zeta^{(\ell)}.$$

We define $ZD_\zeta^{(\ell)}$ to be the central subalgebra of $D_\zeta^{(\ell)}$ generated by the elements of the form

$$\ell_\varphi, \qquad \partial_u, \qquad \sigma_\lambda \quad \big(\varphi \in A_1, u \in Z_{\mathrm{Fr}}(U_\zeta), \lambda \in \Lambda\big),$$

and we set

$$\mathcal{Z}_\zeta = \omega_{\mathcal{B}}^* ZD_\zeta^{(\ell)}.$$

It is a central subalgebra of $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$. Define a subvariety $\mathcal{V}$ of $\mathcal{B} \times K \times H$ by

$$\mathcal{V} = \big\{(B^- g, k, t) \in \mathcal{B} \times K \times H \;\big|\; g\kappa(k)g^{-1} \in t^{2\ell} N^-\big\},$$

where $\kappa : K \to G$ is given by $\kappa(k_1, k_2) = k_1 k_2^{-1}$. We denote by

$$p_{\mathcal{V}} : \mathcal{V} \to \mathcal{B}$$

the projection. Now we can state the main results of [20].

THEOREM 2.3 ([20, Theorem 5.2]). *The $\mathcal{O}_{\mathcal{B}}$-algebra $\mathcal{Z}_\zeta$ is naturally isomorphic to $p_{\mathcal{V}*}\mathcal{O}_{\mathcal{V}}$.*

Define an $\mathcal{O}_{\mathcal{V}}$-algebra $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ by

$$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = p_{\mathcal{V}}^{-1}\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{p_{\mathcal{V}}^{-1} p_{\mathcal{V}*}\mathcal{O}_{\mathcal{V}}} \mathcal{O}_{\mathcal{V}}.$$

THEOREM 2.4 ([20, Theorem 6.1]). *Here $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ is an Azumaya algebra of rank $\ell^{2|\Delta^+|}$.*

Define

$$\delta : \mathcal{V} \to K \times_{H/W} H$$

by $\delta(B^- g, k, t) = (k, t)$, where $K \to H/W$ is given by $k \mapsto [h]$, where h is an element of H conjugate to the semisimple part of $\kappa(k)$, and $H \to H/W$ is given by $t \mapsto [t^{2\ell}]$.

THEOREM 2.5 ([20, Theorem 6.10]). *For any $(k, t) \in K \times_{H/W} H$, the restriction of $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ to $\delta^{-1}(k, t)$ is a split Azumaya algebra.*

2.2. Category of D-modules

We define an abelian category $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ by

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) = \mathcal{C}(A_\zeta, D_\zeta).$$

By Lemmas 1.6 and 1.7, we have an equivalence

$$\mathrm{Fr}_* : \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}) \tag{2.2}$$

of abelian categories, where $\mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})$ denotes the category of quasicoherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$-modules. Moreover, for $M \in \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ we have

$$R^i\Gamma_{(A_\zeta, D_\zeta)}(M) = R^i\Gamma\big(\mathcal{B}, \mathrm{Fr}_*(M)\big) \in \mathrm{Mod}\big(D_\zeta(0)\big), \tag{2.3}$$

where $\Gamma(\mathcal{B}, \,)$ on the right-hand side is the global section functor for the ordinary flag variety $\mathcal{B}$.

For $t \in H$ we define an abelian category $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ by

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) = \mathrm{Mod}_{\Lambda,t}(D_\zeta)/\big(\mathrm{Mod}_{\Lambda,t}(D_\zeta) \cap \mathrm{Tor}_{\Lambda^+}(A_\zeta, D_\zeta)\big),$$

where $\mathrm{Mod}_{\Lambda,t}(D_\zeta)$ is the full subcategory of $\mathrm{Mod}_\Lambda(D_\zeta)$ consisting of $M \in \mathrm{Mod}_\Lambda(D_\zeta)$ so that $\sigma_\lambda|_{M(\mu)} = \theta_\lambda(t)\zeta^{(\lambda,\mu)}\,\mathrm{id}$ for any $\lambda, \mu \in \Lambda$. Then we can regard $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ as a full subcategory of $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ (see [19, Lemma 4.6]). Set

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t} = \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t,$$

where $\mathbb{C}_t$ denotes the 1-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda) \mapsto \theta_\lambda(t)$ for $\lambda \in \Lambda$. The equivalence (2.2) induces the equivalence

$$\mathrm{Fr}_* : \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) \to \mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}), \tag{2.4}$$

where $\mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ denotes the category of quasicoherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. In particular, for $M \in \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ we have

$$R^i\Gamma_{(A_\zeta, D_\zeta)}(M) = R^i\Gamma(\mathcal{B}, \mathrm{Fr}_*M) \in \mathrm{Mod}\big(D_{\zeta,t}(0)\big),$$

where $D_{\zeta,t}(0) = D_\zeta(0)/\sum_{\lambda\in\Lambda} D_\zeta(0)(\sigma_\lambda - \theta_\lambda(t))$.

2.3. Conjecture
By Lemma 2.1, the natural algebra homomorphism

$$U_\zeta \otimes_{\mathbb{C}} \mathbb{C}[\Lambda] \to D_\zeta(0)$$

factors through

$$U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \to D_\zeta(0),$$

where $Z_{\mathrm{Har}}(U_\zeta)$ is identified with $\mathbb{C}[2\Lambda]^{W\circ}$ by (1.15). Hence, we have a natural algebra homomorphism

$$U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \to \Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}). \tag{2.5}$$

For $t \in H$ we denote by $\mathbb{C}_t$ the 1-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda)v = \theta_\lambda(t)v$ ($v \in \mathbb{C}_t$). Then (2.5) induces an algebra homomorphism

$$U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t \to \Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta, t}), \tag{2.6}$$

where $\mathbb{C}_t$ is regarded as a $Z_{\mathrm{Har}}(U_\zeta)$-module by $Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[2\Lambda]^{W\circ} \subset \mathbb{C}[\Lambda]$. Denote by h_G the Coxeter number for G.

CONJECTURE 2.6. *Assume that $\ell > h_G$. The algebra homomorphism (2.5) is an isomorphism, and we have*

$$R^i\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}) = 0$$

for $i \neq 0$.

PROPOSITION 2.7. *Let $\ell > h_G$, and assume that Conjecture 2.6 is valid. Then for $t \in H$ we have*

$$\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta, t}) \cong U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t$$

and

$$R^i\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta, t}) = 0 \quad (i \neq 0).$$

Proof. Define $f : \mathcal{V} \to H$ to be the composite of the embedding $\mathcal{V} \to \mathcal{B} \times K \times H$ and the projection $\mathcal{B} \times K \times H \to H$ onto the third factor. Since $p_{\mathcal{V}}$ is an affine morphism, we have $Rp_{\mathcal{V}*}\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = p_{\mathcal{V}*}\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}$. Hence, we have

$$U_\zeta \otimes^L_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] = U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \cong R\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}) = R\Gamma(\mathcal{V}, \tilde{\mathcal{D}}_{\mathcal{B}_\zeta}).$$

Here we use the fact that $\mathbb{C}[\Lambda]$ is a free $Z_{\mathrm{Har}}(U_\zeta)$-module (see [17]). Denote by $\mathcal{O}_t$ the $\mathcal{O}_H$-module corresponding to the $\mathbb{C}[\Lambda]$-module $\mathbb{C}_t$. Similarly, we have

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t} = p_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t) = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t).$$

Since f is flat, we have $Lf^*\mathcal{O}_t = f^*\mathcal{O}_t$. Hence, by Theorem 2.4 we have

$$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t = \tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} f^*\mathcal{O}_t = \tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathcal{O}_\mathcal{V}} f^*\mathcal{O}_t.$$

It follows that

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t} = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t) = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathcal{O}_H} \mathcal{O}_t.$$

Hence we have

$$\begin{aligned} R\Gamma(\mathcal{B}, \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}) &= R\Gamma\big(H, Rf_*(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t)\big) \\ &= R\Gamma(H, Rf_*\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_H} \mathcal{O}_t) = R\Gamma(H, Rf_*\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t \\ &= R\Gamma(\mathcal{V}, \tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t = U_\zeta \otimes^L_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t \\ &= U_\zeta \otimes^L_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t. \end{aligned}$$

□

2.4. Derived Beilinson–Bernstein equivalence

We show that Conjecture 2.6 implies a variant of the Beilinson–Bernstein equivalence for derived categories.

Recall that we have an identification

$$Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[2\Lambda]^{W\circ} \subset \mathbb{C}[2\Lambda] \subset \mathbb{C}[\Lambda].$$

Recall also that we identify $\mathbb{C}[\Lambda]$ with the coordinate algebra $\mathbb{C}[H]$ of H. Set $H^{(2)} = H/\mathrm{Ker}(H \ni t \mapsto t^2 \in H)$, and let $\pi : H \to H^{(2)}$ be the canonical homomorphism. Then we have a natural identification $\mathbb{C}[H^{(2)}] = \mathbb{C}[2\Lambda]$ so that $\pi^* : \mathbb{C}[H^{(2)}] \to \mathbb{C}[H]$ is identified with the inclusion $\mathbb{C}[2\Lambda] \subset \mathbb{C}[\Lambda]$. Denote the isomorphism $H \cong H^{(2)}$ corresponding to $\mathbb{C}[\Lambda] \ni e(\lambda) \leftrightarrow e(2\lambda) \in \mathbb{C}[2\Lambda]$ by $t \leftrightarrow t^{1/2}$. Then we have $\pi(t) = (t^2)^{1/2}$. The shifted action (1.13) of W on $\mathbb{C}[2\Lambda]$ induces an action of W on $H^{(2)}$ given by

$$w \circ t^{1/2} = \big(w(tt_{2\rho})t_{2\rho}^{-1}\big)^{1/2} \quad (w \in W, t \in H),$$

where $t_{2\rho} \in H$ is given by $\theta_\mu(t_{2\rho}) = \zeta^{2(\mu,\rho)}$ for any $\mu \in \Lambda$ (note that $2(\mu,\rho) \in \mathbb{Z}$), and $Z_{\mathrm{Har}}(U_\zeta)$ is regarded as the coordinate algebra of the quotient variety $(W\circ)\backslash H^{(2)}$. For $t \in H$ we denote by $\chi_t : \mathbb{C}[\Lambda] \to \mathbb{C}$ the corresponding algebra homomorphism. By the above argument, we have

$$\chi_{t_1}|_{Z_{\mathrm{Har}}(U_\zeta)} = \chi_{t_2}|_{Z_{\mathrm{Har}}(U_\zeta)} \iff (t_1^2)^{1/2} \in W \circ t_2^{1/2}.$$

We say that $t \in H$ is *regular* if

$$\left\{ w \in W \;\middle|\; w \circ (t^2)^{1/2} = (t^2)^{1/2} \right\} = \{1\}.$$

We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta,t})$ (resp., $\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)$) the category of coherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules (resp., finitely generated $U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t$-modules). We also denote by $\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})$ (resp., $\mathrm{Mod}_{f,t}(U_\zeta)$) the category of coherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$-modules (resp., finitely generated U_ζ-modules) killed by some power of the maximal ideal of $\mathbb{C}[\Lambda]$ (resp., $Z_{\mathrm{Har}}(U_\zeta)$) corresponding to $t \in H$.

THEOREM 2.8. *Let $\ell > h_G$, and assume that Conjecture 2.6 is valid. If $t \in H$ is regular, then the natural functors*

$$R\Gamma_{\hat{t}} : D^b\big(\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})\big) \to D^b\big(\mathrm{Mod}_{f,t}(U_\zeta)\big),$$
$$R\Gamma_t : D^b\big(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})\big) \to D^b\big(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)\big)$$

give equivalences of derived categories.

The proof of this result is completely similar to that of the corresponding fact for Lie algebras in positive characteristics given in [6, Theorem 5.3.1]. We give below only an outline of it. First note the following.

PROPOSITION 2.9 ([7, Theorem B]). *Here U_ζ has finite homological dimension.*

The functors

$$R\Gamma_{\hat{t}} : D^b\big(\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})\big) \to D^b\big(\mathrm{Mod}_{f,t}(U_\zeta)\big),$$
$$R\Gamma_t : D^-\big(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})\big) \to D^-\big(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)\big)$$

have left adjoints

$$\mathcal{L}_{\hat{t}} : D^b\big(\mathrm{Mod}_{f,t}(U_\zeta)\big) \to D^b\big(\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})\big),$$
$$\mathcal{L}_t : D^-\big(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)\big) \to D^-\big(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})\big).$$

Arguing exactly as in [6, Sections 3.3, 3.4] using Theorem 2.4 and Proposition 2.9, we obtain the following.

Proposition 2.10.

(i) *If t is regular, the adjunction morphism* $\mathrm{Id} \to R\Gamma_{\hat{t}} \circ \mathcal{L}_{\hat{t}}$ *is an isomorphism on* $D^b(\mathrm{Mod}_{f,t}(U_\zeta))$.

(ii) *For any t, the adjunction morphism* $\mathrm{Id} \to R\Gamma_t \circ \mathcal{L}_t$ *is an isomorphism on* $D^-(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t))$.

Arguing exactly as in [6, Section 3.5] using Theorem 2.4, Proposition 2.10, and Lemma 2.11 below, we obtain Theorem 2.8. Details are omitted.

Lemma 2.11 ([21, Section 2.4]). *The variety $\mathcal{V}$ is a symplectic manifold.*

2.5. Finite part

2.5.1. In [20, Section 4], we also introduced a quotient algebra D'_ζ of E_ζ, which is closely related to D_ζ. Let us recall its definition. Take bases $\{x_p\}_p$, $\{y_p\}_p$, $\{x^L_p\}_p$, $\{y^L_p\}_p$ of U^+_ζ, U^-_ζ, $U^{L,+}_\zeta$, $U^{L,-}_\zeta$, respectively, such that

$$\tau^L_\zeta(x_{p_1}, y^L_{p_2}) = \delta_{p_1,p_2}, \qquad {}^L\tau_\zeta(x^L_{p_1}, y_{p_2}) = \delta_{p_1,p_2}.$$

We assume that

$$x_p \in U^+_{\zeta,\beta_p}, \qquad y_p \in U^-_{\zeta,-\beta_p}, \qquad x^L_p \in U^{L,+}_{\zeta,\beta_p}, \qquad y^L_p \in U^{L,-}_{\zeta,-\beta_p}$$

for $\beta_p \in Q^+$.

For $\varphi \in A_\zeta(\lambda)_\xi$ with $\lambda \in \Lambda^+, \xi \in \Lambda$, we set

$$\Omega'_1(\varphi) = \sum_p (y^L_p \cdot \varphi) x_p \in E_{\zeta,\diamondsuit},$$

$$\Omega'_2(\varphi) = \sum_p \big((Sx^L_p) \cdot \varphi\big) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \in E_{\zeta,\diamondsuit},$$

$$\Omega'(\varphi) = \Omega'_1(\varphi) - \Omega'_2(\varphi) \in E_{\zeta,\diamondsuit}.$$

We extend Ω' to whole A_ζ by linearity. Then D'_ζ is defined by

$$D'_\zeta = E_\zeta \Big/ \sum_{\varphi \in A_\zeta} A_\zeta \Omega'(\varphi) U_\zeta \mathbb{C}[\Lambda].$$

We have a sequence

$$E_\zeta \to D'_\zeta \to D_\zeta$$

of surjective homomorphisms of Λ-graded algebras. Moreover, $D'_\zeta \to D_\zeta$ induces an isomorphism

$$\omega^* D'_\zeta \cong \omega^* D_\zeta \tag{2.7}$$

in $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ (see [20, Corollary 6.6]).

2.5.2. We set

$$U^0_{\mathbb{F},\diamondsuit}=\bigoplus_{\lambda\in\Lambda}\mathbb{F}k_{2\lambda}\subset U^0_{\mathbb{F}},\qquad U_{\mathbb{F},\diamondsuit}=S(U^-_{\mathbb{F}})U^0_{\mathbb{F},\diamondsuit}U^+_{\mathbb{F}}\subset U_{\mathbb{F}}.$$

Then we see easily the following.

LEMMA 2.12. *The subspace $U_{\mathbb{F},\diamondsuit}$ is an $\mathrm{ad}(U_{\mathbb{F}})$-stable subalgebra of $U_{\mathbb{F}}$.*

Set

$$U_{\mathbb{F},f}=\left\{u\in U_{\mathbb{F}}\;\middle|\;\dim\mathrm{ad}(U_{\mathbb{F}})(u)<\infty\right\}. \tag{2.8}$$

Then $U_{\mathbb{F},f}$ is a subalgebra of $U_{\mathbb{F}}$. Moreover, by [12] we have

$$U_{\mathbb{F},f}=\sum_{\lambda\in\Lambda^+}\mathrm{ad}(U_{\mathbb{F}})(k_{-2\lambda}), \tag{2.9}$$

and hence $U_{\mathbb{F},f}$ is a subalgebra of $U_{\mathbb{F},\diamondsuit}$. Note that $U_{\mathbb{F},\diamondsuit}$ and $U_{\mathbb{F},f}$ are not Hopf subalgebras of $U_{\mathbb{F}}$; nevertheless, they satisfy the following.

LEMMA 2.13. *We have*

$$\Delta(U_{\mathbb{F},f})\subset U_{\mathbb{F}}\otimes U_{\mathbb{F},f},\qquad \Delta(U_{\mathbb{F},\diamondsuit})\subset U_{\mathbb{F}}\otimes U_{\mathbb{F},\diamondsuit}.$$

Proof. For $u\in U_{\mathbb{F}}$ and $\lambda\in\Lambda^+$, we have

$$\begin{aligned}\Delta\big(\mathrm{ad}(u)(k_{-2\lambda})\big)&=\sum_{(u)}\Delta\big(u_{(0)}k_{-2\lambda}(Su_{(1)})\big)\\&=\sum_{(u)_3}u_{(0)}k_{-2\lambda}(Su_{(3)})\otimes u_{(1)}k_{-2\lambda}(Su_{(2)})\\&=\sum_{(u)_2}u_{(0)}k_{-2\lambda}(Su_{(2)})\otimes \mathrm{ad}(u_{(1)})(k_{-2\lambda}).\end{aligned}$$

Hence, the first formula follows from (2.9). Since $U_{\mathbb{F},\diamondsuit}$ is generated by e_i, Sf_i for $i\in I$ and $k_{2\lambda}$ for $\lambda\in\Lambda$, the second formula is a consequence of the fact that $\Delta(e_i)$, $\Delta(Sf_i)$, $\Delta(k_{2\lambda})$ belong to $U_{\mathbb{F}}\otimes U_{\mathbb{F},\diamondsuit}$. ∎

We set

$$\begin{aligned}E_{\mathbb{F},\diamondsuit}&=A_{\mathbb{F}}\otimes U_{\mathbb{F},\diamondsuit}\otimes\mathbb{F}[\Lambda]\subset E_{\mathbb{F}},\\E_{\mathbb{F},f}&=A_{\mathbb{F}}\otimes U_{\mathbb{F},f}\otimes\mathbb{F}[\Lambda]\subset E_{\mathbb{F}}.\end{aligned}$$

By Lemma 2.13, they are subalgebras of $E_{\mathbb{F}}$.

We set

$$U^0_{\mathbb{A},\diamondsuit} = U^0_{\mathbb{F},\diamondsuit} \cap U_{\mathbb{A}} = \bigoplus_{\lambda\in\Lambda} \mathbb{A}k_{2\lambda}, \qquad U_{\mathbb{A},\diamondsuit} = U_{\mathbb{F},\diamondsuit} \cap U_{\mathbb{A}} = S(U^-_{\mathbb{A}})U^0_{\mathbb{A},\diamondsuit}U^+_{\mathbb{A}},$$

$$U_{\mathbb{A},f} = U_{\mathbb{A}} \cap U_{\mathbb{F},f},$$

and

$$E_{\mathbb{A},\diamondsuit} = E_{\mathbb{A}} \cap E_{\mathbb{F},\diamondsuit} = A_{\mathbb{A}} \otimes U_{\mathbb{A},\diamondsuit} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{F},\diamondsuit},$$
$$E_{\mathbb{A},f} = E_{\mathbb{A}} \cap E_{\mathbb{F},f} = A_{\mathbb{A}} \otimes U_{\mathbb{A},f} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{F},f}.$$

We also set

$$E_{\zeta,\diamondsuit} = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A},\diamondsuit} = A_\zeta \otimes U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda] \subset E_\zeta,$$
$$E_{\zeta,f} = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A},f} = A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \subset E_\zeta,$$

and

$$D_{\zeta,\diamondsuit} = \mathrm{Im}(E_{\zeta,\diamondsuit} \to D_\zeta), \qquad D_{\zeta,f} = \mathrm{Im}(E_{\zeta,f} \to D_\zeta),$$
$$D'_{\zeta,\diamondsuit} = \mathrm{Im}(E_{\zeta,\diamondsuit} \to D'_\zeta), \qquad D'_{\zeta,f} = \mathrm{Im}(E_{\zeta,f} \to D'_\zeta).$$

By

$$E_\zeta \cong E_{\zeta,\diamondsuit} \otimes_{U_{\zeta,\diamondsuit}} U_\zeta$$

we obtain

$$D'_{\zeta,\diamondsuit} = E_{\zeta,\diamondsuit} \Big/ \sum_{\varphi\in A_\zeta} A_\zeta \Omega'(\varphi) U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda], \tag{2.10}$$

$$D'_\zeta \cong D'_{\zeta,\diamondsuit} \otimes_{U_{\zeta,\diamondsuit}} U_\zeta. \tag{2.11}$$

2.5.3. Since U_ζ is a free $U_{\zeta,\diamondsuit}$-module, we have

$$R^i\Gamma(\omega^* D'_\zeta) \cong R^i\Gamma(\omega^* D'_{\zeta,\diamondsuit}) \otimes_{U_{\zeta,\diamondsuit}} U_\zeta$$

for any $i \in \mathbb{Z}$. Since $U_{\zeta,\diamondsuit}$ is a localization of $U_{\zeta,f}$ with respect to the Ore subset $\{k_{-2\lambda} \mid \Lambda \in \Lambda^+\}$, we have

$$R^i\Gamma(\omega^* D'_{\zeta,\diamondsuit}) \cong R^i\Gamma(\omega^* D'_{\zeta,f}) \otimes_{U_{\zeta,f}} U_{\zeta,\diamondsuit}$$

for any $i \in \mathbb{Z}$. It follows that

$$R^i\Gamma(\omega^* D'_\zeta) \cong R^i\Gamma(\omega^* D'_{\zeta,f}) \otimes_{U_{\zeta,f}} U_\zeta \tag{2.12}$$

for any $i \in \mathbb{Z}$. Note that

$$R^i\Gamma(\mathcal{B}, \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}) \cong R^i\Gamma(\omega^* D'_\zeta)$$

by Lemma 1.8 and (2.7). Hence Conjecture 2.6 is a consequence of the following stronger conjecture.

CONJECTURE 2.14. *Assume that $\ell > h_G$. We have*

$$\Gamma(\omega^* D'_{\zeta,f}) \cong U_{\zeta,f} \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda],$$

and

$$R^i\Gamma(\omega^* D'_{\zeta,f}) = 0$$

for $i \neq 0$.

In the rest of this article, we give a reformulation of Conjecture 2.14 in terms of the induction functor.

§3. Representations

3.1.

For simplicity, we introduce a new notation, $\tilde{U}^-_{\mathbb{F}} = S(U^-_{\mathbb{F}})$. Then we have $\tilde{U}^-_{\mathbb{F}} = \langle \tilde{f}_i \mid i \in I \rangle$, where $\tilde{f}_i = f_i k_i$ for $i \in I$. Moreover, setting

$$\tilde{U}^-_{\mathbb{F},\gamma} = \left\{ u \in \tilde{U}^-_{\mathbb{F}} \;\middle|\; k_\mu u k_{-\mu} = q^{(\gamma,\mu)} u \ (\mu \in \Lambda) \right\}$$

for $\gamma \in Q$, we have

$$\tilde{U}^-_{\mathbb{F}} = \bigoplus_{\gamma \in Q^+} \tilde{U}^-_{\mathbb{F},-\gamma}, \qquad \tilde{U}^-_{\mathbb{F},-\gamma} = U^-_{\mathbb{F},-\gamma} k_\gamma \quad (\gamma \in Q^+).$$

We also set

$$\tilde{U}_{\mathbb{A}} = U_{\mathbb{A}} \cap \tilde{U}_{\mathbb{F}}, \qquad \tilde{U}_{\mathbb{A},-\gamma} = U_{\mathbb{A}} \cap \tilde{U}_{\mathbb{F},-\gamma} \quad (\gamma \in Q^+),$$

$$\tilde{U}_\zeta = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}_{\mathbb{A}}, \qquad \tilde{U}_{\zeta,-\gamma} = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}_{\mathbb{A},-\gamma} \quad (\gamma \in Q^+).$$

Then we have

$$\tilde{U}^-_{\mathbb{A}} = \bigoplus_{\gamma \in Q^+} \tilde{U}^-_{\mathbb{A},-\gamma}, \qquad \tilde{U}^-_\zeta = \bigoplus_{\gamma \in Q^+} \tilde{U}^-_{\zeta,-\gamma}.$$

3.2.

For $\lambda \in \Lambda$, we define an algebra homomorphism $\chi_\lambda : U^0_{\mathbb{F}} \to \mathbb{F}$ by $\chi_\lambda(k_\mu) = q^{(\lambda,\mu)}$ $(\mu \in \Lambda)$. For $M \in \operatorname{Mod}(U_{\mathbb{F}})$ and $\lambda \in \Lambda$, we set

$$M_\lambda = \{ m \in M \mid hm = \chi_\lambda(h)m \ (h \in U^0_{\mathbb{F}}) \}.$$

For $\lambda \in \Lambda$, we define $M_{+,\mathbb{F}}(\lambda), M_{-,\mathbb{F}}(\lambda) \in \operatorname{Mod}(U_{\mathbb{F}})$ by

$$M_{+,\mathbb{F}}(\lambda) = U_{\mathbb{F}} \Big/ \sum_{y \in U^-_{\mathbb{F}}} U_{\mathbb{F}}\big(y - \varepsilon(y)\big) + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_\lambda(h)\big),$$
$$M_{-,\mathbb{F}}(\lambda) = U_{\mathbb{F}} \Big/ \sum_{x \in U^+_{\mathbb{F}}} U_{\mathbb{F}}\big(x - \varepsilon(x)\big) + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_\lambda(h)\big),$$

where $M_{+,\mathbb{F}}(\lambda)$ is a lowest-weight module with lowest-weight λ, and $M_{-,\mathbb{F}}(\lambda)$ is a highest-weight module with highest-weight λ. We have isomorphisms

$$M_{+,\mathbb{F}}(\lambda) \cong U^+_{\mathbb{F}} \quad (\overline{u} \leftrightarrow u), \qquad M_{-,\mathbb{F}}(\lambda) \cong U^-_{\mathbb{F}} \quad (\overline{u} \leftrightarrow u)$$

of $\mathbb{F}$-modules. Moreover, we have weight-space decompositions

$$M_{+,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda + Q^+} M_{+,\mathbb{F}}(\lambda)_\mu, \qquad M_{-,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} M_{-,\mathbb{F}}(\lambda)_\mu.$$

For $\lambda \in \Lambda^+$ we define $L_{+,\mathbb{F}}(-\lambda), L_{-,\mathbb{F}}(\lambda) \in \operatorname{Mod}_f(U_{\mathbb{F}})$ by

$$\begin{aligned} L_{+,\mathbb{F}}(-\lambda) = U_{\mathbb{F}} \Big/ & \sum_{y \in U^-_{\mathbb{F}}} U_{\mathbb{F}}\big(y - \varepsilon(y)\big) \\ & + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_{-\lambda}(h)\big) + \sum_{i \in I} U_{\mathbb{F}} e_i^{((\lambda,\alpha_i^\vee)+1)}, \\ L_{-,\mathbb{F}}(\lambda) = U_{\mathbb{F}} \Big/ & \sum_{x \in U^+_{\mathbb{F}}} U_{\mathbb{F}}\big(x - \varepsilon(x)\big) \\ & + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_{\lambda}(h)\big) + \sum_{i \in I} U_{\mathbb{F}} f_i^{((\lambda,\alpha_i^\vee)+1)}. \end{aligned}$$

While $L_{+,\mathbb{F}}(-\lambda)$ is a finite-dimensional irreducible lowest-weight module with lowest-weight $-\lambda$, here $L_{-,\mathbb{F}}(\lambda)$ is a finite-dimensional irreducible highest-weight module with highest-weight λ. We have weight-space decompositions

$$L_{+,\mathbb{F}}(-\lambda) = \bigoplus_{\mu \in -\lambda + Q^+} L_{+,\mathbb{F}}(-\lambda)_\mu, \qquad L_{-,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} L_{-,\mathbb{F}}(\lambda)_\mu.$$

For $\lambda \in \Lambda^+$ we have isomorphisms

$$L_{+,\mathbb{F}}(-\lambda) \cong U_{\mathbb{F}}^{L,+} \Big/ \sum_{i \in I} U_{\mathbb{F}}^{L,+} e_i^{((\lambda,\alpha_i^\vee)+1)} \quad (\overline{u} \leftrightarrow \overline{u}),$$

$$L_{-,\mathbb{F}}(\lambda) \cong \tilde{U}_{\mathbb{F}}^{L,-} \Big/ \sum_{i \in I} \tilde{U}_{\mathbb{F}}^{L,-} \tilde{f}_i^{((\lambda,\alpha_i^\vee)+1)} \quad (\overline{u} \leftrightarrow \overline{u})$$

of vector spaces (see [13]).

Let M be a $U_{\mathbb{F}}$-module with weight-space decomposition $M = \bigoplus_{\mu \in \Lambda} M_\mu$ such that $\dim M_\mu < \infty$ for any $\mu \in \Lambda$. We define a $U_{\mathbb{F}}$-module $M^\bigstar$ by

$$M^\bigstar = \bigoplus_{\mu \in \Lambda} M_\mu^* \subset M^* = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F}),$$

where the action of $U_{\mathbb{F}}$ is given by

$$\langle um^*, m\rangle = \big\langle m^*, (Su)m\big\rangle \quad (u \in U_{\mathbb{F}}, m^* \in M^\bigstar, m \in M).$$

Here $\langle\, ,\, \rangle : M^\bigstar \times M \to \mathbb{F}$ is the natural pairing.

We set

$$M_{\pm,\mathbb{F}}^*(\lambda) = \big(M_{\mp,\mathbb{F}}(-\lambda)\big)^\bigstar \quad (\lambda \in \Lambda),$$

$$L_{\pm,\mathbb{F}}^*(\mp\lambda) = \big(L_{\mp,\mathbb{F}}(\pm\lambda)\big)^\bigstar \quad (\lambda \in \Lambda^+).$$

Since $L_{\mp,\mathbb{F}}(\pm\lambda)$ is irreducible, we have

$$L_{\pm,\mathbb{F}}^*(\mp\lambda) \cong L_{\pm,\mathbb{F}}(\mp\lambda) \quad (\lambda \in \Lambda^+).$$

We define isomorphisms

$$\Phi_\lambda : U_{\mathbb{F}}^+ \to M_{+,\mathbb{F}}^*(\lambda), \qquad \Psi_\lambda : \tilde{U}_{\mathbb{F}}^- \to M_{-,\mathbb{F}}^*(\lambda) \tag{3.1}$$

of vector spaces by

$$\big\langle \Phi_\lambda(x), \overline{v}\big\rangle = \tau(x, v) \quad (x \in U_{\mathbb{F}}^+, v \in \tilde{U}_{\mathbb{F}}^-),$$

$$\big\langle \Psi_\lambda(y), \overline{Su}\big\rangle = \tau(u, y) \quad (y \in \tilde{U}_{\mathbb{F}}^-, u \in U_{\mathbb{F}}^+).$$

LEMMA 3.1.

(i) *The $U_{\mathbb{F}}$-module structure of $M^*_{+,\mathbb{F}}(\lambda)$ is given by*

$$h\cdot\Phi_\lambda(x)=\chi_{\lambda+\gamma}(h)\Phi_\lambda(x)\quad(x\in U^+_{\mathbb{F},\gamma},h\in U^0_{\mathbb{F}}),\tag{3.2}$$

$$v\cdot\Phi_\lambda(x)=\sum_{(x)}\tau(x_{(0)},Sv)\Phi_\lambda(x_{(1)})\quad(x\in U^+_{\mathbb{F}},v\in U^-_{\mathbb{F}}),\tag{3.3}$$

$$u\cdot\Phi_\lambda(x)=\Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda xk_\lambda)\big)k_{-\lambda}\big)\quad(x\in U^+_{\mathbb{F}},u\in U^+_{\mathbb{F}}).\tag{3.4}$$

(ii) *The $U_{\mathbb{F}}$-module structure of $M^*_{-,\mathbb{F}}(\lambda)$ is given by*

$$h\cdot\Psi_\lambda(y)=\chi_{\lambda-\gamma}(h)\Psi_\lambda(y)\quad(y\in\tilde{U}^-_{\mathbb{F},-\gamma},h\in U^0_{\mathbb{F}}),\tag{3.5}$$

$$u\cdot\Psi_\lambda(y)=\sum_{(y)}\tau(u,y_{(0)})\Psi_\lambda(y_{(1)})\quad(y\in\tilde{U}^-_{\mathbb{F}},u\in U^+_{\mathbb{F}}),\tag{3.6}$$

$$v\cdot\Psi_\lambda(y)=\Psi_\lambda\big(k_\lambda\big(\mathrm{ad}(v)(k_{-\lambda}yk_{-\lambda})\big)k_\lambda\big)\quad(y\in\tilde{U}^-_{\mathbb{F}},v\in U^-_{\mathbb{F}}).\tag{3.7}$$

Proof. We will prove only (i). The proof of (ii) is similar and omitted. Note that for $x\in U^+_{\mathbb{F}}$, $a\in U_{\mathbb{F}}$, and $v\in\tilde{U}^-_{\mathbb{F}}$, we have

$$\big\langle a\cdot\Phi_\lambda(x),\overline{v}\big\rangle=\big\langle\Phi_\lambda(x),\overline{(Sa)v}\big\rangle.$$

Let us show (3.2). For $v\in\tilde{U}^-_{\mathbb{F},-\delta}$, we have

$$\begin{aligned}\big\langle h\cdot\Phi_\lambda(x),\overline{v}\big\rangle&=\big\langle\Phi_\lambda(x),\overline{(Sh)v}\big\rangle=\delta_{\gamma,\delta}\big\langle\Phi_\lambda(x),\overline{(Sh)v}\big\rangle\\&=\delta_{\gamma,\delta}\chi_{\lambda+\gamma}(h)\big\langle\Phi_\lambda(x),\overline{v}\big\rangle=\chi_{\lambda+\gamma}(h)\big\langle\Phi_\lambda(x),\overline{v}\big\rangle.\end{aligned}$$

Hence, (3.2) holds. Let us next show (3.3). For $v\in\tilde{U}^-_{\mathbb{F}}$, we have

$$\begin{aligned}\big\langle y\cdot\Phi_\lambda(x),\overline{v}\big\rangle&=\big\langle\Phi_\lambda(x),\overline{(Sy)v}\big\rangle=\tau\big(x,(Sy)v\big)=\sum_{(x)}\tau(x_{(0)},Sy)\tau(x_{(1)},v)\\&=\Big\langle\Phi_\lambda\Big(\sum_{(x)}\tau(x_{(0)},Sy)x_{(1)}\Big),\overline{v}\Big\rangle.\end{aligned}$$

Hence, (3.3) also holds. Let us finally show (3.4). We may assume that $u\in U^+_{\mathbb{F},\beta}$ for some $\beta\in Q^+$. Then we can write

$$\Delta u=\sum_j u_jk_{\beta'_j}\otimes u'_j\quad(\beta_j,\beta'_j\in Q^+,\beta_j+\beta'_j=\beta,u_j\in U^+_{\mathbb{F},\beta_j},u'_j\in U^+_{\mathbb{F},\beta'_j}).$$

For $v \in \tilde{U}_{\mathbb{F}}^{-}$, we have

$$\begin{aligned}
\langle u \cdot \Phi_\lambda(x), \overline{v}\rangle &= \langle \Phi_\lambda(x), \overline{(Su)v}\rangle \\
&= \sum_{(u)_2,(v)_2} \tau(Su_{(2)}, v_{(0)})\tau(Su_{(0)}, Sv_{(2)})\langle \Phi_\lambda(x), \overline{v_{(1)}(Su_{(1)})}\rangle \\
&= \sum_{j,(v)_2} \tau(Su'_j, v_{(0)})\tau(u_j k_{\beta'_j}, v_{(2)})\langle \Phi_\lambda(x), \overline{v_{(1)}(Sk_{\beta'_j})}\rangle \\
&= \sum_{j,(v)_2} q^{(\lambda,\beta'_j-\beta_j)}\tau(Su'_j, v_{(0)})\tau(u_j k_{\beta'_j}, v_{(2)})\langle \Phi_\lambda(x), \overline{v_{(1)}k_{-\beta_j}}\rangle \\
&= \sum_{j,(v)_2} q^{(\lambda,\beta'_j-\beta_j)}\tau(Su'_j, v_{(0)})\tau(u_j k_{\beta'_j}, v_{(2)})\tau(x, v_{(1)}k_{-\beta_j}) \\
&= \sum_{j,(v)_2} q^{(\lambda,\beta'_j-\beta_j)}\tau(Su'_j, v_{(0)})\tau(x, v_{(1)})\tau(u_j k_{\beta'_j}, v_{(2)}) \\
&= \sum_{j} q^{(\lambda,\beta'_j-\beta_j)}\tau\big(u_j k_{\beta'_j} x(Su'_j), v\big) \\
&= \big\langle \Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda x k_\lambda)\big)k_{-\lambda}\big), \overline{v}\big\rangle.
\end{aligned}$$

Here, we have used Lemma 1.1. Note also that $\Delta\tilde{U}_{\mathbb{F}}^{-} \subset \sum_{\gamma\in Q^+}\tilde{U}_{\mathbb{F}}^{-}k_\gamma \otimes \tilde{U}_{\mathbb{F},-\gamma}^{-}$, and hence we have $\Delta_2\tilde{U}_{\mathbb{F}}^{-} \subset \sum_{\gamma,\delta\in Q^+}\tilde{U}_{\mathbb{F}}^{-}k_{\gamma+\delta}\otimes\tilde{U}_{\mathbb{F},-\gamma}^{-}k_\delta\otimes\tilde{U}_{\mathbb{F},-\delta}^{-}$. Thus, (3.4) is proved. □

For $\lambda \in \Lambda$ we denote by $\mathbb{F}_\lambda^{\geqq 0} = \mathbb{F}1_\lambda^{\geqq 0}$ (resp., $\mathbb{F}_\lambda^{\leqq 0} = \mathbb{F}1_\lambda^{\leqq 0}$) the 1-dimensional $U_{\mathbb{F}}^{\geqq 0}$-module(resp., $U_{\mathbb{F}}^{\leqq 0}$-module) such that $h1_\lambda^{\geqq 0} = \chi_\lambda(h)1_\lambda^{\geqq 0}$, $u1_\lambda^{\geqq 0} = \varepsilon(u)1_\lambda^{\geqq 0}$ for $h \in U_{\mathbb{F}}^0$ and $u \in U_{\mathbb{F}}^{+}$ (resp., $h1_\lambda^{\leqq 0} = \chi_\lambda(h)1_\lambda^{\leqq 0}$, $u1_\lambda^{\leqq 0} = \varepsilon(u)1_\lambda^{\leqq 0}$ for $h \in U_{\mathbb{F}}^0$ and $u \in U_{\mathbb{F}}^{-}$).

Note that for any $\lambda \in \Lambda$, $k_{-2\lambda}U_{\mathbb{F}}^{+}$ (resp., $\tilde{U}_{\mathbb{F}}^{-}k_{-2\lambda}$) is $\mathrm{ad}(U_{\mathbb{F}}^{\geqq 0})$-stable (resp., $\mathrm{ad}(U_{\mathbb{F}}^{\leqq 0})$-stable). We see easily from Lemma 3.1 the following.

LEMMA 3.2. *Let $\lambda \in \Lambda$.*

(i) *The linear map*

$$k_{-2\lambda}U_{\mathbb{F}}^{+} \to M_{+,\mathbb{F}}^{*}(-\lambda)\otimes\mathbb{F}_\lambda^{\geqq 0} \quad \big(k_{-\lambda}xk_{-\lambda} \mapsto \Phi_{-\lambda}(x)\otimes 1_\lambda^{\geqq 0}\big)$$

is an isomorphism of $U_{\mathbb{F}}^{\geqq 0}$-modules, where $k_{-2\lambda}U_{\mathbb{F}}^{+}$ is regarded as a $U_{\mathbb{F}}^{\leqq 0}$-module by the adjoint action.

(ii) *The linear map*

$$\tilde{U}_{\mathbb{F}}^{-}k_{-2\lambda} \to \mathbb{F}_{-\lambda}^{\leqq 0} \otimes M_{-,\mathbb{F}}^{*}(\lambda) \quad \big(k_{-\lambda}yk_{-\lambda} \mapsto 1_{-\lambda}^{\leqq 0} \otimes \Psi_{\lambda}(y)\big)$$

is an isomorphism of $U_{\mathbb{F}}^{\leqq 0}$*-modules, where* $\tilde{U}_{\mathbb{F}}^{-}k_{-2\lambda}$ *is regarded as a* $U_{\mathbb{F}}^{\leqq 0}$*-module by the adjoint action.*

We have an injective $U_{\mathbb{F}}$-homomorphism

$$L_{\pm,\mathbb{F}}^{*}(\mp\lambda) \to M_{\pm,\mathbb{F}}^{*}(\mp\lambda) \quad (\lambda \in \Lambda^{+}) \tag{3.8}$$

induced by the natural homomorphism $M_{\pm,\mathbb{F}}(\mp\lambda) \to L_{\pm,\mathbb{F}}(\mp\lambda)$. For $\lambda \in \Lambda^{+}$ we define subspaces $U_{\mathbb{F}}^{+}(\lambda)$, $\tilde{U}_{\mathbb{F}}^{-}(\lambda)$ of $U_{\mathbb{F}}^{+}$, $\tilde{U}_{\mathbb{F}}^{-}$, respectively, by

$$U_{\mathbb{F}}^{+}(\lambda) = \Phi_{-\lambda}^{-1}\big(L_{+,\mathbb{F}}^{*}(-\lambda)\big), \qquad \tilde{U}_{\mathbb{F}}^{-}(\lambda) = \Psi_{\lambda}^{-1}\big(L_{-,\mathbb{F}}^{*}(\lambda)\big).$$

Lemma 3.3.

(i) *For* $\lambda, \mu \in \Lambda^{+}$ *we have*

$$U_{\mathbb{F}}^{+}(\lambda) \subset U_{\mathbb{F}}^{+}(\lambda+\mu), \qquad \tilde{U}_{\mathbb{F}}^{-}(\lambda) \subset \tilde{U}_{\mathbb{F}}^{-}(\lambda+\mu).$$

(ii) *We have*

$$U_{\mathbb{F}}^{+} = \sum_{\lambda \in \Lambda^{+}} U_{\mathbb{F}}^{+}(\lambda), \qquad \tilde{U}_{\mathbb{F}}^{-} = \sum_{\lambda \in \Lambda^{+}} \tilde{U}_{\mathbb{F}}^{-}(\lambda).$$

Proof. We will prove only the statements for $U_{\mathbb{F}}^{+}$. By definition, we have $U_{\mathbb{F}}^{+}(\lambda) = \{x \in U_{\mathbb{F}}^{+} \mid \tau(x, I_{\lambda}) = \{0\}\}$, where $I_{\lambda} = \sum_{i \in I} \tilde{U}_{\mathbb{F}}^{-} \tilde{f}_i^{((\lambda,\alpha_i^{\vee})+1)}$.

Hence, (i) is a consequence of $I_{\lambda} \supset I_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda^{+}$. To show (ii) it is sufficient to show that for any $\beta \in Q^{+}$ there exists some $\lambda \in \Lambda^{+}$ such that $U_{\mathbb{F},\beta}^{+} \subset U_{\mathbb{F}}^{+}(\lambda)$. Set $m = \mathrm{ht}(\beta)$. If $\lambda \in \Lambda^{+}$ satisfies $(\lambda, \alpha_i^{\vee}) \geqq m$ for any $i \in I$, then we have $I_{\lambda} \subset \bigoplus_{\gamma \in Q^{+}, \mathrm{ht}(\gamma) > m} \tilde{U}_{\mathbb{F},-\gamma}^{-}$. From this we obtain $\tau(U_{\mathbb{F},\beta}^{+}, I_{\lambda}) = \{0\}$, and hence $U_{\mathbb{F},\beta}^{+} \subset U_{\mathbb{F}}^{+}(\lambda)$. □

Lemma 3.4. *For* $\lambda \in \Lambda^{+}$*, we have*

$$\tilde{U}_{\mathbb{F}}^{-}(\lambda)k_{-2\lambda} \subset U_{\mathbb{F},f}, \qquad k_{-2\lambda}U_{\mathbb{F}}^{+}(\lambda) \subset U_{\mathbb{F},f}.$$

Proof. By Lemma 3.2, we have an isomorphism

$$k_{-2\lambda}U_{\mathbb{F}}^{+}(\lambda) \to L_{+,\mathbb{F}}^{*}(-\lambda) \otimes \mathbb{F}_{\lambda}^{\geqq 0} \quad \big(k_{-\lambda}xk_{-\lambda} \mapsto \Phi_{-\lambda}(x) \otimes 1_{\lambda}^{\geqq 0}\big)$$

of $U_{\mathbb{F}}^{\geqq 0}$-modules. We have $L_{+,\mathbb{F}}^*(-\lambda) \cong L_{+,\mathbb{F}}(-\lambda)$, and hence $L_{+,\mathbb{F}}^*(-\lambda) \otimes \mathbb{F}_\lambda^{\geqq 0}$ is generated by $\Phi_{-\lambda}(1) \otimes 1_\lambda^{\geqq 0}$ as a $U_{\mathbb{F}}^{\geqq 0}$-module. It follows that

$$k_{-2\lambda} U_{\mathbb{F}}^+(\lambda) = \mathrm{ad}(U_{\mathbb{F}}^{\geqq 0})(k_{-2\lambda}) \subset U_{\mathbb{F},f}$$

by (2.9). The proof of $\tilde{U}_{\mathbb{F}}^-(\lambda) k_{-2\lambda} \subset U_{\mathbb{F},f}$ is similar. □

3.3.

It is well known that, for $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$, there exists $h \in U_{\mathbb{A}}^{L,0}$ such that $\chi_\lambda(h) = 1$ and $\chi_\mu(h) = 0$. In particular, we have $\chi_\lambda \neq \chi_\mu$ (see, e.g., [20, Lemma 2.3]).

For $M \in \mathrm{Mod}(U_{\mathbb{A}}^L)$ and $\lambda \in \Lambda$, we set

$$M_\lambda = \{ m \in M \mid hm = \chi_\lambda(h) m \ (h \in U_{\mathbb{A}}^{L,0}) \}.$$

For $\lambda \in \Lambda$, we define $M_{+,\mathbb{A}}(\lambda), M_{-,\mathbb{A}}(\lambda) \in \mathrm{Mod}(U_{\mathbb{A}}^L)$ by

$$M_{+,\mathbb{A}}(\lambda) = U_{\mathbb{A}}^L \Big/ \sum_{y \in U_{\mathbb{A}}^{L,-}} U_{\mathbb{A}}^L (y - \varepsilon(y)) + \sum_{h \in U_{\mathbb{A}}^{L,0}} U_{\mathbb{A}}^L (h - \chi_\lambda(h)),$$

$$M_{-,\mathbb{A}}(\lambda) = U_{\mathbb{A}}^L \Big/ \sum_{x \in U_{\mathbb{A}}^{L,+}} U_{\mathbb{A}}^L (x - \varepsilon(x)) + \sum_{h \in U_{\mathbb{A}}^{L,0}} U_{\mathbb{A}}^L (h - \chi_\lambda(h)).$$

By the triangular decomposition we have isomorphisms

$$M_{+,\mathbb{A}}(\lambda) \cong U_{\mathbb{A}}^{L,+} \quad (\overline{u} \leftrightarrow u), \qquad M_{-,\mathbb{A}}(\lambda) \cong U_{\mathbb{A}}^{L,-} \quad (\overline{u} \leftrightarrow u)$$

of $\mathbb{A}$-modules. In particular, $M_{\pm,\mathbb{A}}(\lambda)$ is a free $\mathbb{A}$-module, and we have $\mathbb{F} \otimes_{\mathbb{A}} M_{\pm,\mathbb{A}}(\lambda) \cong M_{\pm,\mathbb{F}}(\lambda)$. Moreover, we have weight-space decompositions

$$M_{+,\mathbb{A}}(\lambda) = \bigoplus_{\mu \in \lambda + Q^+} M_{+,\mathbb{A}}(\lambda)_\mu, \qquad M_{-,\mathbb{A}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} M_{-,\mathbb{A}}(\lambda)_\mu.$$

For $\lambda \in \Lambda^+$, we define $L_{+,\mathbb{A}}(-\lambda) \in \mathrm{Mod}(U_{\mathbb{A}}^L)$ (resp., $L_{-,\mathbb{A}}(\lambda) \in \mathrm{Mod}(U_{\mathbb{A}}^L)$) to be the $U_{\mathbb{A}}^L$-submodule of $L_{+,\mathbb{F}}(-\lambda)$ (resp., $L_{-,\mathbb{F}}(\lambda)$) generated by $\overline{1} \in L_{+,\mathbb{F}}(-\lambda)$ (resp., $\overline{1} \in L_{-,\mathbb{F}}(\lambda)$). By definition, $L_{\pm,\mathbb{A}}(\mp\lambda)$ is a free $\mathbb{A}$-module, and we have $\mathbb{F} \otimes_{\mathbb{A}} L_{\pm,\mathbb{A}}(\mp\lambda) \cong L_{\pm,\mathbb{F}}(\mp\lambda)$. Moreover, we have weight-space decompositions

$$L_{+,\mathbb{A}}(-\lambda) = \bigoplus_{\mu \in -\lambda + Q^+} L_{+,\mathbb{A}}(-\lambda)_\mu, \qquad L_{-,\mathbb{A}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} L_{-,\mathbb{A}}(\lambda)_\mu.$$

The canonical surjective $U_{\mathbb{F}}$-homomorphism $M_{\pm,\mathbb{F}}(\mp\lambda) \to L_{\pm,\mathbb{F}}(\mp\lambda)$ induces a surjective $U^L_{\mathbb{A}}$-homomorphism

$$M_{\pm,\mathbb{A}}(\mp\lambda) \to L_{\pm,\mathbb{A}}(\mp\lambda) \quad (\lambda \in \Lambda^+). \tag{3.9}$$

Note that (3.9) is a split epimorphism of $\mathbb{A}$-modules since $\mathbb{A}$ is a PID (Principal Ideal Domain), and note that $M_{\pm,\mathbb{A}}(\mp\lambda)_\mu$, $L_{\pm,\mathbb{A}}(\mp\lambda)_\mu$ are torsion-free finitely generated $\mathbb{A}$-modules for each $\mu \in \Lambda$.

Let M be a $U^L_{\mathbb{A}}$-module with weight-space decomposition $M = \bigoplus_{\mu\in\Lambda} M_\mu$ such that M_μ is a free $\mathbb{A}$-module of finite rank for any $\mu \in \Lambda$. We define a $U^L_{\mathbb{A}}$-module $M^\bigstar$ by

$$M^\bigstar = \bigoplus_{\mu\in\Lambda} \mathrm{Hom}_{\mathbb{A}}(M_\mu, \mathbb{A}) \subset \mathrm{Hom}_{\mathbb{A}}(M, \mathbb{A}),$$

where the action of $U^L_{\mathbb{A}}$ is given by

$$\langle um^*, m\rangle = \big\langle m^*, (Su)m\big\rangle \quad (u \in U^L_{\mathbb{A}}, m^* \in M^\bigstar, m \in M).$$

Here $\langle\,,\,\rangle : M^\bigstar \times M \to \mathbb{A}$ is the natural pairing.

We set

$$M^*_{\pm,\mathbb{A}}(\lambda) = \big(M_{\mp,\mathbb{A}}(-\lambda)\big)^\bigstar \quad (\lambda \in \Lambda),$$
$$L^*_{\pm,\mathbb{A}}(\mp\lambda) = \big(L_{\mp,\mathbb{A}}(\pm\lambda)\big)^\bigstar \quad (\lambda \in \Lambda^+).$$

Then $M^*_{\pm,\mathbb{A}}(\lambda)$ for $\lambda \in \Lambda$ and $L^*_{\pm,\mathbb{A}}(\mp\lambda)$ for $\lambda \in \Lambda^+$ are free $\mathbb{A}$-modules satisfying

$$\mathbb{F} \otimes_{\mathbb{A}} M^*_{\pm,\mathbb{A}}(\lambda) \cong M^*_{\pm,\mathbb{F}}(\lambda), \qquad \mathbb{F} \otimes_{\mathbb{A}} L^*_{\pm,\mathbb{A}}(\mp\lambda) \cong L^*_{\pm,\mathbb{F}}(\mp\lambda).$$

Moreover, we can identify $M^*_{\pm,\mathbb{A}}(\lambda)$ and $L^*_{\pm,\mathbb{A}}(\mp\lambda)$ with $\mathbb{A}$-submodules of $M^*_{\pm,\mathbb{F}}(\lambda)$ and $L^*_{\pm,\mathbb{F}}(\mp\lambda)$, respectively. Under this identification we have

$$L^*_{\pm,\mathbb{A}}(\mp\lambda) = L^*_{\pm,\mathbb{F}}(\mp\lambda) \cap M^*_{\pm,\mathbb{A}}(\mp\lambda) \quad (\lambda \in \Lambda^+). \tag{3.10}$$

In particular, the $U^L_{\mathbb{A}}$-homomorphism

$$L^*_{\pm,\mathbb{A}}(\mp\lambda) \to M^*_{\pm,\mathbb{A}}(\mp\lambda) \quad (\lambda \in \Lambda^+) \tag{3.11}$$

is a split monomorphism of $\mathbb{A}$-modules.

By abuse of notation we write

$$\Phi_\lambda : U^+_{\mathbb{A}} \to M^*_{+,\mathbb{A}}(\lambda), \qquad \Psi_\lambda : \tilde{U}^-_{\mathbb{A}} \to M^*_{-,\mathbb{A}}(\lambda) \tag{3.12}$$

for the isomorphisms of $\mathbb{A}$-modules induced by (3.1). By Lemma 3.1 we have the following.

LEMMA 3.5.

(i) *The* $U_{\mathbb{A}}^{L}$*-module structure of* $M_{+,\mathbb{A}}^{*}(\lambda)$ *is given by*

$$h\cdot\Phi_\lambda(x)=\chi_{\lambda+\gamma}(h)\Phi_\lambda(x)\quad(x\in U_{\mathbb{A},\gamma}^{+},h\in U_{\mathbb{A}}^{L,0}),\tag{3.13}$$

$$v\cdot\Phi_\lambda(x)=\sum_{(x)}\tau_{\mathbb{A}}^{L}(x_{(0)},Sv)\Phi_\lambda(x_{(1)})\quad(x\in U_{\mathbb{A}}^{+},v\in U_{\mathbb{A}}^{L,-}),\tag{3.14}$$

$$u\cdot\Phi_\lambda(x)=\Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda xk_\lambda)\big)k_{-\lambda}\big)\quad(x\in U_{\mathbb{A}}^{+},u\in U_{\mathbb{A}}^{L,+}).\tag{3.15}$$

(ii) *The* $U_{\mathbb{A}}^{L}$*-module structure of* $M_{-,\mathbb{A}}^{*}(\lambda)$ *is given by*

$$h\cdot\Psi_\lambda(y)=\chi_{\lambda-\gamma}(h)\Psi_\lambda(y)\quad(y\in\tilde{U}_{\mathbb{A},-\gamma}^{-},h\in U_{\mathbb{A}}^{L,0}),\tag{3.16}$$

$$u\cdot\Psi_\lambda(y)=\sum_{(y)}{}^{L}\tau_{\mathbb{A}}(u,y_{(0)})\Psi_\lambda(y_{(1)})\quad(y\in\tilde{U}_{\mathbb{A}}^{-},u\in U_{\mathbb{A}}^{L,+}),\tag{3.17}$$

$$v\cdot\Psi_\lambda(y)=\Psi_\lambda\big(k_\lambda\big(\mathrm{ad}(v)(k_{-\lambda}yk_{-\lambda})\big)k_\lambda\big)\quad(y\in\tilde{U}_{\mathbb{A}}^{-},v\in U_{\mathbb{A}}^{L,-}).\tag{3.18}$$

For $\lambda\in\Lambda^+$ we define $\mathbb{A}$-submodules $U_{\mathbb{A}}^{+}(\lambda)$, $\tilde{U}_{\mathbb{A}}^{-}(\lambda)$ of $U_{\mathbb{A}}^{+}$, $\tilde{U}_{\mathbb{A}}^{-}$, respectively, by

$$U_{\mathbb{A}}^{+}(\lambda)=\Phi_{-\lambda}^{-1}\big(L_{+,\mathbb{A}}^{*}(-\lambda)\big),\qquad\tilde{U}_{\mathbb{A}}^{-}(\lambda)=\Psi_{\lambda}^{-1}\big(L_{-,\mathbb{A}}^{*}(\lambda)\big).$$

The embeddings

$$U_{\mathbb{A}}^{+}(\lambda)\hookrightarrow U_{\mathbb{A}}^{+},\qquad\tilde{U}_{\mathbb{A}}^{-}(\lambda)\hookrightarrow\tilde{U}_{\mathbb{A}}^{-}\quad(\lambda\in\Lambda^+)\tag{3.19}$$

are split monomorphisms of $\mathbb{A}$-modules. By (3.10), we have

$$U_{\mathbb{A}}^{+}(\lambda)=U_{\mathbb{F}}^{+}(\lambda)\cap U_{\mathbb{A}}^{+},\qquad\tilde{U}_{\mathbb{A}}^{-}(\lambda)=\tilde{U}_{\mathbb{F}}^{-}(\lambda)\cap\tilde{U}_{\mathbb{A}}^{-}\quad(\lambda\in\Lambda^+).\tag{3.20}$$

In particular, we have

$$U_{\mathbb{A}}^{+}(\lambda)\subset U_{\mathbb{A}}^{+}(\lambda+\mu),\qquad\tilde{U}_{\mathbb{A}}^{-}(\lambda)\subset\tilde{U}_{\mathbb{A}}^{-}(\lambda+\mu)\quad(\lambda,\mu\in\Lambda^+),\tag{3.21}$$

$$U_{\mathbb{A}}^{+}=\sum_{\lambda\in\Lambda^+}U_{\mathbb{A}}^{+}(\lambda),\qquad\tilde{U}_{\mathbb{A}}^{-}=\sum_{\lambda\in\Lambda^+}\tilde{U}_{\mathbb{A}}^{-}(\lambda),\tag{3.22}$$

$$\tilde{U}_{\mathbb{A}}^{-}(\lambda)k_{-2\lambda}\subset U_{\mathbb{A},f},\qquad k_{-2\lambda}U_{\mathbb{A}}^{+}(\lambda)\subset U_{\mathbb{A},f}\quad(\lambda\in\Lambda^+)\tag{3.23}$$

by Lemmas 3.3 and 3.4.

3.4.

Let $\lambda \in \Lambda$. By abuse of notation we also denote by $\chi_\lambda : U_\zeta^{L,0} \to \mathbb{C}$ the $\mathbb{C}$-algebra homomorphism induced by $\chi_\lambda : U_{\mathbb{A}}^{L,0} \to \mathbb{A}$. Then $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is a linearly independent subset of the $\mathbb{C}$-module $\mathrm{Hom}_{\mathbb{C}}(U_\zeta^{L,0}, \mathbb{C})$. For $M \in \mathrm{Mod}(U_\zeta^L)$ and $\lambda \in \Lambda$, we set

$$M_\lambda = \left\{ m \in M \;\middle|\; hm = \chi_\lambda(h)m \ (h \in U_\zeta^{L,0}) \right\}.$$

For $\lambda \in \Lambda$ we set

$$M_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} M_{\pm,\mathbb{A}}(\lambda), \qquad M^*_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} M^*_{\pm,\mathbb{A}}(\lambda).$$

For $\lambda \in \Lambda^+$ we set

$$L_{\pm,\zeta}(\mp\lambda) = \mathbb{C} \otimes_{\mathbb{A}} L_{\pm,\mathbb{A}}(\mp\lambda), \qquad L^*_{\pm,\zeta}(\mp\lambda) = \mathbb{C} \otimes_{\mathbb{A}} L^*_{\pm,\mathbb{A}}(\mp\lambda).$$

We have canonical U_ζ^L-homomorphisms

$$M_{\pm,\zeta}(\mp\lambda) \to L_{\pm,\zeta}(\mp\lambda) \quad (\lambda \in \Lambda^+), \tag{3.24}$$

$$L^*_{\pm,\zeta}(\mp\lambda) \to M^*_{\pm,\zeta}(\mp\lambda) \quad (\lambda \in \Lambda^+). \tag{3.25}$$

Note that (3.24) is surjective and that (3.25) is injective.

For any $\lambda \in \Lambda^+$ we have an isomorphism

$$A_\zeta(\lambda) \cong L^*_{-,\zeta}(\lambda) \tag{3.26}$$

of U_ζ^L-modules (see, e.g., [11, Chapter 9], [20, Section 3.1]).

Let $\lambda \in \Lambda$. By abuse of notation we also denote by

$$\Phi_\lambda : U_\zeta^+ \to M^*_{+,\zeta}(\lambda), \qquad \Psi_\lambda : \tilde{U}_\zeta^- \to M^*_{-,\zeta}(\lambda)$$

the isomorphisms of $\mathbb{C}$-modules given by

$$\left\langle \Phi_\lambda(x), \overline{v} \right\rangle = \tau_\zeta^L(x, v) \quad (x \in U_\zeta^+, v \in \tilde{U}_\zeta^{L,-}),$$
$$\left\langle \Psi_\lambda(y), \overline{Su} \right\rangle = {}^L\tau_\zeta(u, y) \quad (y \in \tilde{U}_\zeta^-, u \in U_\zeta^{L,+}).$$

By Lemma 3.5, we have the following.

LEMMA 3.6.

(i) *The U_ζ^L-module structure of $M_{+,\zeta}^*(\lambda)$ is given by*

$$h \cdot \Phi_\lambda(x) = \chi_{\lambda+\gamma}(h)\Phi_\lambda(x) \quad (x \in U_{\zeta,\gamma}^+, h \in U_\zeta^{L,0}), \tag{3.27}$$

$$v \cdot \Phi_\lambda(x) = \sum_{(x)} \tau_\zeta^L(x_{(0)}, Sv)\Phi_\lambda(x_{(1)}) \quad (x \in U_\zeta^+, v \in U_\zeta^{L,-}), \tag{3.28}$$

$$u \cdot \Phi_\lambda(x) = \Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda x k_\lambda)\big)k_{-\lambda}\big) \quad (x \in U_\zeta^+, u \in U_\zeta^{L,+}). \tag{3.29}$$

(ii) *The U_ζ^L-module structure of $M_{-,\zeta}^*(\lambda)$ is given by*

$$h \cdot \Psi_\lambda(y) = \chi_{\lambda-\gamma}(h)\Psi_\lambda(y) \quad (y \in \tilde{U}_{\zeta,-\gamma}^-, h \in U_\zeta^{L,0}), \tag{3.30}$$

$$u \cdot \Psi_\lambda(y) = \sum_{(y)} {}^L\tau_\zeta(u, y_{(0)})\Psi_\lambda(y_{(1)}) \quad (y \in \tilde{U}_\zeta^-, u \in U_\zeta^{L,+}), \tag{3.31}$$

$$v \cdot \Psi_\lambda(y) = \Psi_\lambda\big(k_\lambda\big(\mathrm{ad}(v)(k_{-\lambda} y k_{-\lambda})\big)k_\lambda\big) \quad (y \in \tilde{U}_\zeta^-, v \in U_\zeta^{L,-}). \tag{3.32}$$

For $\lambda \in \Lambda^+$, we set

$$U_\zeta^+(\lambda) = \mathbb{C} \otimes_\mathbb{A} U_\mathbb{A}^+(\lambda),$$
$$\tilde{U}_\zeta^-(\lambda) = \mathbb{C} \otimes_\mathbb{A} \tilde{U}_\mathbb{A}^-(\lambda).$$

Then $U_\zeta^+(\lambda)$ and $\tilde{U}_\zeta^-(\lambda)$ are the $\mathbb{C}$-submodules of U_ζ^+ and $\tilde{U}_\zeta^-$, respectively, satisfying $\Phi_{-\lambda}(U_\zeta^+(\lambda)) = L_{+,\zeta}^*(-\lambda)$ and $\Psi_\lambda(\tilde{U}_\zeta^-(\lambda)) = L_{-,\zeta}^*(\lambda)$. We have linear isomorphisms

$$\Phi_{-\lambda} : U_\zeta^+(\lambda) \to L_{+,\zeta}^*(-\lambda), \qquad \Psi_\lambda : \tilde{U}_\zeta^-(\lambda) \to L_{-,\zeta}^*(\lambda) \quad (\lambda \in \Lambda^+). \tag{3.33}$$

By (3.21), (3.22), and (3.23), we have

$$U_\zeta^+(\lambda) \subset U_\zeta^+(\lambda+\mu), \qquad \tilde{U}_\zeta^-(\lambda) \subset \tilde{U}_\zeta^-(\lambda+\mu) \quad (\lambda, \mu \in \Lambda^+), \tag{3.34}$$

$$U_\zeta^+ = \sum_{\lambda\in\Lambda^+} U_\zeta^+(\lambda), \qquad \tilde{U}_\zeta^- = \sum_{\lambda\in\Lambda^+} \tilde{U}_\zeta^-(\lambda), \tag{3.35}$$

$$\tilde{U}_\zeta^-(\lambda)k_{-2\lambda} \subset U_{\zeta,f}, \qquad k_{-2\lambda}U_\zeta^+(\lambda) \subset U_{\mathbb{A},f} \quad (\lambda \in \Lambda^+). \tag{3.36}$$

By (3.35) and (3.36), we can easily see the following.

Lemma 3.7. *For any $u \in U_\zeta$ there exists some $\lambda \in \Lambda^+$ such that $uk_{-2\lambda} \in U_{\zeta,f}$.*

§4. Induction functor

We set

$$C_\zeta^{\leqq 0} = C_\zeta / I, \quad I = \{\varphi \in C_\zeta \mid \langle \varphi, U_\zeta^{L,\leqq 0} \rangle = \{0\}\}.$$

Then $C_\zeta^{\leqq 0}$ is a Hopf algebra, and we have a Hopf pairing

$$\langle \, , \, \rangle : C_\zeta^{\leqq 0} \times U_\zeta^{L,\leqq 0} \to \mathbb{C}.$$

We have a canonical Hopf algebra homomorphism

$$\mathrm{res} : C_\zeta \to C_\zeta^{\leqq 0}.$$

Following Backelin and Kremnizer [2, Section 3], we define abelian categories $\mathcal{M}_\zeta$ and $\mathcal{M}_\zeta^{\mathrm{eq}}$ as follows.

An object of $\mathcal{M}_\zeta$ is a triplet (M, α, β) with

(1) M a vector space over $\mathbb{C}$,
(2) $\alpha : C_\zeta \otimes M \to M$ a left C_ζ-module structure of M,
(3) $\beta : M \to C_\zeta^{\leqq 0} \otimes M$ a left $C_\zeta^{\leqq 0}$-comodule structure of M

such that β is a morphism of C_ζ-modules. (Or, equivalently, α is a morphism of $C_\zeta^{\leqq 0}$-comodules.) A morphism from (M, α, β) to (M', α', β') is a linear map $\varphi : M \to M'$ which is a morphism of C_ζ-modules as well as that of $C_\zeta^{\leqq 0}$-comodules.

An object of $\mathcal{M}_\zeta^{\mathrm{eq}}$ is a quadruple $(M, \alpha, \beta, \gamma)$ with

(1) M a vector space over $\mathbb{C}$,
(2) $\alpha : C_\zeta \otimes M \to M$ a left C_ζ-module structure of M,
(3) $\beta : M \to C_\zeta^{\leqq 0} \otimes M$ a left $C_\zeta^{\leqq 0}$-comodule structure of M,
(4) $\gamma : M \to M \otimes C_\zeta$ a right C_ζ-comodule structure of M

subject to the conditions that $(M, \alpha, \beta) \in \mathcal{M}_\zeta$, that β and γ commute with each other, and that γ is a homomorphism of left C_ζ-modules. A morphism from $(M, \alpha, \beta, \gamma)$ to $(M', \alpha', \beta', \gamma')$ is a linear map $\varphi : M \to M'$ which is compatible with the left C_ζ-module structure, the left $C_\zeta^{\leqq 0}$-comodule structure, and the right C_ζ-comodule structure.

For a coalgebra $\mathcal{C}$ we denote by $\mathrm{Comod}(\mathcal{C})$ (resp., $\mathrm{Comod}^r(\mathcal{C})$) the category of left $\mathcal{C}$-comodules (resp., right $\mathcal{C}$-comodules). We define functors

$$\Xi : \mathcal{M}_\zeta^{\mathrm{eq}} \to \mathrm{Comod}(C_\zeta^{\leqq 0}),$$
$$\Upsilon : \mathrm{Comod}(C_\zeta^{\leqq 0}) \to \mathcal{M}_\zeta^{\mathrm{eq}}$$

by

$$\Xi(M) = \{M \in M \mid \gamma(m) = m \otimes 1\},$$
$$\Upsilon(L) = C_\zeta \otimes L.$$

By Backelin and Kremnizer [2, Section 3.5], we have the following.

PROPOSITION 4.1. *The functor* $\Xi : \mathcal{M}_\zeta^{\mathrm{eq}} \to \mathrm{Comod}(C_\zeta^{\leqq 0})$ *gives an equivalence of categories, and its quasi-inverse is given by* Υ*.*

REMARK 4.2. For $M \in \mathcal{M}_\zeta^{\mathrm{eq}}$ we have an isomorphism

$$\Xi(M) \cong \mathbb{C} \otimes_{C_\zeta} M$$

of vector spaces by Proposition 4.1. Here $C_\zeta \to \mathbb{C}$ is given by ε.

For $\lambda \in \Lambda$ we define $\chi_\lambda^{\leqq 0} \in C_\zeta^{\leqq 0} \subset \mathrm{Hom}_{\mathbb{C}}(U_\zeta^{L,\leqq 0}, \mathbb{C})$ by

$$\chi_\lambda^{\leqq 0}(hu) = \chi_\lambda(h)\varepsilon(u) \quad (h \in U_\zeta^{L,0}, u \in U_\zeta^{L,-}).$$

We define left exact functors

$$\omega_{\mathcal{M}*} : \mathcal{M}_\zeta \to \mathrm{Mod}_\Lambda(A_\zeta), \tag{4.1}$$
$$\Gamma_{\mathcal{M}} : \mathcal{M}_\zeta \to \mathrm{Mod}(\mathbb{C}) \tag{4.2}$$

by

$$\omega_{\mathcal{M}*}(M) = \bigoplus_{\lambda \in \Lambda} \big(\omega_{\mathcal{M}*}(M)\big)(\lambda) \subset M,$$
$$\big(\omega_{\mathcal{M}*}(M)\big)(\lambda) = \{m \in M \mid \beta(m) = \chi_\lambda^{\leqq 0} \otimes m\},$$
$$\Gamma_{\mathcal{M}}(M) = \big(\omega_{\mathcal{M}*}(M)\big)(0).$$

We denote by $\mathrm{Mod}_\Lambda^{\mathrm{eq}}(A_\zeta)$ the category consisting of $N \in \mathrm{Mod}_\Lambda(A_\zeta)$ equipped with a right C_ζ-comodule structure $\gamma : N \to N \otimes C_\zeta$ such that

$\gamma(N(\lambda)) \subset N(\lambda) \otimes C_\zeta$ for any $\lambda \in \Lambda$ and $\gamma(\varphi n) = \Delta(\varphi)\gamma(n)$ for any $\varphi \in A_\zeta$ and $n \in N$. (Note that $\Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta$.) By definition, (4.1) and (4.2) induce left exact functors

$$\omega^{\mathrm{eq}}_{\mathcal{M}*} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta), \tag{4.3}$$

$$\Gamma^{\mathrm{eq}}_{\mathcal{M}} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Comod}^r(C_\zeta). \tag{4.4}$$

We also define a left exact functor

$$\mathrm{Ind} : \mathrm{Comod}(C^{\leqq 0}_\zeta) \to \mathrm{Comod}^r(C_\zeta) \tag{4.5}$$

by $\mathrm{Ind} = \Gamma^{\mathrm{eq}}_{\mathcal{M}} \circ \Upsilon$.

The abelian categories $\mathcal{M}_\zeta$, $\mathcal{M}^{\mathrm{eq}}_\zeta$, $\mathrm{Comod}^r(C_\zeta)$ have enough injectives, and the forgetful functor $\mathcal{M}^{\mathrm{eq}}_\zeta \to \mathcal{M}_\zeta$ sends injective objects to $\Gamma_{\mathcal{M}}$-acyclic objects (see [2, Section 3.4]). Hence, we have the following.

Lemma 4.3. *We have*

$$\mathrm{For} \circ R^i\Gamma^{\mathrm{eq}}_{\mathcal{M}} = R^i\Gamma_{\mathcal{M}} \circ \mathrm{For} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Mod}(\mathbb{C}),$$
$$R^i\mathrm{Ind} \circ \Xi = R^i\Gamma^{\mathrm{eq}}_{\mathcal{M}} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Comod}^r(C_\zeta)$$

for any i, where $\mathrm{For} : \mathrm{Comod}^r(C_\zeta) \to \mathrm{Mod}(\mathbb{C})$ *and* $\mathrm{For} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathcal{M}_\zeta$ *are forgetful functors.*

We define an exact functor

$$\mathrm{res} : \mathrm{Comod}^r(C_\zeta) \to \mathrm{Comod}(C^{\leqq 0}_\zeta) \tag{4.6}$$

as follows. For $V \in \mathrm{Comod}^r(C_\zeta)$ with right C_ζ-comodule structure $\beta : V \to V \otimes C_\zeta$, we have $\mathrm{res}(V) = V$ as a $\mathbb{C}$-module, and the left $C^{\leqq 0}_\zeta$-comodule structure $\mathrm{res}(V) \to C^{\leqq 0}_\zeta \otimes \mathrm{res}(V)$ of $\mathrm{res}(V)$ is given by

$$\beta(v) = \sum_k v_k \otimes \varphi_k \quad \Longrightarrow \quad \gamma(v) = \sum_k \mathrm{res}(S^{-1}\varphi_k) \otimes v_k.$$

The following fact is standard.

Lemma 4.4. *For $V \in \mathrm{Comod}^r(C_\zeta)$, $M \in \mathrm{Comod}(C^{\leqq 0}_\zeta)$, we have an isomorphism*

$$F : \mathrm{Ind}(M) \otimes V \to \mathrm{Ind}\big(\mathrm{res}(V) \otimes M\big)$$

of right C_ζ-comodules given by

$$F\Big(\Big(\sum_i \varphi_i \otimes m_i\Big) \otimes v\Big) = \sum_{i,(v)} \varphi_i v_{(1)} \otimes v_{(0)} \otimes m_i,$$

where we write the right C_ζ-comodule structure of V by

$$V \ni v \mapsto \sum_{(v)} v_{(0)} \otimes v_{(1)} \in V \otimes C_\zeta.$$

For $\lambda \in \Lambda$ we denote by $\mathbb{C}_\lambda^{\leqq 0} = \mathbb{C}1_\lambda^{\leqq 0}$ the object of $\mathrm{Comod}(C_\zeta^{\leqq 0})$ corresponding to the 1-dimensional right $U_\zeta^{L,\leqq 0}$-module given by $1_\lambda^{\leqq 0} u = \chi_\lambda^{\leqq 0}(u) 1_\lambda^{\leqq 0}$ for $u \in U_\zeta^{L,\leqq 0}$. By definition, we have an isomorphism

$$\mathrm{Ind}(\mathbb{C}_{-\lambda}^{\leqq 0}) \cong A_\zeta(\lambda) \quad (\lambda \in \Lambda^+)$$

of right C_ζ-comodules.

Let $N \in \mathrm{Mod}_\Lambda(A_\zeta)$. Then $C_\zeta \otimes_{A_\zeta} N$ turns out to be an object of $\mathcal{M}_\zeta$ by

$$\alpha\big(f \otimes (f' \otimes n)\big) = ff' \otimes n \quad (f, f' \in C_\zeta, n \in N),$$
$$\beta(f \otimes n) = \sum_{(f)} \mathrm{res}(f_{(0)})\chi_\lambda \otimes (f_{(1)} \otimes n) \quad \big(f \in C_\zeta, n \in N(\lambda)\big).$$

Hence, we have a functor $\mathrm{Mod}_\Lambda(A_\zeta) \to \mathcal{M}_\zeta$ sending N to $C_\zeta \otimes_{A_\zeta} N$.

LEMMA 4.5. *The functor* $\mathrm{Mod}_\Lambda(A_\zeta) \to \mathcal{M}_\zeta$ *as above induces a functor*

$$\Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}_\zeta.$$

Proof. It is sufficient to show that $C_\zeta \otimes_{A_\zeta} A_\zeta / A_\zeta(\lambda + \Lambda^+) = \{0\}$ for any $\lambda \in \Lambda$. Hence, we have only to show that $C_\zeta A_\zeta(\lambda) = C_\zeta$ for any $\lambda \in \Lambda^+$. Take $\varphi \in A_\zeta(\lambda)$ such that $\varepsilon(\varphi) = 1$. We have $\Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta$, and hence we can write $\Delta(\varphi) = \sum_i \varphi_i \otimes \varphi'_i$ with $\varphi_i \in A_\zeta(\lambda)$, $\varphi'_i \in C_\zeta$. Then we have $C_\zeta A_\zeta(\lambda) \ni \sum_i (S^{-1}\varphi'_i)\varphi_i = 1$. ∎

We set

$$\Psi = \omega^* \circ \omega_{\mathcal{M}*} : \mathcal{M}_\zeta \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}).$$

Backelin and Kremnizer [2, Section 3.3] obtained the following result using a result of Artin and Zhang [1, Theorem 4.5].

Proposition 4.6. *The functor* $\Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}_\zeta$ *gives an equivalence of categories, and its quasi-inverse is given by* Ψ*. Moreover, we have an identification*

$$\omega_{\mathcal{M}*} \circ \Phi = \omega_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}_\Lambda(A_\zeta)$$

of functors.

Hence we have the following.

Lemma 4.7. *We have*

$$R^i\Gamma = R^i\Gamma_{\mathcal{M}} \circ \Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C})$$

for any i*.*

We set

$$\mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) = \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta)/\mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta) \cap \mathrm{Tor}_{\Lambda^+}(A_\zeta).$$

Let $N \in \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta)$. We denote the right C_ζ-comodule structure of N by $\gamma' : N \to N \otimes C_\zeta$. Then we have a right C_ζ-comodule structure $\gamma : C_\zeta \otimes_{A_\zeta} N \to (C_\zeta \otimes_{A_\zeta} N) \otimes C_\zeta$ of $C_\zeta \otimes_{A_\zeta} N$ given by

$$\gamma'(n) = \sum_k n_k \otimes \varphi_k \quad \Longrightarrow \quad \gamma(f \otimes n) = \sum_{k,(f)} (f_{(0)} \otimes n_k) \otimes f_{(1)}\varphi_k.$$

This gives a functor $\mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta) \to \mathcal{M}^{\mathrm{eq}}_\zeta$. Hence, by Lemma 4.5 we have a functor

$$\Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}^{\mathrm{eq}}_\zeta \tag{4.7}$$

induced by Φ. Let $M \in \mathcal{M}^{\mathrm{eq}}_\zeta$. The right C_ζ-comodule structure of M restricts to that of $\omega_{\mathcal{M}*}M$ so that $\omega_{\mathcal{M}*}M \in \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta)$. Hence, we have a functor

$$\Psi^{\mathrm{eq}} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \tag{4.8}$$

induced by Ψ. By Proposition 4.6, we have the following.

Proposition 4.8. *The functor* $\Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}^{\mathrm{eq}}_\zeta$ *gives an equivalence of categories, and its quasi-inverse is given by* Ψ^{eq}*.*

By Proposition 4.8 we see that (4.1) and (4.2) induce

$$\omega^{\mathrm{eq}}_* = \omega^{\mathrm{eq}}_{\mathcal{M}*} \circ \Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta), \tag{4.9}$$

$$\Gamma^{\mathrm{eq}} = \Gamma^{\mathrm{eq}}_{\mathcal{M}} \circ \Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Comod}^r(C_\zeta). \tag{4.10}$$

By Lemma 4.3, we have the following.

LEMMA 4.9. *We have*

$$\mathrm{For}\circ R^i\Gamma^{\mathrm{eq}} = R^i\Gamma\circ\mathrm{For} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C})$$

for any i, where $\mathrm{For} : \mathrm{Comod}^r(C_\zeta) \to \mathrm{Mod}(\mathbb{C})$ *and* $\mathrm{For} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ *are forgetful functors.*

§5. Reformulation of Conjecture 2.14

5.1. Adjoint action of U_ζ^L on D'_ζ

Define a left $U_{\mathbb{F}}$-module structure of $E_{\mathbb{F}}$ by

$$\mathrm{ad}(u)(P) = \sum_{(u)} u_{(0)}P(Su_{(1)}) \quad (u\in U_{\mathbb{F}}, P\in E_{\mathbb{F}}).$$

Then we have

$$\mathrm{ad}(u)(P_1P_2) = \sum_{(u)} \mathrm{ad}(u_{(0)})(P_1)\mathrm{ad}(u_{(1)})(P_2) \quad (P_1,P_2\in E_{\mathbb{F}}),$$

$$\mathrm{ad}(u)(\varphi) = u\cdot\varphi \quad (\varphi\in A_{\mathbb{F}}\subset E_{\mathbb{F}}),$$

$$\mathrm{ad}(u)(v) = \sum_{(u)} u_{(0)}v(Su_{(1)}) \quad (v\in U_{\mathbb{F}}\subset E_{\mathbb{F}}),$$

$$\mathrm{ad}(u)\big(e(\lambda)\big) = \varepsilon(u)e(\lambda) \quad \big(\lambda\in\Lambda, e(\lambda)\in\mathbb{F}[\Lambda]\subset E_{\mathbb{F}}\big)$$

for $u\in U_{\mathbb{F}}$. We see from [20, Lemma 4.2] that this induces a left $U_{\mathbb{F}}$-module structure of $D'_{\mathbb{F}}$. Moreover, the $U_{\mathbb{F}}$-module structures of $E_{\mathbb{F}}$ and $D'_{\mathbb{F}}$ induce $U_{\mathbb{A}}^L$-module structures of $E_{\mathbb{A}}$, $D'_{\mathbb{A}}$, $E_{\mathbb{A},\diamondsuit}$, $D'_{\mathbb{A},\diamondsuit}$, $E_{\mathbb{A},f}$, and $D'_{\mathbb{A},f}$ by Lemmas 1.2 and 2.12. Hence, by specialization we obtain U_ζ^L-module structures of E_ζ, D'_ζ, $E_{\zeta,\diamondsuit}$, $D'_{\zeta,\diamondsuit}$, $E_{\zeta,f}$, and $D'_{\zeta,f}$ also denoted by ad.

5.2.

We will regard $E_{\zeta,f}, D'_{\zeta,f}\in\mathrm{Mod}_\Lambda(A_\zeta)$ as objects of $\mathrm{Mod}_\Lambda^{\mathrm{eq}}(A_\zeta)$ by the right C_ζ-comodule structures induced from the left U_ζ^L-module structures

$$(u,P)\mapsto \mathrm{ad}(u)(P) \quad (u\in U_\zeta^L, P\in E_{\zeta,f} \text{ or } D'_{\zeta,f}).$$

Then for

$$(\Xi\circ\Phi^{\mathrm{eq}})(\omega^*D'_{\zeta,f})\in\mathrm{Comod}(C_\zeta^{\leqq 0})$$

we have

$$R^i\Gamma(\omega^*D'_{\zeta,f}) = R^i\mathrm{Ind}\big((\Xi\circ\Phi^{\mathrm{eq}})(\omega^*D'_{\zeta,f})\big)$$

by Lemmas 4.3 and 4.9 and by (4.10).

Define a right $(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])$-module V by

$$V=\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big)/\mathcal{I},$$

where

$$\mathcal{I}=\big(\tilde{U}_\zeta^-\cap\mathrm{Ker}(\varepsilon)\big)U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda]+\sum_{\lambda\in\Lambda}\big(k_{2\lambda}-e(2\lambda)\big)U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda].$$

By the triangular decomposition $\tilde{U}_\zeta^-\otimes U^0_{\zeta,\diamondsuit}\otimes U_\zeta^+\cong U_{\zeta,\diamondsuit}$ we have

$$V\cong U_\zeta^+\otimes\mathbb{C}[\Lambda]$$

as a vector space. Define a right action of $U_\zeta^{L,\leqq 0}$ on $U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]$ by

$$\big(u\otimes e(\lambda)\big)\star v=\mathrm{ad}(Sv)(u)\otimes e(\lambda)\quad (u\in U_{\zeta,\diamondsuit},\lambda\in\Lambda,v\in U_\zeta^{L\leqq 0}).$$

It induces a right action of $U_\zeta^{L,\leqq 0}$ on V. Moreover, we see easily that this right $U_\zeta^{L,\leqq 0}$-module structure gives a left $C_\zeta^{\leqq 0}$-comodule structure of V.

Proposition 5.1. *We have*

$$(\Xi\circ\Phi^{\mathrm{eq}})(\omega^*D'_{\zeta,f})\cong V$$

as a left $C_\zeta^{\leqq 0}$-comodule.

The proof is given in Section 5.3.

It follows from Proposition 5.1 that Conjecture 2.14 is equivalent to the following conjecture.

Conjecture 5.2. *Assume that $\ell>h_G$. We have*

$$\mathrm{Ind}(V)\cong U_{\zeta,f}\otimes_{Z_{\mathrm{Har}}(U_\zeta)}\mathbb{C}[\Lambda],$$

and

$$R^i\mathrm{Ind}(V)=0$$

for $i\neq 0$.

Remark 5.3. We can show that

$$U_{\zeta,f}\cong(C_\zeta)_{\mathrm{ad}},\qquad V\cong{}_{\mathrm{ad}}(C_\zeta^{\leqq 0})\otimes_{\mathbb{C}[2\Lambda]}\mathbb{C}[\Lambda],$$

where $(C_\zeta)_{\mathrm{ad}}$ (resp., ${}_{\mathrm{ad}}(C_\zeta^{\leqq 0})$) is given by the right (resp., left) adjoint coaction of C_ζ (resp., $C_\zeta^{\leqq 0}$) on itself. Hence, Conjecture 5.2 is equivalent to

$$R\,\mathrm{Ind}\big({}_{\mathrm{ad}}(C_\zeta^{\leqq 0})\big) \cong (C_\zeta)_{\mathrm{ad}} \otimes_{\mathbb{C}[2\Lambda]^W} \mathbb{C}[2\Lambda].$$

The corresponding statement for $q = 1$ is

$$R\,\mathrm{Ind}\big({}_{\mathrm{ad}}\mathbb{C}[B^-]\big) \cong \mathbb{C}[G]_{\mathrm{ad}} \otimes_{\mathbb{C}[H/W]} \mathbb{C}[H].$$

We can prove this by a geometric method.

REMARK 5.4.† A proof of Conjecture 5.2, when ℓ is a prime greater than the Coxeter number, is given by Backelin and Kremnizer in [3, Proposition 3.25]; however, in a more recent article they admit that there are gaps in [3] (see [4, Version 3, Section 1.1.2]) and propose different proofs. But it is likely that problems still remain in the new proofs given in [4], as explained below.

The proof in [4, Versions 1 and 2] is wrong because all positive roots are assumed there to be dominant (see [4, Version 2, proof of Theorem 2.1]).

Another proof given in [4, Version 3] also has problems. In Step (b) of [4, Version 3, proof of Theorem 2.2.1], the authors compare certain weight multiplicities $a_{q,\mu}$ and $b_{q,\mu}$. But since those multiplicities are infinite, the argument there should be modified using multiplicities as U_q-modules. Let us assume for simplicity that q is generic and try to modify the original argument by replacing $a_{q,\mu}, b_{q,\mu}, b'_{q,\mu}$ with their counterparts as multiplicities of U_q-modules. This even fails since $a_{1,\mu}$ (resp., $b'_{1,\mu}$) is the dimension of the 0-weight space of the irreducible module (resp., Verma module) with highest-weight μ. We also point out that the reason that U_q^λ is an integral domain is not given in Step (a).

Note that the arguments in [4, Version 3, proof of Theorem 2.2.1] are partially similar to those in the earlier manuscripts (see [2, Proposition 4.8], [3, Proposition 3.25]). The main difference is that [4, Version 3] relies on a B_q-stable filtration with 1-dimensional subquotients instead of the Joseph–Letzter filtration used in [2] and [3]. For us, the original argument in [2] and [3] for generic q using the Joseph–Letzter filtration is not comprehensible either. In the notation of [2, proof of Proposition 4.8], the validity of the formula $m_j(1) = \tilde{n}_j(1)$ is not clear to us since the Joseph–Letzter filtration does not induce at $q = 1$ the ordinary filtration for enveloping algebras and differential operators in general.

†This remark is added at the editor's request.

5.3.

We will give a proof of Proposition 5.1 in the rest of this article. By Remark 4.2, we have

$$(\Xi\circ\Phi^{\mathrm{eq}})(\omega^* D'_{\zeta,f})\cong \mathbb{C}\otimes_{A_\zeta} D'_{\zeta,f}$$

as a vector space, where $A_\zeta\to\mathbb{C}$ is given by ε. Note that

$$\mathbb{C}\otimes_{A_\zeta} E_{\zeta,\diamondsuit}\cong U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda].$$

We first show the following.

Lemma 5.5. *We have*

$$\mathbb{C}\otimes_{A_\zeta} D'_{\zeta,\diamondsuit}\cong V.$$

Proof. By (2.10) we obtain

$$\mathbb{C}\otimes_{A_\zeta} D'_{\zeta,\diamondsuit}\cong \big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big)\Big/\sum_{\varphi\in A_\zeta}\big(1\otimes\Omega'(\varphi)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big),$$

where $1\otimes\Omega'(\varphi)$ is the image of $\Omega'(\varphi)$ in $\mathbb{C}\otimes_{A_\zeta} E_{\zeta,\diamondsuit}=U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]$. Note that $\varepsilon(A_\zeta(\lambda)_\xi)=\{0\}$ for $\lambda\in\Lambda^+$, $\xi\in\Lambda$ with $\lambda\neq\xi$, and that $\varepsilon(A_\zeta(\lambda)_\lambda)=\mathbb{C}$ for $\lambda\in\Lambda^+$. Hence, for $\varphi\in A_\zeta(\lambda)_\xi$ with $\lambda\in\Lambda^+$, $\xi\in\Lambda$ we have

$$1\otimes\Omega'_1(\varphi)=\begin{cases}0 & (\lambda\neq\xi),\\ \varepsilon(\varphi) & (\lambda=\xi).\end{cases}$$

Let us also compute $1\otimes\Omega'_2(\varphi)$. Let

$$\tilde\Psi_\lambda:\tilde U^-_\zeta(\lambda)\to A_\zeta(\lambda)$$

be the composite of the linear isomorphism $\Psi_\lambda:\tilde U^-_\zeta(\lambda)\to L^*_{-,\zeta}(\lambda)$ (see (3.33)) and an isomorphism $f:L^*_{-,\zeta}(\lambda)\to A_\zeta(\lambda)$ of U^L_ζ-modules. We have $\tilde\Psi_\lambda(\tilde U^-_\zeta(\lambda)_{-(\lambda-\xi)})=A_\zeta(\lambda)_\xi$ for any $\xi\in\Lambda$. Hence, we may assume that $\varepsilon=\varepsilon\circ\tilde\Psi_\lambda$ on $\tilde U^-_\zeta(\lambda)$. Let $\varphi\in A_\zeta(\lambda)_\xi$, and take $v\in\tilde U^-_\zeta(\lambda)_{-(\lambda-\xi)}$ satisfying $\tilde\Psi_\lambda(v)=\varphi$. Then we have

$$\begin{aligned}\sum_p(Sx^L_p)\cdot\varphi\otimes y_pk_{\beta_p}&=\sum_p f\big((Sx^L_p)\cdot\Psi_\lambda(v)\big)\otimes y_pk_{\beta_p}\\&=\sum_p\zeta^{-(\beta_p,\xi)}f\big((Sx^L_p)k_{\beta_p}\cdot\Psi_\lambda(v)\big)\otimes y_pk_{\beta_p}\end{aligned}$$

$$= \sum_{p,(v)} \zeta^{-(\beta_p,\xi)} f\big({}^L\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v_{(0)}\big)\Psi_\lambda(v_{(1)})\big) \otimes y_p k_{\beta_p}$$

$$= \sum_{p,(v)} \zeta^{-(\beta_p,\xi)} {}^L\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v_{(0)}\big)\tilde{\Psi}_\lambda(v_{(1)}) \otimes y_p k_{\beta_p},$$

and hence

$$\begin{aligned}
1 \otimes \Omega_2'(\varphi) &= \sum_p \varepsilon\big((Sx_p^L)\cdot\varphi\big) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_{p,(v)} \zeta^{-(\beta_p,\xi)} {}^L\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v_{(0)}\big)\varepsilon(v_{(1)}) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\beta_p,\xi)} {}^L\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v\big) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\beta_p,\xi)} {}^L\tau_\zeta(k_{-\beta_p}x_p^L, S^{-1}v) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\beta_p,\xi)-(\beta_p,\beta_p)} {}^L\tau_\zeta(x_p^L, S^{-1}v) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\lambda-\xi,\lambda)} {}^L\tau_\zeta(x_p^L, S^{-1}v) y_p k_{\lambda-\xi} k_{2\xi} e(-2\lambda) \\
&= \zeta^{-(\lambda-\xi,\lambda)}(S^{-1}v) k_{\lambda-\xi} k_{2\xi} e(-2\lambda).
\end{aligned}$$

(Note that $(S^{-1}v)k_{\lambda-\xi} \in \tilde{U}_\zeta^-(\lambda)_{-(\lambda-\xi)}$.) It follows that

$$1 \otimes \Omega'(\varphi) = \begin{cases} -\zeta^{-(\lambda-\xi,\lambda)}(S^{-1}v)k_{\lambda-\xi}k_{2\xi}e(-2\lambda) & (\lambda \neq \xi), \\ \varepsilon(\varphi)(1 - k_{2\lambda}e(-2\lambda)) & (\lambda = \xi). \end{cases}$$

Hence, we have

$$\begin{aligned}
&\sum_{\substack{\lambda\in\Lambda^+,\\ \gamma\in Q^+}} \sum_{\varphi\in A_\zeta(\lambda)_{\lambda-\gamma}} \big(1\otimes\Omega'(\varphi)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) \\
&\quad = \sum_{\substack{\lambda\in\Lambda^+,\\ \gamma\in Q^+\setminus\{0\}}} \tilde{U}_\zeta^-(\lambda)_{-\gamma}\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) + \sum_{\lambda\in\Lambda^+} \big(1-k_{2\lambda}e(-2\lambda)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) \\
&\quad = \big(\tilde{U}_\zeta^- \cap \mathrm{Ker}(\varepsilon)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) + \sum_{\lambda\in\Lambda} \big(k_{2\lambda}-e(2\lambda)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big)
\end{aligned}$$

by (3.35). ∎

Lemma 5.6. *We have*

$$\mathbb{C} \otimes_{A_\zeta} D'_{\zeta,f} \cong V.$$

Proof. We need to show that the canonical homomorphism $\mathbb{C} \otimes_{A_\zeta} D'_{\zeta,f} \to \mathbb{C} \otimes_{A_\zeta} D'_{\zeta,\diamondsuit}$ is bijective. The surjectivity is a consequence of (3.35) and (3.36). Let us give a proof of the injectivity. Set

$$\mathcal{K} = A_\zeta U_{\zeta,f}\mathbb{C}[\Lambda] \cap \sum_{\varphi \in A_\zeta} A_\zeta \Omega'(\varphi) U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda] \subset A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda].$$

Then it is sufficient to show that the natural map

$$\mathbb{C} \otimes_{A_\zeta} \big((A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda])/\mathcal{K}\big) \to \big(U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]\big)/\mathcal{I}$$

is injective. Let $F : A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \to U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$ be the natural map. Then it is sufficient to show that

$$\mathcal{I} \cap \big(U_{\zeta,f} \otimes \mathbb{C}[\Lambda]\big) \subset F(\mathcal{K}). \tag{5.1}$$

Indeed, assume that (5.1) holds. Denote by

$$p : A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \to \mathbb{C} \otimes_{A_\zeta} \big((A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda])/\mathcal{K}\big),$$
$$\pi : U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda] \to \big(U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]\big)/\mathcal{I}$$

the natural maps. We have to show that $\mathrm{Ker}(\pi \circ F) \subset \mathrm{Ker}(p)$. Take $x \in \mathrm{Ker}(\pi \circ F)$. Then $F(x) \in \mathcal{I} \cap (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$. Hence, by (5.1) there exists some $v \in \mathcal{K}$ such that $F(x) = F(v)$. Then $p(x) = p(x-v) + p(v) = p(x-v)$. Hence, we may assume that $F(x) = 0$ from the beginning. Note that p factors through

$$p' : A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \to \mathbb{C} \otimes_{A_\zeta} \big(A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda]\big)\big(= U_{\zeta,f} \otimes \mathbb{C}[\Lambda]\big).$$

By $F(x) = 0$ we have $p'(x) = 0$, and hence $p(x) = 0$, as desired.

It remains to show (5.1). Let $\lambda \in \Lambda^+$, and let $\varphi \in A_\zeta(\lambda)_\lambda$. Then we have

$$\Omega'_1(\varphi) = \sum_p (y_p^L \cdot \varphi) x_p \in A_\zeta U_\zeta^+, \qquad \Omega'_2(\varphi) = \varphi k_{2\lambda} e(-2\lambda).$$

Let us show that

$$\Omega'_1(\varphi) = \sum_p (y_p^L \cdot \varphi) x_p \in A_\zeta U_\zeta^+(\lambda). \tag{5.2}$$

This is equivalent to

$$\sum_p (y_p^L \cdot \varphi) \otimes \Phi_{-\lambda}(x_p) \in A_\zeta \otimes L^*_{+,\zeta}(-\lambda).$$

This follows from

$$\sum_p \langle \Phi_{-\lambda}(x_p), \overline{uf_i^{((\lambda,\alpha_i^\vee)+1)}} \rangle y_p^L \cdot \varphi = \sum_p \tau_\zeta^L(x_p, uf_i^{((\lambda,\alpha_i^\vee)+1)}) y_p^L \cdot \varphi$$
$$= (uf_i^{((\lambda,\alpha_i^\vee)+1)}) \cdot \varphi = 0$$

for $u \in U_\zeta^{L,-}, i \in I$. Thus, (5.2) is verified. Hence, we have

$$\Omega'(\varphi) k_{-2\lambda} \in \mathcal{K}.$$

It follows that

$$F(\mathcal{K}) \supset \bigl(k_{-2\lambda} - e(-2\lambda)\bigr) U_{\zeta,f} \mathbb{C}[\Lambda] \quad (\lambda \in \Lambda^+). \tag{5.3}$$

Now let $u \in \mathcal{I} \cap (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$. If we can show that $k_{-2\mu} u \in F(\mathcal{K})$ for some $\mu \in \Lambda^+$, then we obtain

$$u = e(2\mu)\bigl(e(-2\mu) - k_{-2\mu}\bigr) u + e(2\mu) k_{-2\mu} u \in F(\mathcal{K})$$

by (5.3). Hence, it is sufficient to show that for any $u \in \mathcal{I}$ there exists some $\mu \in \Lambda^+$ such that $k_{-2\mu} u \in F(\mathcal{K})$. We may assume that there exists $\nu \in Q$ such that $k_{-2\mu} u = \zeta^{(\mu,\nu)} u k_{-2\mu}$ for any $\mu \in \Lambda$. Therefore, we have only to show that for any $u \in \mathcal{I}$ there exists some $\mu \in \Lambda^+$ such that $uk_{-2\mu} \in F(\mathcal{K})$. By Lemma 5.5 we can take $\varphi_i \in A_\zeta$, $x_i \in U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$ $(i = 1, \dots, N)$ such that

$$u = 1 \otimes \sum_{i=1}^N \Omega'(\varphi_i) x_i.$$

By Lemma 3.7 we can take $\mu \in \Lambda^+$ such that $\Omega'(\varphi_i) x_i k_{-2\mu} \in A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ for any i. Then we have

$$uk_{-2\mu} = \sum_{i=1}^N F\bigl(\Omega'(\varphi_i) x_i k_{-2\mu}\bigr) \in F(\mathcal{K}).$$

∎

By Lemma 5.6 we obtain an isomorphism

$$(\Xi \circ \Phi^{\mathrm{eq}})(\omega^* D'_{\zeta,f}) \cong V$$

of vector spaces. We need to show that it is in fact an isomorphism of left $C_\zeta^{\leqq 0}$-comodules. This is a consequence of the corresponding fact for $E_{\zeta,f}$. Note that we have

$$\mathbb{C} \otimes_{A_\zeta} E_{\zeta,f} \cong U_{\zeta,f} \otimes \mathbb{C}[\Lambda],$$

and hence we have an isomorphism

$$(\Xi \circ \Phi^{\mathrm{eq}})(\omega^* E_{\zeta,f}) \cong U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \tag{5.4}$$

of vector spaces. Hence, we have only to show the following.

Lemma 5.7. *Under identification (5.4), the left $C_\zeta^{\leqq 0}$-comodule structure of $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ is associated to the right $U_\zeta^{L,\leqq 0}$-module structure given by*

$$\big(u \otimes e(\lambda)\big) \cdot v = \mathrm{ad}(Sv)(u) \otimes e(\lambda) \quad (u \in U_{\zeta,f}, \lambda \in \Lambda, v \in U_\zeta^{L,\leqq 0}).$$

Proof. Note that the left $C_\zeta^{\leqq 0}$-comodule structure of $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ is given by

$$U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \cong \Xi\big(C_\zeta \otimes \big(U_{\zeta,f} \otimes \mathbb{C}[\Lambda]\big)\big),$$

where $C_\zeta \otimes (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$ is regarded as a left $C_\zeta^{\leqq 0}$-comodule by the tensor product of C_ζ (with left $C_\zeta^{\leqq 0}$-comodule structure $(\mathrm{res} \otimes 1) \circ \Delta : C_\zeta \to C_\zeta^{\leqq 0} \otimes C_\zeta$) and $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ with trivial left $C_\zeta^{\leqq 0}$-comodule structure. Hence, it is sufficient to show that for a right C_ζ-comodule M the right $U_\zeta^{L,\leqq 0}$-module structure of

$$M \cong \Xi(C_\zeta \otimes M) \in \mathrm{Comod}(C_\zeta^{\leqq 0})$$

is given by

$$m \cdot v = (Sv) \cdot m \quad (m \in M, v \in U_\zeta^{L,\leqq 0}).$$

Denote by M^{triv} the trivial right C_ζ-comodule which coincides with M as a vector space. We denote by $M \ni m \leftrightarrow \overline{m} \in M^{\mathrm{triv}}$ the canonical linear isomorphism. We have $C_\zeta \otimes M^{\mathrm{triv}} \in \mathrm{Comod}^r(C_\zeta)$ as the tensor product of $C_\zeta \in \mathrm{Comod}^r(C_\zeta)$ and $M^{\mathrm{triv}} \in \mathrm{Comod}^r(C_\zeta)$. We can also define a left $C_\zeta^{\leqq 0}$-comodule structure of $C_\zeta \otimes M^{\mathrm{triv}}$ as the tensor product of the left

$C_\zeta^{\leqq 0}$-comodules C_ζ and M^{triv}, where the left $C_\zeta^{\leqq 0}$-comodule structure of M^{triv} is given by the right $U_\zeta^{L,\leqq 0}$-module structure

$$\overline{m} \cdot v = \overline{(Sv) \cdot m} \quad (m \in M, v \in U_\zeta^{L,\leqq 0}).$$

Then we have a linear isomorphism

$$C_\zeta \otimes M \ni \varphi \otimes m \mapsto \sum_{(m)} \varphi m_{(1)} \otimes \overline{m_{(0)}} \in C_\zeta \otimes M^{\mathrm{triv}}$$

preserving the right C_ζ-comodule structures and the left $C_\zeta^{\leqq 0}$-comodule structures. It follows that

$$\Xi(C_\zeta \otimes M) \cong \Xi(C_\zeta \otimes M^{\mathrm{triv}}) = M^{\mathrm{triv}} \in \mathrm{Comod}(C_\zeta^{\leqq 0}). \qquad \square$$

The proof of Proposition 5.1 is complete.

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