ON THE BOUNDARY OF MODULI SPACES OF LOG HODGE STRUCTURES, II: NONTRIVIAL TORSORS

TATSUKI HAYAMA

Abstract. In this paper, we determine when a natural torsor arising in the work of Kato and Usui on partial compactification of period domains of pure Hodge structure is trivial, and we give an application to cycle spaces.

§1. Introduction

Let D be a period domain of pure Hodge structures defined by Griffiths [G]. A variation of $\mathbb{Z}$-Hodge structure over the n-product of punctured disk $(\Delta^*)^n$ gives the period map $(\Delta^*)^n \to \Gamma \backslash D$, where Γ is the monodromy group, that is, the $\mathbb{Z}$-module generated by the monodromy transformations. We assume that the monodromy transformations are unipotent. In this article, we treat partial compactifications of $\Gamma \backslash D$ so that the period map is extended over Δ^n.

In the case where D is Hermitian symmetric, Ash, Mumford, Rapoport, and Tai in [AMRT] gave partial compactifications of $\Gamma \backslash D$ (and also compactifications of arithmetic quotients of D). Later, Kato and Usui in [KU] generalized toroidal partial compactifications for any period domain D which is not Hermitian symmetric in general, and showed that these are moduli spaces of log Hodge structures. This partial compactification is given by using toroidal embedding associated to the cone generated by the data of the monodromy. In fact, for generators $T_1, \ldots, T_n$ of the monodromy group Γ, the partial compactification $\Gamma \backslash D_\sigma$ is given by the cone $\sigma = \sum_{j=1}^n \mathbb{R}_{\geq 0} N_j$ ($N_j = \log T_j$) in the Lie algebra. Here a boundary point is a nilpotent orbit associated to a face of σ (see Section 3.1).

In the classical situation (i.e., D is Hermitian symmetric), $\Gamma \backslash D_\sigma$ is an analytic space. In contrast, for general period domains, $\Gamma \backslash D_\sigma$ may not be

an analytic space. In fact, the boundary components of the partial compact-ification $\Gamma\backslash D_\sigma$ can have codimension greater than 1, although it is 1 in the classical situation (see Example 3.3). This means that there can be *slits* on the boundary of $\Gamma\backslash D_\sigma$. Kato and Usui [KU] define logarithmic manifolds as a generalization of analytic spaces and state that $\Gamma\backslash D_\sigma$ is a logarithmic manifold.

A part of the geometric structure of $\Gamma\backslash D_\sigma$ is given by a torsor $E_\sigma \to \Gamma\backslash D_\sigma$ constructed in [KU] (see Section 3.1). We will discuss these torsors. Our main result is the following.

THEOREM 1.1 (see Theorem 3.4). *The torsor $E_\sigma \to \Gamma\backslash D_\sigma$ is trivial if and only if D is Hermitian symmetric or $\sigma = \{0\}$.*

In [H1], we proved this result in the case where D is Hermitian symmet-ric. In this paper, we treat the case where D is not Hermitian symmetric. To prove Theorem 1.1, we use a result from Fels, Huckleberry, and Wolf [FHW, Theorem 4.4.3] to show that, unless D is Hermitian symmetric, any holomorphic function on D is constant.

We used a different strategy to prove the nontriviality of the torsors for the one example in [H1, Proposition 5.8]. Generalizing this approach, we give another proof of the nontriviality result in Proposition 4.3. This second proof is stronger than the first one since it gives a nontriviality on some open sets around a boundary point. We use cycle spaces and the SL(2)-orbit theorem there. Some property of cycle spaces induces the nontriviality.

In conclusion, the properties of cycle spaces induce the above nontrivial-ity results. Cycle spaces can have significance in the study of moduli spaces of log Hodge structures in a wider framework. In fact, the property of cycle spaces of Lemma 4.1 induces our later work [H2]. On the other hand, Green, Griffiths, and Kerr [GGK2] have introduced Mumford–Tate domains as a generalization of period domains, and they also indicate the importance of cycle space in relation to cohomology groups of Mumford–Tate domains. Moreover, Kerr and Pearlstein [KP] have constructed partial compactifi-cations of Mumford–Tate domains in the same manner as Kato and Usui [KU]. We expect that our results will fit into the case for the boundaries of the Mumford–Tate domains.

This article has the following organization. In Section 2.1, we review period domains of Hodge structures. In Sections 2.2 and 2.3, we discuss cycle spaces of period domains. In Section 3, we review moduli spaces of

polarized log Hodge structures. In Section 4, we reformulate our prior results from [H1] in terms of cycle spaces.

§2. Cycle spaces of period domains

2.1. Polarized Hodge structures and period domains

We recall the definition of polarized Hodge structures and of period domains. A Hodge structure of weight w with Hodge numbers $(h^{p,q})_{p,q}$ is a pair $(H_{\mathbb{Z}}, F)$ consisting of a free $\mathbb{Z}$-module of rank $\sum_{p,q} h^{p,q}$ and a decreasing filtration on $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$ satisfying the following conditions:

(H1) $\dim_{\mathbb{C}} F^p = \sum_{r \geq p} h^{r,w-r}$ for all p;

(H2) $H_{\mathbb{C}} = \bigoplus_{p+q=w} H^{p,q}$ $(H^{p,q} := F^p \cap \overline{F^{w-p}})$.

A polarization $\langle\,,\,\rangle$ for a Hodge structure $(H_{\mathbb{Z}}, F)$ of weight w is a nondegenerate bilinear form on $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$, symmetric if w is even and skew-symmetric if w is odd, satisfying the following conditions:

(P1) $\langle F^p, F^q \rangle = 0$ for $p + q > w$;

(P2) $i^{p-q} \langle v, \bar{v} \rangle > 0$ for $0 \neq v \in H^{p,q}$.

We fix a polarized Hodge structure $(H_{\mathbb{Z},0}, F_0, \langle\,,\,\rangle_0)$ of weight w with Hodge numbers $(h^{p,q})_{p,q}$. We define the set of all Hodge structures of this type

$$
D := \left\{ F \ \middle| \ \begin{array}{c} (H_{\mathbb{Z},0}, F, \langle\,,\,\rangle_0) \text{ is a polarized Hodge structure} \\ \text{of weight } w \text{ with Hodge numbers } (h^{p,q})_{p,q} \end{array} \right\}.
$$

The space D is called the *period domain*. Moreover, we have the flag manifold

$$
\check{D} := \left\{ F \ \middle| \ \begin{array}{c} (H_{\mathbb{Z},0}, F, \langle\,,\,\rangle_0) \text{ satisfies the conditions} \\ \text{(H1), (H2), and (P1)} \end{array} \right\}.
$$

The space $\check{D}$ is called the *compact dual* of D, and it contains D as an open subset. Let $G_A := \operatorname{Aut}(H_{A,0}, \langle\,,\,\rangle_0)$. Then, $G_{\mathbb{R}}$ acts transitively on D, and $G_{\mathbb{C}}$ acts transitively on $\check{D}$. The group $G_{\mathbb{R}}$ is a classical group such that

$$
G_{\mathbb{R}} \cong \begin{cases} \operatorname{Sp}(h, \mathbb{R}) & \text{if } w \text{ is odd,} \\ \operatorname{SO}(h_{\text{odd}}, h_{\text{even}}) & \text{if } w \text{ is even,} \end{cases}
$$

where $2h = \operatorname{rank} H_{\mathbb{Z}}$, $\operatorname{Sp}(h, \mathbb{R})$ is the $(2h \times 2h)$-matrix symplectic group, $h_{\text{odd}} = \sum_{p:\ \text{odd}} h^{p,q}$, and $h_{\text{even}} = \sum_{p:\ \text{even}} h^{p,q}$.

Let $\mathfrak{g}_A = \mathrm{Lie} G_A$ $(A = \mathbb{R}, \mathbb{C})$. We then have the decomposition $\mathfrak{g}_\mathbb{C} = \bigoplus_{p+q=0} \mathfrak{g}^{p,q}$ given by

$$\mathfrak{g}^{p,q} = \left\{ \alpha \in \mathfrak{g}_\mathbb{C} \mid \alpha H^{p',q'} \subset H^{p+p',q+q'} \text{ for } p', q' \in \mathbb{Z} \right\}$$

with respect to a Hodge decomposition $H_\mathbb{C} = \bigoplus H^{p,q}$.

EXAMPLE 2.1 (Upper half-plane). Let us consider the case where the Hodge numbers $h^{1,0} = h^{0,1} = 1$, 0 otherwise. Then the corresponding classifying space D is the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$, and $\check{D} \cong \mathbb{P}^1$. We have $G_A \cong \mathrm{SL}(2, A)$ $(A = \mathbb{Z}, \mathbb{R}, \mathbb{C})$, where the action of $\mathrm{SL}(2, \mathbb{C})$ on $\check{D}$ is given by the linear fractional transformation. Here $\mathfrak{g}_\mathbb{R} = \mathfrak{sl}(2, \mathbb{R})$ is generated by

$$\mathbf{n}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{h} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{n}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We call the triple the sl_2-triple. The sl_2-triple satisfies the following conditions:

$$[\mathbf{n}_+, \mathbf{n}_-] = \mathbf{h}, \qquad [\mathbf{n}_\pm, \mathbf{h}] = \pm 2 \mathbf{n}_\pm.$$

The Hodge decomposition of $\mathfrak{g}_\mathbb{C}$ with respect to $i \in D$ is given by

$$(2.1) \ \ \mathfrak{g}^{-1,1} = \mathbb{C}(i\mathbf{n}_- - \mathbf{h} + i\mathbf{n}_+), \qquad \mathfrak{g}^{0,0} = \mathbb{C}(\mathbf{n}_- - \mathbf{n}_+), \qquad \mathfrak{g}^{1,-1} = \overline{\mathfrak{g}^{-1,1}}.$$

Returning to the general case, the isotropy subgroup L of $G_\mathbb{R}$ at F_0 is given by

$$L = \{ g \in G_\mathbb{R} \mid gF_0 = F_0 \}$$

$$\cong \begin{cases} \prod_{p \le m} U(h^{p,q}) & \text{if } w = 2m+1, \\ \prod_{p < m} U(h^{p,q}) \times \mathrm{SO}(h^{m,m}) & \text{if } w = 2m. \end{cases}$$

They are compact subgroups of $G_\mathbb{R}$ but not maximal compact unless D is Hermitian symmetric. We define

$$H^{\mathrm{even}} = \bigoplus_{p: \text{ even}} H_0^{p,q}, \qquad H^{\mathrm{odd}} = \bigoplus_{p: \text{ odd}} H_0^{p,q},$$

where $H_{\mathbb{C}} = \bigoplus H_0^{p,q}$ is the Hodge decomposition for F_0. Here

$$K = \{g \in G_{\mathbb{R}} \mid gH^{\mathrm{even}} = H^{\mathrm{even}}\}$$

$$\cong \begin{cases} U(h) & \text{if } w \text{ is odd,} \\ S\big(O(h_{\mathrm{odd}}) \times O(h_{\mathrm{even}})\big) & \text{if } w \text{ is even} \end{cases}$$

is the maximal subgroup containing L (see [CMP, Example 4.3.6], [LS, Lemma 2.8]). By the connectivity of $G_{\mathbb{R}}$, D is connected if w is odd, D has two connected components if w is even, and $h_{\mathrm{even}}, h_{\mathrm{odd}} > 0$. Here D is Hermitian symmetric if and only if the isotropy subgroup is a maximally compact subgroup, that is, if one of the following is satisfied (see [U1, (1.8)]):

(1) $w = 2m+1, h^{p,q} = 0$ unless $p = m+1, m$;
(2) $w = 2m, h^{p,q} = 1$ for $p = m+1, m-1, h^{m,m}$ is arbitrary, $h^{p,q} = 0$ otherwise;
(3) $w = 2m, h^{p,q} = 1$ for $p = m+a, m+a-1, m-a, m-a+1$ for some $a \geq 2, h^{p,q} = 0$ otherwise.

In the case (1), D is a Hermitian symmetric domain of type III. In the case (2) or (3), an irreducible component of D is a Hermitian symmetric domain of type IV. We call the cases (1)–(3) the *classical situation*.

EXAMPLE 2.2 (The weight 1 case). We give an example of period domains of weight 1 and $h^{1,0} = h^{0,1} = n$, 0 otherwise. This case corresponds to case (1) above. Now $G_A = \mathrm{Sp}(n, A)$ ($A = \mathbb{Z}, \mathbb{R}, \mathbb{C}$) and

$$D = \big\{W : \langle\,,\,\rangle\text{-isotropic } n\text{-planes} \,\big|\, W > 0 \text{ for } i\langle \bullet, \bar{\bullet}\rangle\big\}$$

(2.2) $$\cong \{Z \in \mathbb{C}^{n \times n} \mid I - ZZ^* > 0\}$$

$$\cong \mathrm{Sp}(n, \mathbb{R})/U(n),$$

where > 0 means positive definite. The space D is called the *Siegel space* of degree n (see [N] for details).

2.2. Cycle spaces of period domains

Let D_0 be an irreducible component including F_0 of a period domain D. Then the identity component $G_{\mathbb{R},0}$ acts on D_0 transitively. Let K_0 be the maximal compact subgroup of $G_{\mathbb{R},0}$ containing the isotropy subgroup L_0 at F_0. We then have the real analytic projection

$$p : D_0 \cong G_{\mathbb{R},0}/L_0 \to G_{\mathbb{R},0}/K_0.$$

If w is odd,

$$(2.3) \qquad\qquad G_{\mathbb{R},0}/K_0 = \mathrm{Sp}(h,\mathbb{R})/U(h)$$

is the Hermitian symmetric domain of Example 2.2. If w is even,

$$G_{\mathbb{R},0}/K_0 = \mathrm{SO}_0(h_{\mathrm{odd}}, h_{\mathrm{even}})/\mathrm{SO}(h_{\mathrm{odd}}) \times \mathrm{SO}(h_{\mathrm{even}})$$

is a symmetric space which does not have any complex structure unless the projection p is trivial (see [CMP, Example 4.3.6]). Moreover, $G_{\mathbb{R},0}/K_0$ is written as the set of all H_{even} for $F \in D_0$ (see [LS, Lemma 2.10]), and the projection p is given by

$$F \mapsto H_{\mathrm{even}}.$$

Then it is not holomorphic even if w is odd. Therefore, we have the following theorem applying [FHW] to D_0.

THEOREM 2.3 ([FHW, Theorem 4.4.3]). *If D_0 is not a Hermitian symmetric domain (i.e., $L_0 \neq K_0$), any holomorphic function on D_0 is constant.*

Now the fiber of $p(F_0)$ is the K_0-orbit $C_0 = K_0 \cdot F_0$. We call C_0 the *base cycle* of F_0. By [FHW, Theorem 4.3.1], $K_{0,\mathbb{C}}$ acts on C_0 transitively, and then $C_0 = K_{0,\mathbb{C}} \cdot F_0$ is a compact submanifold of D_0.

PROPOSITION 2.4 ([FHW, Lemma 5.1.3]). *Let $J = \{g \in G_{\mathbb{C}} \mid gC_0 = C_0\}$. Then J is a closed complex subgroup of $G_{\mathbb{C}}$. The quotient manifold $\mathcal{M}_{\check{D}} = \{gC_0 \mid g \in G_{\mathbb{C}}\} \cong G_{\mathbb{C}}/J$ has a natural structure of $G_{\mathbb{C}}$-homogeneous complex manifold, and the subset $\{gC_0 \mid g \in G_{\mathbb{C}} \text{ and } gC_0 \subset D\}$ is open in $\mathcal{M}_{\check{D}}$.*

The topological component of C_0 in $\{gC_0 \mid g \in G_{\mathbb{C}} \text{ and } gC_0 \subset D_0\}$ is called the *cycle space* of D_0. We denote the cycle space of D_0 by $\mathcal{M}_{D_0}$. If D_0 is Hermitian symmetric, the projection p is trivial; therefore, $\mathcal{M}_{D_0} = D_0$.

2.3. Cycle spaces for odd weight cases

We describe cycle spaces explicitly in the odd weight case when D_0 is not Hermitian symmetric according to [FHW, Section 5.5B]. In this case $D = D_0$. For a base point $F_0 \in D$, we define

$$f^p_{\mathrm{even}} = \sum_{\substack{r \geq p, \\ r:\ \mathrm{even}}} h^{r,s}, \qquad f^p_{\mathrm{odd}} = \sum_{\substack{r \geq p, \\ r:\ \mathrm{odd}}} h^{r,s}.$$

Let V and W be $\langle\,,\rangle$-isotropic subspaces, and let

$$C_{V,W} = \left\{ F \in \check{D} \;\middle|\; \dim(F^p \cap V) = f^p_{\mathrm{even}}, \dim(F^p \cap W) = f^p_{\mathrm{odd}} \right\}.$$

Here $gC_{V,W} = C_{gV,gW}$ for $g \in G_{\mathbb{C}}$ by the definition. Now

$$F_0^p \cap H_{\text{even}} = \bigoplus_{\substack{r \geq p, \\ r: \text{ even}}} H^{r,s},$$

$$gF_0^p \cap H_{\text{even}} = g(F_0^p \cap H_{\text{even}}) = \bigoplus_{\substack{r \geq p, \\ r: \text{ even}}} gH^{r,s}$$

for $g \in K$. Then $C_0 = C_{H^{\text{even}}, H^{\text{odd}}}$.

By (2.3), $G_{\mathbb{R}}/K$ is isomorphic to the Siegel space $\mathcal{B}$. In this case, the cycle space $\mathcal{M}_D$ is described as follows.

PROPOSITION 2.5. *We have*

$$\mathcal{M}_D = \left\{ C_{V,W} \mid V < 0 \text{ and } W > 0 \text{ for } i^w \langle \bullet, \bar{\bullet} \rangle \right\} \cong \mathcal{B} \times \bar{\mathcal{B}},$$

where w is the weight.

Proof. Now the $G_{\mathbb{R}}$-orbit $G_{\mathbb{R}}H^{\text{even}}$ is isomorphic to $\mathcal{B}$. Then $G_{\mathbb{R}}H^{\text{odd}}$ is the complex conjugate $\bar{\mathcal{B}}$. Since $\mathcal{B}$ (resp., $\bar{\mathcal{B}}$) is an open subset of the flag manifold $G_{\mathbb{C}}H^{\text{even}}$ (resp., $G_{\mathbb{C}}H^{\text{odd}}$), we have $\mathcal{B} \times \bar{\mathcal{B}} \subset G_{\mathbb{C}}H^{\text{even}} \times G_{\mathbb{C}}H^{\text{odd}}$. For $(H^{\text{even}}, H^{\text{odd}}) \in G_{\mathbb{C}}H^{\text{even}} \times G_{\mathbb{C}}H^{\text{odd}}$, the $G_{\mathbb{C}}$-orbit $G_{\mathbb{C}}(H^{\text{even}}, H^{\text{odd}})$ includes $\mathcal{B} \times \bar{\mathcal{B}}$ by [FHW, Lemma 5.4.1]. Now the isotropy subgroup of $G_{\mathbb{C}}$ at $(H^{\text{even}}, H^{\text{odd}})$ is $K_{\mathbb{C}}$. Then we have $G_{\mathbb{C}}/K_{\mathbb{C}} \supset \mathcal{B} \times \bar{\mathcal{B}}$.

Since $\mathcal{M}_{\check{D}} \cong G_{\mathbb{C}}/J$ and $K_{\mathbb{C}} \subset J$, we have the projection

$$\pi : G_{\mathbb{C}}/K_{\mathbb{C}} \to \mathcal{M}_{\check{D}}; \qquad g \pmod{K_{\mathbb{C}}} \mapsto gC_0.$$

By [FHW, Proposition 5.4.3], π is injective on $\mathcal{B} \times \bar{\mathcal{B}}$. Moreover, by [FHW, Theorem 5.5.1], $\pi(\mathcal{B} \times \bar{\mathcal{B}}) = \mathcal{M}_D \subset \mathcal{M}_{\check{D}}$. □

§3. Moduli spaces of polarized log Hodge structures

In this section, we review the construction of moduli spaces of log Hodge structures and state the fundamental properties following [KU] in Section 3.1. We state the main result in Section 3.2.

3.1. Construction and fundamental properties

We call $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ a *nilpotent cone* if it satisfies the following conditions:

(1) σ is a closed cone generated by finitely many elements of $\mathfrak{g}_{\mathbb{Q}}$;

(2) $N \in \sigma$ is nilpotent as an endomorphism of $H_{\mathbb{R}}$;

(3) $NN' = N'N$ for any $N, N' \in \sigma$.

For $A = \mathbb{R}, \mathbb{C}$, we denote by σ_A the A-linear span of σ in $\mathfrak{g}_A$.

DEFINITION 3.1. Let $\sigma = \sum_{j=1}^{n} \mathbb{R}_{\geq 0} N_j$ be a nilpotent cone, and let $F \in \check{D}$. Then

$$\exp(\sigma_{\mathbb{C}})F \subset \check{D}$$

is called a σ-*nilpotent orbit* if it satisfies the following conditions:

(1) $\exp(\sum_j iy_j N_j)F \in D$ for all $y_j \gg 0$;

(2) $NF^p \subset F^{p-1}$ for all $p \in \mathbb{Z}$ and for all $N \in \sigma$.

We define the set of nilpotent orbits

$$D_{\sigma} := \big\{ (\tau, Z) \,\big|\, \tau : \text{ face of } \sigma, Z \text{ is a } \tau\text{-nilpotent orbit} \big\}.$$

For a nilpotent cone σ, we have the abelian group and the monoid

$$\Gamma(\sigma)^{\mathrm{gp}} = \exp(\sigma_{\mathbb{R}}) \cap G_{\mathbb{Z}}, \qquad \Gamma(\sigma) = \exp(\sigma) \cap G_{\mathbb{Z}}.$$

We define a geometric structure on $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\sigma}$. First, we review some basic facts about toric varieties. The monoid $\Gamma(\sigma)$ defines the toric varieties

$$\mathrm{toric}_{\sigma} := \mathrm{Spec}\big(\mathbb{C}\big[\Gamma(\sigma)^{\vee}\big]\big)_{\mathrm{an}} \cong \mathrm{Hom}\big(\Gamma(\sigma)^{\vee}, \mathbb{C}\big),$$

$$\mathrm{torus}_{\sigma} := \mathrm{Spec}\big(\mathbb{C}\big[\Gamma(\sigma)^{\vee\mathrm{gp}}\big]\big)_{\mathrm{an}} \cong \mathrm{Hom}\big(\Gamma(\sigma)^{\vee\mathrm{gp}}, \mathbb{G}_m\big) \cong \mathbb{G}_m \otimes \Gamma(\sigma)^{\mathrm{gp}},$$

where $\mathbb{C}$ in the right-hand side of the first line is regarded as a semigroup via multiplication and the above homomorphisms are of semigroups. As in [F, Section 2.1], we choose for a face τ of σ the distinguished point

$$x_{\tau} : \Gamma(\sigma)^{\vee} \to \mathbb{C}; \qquad u \mapsto \begin{cases} 1 & \text{if } u \in \Gamma(\tau)^{\perp}, \\ 0 & \text{otherwise.} \end{cases}$$

Then toric_{σ} can be decomposed by torus orbits as

$$\mathrm{toric}_{\sigma} = \bigsqcup_{\tau : \text{ face of } \sigma} (\mathrm{torus}_{\sigma} \cdot x_{\tau}).$$

For $q \in \mathrm{toric}_{\sigma}$, there exists the face $\sigma(q)$ of σ such that $q \in \mathrm{torus}_{\sigma} \cdot x_{\sigma(q)}$. By a surjective homomorphism

$$\mathbf{e} : \sigma_{\mathbb{C}} \to \mathrm{torus}_{\sigma} \cong \mathbb{G}_m \otimes \Gamma(\sigma)^{\mathrm{gp}}; \qquad w\log(\gamma) \mapsto \exp(2\pi\sqrt{-1}w) \otimes \gamma,$$

q can be written as $q = \mathbf{e}(z) \cdot x_{\sigma(q)}$. Here $\ker(\mathbf{e}) = \log(\Gamma(\sigma)^{\mathrm{gp}})$, and z is determined uniquely modulo $\log(\Gamma(\sigma)^{\mathrm{gp}}) + \sigma(q)_{\mathbb{C}}$.

We define the analytic space $\check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D}$ and the subset

$$E_\sigma := \left\{ (q, F) \in \check{E}_\sigma \;\middle|\; \begin{array}{c} \exp\left(\sigma(q)_{\mathbb{C}}\right) \exp\left(z\right) F \text{ is a } \sigma(q)\text{-nilpotent orbit,} \\ \text{where } q = \mathbf{e}(z) \cdot x_{\sigma(q)} \end{array} \right\}.$$

Here we endow E_σ with the *strong topology* (see [KU, Section 3.1]) in $\check{E}_\sigma$. We then define the canonical map

$$\pi : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma,$$

$$(q, F) \mapsto \left(\sigma(q), \exp\left(\sigma(q)_{\mathbb{C}}\right) \exp\left(z\right) F\right) \quad \mathrm{mod}\ \Gamma(\sigma)^{\mathrm{gp}}.$$

We endow $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ with the strongest topology for which the maps π are continuous. Kato and Usui give the geometric properties of E_σ, $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ and $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ by using the language *log manifolds* (see [KU, Section 3.5]).

THEOREM 3.2 ([KU, Theorem A]). *We have the following:*
(1) *E_σ and $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ are logarithmic manifolds;*
(2) *we have that the $\sigma_{\mathbb{C}}$-action on E_σ over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ by*

$$a \cdot (q, F) := \left(\mathbf{e}(a)q, \exp\left(-a\right) F\right) \quad \left(a \in \sigma_{\mathbb{C}}, (q, F) \in E_\sigma\right),$$

and $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is a $\sigma_{\mathbb{C}}$-torsor in the category of logarithmic manifolds.

Log manifolds are roughly analytic spaces with *slits*. A typical example of a log manifold is

$$\mathbb{C}^2 \supset \left\{ (x, y) \;\middle|\; x = 0 \Rightarrow y = 0 \right\},$$

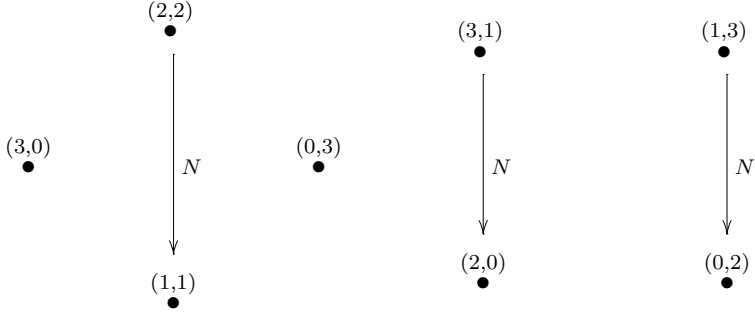which is defined by the log differential 1-form $y\, d\log x$ of the log analytic space $\mathbb{C}^2$.

Moreover, Kato and Usui in [KU] define *polarized log Hodge structures* (see [KU, Section 2.4]), and they show that $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is a fine moduli space of polarized log Hodge structures (see [KU, Theorem B]).

In the classical situation, $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is just a toroidal partial compactification and the boundary is of codimension 1 (see [N]). However, the codimension may be greater than 1 in the nonclassical situation.

EXAMPLE 3.3 (The $(1, 1, 1, 1)$-case). Nilpotent orbits in the case where the Hodge numbers are $h^{3,0} = h^{0,3} = 1$ and $h^{1,2} = h^{2,1} = 1$, 0 otherwise (we

call it the $(1,1,1,1)$-*case*) are classified by [KU, Section 12.3] or [GGK1]. In this case, $D \cong \mathrm{Sp}(2,\mathbb{R})/(U(1) \times U(1))$ and $\dim D = 4$. Here D is not a Hermitian symmetric space. All possible nilpotent cones are of rank 1. For a nilpotent orbit $(\mathbb{R}_{\geq 0}N, \exp(\mathbb{C}N)F)$, we have the limiting mixed Hodge structure $(W(N), F)$ by [S] twisting $W(N)$. Here $(W(N), F)$ is one of the following types:

Type-I: $N^2 = 0, \dim(\mathrm{Im}N) = 1$.   Type-II: $N^2 = 0, \dim(\mathrm{Im}N) = 2$.

$$
\begin{array}{ccc}
(2,2) & & \\
\bullet & & \\
(3,0) & \Big\downarrow N & (0,3) \\
\bullet & & \bullet \\
& (1,1) & \\
& \bullet &
\end{array}
\qquad
\begin{array}{cc}
(3,1) & (1,3) \\
\bullet & \bullet \\
\Big\downarrow N & \Big\downarrow N \\
(2,0) & (0,2) \\
\bullet & \bullet
\end{array}
$$

Type-III: $N^3 \neq 0, N^4 = 0$.   Dimensions of boundaries

$$
\begin{array}{c}
(3,3) \\
\bullet \\
\Big\downarrow N \\
(2,2) \\
\bullet \\
\Big\downarrow N \\
(1,1) \\
\bullet \\
\Big\downarrow N \\
(0,0) \\
\bullet
\end{array}
$$

	$\dim(D_\sigma - D)$
Type-I	2
Type-II	1
Type-III	1

Geometrically, type-I or type-III degeneration occurs in the quintic-mirror family, and type-II degeneration occurs in the Borcea–Voisin mirror family (see [GGK1, Part III.A], [U2]).

3.2. Whether the torsors are trivial

By Theorem 3.2, we have the torsor $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ for a period domain D and a nilpotent cone σ. In [H1], we showed the triviality of torsors in the classical situation. We show a nontriviality of the torsors in the nonclassical

situation by using the fact that any holomorphic function on D is constant in the nonclassical case (see Theorem 2.3).

THEOREM 3.4. *Let D be a period domain (for pure Hodge structures), and let (σ, Z) be a nilpotent orbit. Then $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is trivial if and only if D is Hermitian symmetric or $\sigma = \{0\}$.*

Proof. By [H1, Theorem 5.6], the torsors are trivial if D is a Hermitian symmetric space. If $\sigma = \{0\}$, the torsor is just the identity map $D \to D$; therefore, the torsor is trivial. Thus it suffices to show that the torsor is nontrivial if D is not Hermitian symmetric.

We assume that $\pi : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is trivial for a non-Hermitian symmetric space D and for a nilpotent cone $\sigma \neq \{0\}$. Now

$$\pi^{-1}\big(\Gamma(\sigma)^{\mathrm{gp}}\backslash D\big) = E_\sigma \cap (\mathrm{torus}_\sigma \times \check{D})$$

by the definition of E_σ, and this is a complex analytic space since $\mathrm{torus}_\sigma \times \check{D}$ has trivial log structure. Thus the restriction of the torsor to $\pi^{-1}(\Gamma(\sigma)^{\mathrm{gp}}\backslash D)$ is a torsor in the category of complex analytic spaces, and we have a section $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to E_\sigma$ and a holomorphic map $\Phi : D \to (\mathbb{C}^*)^l$ such that we have the following diagram:

(3.1)
$$\begin{array}{ccc} \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma & \longrightarrow & E_\sigma \\ \cup & & \cup \\ \Phi : D \xrightarrow{\mathrm{quot.}} \Gamma(\sigma)^{\mathrm{gp}}\backslash D \longrightarrow & E_\sigma \cap (\mathrm{torus}_\sigma \times \check{D}) \xrightarrow{\mathrm{proj.}} & \mathrm{torus}_\sigma \cong (\mathbb{C}^*)^l, \end{array}$$

where $l = \mathrm{rank}\Gamma(\sigma)^{\mathrm{gp}}$.

For a nilpotent N in the relative interior of σ, we have

(3.2)
$$\lim_{y \to \infty} \exp(iyN)F = \big(\sigma, \exp(\sigma_{\mathbb{C}})F\big)$$

through $D \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D \hookrightarrow \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ by [KU, Proposition 3.4.4]. Then $\Phi(\exp(iyN)F)$ has to converge to $0 \in \mathrm{toric}_\sigma$ as $y \to \infty$. This contradicts Theorem 2.3. □

§4. Remarks on [H1]

We showed the nontriviality of the torsor in [H1, Proposition 5.8] using a method different from that used in Theorem 3.4. We formulate it by using

the SL(2)-orbit theorem and cycle spaces, and we give a second proof of the nontriviality result for a special case. While this second proof requires some special conditions, the result is stronger than the first one since it says that there exists no section over certain open sets around a boundary point. A property of some cycle spaces induces this result. We observe the property of cycle spaces in the $(1,1,1,1)$-case explicitly in Section 4.3. In this section, we assume that D is not Hermitian symmetric.

4.1. SL(2)-orbits and cycle spaces

Let $(\mathbb{R}_{\geq 0}N, \exp(\mathbb{C}N)F)$ be a nilpotent orbit. By [S] there exists the monodromy weight filtration $W(N)$, and $(W(N), F)$ is a mixed Hodge structure. By [CKS, Proposition 2.20] there exists the $\mathbb{R}$-split mixed Hodge structure $(W(N), \hat{F})$ associated to it. We then have the Deligne decomposition $H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$ for $(W(N), \hat{F})$, where

$$\hat{F}^p = \bigoplus_{r \leq p} I^{r,s}, \qquad W(N)_k = \bigoplus_{r+s=k} I^{r,s}, \qquad \overline{I^{p,q}} = I^{q,p}.$$

By the SL(2)-orbit theorem (see [S, Theorem 5.13], [CKS, Section 3]), there exists the Lie group homomorphism $\rho : \mathrm{SL}(2,\mathbb{C}) \to G_{\mathbb{C}}$ defined over $\mathbb{R}$ and the holomorphic map $\phi : \mathbb{P}^1 \to \check{D}$ satisfying the following conditions:

(S1) $\rho(g)\phi(z) = \phi(gz)$;

(S2) $\phi(0) = \hat{F}$;

(S3) $\rho_*(\mathbf{n}_-) = N$;

(S4) $Hv = (p + q - w)v$ for $v \in I^{p,q}$, where $\rho_*(\mathbf{h}) = H$;

(S5) $\rho_* : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$ is a $(0,0)$-morphism of Hodge structure, where $\mathfrak{g}_{\mathbb{R}}$ (resp., $\mathfrak{sl}(2,\mathbb{R})$) has a Hodge structure of weight 0 relative to $\phi(i)$ (resp., i),

where $\{\mathbf{n}_-, \mathbf{h}, \mathbf{n}_+,\}$ is the sl_2-triple (see Example 2.1).

Let $F_0 = \phi(i)$ be a base point of D. We write

$$(4.1) \qquad \rho_*(\mathbf{n}_+) = N^+, \qquad X = \frac{1}{2}(iN - H + iN^+).$$

Then $X \in \mathfrak{g}_0^{-1,1}$ by (S5) and (2.1), where $\mathfrak{g}_0^{-1,1}$ is the $(-1,1)$-component of the Hodge decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to F_0. By (S1) we have

$$\exp(zX)\phi(i) = \phi\Big(\frac{1+z}{1-z}i\Big);$$

therefore,

$$(4.2) \qquad \exp\left(\frac{y}{2+y}X\right)\phi(i) = \phi\big((1+y)i\big) = \exp\left(iyN\right)\phi(i).$$

LEMMA 4.1. *Let C_0 be the base cycle of F_0. If $\dim(\mathrm{Im}N) = 1$, then both (A) and (B) hold:*

(A) *there exists $F_{\mathrm{fix}} \in C_0$ such that $\exp(X)F_{\mathrm{fix}} = F_{\mathrm{fix}}$;*

(B) *$\exp(zX)C_0 \subset D$ (i.e., $\exp(zX)C_0 \in \mathcal{M}_D$) for $|z| < 1$.*

Proof. At first we write X explicitly. Considering the type of the limiting Hodge structure $(W(N), F)$, the case where $\dim(\mathrm{Im}N) = 1$ is possible only if the weight is $2m - 1$ and $\dim(\mathrm{Gr}_{2m}^W) = \dim(\mathrm{Gr}_{2m-2}^W) = 1$. We then have an $\mathbb{R}$-element e in the (m, m)-component $I^{m,m}$ of the Deligne decomposition of $(W(N), \hat{F})$. Here X is given by

$$e \mapsto \frac{1}{2}(-e + iNe), \qquad Ne \mapsto \frac{1}{2}(ie + Ne) = -iXe,$$

$$I^{p,q} \to 0 \quad \text{for } p + q = 2m - 1.$$

We write $u = \exp(iN)e$. Since $e \in \hat{F}^m$, $u \in \exp(iN)\hat{F}^m = F_0^m$. Moreover, since $Ne \in \hat{F}^{m-1}$,

$$Ne = \exp(iN)Ne \in \exp(iN)\hat{F}^{m-1} = F_0^{m-1}.$$

Then

$$\bar{u} = e - iNe = u - 2iNe \in F_0^{m-1}.$$

Hence u is in the $(m, m-1)$-component $H_0^{m,m-1}$ of the Hodge decomposition for F_0. Here

$$(4.3) \qquad Xu = -e + iNe = -\bar{u}, \qquad Xw = 0 \quad \text{if } \langle w, \bar{u} \rangle = 0.$$

First we show (A). We denote by $\|\bullet\|$ the norm induced by the positive definite Hermitian form $\langle C_{F_0}\bullet, \bar{\bullet}\rangle$, where C_{F_0} is the Weil operator for F_0. Scaling u, we may assume that $\|u\| = 1$. We take $v \in H_0^{m-2,m+1}$ such that $\|v\| = 1$. We define $g \in \mathrm{Aut}(H_\mathbb{C})$ by

$$gu = v, \qquad gv = u, \qquad g\bar{v} = \bar{u}, \qquad g\bar{u} = \bar{v},$$

and $gw = w$ if w is vertical to $u, v, \bar{u}$, and $\bar{v}$ for $\langle\,,\,\rangle$. Then

$$gu = v = \overline{g\bar{u}} = \bar{g}u, \qquad gv = u = \overline{g\bar{v}} = \bar{g}v.$$

Therefore, g is defined over $\mathbb{R}$ and preserves the polarization $\langle\,,\,\rangle$; that is, $g \in G_\mathbb{R}$. Moreover, $g \in K$ since g preserves H^{even}.

CLAIM. *We have that $gF_0 \in C_0$ is a fixed point for $\exp(X)$.*

Proof. Now $u = gv \in gH_0^{m-2,m+1}$. By (4.3), it suffices to show that $Xu \in gF_0^{m-2}$. In fact,

$$Xu = -\bar{u} = -g\bar{v} \in gH_0^{m+1,m-2}. \qquad \Box$$

Next we show (B). We take a unitary basis $\{u_1, \ldots, u_l\}$ of $H_0^{m,m-1}$. We may assume that $u_1 = u$. Then $\exp(X)u_j = u_j$ if $j \neq 1$ and $\exp(zX)u_1 = u_1 - z\bar{u}_1$. Here

$$i\langle \exp(zX)u_1, \overline{\exp(zX)u_1} \rangle = \|u_1\|^2 - |z|^2\|u_1\|^2 = 1 - |z|^2.$$

By Proposition 2.5, $\exp(zX)C_0 \subset D$ if and only if $|z| < 1$. $\qquad \Box$

REMARK 4.2. In the $(1,1,1,1)$-case, a type-I nilpotent N satisfies $\dim(\mathrm{Im}N) = 1$, however other types do not. Above, (4.2) and items (A) and (B) of Lemma 4.1 correspond to the conditions (5.4), (5.6), and (5.5) of [H1, Section 5], respectively.

4.2. Nontriviality on some open sets

Let $(\mathbb{R}_{\geq 0}N, \exp(\mathbb{C}N)F)$ be a nilpotent orbit. Let (ρ, ϕ) be the SL(2)-orbit associated to (N, F). Taking $F_0 = \phi(i)$ as a base point, we have the base cycle C_0 and $X \in \mathfrak{g}_0^{-1,1}$ as in (4.1). We define the subset

$$\mathcal{M}(\varepsilon) = \left\{ \exp(\alpha X)C_0 \,\middle|\, 1 - \varepsilon < \alpha < 1 \right\} \subset \mathcal{M}_{\check{D}}$$

for $0 < \varepsilon$. If $\dim(\mathrm{Im}N) = 1$, by Lemma 4.1 (B),

$$\exp(\alpha X)C_0 \in \mathcal{M}_D \quad \text{for } -1 < \alpha < 1, \exp(X)C_0 \notin \mathcal{M}_D.$$

Then $\mathcal{M}(\varepsilon)$ is a nearby set of the boundary point $\exp(X)C_0 \in \overline{\mathcal{M}_D}$.

PROPOSITION 4.3. *Let U be an open set including the boundary point $(\sigma, \exp(\sigma_{\mathbb{C}})\hat{F})$ in $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$, where $\sigma = \mathbb{R}_{\geq 0}N$ with $\dim(\mathrm{Im}N) = 1$. If there exists $0 < \varepsilon < 1$ such that $q(C) \subset U$ for any $C \in \mathcal{M}(\varepsilon)$ and the quotient map $q: D \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D$, then no section over the open set U exists.*

Proof. We assume that there exists a local trivialization over U. Similar to the proof of Theorem 3.4, we have a section $U \to E_\sigma$ and the holomorphic map $\Phi: q^{-1}(U) \to \mathbb{C}^*$ given by the following diagram

$$U \longrightarrow E_\sigma$$

$$\cup \qquad\qquad \cup$$

$$\Phi: q^{-1}(U) \longrightarrow U \cap \left(\Gamma(\sigma)^{\mathrm{gp}} \backslash D\right) \longrightarrow E_\sigma \cap (\mathbb{C}^* \times \check{D}) \longrightarrow \mathbb{C}^*.$$

By (4.2) and the assumption, we have

$$q\Big(\exp\Big(\frac{y}{2+y}X\Big)F_0\Big) = q\big(\exp\,(iyN)F_0\big) \subset U$$

for

$$1-\varepsilon < \frac{y}{2+y} < 1, \quad \text{that is,} \quad \frac{2(1-\varepsilon)}{\varepsilon} \leq y.$$

By (3.2), $\Phi(\exp\,(iyN)F_0)$ has to converge to $0 \in \mathrm{toric}_\sigma$ as $y \to \infty$.

Now Φ is constant on the compact complex submanifold $C \in \mathcal{M}(\varepsilon)$. By Lemma 4.1, we then have

$$\Phi\big(\exp\,(iyN)F_0\big) = \Phi\Big(\exp\Big(\frac{y}{2+y}X\Big)F_0\Big) = \Phi\Big(\exp\Big(\frac{y}{2+y}X\Big)F_{\mathrm{fix}}\Big)$$

$$= \Phi\Big(\exp\Big(\frac{y'}{2+y'}X\Big)F_{\mathrm{fix}}\Big) = \Phi\big(\exp\,(iy'N)F_0\big)$$

for $y, y' > 2(1-\varepsilon)/\varepsilon$. This contradicts the convergence of $\Phi(\exp\,(iyN)F_0)$.

□

REMARK 4.4. Above X, F_{fix} and F_0 correspond to the notations N', F_∞ and F_0 in [H1, Section 5], respectively.

4.3. The $(1,1,1,1)$-case

The conditions (A) and (B) of Lemma 4.1 induce Proposition 4.3. In our later work [H2], condition (B) also plays an important role in the study of boundary structure. We then expect that $\Gamma \backslash D_\Sigma$ has good properties if Σ satisfies the conditions (A) or (B). Therefore, it is important to determine which cone satisfies (A) or (B). As we saw in Example 3.3, the types of nilpotent orbits in the $(1,1,1,1)$-case are well known. Type-I nilpotents satisfy (A) and (B). We show that (A) or (B) does not hold in other types below.

Let $(\mathbb{R}_{\geq 0}N, \exp\,(\mathbb{C}N)F)$ be a nilpotent orbit, and let (ρ, ϕ) be the $\mathrm{SL}(2)$-orbit associated (N, F). We can choose a unitary basis

$$u_3 \in H_0^{3,0}, \qquad u_2 \in H_0^{2,1}, \qquad \bar{u}_2 \in H_0^{1,2}, \qquad \bar{u}_3 \in H_0^{0,3}$$

for the Hodge decomposition for $F_0 = \phi(i)$. Here the base cycle of F_0 is $C_0 \cong U(2)/(U(1) \times U(1)) \cong \mathbb{P}^1$. The isomorphism $\mathbb{P}^1 \xrightarrow{\sim} C_0 \subset D$ is given by

$$(4.4) \qquad \begin{aligned} F_z^3 &= \mathrm{span}_{\mathbb{C}}\{z\bar{u}_2 + u_3\}, \qquad F_z^2 = \mathrm{span}_{\mathbb{C}}\{z\bar{u}_2 + u_3, u_2 - z\bar{u}_3\}, \\ F_\infty^3 &= \mathrm{span}_{\mathbb{C}}\{\bar{u}_2\}, \qquad F_\infty^2 = \mathrm{span}_{\mathbb{C}}\{\bar{u}_2, \bar{u}_3\}. \end{aligned}$$

The properties (A) and (B) of Lemma 4.1 depend on $X \in \mathfrak{g}_0^{-1,1}$. For a type-I nilpotent, X is given by

$$\overset{(3,0)}{\bullet} \qquad \overset{(2,1)}{\bullet} \overset{X}{\longrightarrow} \overset{(1,2)}{\bullet} \qquad \overset{(0,3)}{\bullet} \ (u_2 \mapsto -\bar{u}_2 \mapsto 0).$$

We determine the type of X in the case for type-II and for type-III, and consider whether (A) or (B) holds or not.

4.3.1. Type-II. We have the following.

PROPOSITION 4.5. *If N is of type-II, then (B) holds; however, (A) does not hold.*

Proof. Let v be a nonzero element in $I^{3,1}$ of the Deligne decomposition of $(W(N), \hat{F})$. Then

$$Nv \in I^{2,0}, \qquad \bar{v} \in I^{1,3}, \qquad N\bar{v} \in I^{0,2}.$$

We write $u_3 = \exp(iN)v$. Since $v \in \hat{F}^3$, $u_3 \in F_0^3 = H_0^{3,0}$. Here the sl_2-triple is given by

$$N^+ Nv = v, \qquad N^+ N\bar{v} = \bar{v}, \qquad N^+ v = N^+ \bar{v} = 0,$$

$$Hv = v, \qquad H\bar{v} = \bar{v}, \qquad HNv = -Nv, \qquad HN\bar{v} = -N\bar{v}.$$

Then we have

$$H_0^{2,1} \ni Xu_3 = -v + iNv = -\exp(-iN)v.$$

We write $u_2 = Xu_3$. Then $Xu_2 = 0$. Moreover, $\bar{u}_2 \in H_0^{1,2}$ and

$$X\bar{u}_2 = \bar{v} - iN\bar{v} = \bar{u}_3.$$

Summarizing these, $X \in \mathfrak{g}_0^{-1,1}$ is given by

$$\overset{(3,0)}{\bullet} \overset{X}{\longrightarrow} \overset{(2,1)}{\bullet} \qquad \overset{(1,2)}{\bullet} \overset{X}{\longrightarrow} \overset{(0,3)}{\bullet} \ (u_3 \mapsto u_2 \mapsto 0, \bar{u}_2 \mapsto \bar{u}_3 \mapsto 0).$$

Since $X(z\bar{u}_2 + u_3) = z\bar{u}_3 + u_2$, $XF_z^3 \not\subset F_z^3$ for $z \in \mathbb{P}^1$ in (4.4). Then there is no fixed point for $\exp(X)$ in C_0.

Next we show that (B) holds. Scaling v, we may assume that $\|u_3\| = 1$.

CLAIM. *We have* $\|u_2\| = 1$.

Proof. Let $a = \langle v, \bar{v} \rangle$, $b = \langle Nv, \bar{v} \rangle$, $c = \langle v, N\bar{v} \rangle$, and $d = \langle Nv, N\bar{v} \rangle$. Then by orthogonality, we have

$$\langle u_3, \bar{u}_3 \rangle = a + ib - ic + d = i, \qquad \langle u_3, \bar{u}_2 \rangle = -a - ib - ic + d = 0,$$
$$\langle u_2, \bar{u}_3 \rangle = -a + ib + ic + d = 0.$$

Since $v \in \hat{F}^3$ and $\bar{v} \in \hat{F}^1$, $a = 0$. Therefore, the simultaneous equation induces $d = 0$, $b - c = 1$, and $\langle u_2, \bar{u}_2 \rangle = a - ib + ic + d = -i$. □

Here $\{u_3, u_2, \bar{u}_3, \bar{u}_2\}$ is a unitary basis. Since

$$-i\langle \exp(zX)u_3, \overline{\exp(zX)u_3} \rangle = \|u_3\|^2 - |z|^2\|u_2\|^2 = 1 - |z|^2,$$
$$-i\langle \exp(zX)\bar{u}_2, \overline{\exp(zX)\bar{u}_2} \rangle = \|u_2\|^2 - |z|^2\|u_3\|^2 = 1 - |z|^2,$$

$\exp(zX)C_0 \subset D$ if and only if $|z| < 1$ by Proposition 2.5. □

4.3.2. *Type-III.* We give an example of type-III which satisfies neither (A) nor (B). All nilpotent orbits of type-III are described in [GGK1] explicitly. We consider the case where $a, b = 1$ and $e, f, \pi = 0$ in the notation of [GGK1, (I.C.2), (I.C.10)]. Let $H_{\mathbb{Z}} = \sum_{j=0}^3 \mathbb{Z}e_j$. We write

$$e_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where the bilinear form is given by

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \hat{F}^p = \{e_3, \ldots, e_p\} \quad (3 \geq p \geq 0).$$

Then N and $\hat{F}$ give a nilpotent orbit of type-III, where the limit mixed Hodge structure $(W(N), \hat{F})$ is $\mathbb{R}$-split.

The sl_2-triple of the SL(2)-orbit associated to this nilpotent orbit is given by

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \qquad N^+ = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$X = \frac{1}{2} \begin{pmatrix} -3 & 3i & 0 & 0 \\ i & -1 & 4i & 0 \\ 0 & i & 1 & -3i \\ 0 & 0 & -i & 3 \end{pmatrix}.$$

PROPOSITION 4.6. *For the above example, both (A) and (B) do not hold.*

Proof. Let

$$u_3 = \frac{\sqrt{3}}{2} \exp(iN)e_3 = \frac{\sqrt{3}}{12} \begin{pmatrix} 6 \\ 6i \\ -3 \\ i \end{pmatrix}.$$

Then $\|u_3\| = 1$. Now

$$X u_3 = \frac{\sqrt{3}}{4} \begin{pmatrix} -6 \\ -2i \\ -1 \\ i \end{pmatrix}, \qquad X^2 u_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} 6 \\ -2i \\ 1 \\ i \end{pmatrix} = -2\overline{X u_3},$$

$$X^3 u_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} -6 \\ 6i \\ 3 \\ i \end{pmatrix} = -6\bar{u}_3.$$

Here $\|X u_3\| = 3$. Letting $u_2 = X u_3/\sqrt{3}$, we then have a unitary basis $\{u_3, u_2, \bar{u}_3, \bar{u}_2\}$. Endomorphism X gives the map

$$\overset{(3,0)}{\bullet} \xrightarrow{\;X\;} \overset{(2,1)}{\bullet} \xrightarrow{\;X\;} \overset{(1,2)}{\bullet} \xrightarrow{\;X\;} \overset{(0,3)}{\bullet}$$

$$(u_3 \mapsto \sqrt{3}u_2 \mapsto -2\sqrt{3}\overline{u_2} \mapsto -6\overline{u_3} \mapsto 0).$$

Then $XF_z^3 \not\subset F_z^3$ for $z \in \mathbb{P}^1$ in (4.4), and so there is no fixed point in C_0. Moreover, for $\bar{u}_2 \in F_\infty^3$,

$$-i\langle \exp(zX)\bar{u}_2, \overline{\exp(zX)\bar{u}_2}\rangle = \|u_2\|^2 - 3|z|^2\|u_3\|^2 = 1 - 3|z|^2.$$

Then $\exp(zX)C_0 \not\subset D$ for $|z| \geq 1/\sqrt{3}$. $\qquad\square$

References

[AMRT] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth Compactification of Locally Symmetric Varieties*, Math. Sci. Press, Brookline, Massachusetts, 1975. MR 0457437.

[CMP] J. Carlson, S. Müller-Stach, and C. Peters, *Period Mappings and Period Domains*, Cambridge Stud. Adv. Math. **85**, Cambridge University Press, Cambridge, 2003. MR 2012297.

[CKS] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), 457–535. MR 0840721. DOI 10.2307/1971333.

[FHW] G. Fels, A. Huckleberry, and J. A. Wolf, *Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint*, Progr. Math. **245**, Birkhäuser, Boston, 2006. MR 2188135.

[F] W. Fulton, *Introduction to Toric Varieties*, Ann. of Math. Stud. **131**, Princeton University Press, Princeton, 1993. MR 1234037.

[GGK1] M. Green, P. Griffiths, and M. Kerr, "Néron models and boundary components for degenerations of Hodge structures of mirror quintic type" in *Curves and Abelian Varieties*, Contemp. Math. **465**, Amer. Math. Soc., 2008, 71–145. MR 2457736. DOI 10.1090/conm/465/09101.

[GGK2] ———, *Mumford-Tate Groups and Domains: Their Geometry and Arithmetic*, Ann. of Math. Stud. **183**, Princeton University Press, Princeton, 2012. MR 2918237.

[G] P. Griffiths, *Periods of integrals on algebraic manifolds, I: Construction and properties of the modular varieties*, Amer. J. Math. **90** (1968), 568–626. MR 0229641.

[H1] T. Hayama, *On the boundary of the moduli spaces of log Hodge structures: Triviality of the torsor*, Nagoya Math. J. **198** (2010), 173–190. MR 2666580.

[H2] ———, *Boundaries of cycle spaces and degenerating Hodge structures*, to appear in Asian J. Math, preprint, `arXiv:1203.6770v3` [math.AG].

[KU] K. Kato and S. Usui, *Classifying Spaces of Degenerating Polarized Hodge Structures*, Ann. of Math. Stud. **169**, Princeton University Press, Princeton, 2009. MR 2465224.

[KP]   M. Kerr and G. Pearlstein, *Boundary components of Mumford-Tate domains*,
        preprint, `arXiv:1210.5301v1` [math.AG].

[LS]   Z. Lu and X. Sun, *Weil-Petersson geometry on moduli space of polarized Calabi-
        Yau manifolds*, J. Inst. Math. Jussieu **3** (2004), 185–229. MR 2055709. DOI 10.
        1017/S1474748004000076.

[N]   Y. Namikawa, *Toroidal Compactification of Siegel Spaces*, Lecture Notes in Math.
        812, Springer, Berlin, 1980. MR 0584625.

[S]   W. Schmid, *Variation of Hodge structure: The singularities of the period map-
        ping*, Invent. Math. **22** (1973), 211–319. MR 0382272.

[U1]   S. Usui, *Complex structures on partial compactifications of arithmetic quotients
        of classifying spaces of Hodge structures*, Tohoku Math. J. (2) **47** (1995), 405–
        429. MR 1344910. DOI 10.2748/tmj/1178225524.

[U2]   ———, *Generic Torelli theorem for quintic-mirror family*, Proc. Japan Acad.
        Ser. A Math. Sci. **84** (2008), 143–146. MR 2457802.

Department of Mathematics
National Taiwan University
Taipei 106
Taiwan
Current:
Mathematical Sciences Center
Tsinghua University
Haidian District
Beijing 100084
China
`tatsuki@math.tsinghua.edu.cn`