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ON THE HOMOLOGY OF BRANCHED COVERINGS OF 3-MANIFOLDS

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Abstract. Following the analogies between 3-manifolds and number rings in arithmetic topology, we study the homology of branched covers of 3-manifolds. In particular, we show some analogues of Iwasawa's theorems on ideal class groups and unit groups, Hilbert's Satz 90, and some genus-theory–type results in the context of 3-dimensional topology. We also prove that the 2-cycles valued Tate cohomology of branched Galois covers is a topological invariant, and we give a new insight into the analogy between 2-cycle groups and unit groups.

§1. Introduction

The analogy between 3-dimensional topology and number theory was first pointed out by Mazur [Ma] in the 1960s, and it has been studied systematically by Kapranov, Reznikov [Re], and Morishita [Mo3], [Mo4]. In their analogies, for example, knots and 3-manifolds correspond to primes and number rings, respectively. The study of these analogies is now called *arithmetic topology*.

The purpose of this article is to study the homology of branched coverings of 3-manifolds by following the analogies in arithmetic topology. In particular, we show the topological analogues of Iwasawa's theorems on ideal class groups and unit groups in number field extensions, and we give some applications to topological analogues of genus theory. In the course of our proof, we show the 3-manifold analogue of Hilbert's Satz 90. In addition, we prove that the 2-cycles valued Tate cohomology of branched covers is a topological invariant, which gives a new insight into the analogy between 2-cycles and units.

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This article is organized in the following manner. In Section 2, we review the basic analogies between 3-dimensional topology and number theory, which will be used throughout this paper, and Hilbert theory for 3-manifold following [Mo4]. In Section 3, we recall Iwasawa's theorems and so forth on ideal class groups and unit groups in number field extensions, and we state their topological analogues as our main theorems. We also give some remarks on the 2-cycles valued Tate cohomology of branched covers. In Section 4, we prove our main theorems. In Section 5, as applications of our theorems, we give genus theory-type results, which give balances between homology groups and branch information.

§2. Hilbert theory for 3-manifolds

Hilbert theory deals with, in a group-theoretic manner, the decomposition of a prime in a finite Galois extension of number fields. In this section, we recollect Hilbert theory for 3-manifolds, which describes the decomposition of a knot in a finite Galois branched cover of 3-manifolds.

First, we recall in the following table some of the basic analogies between number theory and 3-dimensional topology which will be used in this paper. For a number field k, $\mathcal{O}_k$ denotes the ring of integers in k.

Number ring $\mathrm{Spec}(\mathcal{O}_k)$	3-dimensional manifold M
Prime ideal $\mathfrak{p}$ Prime ideals $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$	Knot K Link $L = \{K_1, \ldots, K_r\}$
Étale fundamental group $\pi_1(\mathrm{Spec}(\mathcal{O}_k))$ $\pi_1(\mathrm{Spec}(\mathcal{O}_k) - S)$	$\pi_1(M)$ Link group $\pi_1(M - L)$
Number field extension ℓ/k	(Branched) Cover $h : N \to M$
Ideal group $I(k)$ $k^* \to I(k)$; $a \mapsto (a)$	1-Cycle group $Z_1(M,\mathbb{Z})$ $C_2(M,\mathbb{Z}) \to Z_1(M,\mathbb{Z})$; $S \mapsto \partial S$
Principal ideal group $P(k)$	1-boundary group $B_1(M,\mathbb{Z})$
Ideal class group $Cl(k) = I(k)/P(k)$	First homology $H_1(M,\mathbb{Z}) = Z_1(M,\mathbb{Z})/B_1(M,\mathbb{Z})$
Unit group $\mathcal{O}_k^*$	Second homology $H_2(M,\mathbb{Z})$, or 2-cycles $Z_2(M,\mathbb{Z})$
Artin reciprocity $Cl(k) \cong \mathrm{Gal}(k_{ab}^{ur}/k) \cong \pi_1(\mathrm{Spec}(\mathcal{O}_k))^{ab}$ k_{ab}^{ur}: Hilbert class field of k	Hurewicz isomorphism $H_1(M,\mathbb{Z}) \cong \mathrm{Gal}(M_{ab}/M) \cong \pi_1(M)^{ab}$ M_{ab}: maximal abelian cover of M

Particularly, we have parallel exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_k^* \arrow[r] & k^* \arrow[r] & I(k) \arrow[r] & Cl(k) \arrow[r] & 1, \\
0 \arrow[r] & Z_2(M) \arrow[r] & C_2(M) \arrow[r] & Z_1(M) \arrow[r] & H_1(M) \arrow[r] & 0.
\end{tikzcd}$$

Because of this, in this paper we consider 2-cycles of 3-manifolds, rather than 2-homologies, as an analogue of units of $\mathcal{O}_k$, and we pursue some analogies with number fields (see Theorems 5, 10, and 11 below). Reznikov [Re] also considered surfaces without boundaries for this.

Since $Cl(k)$ is finite, to get the precise analogies for manifold it is natural to assume that $H_1(M,\mathbb{Z})$ is finite, that is, that manifolds are rational homology 3-spheres. However, by considering the torsion subgroup $\operatorname{tor}(H_1(M,\mathbb{Z}))$ as the counterpart of $Cl(k)$, some analogies work (see [Si], [Mn1]).

For more analogies, we refer to [Mo3], [Mo4], or [Mn2].

Now, based on the above dictionary, we present a topological analogue of Hilbert theory following [Mo4, Chapter 3].

Let $h : N \to M$ be a finite Galois covering of connected oriented closed 3-manifold branched over a link $L \subset M$, let $X := M - L$, let $Y := N - h^{-1}(L)$, let $G := \mathrm{Gal}(Y/X) = \mathrm{Gal}(N/M)$, and let $n := \#G(1)$. Let K be a knot in M which is a component of L or disjoint from L, and suppose that $h^{-1}(K) = K_1 \cup \cdots \cup K_r$ ($r = r_K$-component link). For a tubular neighborhood V_K of K, let V_{K_i} be the connected component of $h^{-1}(V_K)$ containing K_i. (They are canonical up to isotopy, in any category.) Note that G acts transitively on the set of knots $S_K := \{K_1, \ldots, K_r\}$ lying over K. We call the stabilizer D_{K_i} of K_i the *decomposition group* of K_i:

$$D_{K_i} := \bigl\{ g \in G \bigm| g(K_i) = K_i \bigr\}.$$

Since we have the bijection $G/D_{K_i} \cong S_K$ for each i, $\#D_{K_i} = n/r$ is independent of K_i. In fact, if $g(K_i) = K_j$, then $D_{K_j} = gD_{K_i}g^{-1}$. Since each $g \in G$ induces a homeomorphism $g|_{\partial V_{K_i}} : \partial V_{K_i} \xrightarrow{\cong} \partial V_{g(K_i)}$, $g|_{\partial V_{K_i}}$ is a covering transformation of ∂V_{K_i} over ∂V_K for each $g \in D_{K_i}$, and the corresponding $g \to g|_{\partial V_{K_i}}$ gives an isomorphism

$$D_{K_i} \cong \mathrm{Gal}(\partial V_{K_i}/\partial V_K).$$

The Fox completion of the subcovering space of Y over X corresponding to D_{K_i} is called the *decomposition covering space* of K_i and is denoted by

Z_{K_i}. The map $g \mapsto \bar{g} = g|_{\partial V_{K_i}}$ induces the homomorphism

$$D_{K_i} \to \mathrm{Gal}(K_i/K),$$

whose kernel is called the *inertia group* of K_i and is denoted by I_{K_i}:

$$I_{K_i} := \{g \in D_{K_i} \mid \bar{g} = \mathrm{id}_{K_i}\}.$$

If $K_j = g(K_i)$ $(g \in G)$, one has $I_{K_j} = gI_{K_i}g^{-1}$, and hence $\#I_{K_i}$ is independent of K_i. Set $e = e_K := \#I_{K_i}$. The Fox completion of the subcovering space of Y over X corresponding to I_{K_i} is called the *inertia covering space* of K_i and is denoted by T_{K_i}:

$$\begin{tikzcd}
N \arrow[r] & T_{K_i} \arrow[r] & Z_{K_i} \arrow[r] & M \\
1 \arrow[r, dash, "e"] & I_{K_i} \arrow[r, dash, "f"] & D_{K_i} \arrow[r, dash, "r"] & G
\end{tikzcd}$$

Here we have the equalities

$$\#D_{K_i} = ef, \qquad \#I_{K_i} = e, \qquad \#\mathrm{Gal}(K_i/K) =: f.$$

By comparing the orders, we see that the homomorphism $D_{K_i} \ni g \mapsto \bar{g} \in \mathrm{Gal}(K_i/K)$ is surjective:

$$1 \longrightarrow I_{K_i} \longrightarrow D_{K_i} \longrightarrow \mathrm{Gal}(K_i/K) \longrightarrow 1 \quad \text{(exact)}.$$

Let $K_{i,T}$ be the image of K_i under $N \to T_{K_i}$, and let $K_{i,Z}$ be the image of $K_{i,T}$ under $T_{K_i} \to Z_{K_i}$. Then one has the following.

THEOREM ([Mo4, Chapter 3]). *The map $N \to T_{K_i}$ is a branched cover of degree e such that the branching index of K_i over $K_{i,T}$ is e. The map $T_{K_i} \to Z_{Ki}$ is a cyclic cover of degree f such that the covering degree of $K_{i,T}$ over $K_{i,Z}$ is f. The map $Z_{K_i} \to M$ is a cover of degree r such that K is completely decomposed into an r-component link containing $K_{i,Z}$ as a component.*

§3. Iwasawa's theorems and their topological analogues

3.1. Iwasawa's theorems

Now we recall Iwasawa's theorems on ideal class groups in number field extensions.

THEOREM 1 ([I2]). *Let ℓ/k be a finite extension of number fields (finite extensions of $\mathbb{Q}$), and suppose that ℓ/k has no nontrivial unramified abelian subextension; that is, if $\ell/m/k$ and m/k are unramified abelian, then $m = k$. (This assumption is obviously satisfied if ℓ/k is totally ramified at some prime $\mathfrak{p}$.) Then the norm map of ideal class groups $N_{\ell/k} : Cl(\ell) \to Cl(k)$ is surjective, and hence $\#Cl(k) \mid \#Cl(\ell)$.*

THEOREM 2 ([I2], [W, Theorem 10.4]). *Let ℓ/k be a Galois extension of number field with degree p^ν, where p is a prime number and ν is a natural number, and suppose that ℓ/k is ramified over at most one prime. Then $p \mid \#Cl(\ell)$ implies that $p \mid \#Cl(k)$. In particular, when $k = \mathbb{Q}$, it follows that $p \nmid \#Cl(\ell)$.*

COROLLARY 3 ([F]). *We have*

$$p \mid \#Cl\big(\mathbb{Q}(\zeta_p)\big) \iff p \mid \#Cl\big(\mathbb{Q}(\zeta_{p^\nu})\big),$$

where p is a prime number, ν is a natural number, and ζ_n is a primitive nth root of unity.

REMARK. For the asymptotic behavior of the order of the p-part of $Cl(\mathbb{Q}(\zeta_{p^\nu}))$ as $\nu \to \infty$, Iwasawa's class number formula [I1, Theorem 12] is known.

We also recall Iwasawa's theorems on unit groups in number field extensions. We denote by $\widehat{H}^n(G, A)$ the Tate cohomology of a group G acting on an abelian group A, which is equal to the Galois cohomology $H^n(G, A)$ if $n > 0$ (see [Se, Chapter 8]). The following lemma proves Theorem 5.

LEMMA 4 (Hilbert's Satz 90). *Let ℓ/k be a finite Galois extension of number fields with Galois group G; then we have*

$$\widehat{H}^1(G, \ell^*) = 0.$$

THEOREM 5 ([I1, Chapter 2]). *Let ℓ/k be a finite Galois extension of number fields with Galois group G; then*

$$\widehat{H}^1(G, \mathcal{O}_k^*) \cong P(\ell)^G / P(k),$$

where $P(k)$ is the group of principal ideals in k and $P(\ell)^G$ is the group of principal ideals in k on which G acts trivially.

3.2. Main theorems

Next, we state our main theorems, which are topological analogues of Iwasawa's theorems and so forth given in Section 3.1. Hereafter, all manifolds in the statements are assumed to be closed (i.e., compact and $\partial M = \phi$), oriented, and connected. Note that a finite branched cover over a closed manifold is always closed.

On the first homologies, we have Theorems 6 and 7 and Corollary 8, which are regarded as analogues of Theorems 1 and 2 and Corollary 3.

Theorem 6. *Let $h : N \to M$ be a (branched) cover of n-manifolds, and suppose that h has no nontrivial unbranched abelian subcover; that is, if $h = g \circ f : N \to S \to M$ for some continuous map f and some unbranched abelian cover g of M, then g is a homeomorphism. (This assumption is satisfied if $n = 3$ and h is totally branched over some knot K.) Then the induced map $h_* : H_1(N, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is surjective. In particular, if N and M are rational homology spheres, then $\#H_1(M) \mid \#H_1(N)$.*

Theorem 7. *Let $h : N \to M$ be a (branched) Galois cover of 3-manifolds branched over at most one knot and of degree p^ν, where p is a prime number and ν is a natural number. If N and M are rational homology spheres, then $p \mid \#H_1(N, \mathbb{Z})$ implies that $p \mid \#H_1(M, \mathbb{Z})$. In particular, when M is S^3 or an integral homology sphere, it follows that $p \nmid \#H_1(N, \mathbb{Z})$.*

Corollary 8. *Let M_n denote the cyclic branched cover of S^3 branched over a knot K, and of positive degree n. Assume that M_n are all rational homology spheres. Let p be a prime number, let m be a positive integer, and let ν be a natural number. Then we can show that*

$$p \mid \#H_1(M_m, \mathbb{Z}) \iff p \mid \#H_1(M_{mp^\nu}, \mathbb{Z}).$$

Remark. For the asymptotic behavior of the order of p-part of $H_1(M_{mp^\nu}, \mathbb{Z})$, we refer to [HMM, Chapter 5] and [KM].

On 2-chains and 2-cycles valued Tate cohomology groups of Galois covers, we have Lemma 9 and Theorem 10, which are 3-manifold analogues of Lemma 4 and Theorem 5.

For the remainder of this article, we assume that manifolds admit finite CW-structures that are compatible to covering maps and include branching sets, and we fix such structures on them.[†] Then we can consider branched Galois covers with Galois group G as G-complexes (see [B, Chapter I.4]). The following Lemma 9 implies our Theorem 10.

LEMMA 9 (Analogue of Hilbert's Satz 90). *Let $h: N \to M$ be a Galois cover of 3-manifolds branched over some link with Galois group G; then*

$$\widehat{H}^1\bigl(G, C_2(N,\mathbb{Z})\bigr) = 0.$$

THEOREM 10. *Let $h: N \to M$ be a Galois cover of 3-manifolds branched over some link; then*

$$\widehat{H}^1\bigl(G, Z_2(N,\mathbb{Z})\bigr) \xleftarrow{\cong} B_1(N,\mathbb{Z})^G / h^! B_1(M,\mathbb{Z}),$$

where $B_1(N,\mathbb{Z})^G$ is the G-invariant subgroup of 1-boundaries.

Here, we denote by $h^!: C_*(M,\mathbb{Z}) \hookrightarrow C_*(N,\mathbb{Z})$ the canonical injection, called *transfer*, which is defined as follows. For any open chain $c \subset M$, take one connected component of $h^{-1}(c)$, say, c_1, and put $h^!(c) = \sum_{\sigma \in G} \sigma c_1$ and extend linearly on the whole $C_*(M,\mathbb{Z})$.

Although $Z_2(N,\mathbb{Z})$ depends on the choice of CW-structure, there is a remarkable fact.

THEOREM 11. *We have the following isomorphism:*

$$\widehat{H}^r\bigl(G, Z_2(N,\mathbb{Z})\bigr) \xrightarrow{\cong} \widehat{H}^r\bigl(G, Z_2^{\text{sing}}(N)/Z_2^{\text{sing}}(h^{-1}(L))\bigr),$$

where $Z_2^{\text{sing}}(N)$ is the singular 2-cycle group of N and L is the branching link of h. Especially, the Tate cohomology $\widehat{H}^r(G, Z_2(N,\mathbb{Z}))$ ($r \in \mathbb{Z}$) is a topological invariant of branched covers, that is, independent of the choice of CW-structure.

REMARKS.

(1) Theorems 10 and 11 strengthen our reason to consider the 2-cycle group $Z_2(N,\mathbb{Z})$, rather than the 2-homology group $H_2(N,\mathbb{Z})$, as an analogue of unit group $\mathcal{O}_k^*$. There are two merits of our point of view.

[†] It is known that compact manifold M always admits CW-structure if dim $M \neq 4$, while the case for dim $M = 4$ is an open problem (see [H, p. 529]). Although a CW-structure on a 3-manifold is not unique, we can discuss closer analogies with number fields, such as Lemma 9 and Theorems 10 and 11 below, once such a structure is chosen.

- (i) $Z_2(N,\mathbb{Z})$ has more information (of the topology of $h : N \to M$) than does $H_2(N,\mathbb{Z})$. Indeed, if N is a rational homology sphere, $H_2(N,\mathbb{Z})$ is always finite, and $\widehat{H}^q(G, H_2(N,\mathbb{Z})) = 0$, while usually $\widehat{H}^q(G, Z_2(N,\mathbb{Z})) \neq 0$.
- (ii) There are parallel exact sequences (see Section 2) that enable us to "translate" cohomological discussions in number theory into topological context word by word. Our Section 4 will be an example. For example, we will consider in Proposition 16 the Herbrand quotient $Q = (\#\widehat{H}^0(G, Z_2(N,\mathbb{Z})))/(\#\widehat{H}^1(G, Z_2(N,\mathbb{Z})))$ as an invariant of branched covers.

(2) Since Hilbert's Satz 90 is a very basic fact, we can expect various applications of this, such as analogues of Kummer theory, for example. An analogue of Hilbert's Satz 94 is also proved in [Mo2, Chapter 1].

(3) Lemma 9 and Theorems 10 and 11 hold for any dimension $n > 1$; that is, we have $\widehat{H}^1(G, Z_{n-2}(N,\mathbb{Z})) \cong B_{n-1}(N,\mathbb{Z})^G / h^! B_{n-1}(M,\mathbb{Z})$ and $\widehat{H}^1(G, C_{n-2}(N,\mathbb{Z})) = 0$ for covers of n-manifolds branched over $(n-2)$-submanifolds.

§4. Proofs of main theorems

In this section we prove our main theorems in Section 3. We will sometimes omit the coefficients $\mathbb{Z}$ to make the notation brief in the following.

Proof of Theorem 6. First, let us paraphrase the conclusion. By Hurewicz's theorem, we have $H_1(M,\mathbb{Z}) \cong \pi_1(M)/D(\pi_1(M))$ for any manifold M, where $D(G) := [G,G]$ denotes the commutator group of a group G. Note also that we have $h_*(D(\pi_1(N))) \lhd D(\pi_1(M))$. Then we see that the following conditions are equivalent:

$$\begin{aligned} &h_* : H_1(N,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \quad \text{is surjective} \\ &\qquad \iff h_* : \pi_1(N)/D\big(\pi_1(N)\big) \to \pi_1(M)/D\big(\pi_1(M)\big) \quad \text{is surjective} \\ &\qquad \iff \pi_1(M) \quad \text{is generated by } h_*\big(\pi_1(N)\big) \text{ and } D\big(\pi_1(M)\big). \end{aligned}$$

Now we prove the contraposition of the theorem. Suppose that $\pi_1(M)$ is not generated by $h_*(\pi_1(N))$ and $D(\pi_1(M))$, that is, that $\pi_1(M) \supsetneqq h_*(\pi_1(N)) \cdot D(\pi_1(M)) \rhd D(\pi_1(M))$. Then by Galois theory for covers, we have a nontrivial subcover $g : S \to M$ of the maximal (unbranched) abelian cover $M_{\mathrm{ab}} \to M$ which satisfies $\pi_1(S) = h_*(\pi_1(N)) \cdot D(\pi_1(M))$, where the last term is the group generated by the elements of $h_*(\pi_1(N))$ and $D(\pi_1(M))$.

Since $g : S \to M$ is a cover and $h : N \to M$ is a continuous map with $h_*(\pi_1(N)) \lhd \pi_1(S)$, we have a lift of h, a continuous map $f : N \to S$ with $h = g \circ f$, by general theory of cover (which is proved by using lifts of homotopies):

$$\begin{tikzcd}
N \arrow[d, "h"'] \arrow[dr, dashed, "f"] \arrow[dr, phantom, "\circlearrowleft"' near start] & & \\
M & S \arrow[l, "g"] & M_{ab}, \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
h_*\big(\pi_1(N)\big) \arrow[d, hook] \arrow[dr, hook] \arrow[dr, phantom, "\circlearrowleft"' near start] & & \\
\pi_1(M) & \pi_1(S) = h_*\big(\pi_1(N)\big) \cdot D\big(\pi_1(M)\big) \arrow[l, hook'] & D\big(\pi_1(M)\big) \arrow[l, hook']
\end{tikzcd}$$

Thus, we obtain a nontrivial abelian subcover g of h, and the contraposition is proved. $\square$

We give here a proof of Theorem 7 which is parallel to [W, proof of Theorem 2], a group-theoretic one. We can observe what is *not* parallel in this proof. We may have another proof by taking contraposition and by using the Wang sequence and the fact that $p \nmid \# H_1(M, \mathbb{Z}) \iff H_1(M, \mathbb{F}_p) = 0$.

Proof of Theorem 7. By Hurewicz's theorem and Galois theory, we see that $p \mid \# H_1(M)$ if and only if M has an unbranched abelian cover of degree p.

When h is a nontrivial unbranched cover, $\mathrm{Gal}(N/M)$ is a p-group, and it has a normal subgroup G_1 with index p, so we obtain a subcover $N/G_1 \to M$, which is an unbranched cover of M of degree p.

If h is branched at a knot K, we denote by $g : H \to N$ the maximal p-abelian (unbranched) cover of N, we put $L := \bigsqcup \{L_i\} = h^{-1}(K)$, and we put $J := g^{-1}(L) = \bigsqcup \{J_{ij}\}$, where $g(J_{ij}) = L_i$.

We claim that the branched cover $f := h \circ g : H \to M$ is Galois $\ldots(\blacklozenge)$. (For number field extensions, this follows immediately from the maximality of H.) We check this later.

Now let G be the Galois group of the branched cover $f : H \to M$, and let $I_{ij} < G$ be the inertia groups of knots J_{ij} (see Section 2 for inertia groups).

By the assumption that $p \mid \#H_1(N)$, the cover $H \to N$ is not trivial, and since $g : H \to N$ is unbranched, we have $\#I_{ij}\deg(h) < \deg(f)$, and hence $I_{ij} \lneqq G$.

Because G is p-group, there is a normal subgroup G_1 of G with index p which includes one of inertia groups, say, I_{11}. But at this moment, since $f : H \to M$ is Galois, all the inertia groups are conjugate of I_{11} in G, and all I_{ij} are included in G_1. Therefore, $H/G_1 \to M$ is an unbranched cover with degree p:

$$\begin{tikzcd}
N \arrow[d, "h"'] & H \arrow[l, "g"'] \arrow[dl, "f", "G"'] \arrow[d, "G_1"] & \\
M & H/G_1 \arrow[l, "(p)"] & M_{ab}, \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_1(N-L) \arrow[r, phantom, "\triangleright"] & \pi_1(H-J) \arrow[r, phantom, "\triangleright"] & D\big(\pi_1(N-L)\big) \\
\pi_1(M-K) \arrow[u, phantom, "\triangle"] \arrow[ur, phantom, "\triangledown"] & &
\end{tikzcd}$$

□

Proof of claim (♦). Let $X = M - K, Y = N - L, Z = H - J$ be the link complement spaces. Notice that $\pi_1(Z)$ is the pullback of the unique Syllow p-group S_p of $H_1(N, \mathbb{Z})$ by the map $\pi_1(Y) \twoheadrightarrow \pi_1(N) \twoheadrightarrow \pi_1(N)^{ab} \overset{\cong}{\to} H_1(N, \mathbb{Z})$.

Since $h : N \to M$ is Galois, $\pi_1(Y) \lhd \pi_1(X)$ is a normal subgroup, and we have an action of $\pi_1(X)$ on $\pi_1(Y)$ by conjugation: $\pi_1(X) \curvearrowright \pi_1(Y)$. Then, since S_p is stable under the induced action of $\pi_1(X)$ on $H_1(N, \mathbb{Z})$, the group $\pi_1(Z)$, which is the pullback of S_p in $\pi_1(Y)$, is also stable under the action $\pi_1(X) \curvearrowright \pi_1(Y)$. Therefore, $\pi_1(Z)$ is a normal subgroup of $\pi_1(X)$, and hence $f : H \to M$ is Galois. □

Proof of Corollary 8. We prove the result when m and p are coprime, from which the other cases follow:

$$M_{mp^\nu} \longrightarrow M_m \longrightarrow S^3.$$

Since the cover $M_{mp^\nu} \to M_m$ has no nontrivial unbranched subcover, Theorem 6 implies that $\#H_1(M_m) \mid \#H_1(M_{mp^\nu})$, and hence that $p \mid \#H_1(M_m) \Longrightarrow p \mid \#H_1(M_{mp^\nu})$.

On the other hand, since this is a cover of degree p^ν and its branching set is a knot K, Theorem 7 implies that $p \mid \#H_1(M_{mp^\nu}) \Longrightarrow p \mid \#H_1(M_m)$. □

Remark. In Corollary 3, only one prime (p) is ramified in $\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}$, and its extension degree is $(p-1)p^{\nu-1}$. If we put $m = p-1$ in Corollary 8, we get the strict analogue of Corollary 3.

Proof of Lemma 9 (Analogue of Hilbert's Satz 90). Since h is Galois and N is a G-complex with its branching set being 1-dimensional subcomplex, $C_2(N,\mathbb{Z})$ is a $\mathbb{Z}[G]$-free module. The result is immediate from this, because $\widehat{H}^q(G,F) = 0$ for any $q \in \mathbb{Z}$, finite group G, and G-free module F (see [B, Chapter 6]). □

Remark. Note that since multiplicative group of number field ℓ^* is not $\mathbb{Z}[G]$-free, Hilbert's Satz 90 for number field extensions is more nontrivial. It is proved by using Dedekind's lemma, or linear independence of automorphisms, which needs both addition and multiplication. We cannot prove the two lemmas in parallel ways.

Proof of Theorem 10. We omit the coefficients $\mathbb{Z}$ in the following. Note that $H^1(G,) = \widehat{H}^1(G,)$.

We consider the following short exact sequence of $\mathbb{Z}[G]$-module:

$$0 \to Z_2(N) \to C_2(N) \xrightarrow{\partial} B_1(N) \to 0.$$

By taking Galois cohomology, we obtain a long exact sequence

$$0 \to Z_2(N)^G \to C_2(N)^G \xrightarrow{\partial} B_1(N)^G \to H^1\big(G, Z_2(N)\big) \to H^1(G, C_2(N) = 0,$$

where the last equality is by Lemma 9.

Now the transfer map $h^! : C_2(M) \xrightarrow{\cong} C_2(N)^G$ and the map $(1/n)h_* : C_2(N)^G \xrightarrow{\cong} C_2(M)$ induce the isomorphism

$$B_1(M) = \partial\big(C_2(M)\big) \xrightarrow[h^!]{\cong} \partial\big(C_2(N)^G\big),$$

and hence we have an exact sequence

$$0 \to B_1(M) \xrightarrow{h^!} B_1(N)^G \to H^1\big(G, Z_2(N)\big) \to 0.$$ □

Proof of Theorem 11. We prove here that the Tate cohomology $\widehat{H}^r(G, Z_2(N))$ is independent of cellular decompositions of N, where $h : N \to M$ is a branched Galois cover branched over a link L with Galois group G and of degree $n = \#G$. We take one G-CW-structure on N which has $h^{-1}(L)$ as a subcomplex. We denote the singular chains of N by $C_*^{\mathrm{sing}}(N)$, and so on.

The inclusion from cellular chains into singular chains leads to a commutative diagram of G-chain complexes with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*\big(h^{-1}(L)\big) \arrow[r, "\alpha_*"] \arrow[d] & C_*^{\mathrm{sing}}\big(h^{-1}(L)\big) \arrow[r] \arrow[d] & \mathrm{Coker}(\alpha_*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(N) \arrow[r, "\beta_*"] \arrow[d] & C_*^{\mathrm{sing}}(N) \arrow[r] \arrow[d] & \mathrm{Coker}(\beta_*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*\big(N, h^{-1}(L)\big) \arrow[r, "\gamma_*"] \arrow[d] & C_*^{\mathrm{sing}}\big(N, h^{-1}(L)\big) \arrow[r] \arrow[d] & \mathrm{Coker}(\gamma_*) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since the inclusions $\alpha_*, \beta_*, \gamma_*$ are chain homotopy equivalences, the cokernels are acyclic complexes. Since G acts freely away from $h^{-1}(L)$, the complexes in the third row are free G-modules. By diagram chasing (use $B_q(\mathrm{Coker}(\alpha_*)) = Z_q(\mathrm{Coker}(\alpha_*))$ by acyclicity), we obtain exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & Z_q\big(h^{-1}(L)\big) \arrow[r] & Z_q^{\mathrm{sing}}\big(h^{-1}(L)\big) \arrow[r] & Z_q\big(\mathrm{Coker}(\alpha_*)\big) \arrow[r] & 0, \\
0 \arrow[r] & Z_q(N) \arrow[r] & Z_q^{\mathrm{sing}}(N) \arrow[r] & Z_q\big(\mathrm{Coker}(\beta_*)\big) \arrow[r] & 0,
\end{tikzcd}$$

and so on. We also have an exact column $0 \longrightarrow Z_q(\mathrm{Coker}(\alpha_*)) \longrightarrow Z_q(\mathrm{Coker}(\beta_*)) \longrightarrow Z_q(\mathrm{Coker}(\gamma_*)) \longrightarrow 0$. Since $h^{-1}(L)$ is 1-dimensional,

$Z_q(h^{-1}(L)) = 0$ for all $q > 1$. Hence, we have following exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & Z_2^{\mathrm{sing}}\bigl(h^{-1}(L)\bigr) \arrow[r, "\cong"] \arrow[d] & Z_2\bigl(\mathrm{Coker}(\alpha_*)\bigr) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_2(N) \arrow[r] & Z_2^{\mathrm{sing}}(N) \arrow[r] & Z_2\bigl(\mathrm{Coker}(\beta_*)\bigr) \arrow[r] \arrow[d] & 0 \\
 & & & Z_2\bigl(\mathrm{Coker}(\gamma_*)\bigr) \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

Hence, $0 \longrightarrow Z_2(N) \longrightarrow Z_2^{\mathrm{sing}}(N)/Z_2^{\mathrm{sing}}(h^{-1}(L)) \longrightarrow Z_2(\mathrm{Coker}(\gamma_*)) \longrightarrow 0$ is exact by the nine lemma, and $0 \longrightarrow \widehat{H}^r(G, Z_2(N)) \longrightarrow \widehat{H}^r(G, Z_2^{\mathrm{sing}}(N)) \longrightarrow \widehat{H}^r(G, Z_2(\mathrm{Coker}(\gamma_*)) \longrightarrow \widehat{H}^{r+1}(G, Z_2(N)) \longrightarrow 0$ is exact by the snake lemma. Since $Z_2(\mathrm{Coker}(\gamma_*))$ is G-free and G is finite, $\widehat{H}^r(G, Z_2(\mathrm{Coker}(\gamma_*))) = 0$ for all $r \in \mathbb{Z}$, and hence $\widehat{H}^r(G, Z_2(N)) \xrightarrow{\cong} \widehat{H}^r(G, Z_2^{\mathrm{sing}}(N))$ for all $r \in \mathbb{Z}$. The last group is independent of the cellular decomposition. (This proof holds for all $q > 1$.) $\square$

§5. Application to genus theory

In this section, we give some genus-theory–type theorems obtained as applications of our Theorem 10. First, we recall the number field case. Here are some applications of Theorem 5, which give balances between ideal class groups and ramification indices.

Lemma 12 ([Y, Lemma 1]). *Let ℓ/k be a finite Galois extension of number fields with Galois group G. Then*

$$\#I(\ell)^G/P(\ell)^G = \#Cl(k)\frac{\prod e}{\#\widehat{H}^1(G, \mathcal{O}_k^*)},$$

where $\prod e$ is the product of the ramification indices of all the finite primes at ℓ/k.

Note that in the proof of Lemma 12 we use Theorem 5, $\widehat{H}^1(G,\mathcal{O}_\ell^*)\cong P(\ell)^G/P(k)$, and the fact that $\prod e=[I(\ell)^G:I(k)]$.

Proposition 13 ([Y, Lemma 6]). *Let ℓ/k be a cyclic extension with Galois group G. Then, for the G-invariant subgroup of ideal class group $Cl(\ell)^G$,*

$$\#Cl(\ell)^G=\frac{\#\widehat{H}^0(G,\mathcal{O}_\ell^*)\cdot\prod e\cdot\#Cl(k)}{\#\widehat{H}^1(G,\mathcal{O}_\ell^*)\cdot[\mathcal{O}_k^*:(\eta)]}.$$

Remark. If ℓ/k is abelian, $\#Cl(\ell)^G$ is equal to what is called the *relative genus number* of ℓ/k.

Proposition 14 ([Y, Lemma 3]). *When a group G is acting on A, $Q(A)=\#\widehat{H}^0(G,A)/\#\widehat{H}^1(G,A)$ is called the* Herbrand quotient. *For a cyclic extension l/k with Galois group G, we have*

$$Q(\mathcal{O}_\ell^*):=\frac{\widehat{H}^0(G,\mathcal{O}_\ell^*)}{\widehat{H}^1(G,\mathcal{O}_\ell^*)}=\frac{\prod e_\infty}{n},$$

where $\prod e_\infty$ is the product of the branching indices at the infinite primes.

Note. In order to see the analogy closely, we sketch the proof of Proposition 13. Precisely, we have

$$\begin{aligned}\#Cl(\ell)^G&=\left[\bigcup Cl(\ell)^G:P(\ell)I(\ell)^G\right]\times\left[P(\ell)I(\ell)^G:P(\ell)I(k)\right]\\&\quad\times\left[P(\ell)I(k):P(\ell)\right]\\&=\frac{\#\widehat{H}^0(G,\mathcal{O}_\ell^*)}{[\mathcal{O}_k^*:(\eta)]}\times\frac{\prod e\cdot h_0}{\#\widehat{H}^1(G,\mathcal{O}_\ell^*)}\times\frac{\#Cl(k)}{h_0},\end{aligned}$$

where $\widehat{H}^0(G,\mathcal{O}_\ell^*)=\mathcal{O}_k^*/N(\mathcal{O}_\ell^*)$, $(\eta):=\mathcal{O}_k^\times\cap N(\ell)$, and $h_0=\#\mathrm{Ker}(Tr:Cl(k)\to Cl(\ell))$. Furthermore,

(i) $[\bigcup Cl(\ell)^G:P(\ell)I(\ell)^G]=[(\eta):N(\mathcal{O}_\ell^*)]$ $(=\#\widehat{H}^0(G,\mathcal{O}_\ell^*)/[\mathcal{O}_k^*:(\eta)])$ is proved by the isomorphism $\bigcup Cl(\ell)^G/P(\ell)I(\ell)^G\cong(\eta)/N(\mathcal{O}_\ell^*)$,

(ii) $[P(\ell)I(k):P(\ell)]=\#Cl(k)/h_0$ is proved by applying the homomorphism theorem for the transfer map, and

(iii) $[P(\ell)I(\ell)^G:P(\ell)I(k)]=\prod e\cdot h_0/\#\widehat{H}^1(G,\mathcal{O}_\ell^*)$ follows from the fact that $P(\ell)I(\ell)^G/P(\ell)I(k)\cong I(\ell)^G/P(\ell)^G$, (ii), and Lemma 12.

Next, we consider the 3-manifold analogues of the propositions above in this section, which are related to genus theory for 3-manifolds (see [Mo1], [Mo3], [Mo4]), as applications of our Theorem 10. They can be seen as more precise versions of theorems in [Mo1]. Note that we omit coefficients $\mathbb{Z}$ in the following.

LEMMA 15 (Analogue of Lemma 12). *Let $h: N \to M$ be a finite Galois cover over a rational homology 3-sphere M, of degree n with Galois group G. Then*

$$\#Z_1(N)^G/B_1(N)^G = \frac{\#H_1(M)\cdot\prod e}{\#\widehat{H}^1(G, Z_2(N))},$$

where $\prod e$ is the product of the branching indices of all the branching knots of h.

Proof of Lemma 15. By our Theorem 10, $\widehat{H}^1(G, Z_2(N)) \cong B_1(N)^G/h^! B_1(M)$, and by the fact that $\prod e = [Z(N)^G : h^! Z(M)]$, we obtain the following:

$$\begin{aligned}\#Z_1(N)^G/B_1^G &= \left[Z_1(N)^G : B_1(N)^G\right] \\ &= \left[Z_1(N)^G : h^! Z_1(M)\right] \\ &\quad \cdot \left[h^! Z_1(M) : h^! B_1(M)\right]/\left[B_1(N)^G : h^! B_1(M)\right] \\ &= \prod e \cdot \#H_1(M)/\#\widehat{H}^1\big(G, Z_2(N)\big).\end{aligned}$$

□

PROPOSITION 16 (Analogue of Proposition 13). *Let $h: N \to M$ be a cyclic cover over a rational homology sphere M, with Galois group $G = \langle \sigma \mid \sigma^n = 1_G\rangle$. Then, for the G-invariant subgroup of the first homology group $H_1(N, \mathbb{Z})^G$, we have*

$$\#H_1(N)^G = \frac{\#\widehat{H}^0(G, Z_2(N))}{\#\widehat{H}^1(G, Z_2(N))}\cdot \prod e \cdot \#H_1(M).$$

In particular, the Herbrand quotient $Q(Z_2(N)) = (\#\widehat{H}^0(G, Z_2(N)))/(\#\widehat{H}^1(G, Z_2(N)))$ is an invariant of the cover.

REMARK. If $h: N \to M$ is abelian, the *relative genus cover* $N^* \to N$ is defined to be the maximal cover of N which is an abelian cover over M and is unbranched over N, and $g_{N/M} := \deg(N^* \to N)$ is called the *relative genus number* with respect to $h: N \to M$.

When $h : N \to M$ is a cyclic cover over an integral homology sphere M (i.e., $H_1(M, \mathbb{Z}) = 0$), we have the following equalities proved by Morishita [Mo1]:

$$\# H_1(N)^G = \big[H_1(N) : (1-\sigma)H_1(N)\big] = \big[H_1(N) : H_1(N^*)\big] = g_{N/M} = \frac{\prod e}{n}.$$

Comparing with our equality in Proposition 16, we obtain the Herbrand quotient of G acting on $Z_2(N)$, as follows.

Proposition 17. *Suppose that M is an integral homology sphere in the assumptions of Proposition 16; then*

$$Q\big(Z_2(N)\big) = \frac{\# \widehat{H}^0(G, Z_2(N))}{\# \widehat{H}^1(G, Z_2(N))} = \frac{1}{n}.$$

This is an analogue of the Herbrand quotient for unit groups of cyclic extension of number fields ℓ/k with $\prod e_\infty = 1$ (i.e., without branching at any infinite prime).

Remarks.

(1) The Herbrand quotient of $Z_2(N)$ for general M is yet to be calculated, because it is difficult to apply the technique used in [Mo1] for general M. However, we have following.
   (i) For any branched Galois cover $h : N \to M$ with Galois group G, by definition we have

$$\widehat{H}^0\big(G, Z_2(N)\big) = Z_2(N)^G / \Big(\sum_{\sigma \in G} \sigma\Big) Z_2(N).$$

   (ii) If G is cyclic, by group cohomology theory and our Lemma 9, we also have

$$\begin{aligned} B_1(N)^G / h^! B_1(M) &\cong \widehat{H}^1\big(G, Z_2(N)\big) \cong \widehat{H}^{-1}\big(G, Z_2(N)\big) \\ &= \mathrm{Ker}\big(h_* : Z_2(N) \to h^! Z_2(M)/(1-\sigma) Z_2(N)\big). \end{aligned}$$

(2) We note that topological analogues of the infinite primes are known to be ends of noncompact 3-manifolds (see [D], [Mo3], [Mo4], [Ra]). Therefore, since M is closed, the term corresponding to the infinite primes disappears in Proposition 17.

Proof of Proposition 16. We have more precise equations:

$$\begin{aligned}\#H_1(N)^G &= \Big[\bigcup H_1(N)^G : B_1(N)+Z_1(N)^G\Big]\\ &\quad\times \big[B_1(N)+Z_1(N)^G : B_1(N)+h^! Z_1(M)\big]\\ &\quad\times \big[B_1(N)+h^! Z_1(M) : B_1(N)\big]\\ &= \#\widehat{H}^0\big(G, Z_2(N)\big) \times \frac{\prod e\cdot h_0}{\#\widehat{H}^1(G, Z_2(N))} \times \frac{\#H_1(M)}{h_0},\end{aligned}$$

and each term coincides (so, they are integers), where $\widehat{H}^0(G, Z_2) = Z_2(M)/h_*(Z_2(N))$ and $h_0 = \#\mathrm{Ker}(h^! : H_1(M) \to H_1(N))$. Indeed,

(i) $\bigcup H_1(N)^G/B_1(N) + Z_1(N)^G \xrightarrow{\cong} Z_2(M)/h_*(Z_2(N))$ proves the coincidence of the first terms,

(ii) $B_1(N) + h^! Z_1(M)/B_1(N)$ is the image of transfer map $h^! : H_1(M) \to H_1(N)$ with $\#\mathrm{Ker}(h^!) = h_0$, and

(iii) $[B_1(N)+Z_1(N)^G : h^!(Z_1(M))+B_1(N)] = (\prod e\cdot h_0)/(\#\widehat{H}^1(G, Z_2(N)))$ since $B_1(N) + Z_1(N)^G/B_1(N) \cong Z_1(N)^G/B_1^G$ and by Lemma 15 and (ii).

Proof of (i). In order to prove the isomorphism $\bigcup(H_1(N)^G)/B_1(N) + Z_1(N)^G \xrightarrow{\cong} Z_2(M)/h_*(Z_2(N))$, we consider the following map:

$$\begin{aligned}\varphi : \bigcup H_1(N)^G &\to Z_2(M)/h_*\big(Z_2(N)\big)\\ c &\mapsto \big[h_*(s)\big],\end{aligned}$$

where $(1-\sigma)c = \partial s$. We will define the map, show it is surjective, and make its kernel explicit.

First, for $c \in Z_1(N)$, the following equivalence and equation show that there is such s:

$$\begin{aligned}c \in \bigcup H_1(N)^G &\iff [c] \in H_1(N)^G\\ &\iff (1-\sigma)c \in B_1(N)\\ &\iff \exists s \in C_2(N), \quad \text{such that } (1-\sigma)c = \partial s,\end{aligned}$$

$\partial h_*(s) = h_*(\partial s) = 0$.

Suppose that $(1-\sigma)c = \partial s = \partial s'$; then $\partial(s-s') = 0$, $s - s' \in Z_2(N)$, and hence

$$h_*(s) - h_*(s') = h_*(s-s') \in h_*\big(Z_2(N)\big).$$

Hence, the map φ exists and is well defined.

Next, for any $s \in Z_2(M)$, there exists $\tilde{s} \in C_2(N)$ such that $h_*(\tilde{s}) = s$. Since $h_*(\partial s) = \partial h_*(s) = 0$ and $\mathrm{Ker}(h_*) = \mathrm{Im}(1-\sigma)$ on $B_1(N)$, we have $\partial s = (1-\sigma)c$ for some $c \in B_1(N)$. Hence, φ is surjective.

Lastly, take $c \in \bigcup H_1(N)^G$, and put $\varphi(c) = [h_*(s)]$ with $(1-\sigma)c = \partial s$; then the following equivalence shows the results:

$$\begin{aligned} c \in \mathrm{Ker}\varphi &\iff h_*(s) \in h_*\big(Z_2(N)\big) \\ &\iff s \in Z_2(N) + \mathrm{Ker}(h_*) = Z_2(N) + (1-\sigma)C_2(N) \\ &\iff (1-\sigma)c = \partial s \in (1-\sigma)\partial C_2(N) = (1-\sigma)B_1(N) \\ &\iff c \in B_1(N) + Z_1(N)^G. \end{aligned}$$

Note that since $\varphi(c) = [h_*(s)]$ is well defined, it is sufficient to consider any $s \in C_2(N)$ with $(1-\sigma)c = \partial s$ above. □

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