PERIODICITIES OF T-SYSTEMS AND Y-SYSTEMS

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Abstract. The unrestricted T-system is a family of relations in the Grothendieck ring of the category of the finite-dimensional modules of Yangian or quantum affine algebra associated with a complex simple Lie algebra. The unrestricted T-system admits a reduction called the restricted T-system. In this paper we formulate the periodicity conjecture for the restricted T-systems, which is the counterpart of the known and partially proved periodicity conjecture for the restricted Y-systems. Then, we partially prove the conjecture by various methods: the cluster algebra and cluster category method for the simply laced case, the determinant method for types A and C, and the direct method for types A, D, and B (level 2).

§1. Introduction

The *Y-system* was introduced as a system of functional relations concerning the solutions of the thermodynamic Bethe ansatz equations for the factorizable scattering theory and the solvable lattice models [Z], [KP], [KN], [RTV]. It was conjectured that the solutions of the Y-system have periodicity [Z], [RTV], [KNS1]. Fomin and Zelevinsky proved it for a special case (level 2 case in our terminology) [FZ3] by the cluster algebra approach [FZ1], [FZ2], [FZ4]. Since then, a remarkable link has been established between cluster algebras and cluster categories of the quiver representations (see [BMRRT], [BMR], [CC], [CK1], [CK2], [Kel2], and references therein). Based on this categorification method, Keller recently proved the periodicity of the Y-system for a more general case [Kel2], [Kel3].

Meanwhile, it has been known that the Y-system is related to other systems of relations called the *T-system* and the *Q-system* [KP], [KNS1]. The T-system is a family of relations in the Grothendieck ring of the category of the finite-dimensional modules of the Yangian $Y(\mathfrak{g})$ or the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ associated with a complex simple Lie algebra $\mathfrak{g}$ [KNS1], [N3], [Her1], [Her2]. As a discrete dynamical system, the T-system can be also

viewed as a discrete analogue of the Toda field equation [KOS], [KLWZ]. The Q-system is a degenerated version of the T-system and plays an important role in the algebraic Bethe ansatz method [Ki], [KR], [HKOTY], [KNT]. As a side remark, it may be worth mentioning at this point that "T" stands for *transfer matrix* and "Q" stands for *quantum character* [Ki2] in the original literature.

As a more recent development, a connection between the Q-systems and cluster algebras is clarified by [Ked], [DiK]. Also, a connection between the T-systems (or q-characters) and cluster algebras is studied while seeking a natural categorification of cluster algebras by abelian monoidal categories [HL].

Having these results as a background, we make three simple observations:

(1) There are actually *two classes* of the Y-systems (resp. T-systems), namely, the *unrestricted* and *restricted* Y-systems (resp. T-systems). The latter is obtained by a certain reduction from the former. The periodicity property above mentioned is for the *restricted* Y-systems.
(2) The cluster algebra structure is simpler in the T-systems than in the Y-systems.
(3) The representation theory of quantum affine algebras is more directly connected with the T-systems than with the Y-systems.

These observations motivate us to ask if there is a similar periodicity property for the *restricted T-systems*, and indeed, there is.

In this paper, we formulate the periodicity conjecture for the restricted T-systems, which is the counterpart of the known and partially proved periodicity conjecture for the restricted Y-systems. Then, we partially prove the conjecture by various methods. We remark that the restricted T-systems are relations in certain *quotients* of the Grothendieck ring $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$, while the T-systems studied in [HL] are relations in certain *subrings* of $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$. Accordingly, the correspondence between the T-systems for the simply laced case and cluster algebras considered here and the one in [HL] are close but slightly different. We also note that the correspondence between the *unrestricted* T-systems for the simply laced case and cluster algebras is described in [DiK, Appendix B].

Let us explain the outline of the paper, whose contents are roughly divided into three parts.

In the first part (Section 2) we introduce the *unrestricted T-systems* together with their associated rings, which we call the *unrestricted T-algebras*.

Then, we establish an isomorphism between a subring of the unrestricted T-algebra and the Grothendieck ring of the category of the finite-dimensional representations of an untwisted quantum affine algebra (Corollary 2.9). The relation between the unrestricted T- and Y-algebras is also given (Theorem 2.12). They provide the representation theoretical background of the periodicity problem we are going to discuss.

In the second part (Sections 3–7) we introduce the *level ℓ restricted T-systems* together with their associated rings, which we call the *level ℓ restricted T-algebras*, where ℓ is an integer greater than or equal to 2. Then, we formulate the *periodicity conjecture* (Conjecture 3.11) of the restricted T-systems in terms of the restricted T-algebras. This is the main claim of the paper. Conjecture 3.11 is completely parallel to that of the restricted Y-systems (Conjecture 3.12). A detailed summary of our methods and results concerning Conjecture 3.11 is given in Section 3.4. In brief, we study and partially prove the periodicity conjecture by three independent methods: the *cluster algebra/category method* for the simply laced case in Section 4, the *determinant method* for types A and C in Sections 5 and 6, and the *direct method* for types A, D, and B (level 2) in Section 7. In particular, for the simply laced case, the relation between the restricted T-algebras and cluster algebras is clarified in Section 4. For the cluster category method, we follow the ideas of Keller [Kel2] based on Amiot's generalized cluster categories [A].

In the third part (Sections 8 and 9) we apply the extensions of the above periodicity property to two classes of T- and Y-systems. In Section 8 we formulate and prove the periodicity property for the restricted T- and Y-systems at *levels 1 and 0*. In Section 9 we formulate the periodicity property also for the restricted T- and Y-systems associated with the *twisted* quantum affine algebras. It turns out that their periodicity property reduces to that of the untwisted case. We remark that the nonsimply laced Y-systems studied in [FZ2], [Kel2] are identified with certain reductions of the restricted Y-systems belonging to this class (Remark 9.22).

We conclude the paper with a brief remark (Section 10) on a formal correspondence between the periodicity of the T-systems and the q-character of the quantum affine algebras *at roots of unity*. This suggests that there is some further connection between the representation theories of quivers and the quantum affine algebras at roots of unity behind this periodicity phenomenon, possibly through the works of [N1], [N2]. The relation between the

restricted T- and Y-algebras and cluster algebras for the nonsimply laced case will be discussed in a separate publication.

§2. Unrestricted T- and Y-systems

In this section we introduce the *unrestricted T- and Y-systems* of [KNS1] as a background of the periodicity problem. We also introduce the associated algebras, which we call the *unrestricted T- and Y-algebras*. They are closely connected to the Grothendieck ring of the category of the finite-dimensional $Y(\mathfrak{g})$-modules or $U_q(\hat{\mathfrak{g}})$-modules. The content of this section is rather independent of the rest of the paper.

2.1. Unrestricted T-systems

Throughout the paper, a "ring" means a *commutative ring (algebra over* $\mathbb{Z}$*) with identity element.* For a ring R, $R^{\times}$ denotes the set of all the invertible elements of R. The set of all the positive integers is denoted by $\mathbb{N}$.

Let X_r be a Dynkin diagram of finite type with rank r, and let $I = \{1, \ldots, r\}$ be the enumeration of the vertices of X_r as in Figure 1. We follow [Ka] *except for* E_6, for which we choose the one naturally corresponding to the enumeration of the twisted affine diagram $E_6^{(2)}$ in Section 9.

Let $C = (C_{ab})$, $C_{ab} = 2(\alpha_a, \alpha_b)/(\alpha_a, \alpha_a)$, be the Cartan matrix of X_r. We set numbers t and t_a $(a \in I)$ by

$$(2.1) \qquad t = \begin{cases} 1 & X_r\text{: simply laced,} \\ 2 & X_r = B_r,\ C_r,\ F_4, \\ 3 & X_r = G_2, \end{cases}$$

$$t_a = \begin{cases} 1 & X_r\text{: simply laced,} \\ 1 & X_r\text{: nonsimply laced, } \alpha_a\text{: long root,} \\ t & X_r\text{: nonsimply laced, } \alpha_a\text{: short root.} \end{cases}$$

Let U be either the complex plane $\mathbb{C}$ or the cylinder $\mathbb{C}_\xi := \mathbb{C}/(2\pi\sqrt{-1}/\xi)\mathbb{Z}$ for some $\xi \in \mathbb{C} \setminus 2\pi\sqrt{-1}\mathbb{Q}$.

DEFINITION 2.1. The *unrestricted T-system* $\mathbb{T}(X_r)$ *of type* X_r is the following system of relations for a family of variables $T = \{T_m^{(a)}(u) \mid a \in I, m \in$
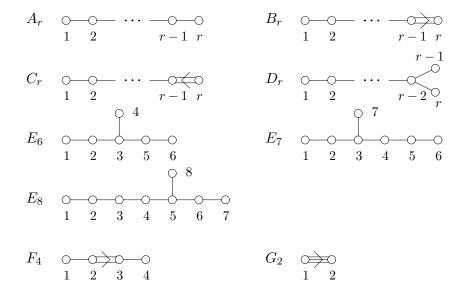
Figure 1: The Dynkin diagrams X_r and their enumerations

$\mathbb{N}, u \in U\}$, where $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$ if they occur in the right-hand sides in the relations.

(Here and throughout the paper, $2m$ (resp. $2m+1$) in the left-hand side, e.g., represents elements $2, 4, \dots$ (resp. $1, 3, \dots$).)

For simply laced X_r,

$$(2.2) \qquad T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{b \in I : C_{ab} = -1} T_m^{(b)}(u).$$

For $X_r = B_r$,

$$(2.3) \qquad T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)$$
$$+ T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \le a \le r-2),$$
$$T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) = T_{m-1}^{(r-1)}(u)T_{m+1}^{(r-1)}(u) + T_m^{(r-2)}(u)T_{2m}^{(r)}(u),$$
$$T_{2m}^{(r)}\Big(u-\frac{1}{2}\Big)T_{2m}^{(r)}\Big(u+\frac{1}{2}\Big) = T_{2m-1}^{(r)}(u)T_{2m+1}^{(r)}(u)$$
$$+ T_m^{(r-1)}\Big(u-\frac{1}{2}\Big)T_m^{(r-1)}\Big(u+\frac{1}{2}\Big),$$
$$T_{2m+1}^{(r)}\Big(u-\frac{1}{2}\Big)T_{2m+1}^{(r)}\Big(u+\frac{1}{2}\Big) = T_{2m}^{(r)}(u)T_{2m+2}^{(r)}(u) + T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u).$$

For $X_r = C_r$,

$$(2.4) \quad T_m^{(a)}\Big(u - \frac{1}{2}\Big)T_m^{(a)}\Big(u + \frac{1}{2}\Big) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)$$
$$+ T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \le a \le r - 2),$$

$$T_{2m}^{(r-1)}\Big(u - \frac{1}{2}\Big)T_{2m}^{(r-1)}\Big(u + \frac{1}{2}\Big) = T_{2m-1}^{(r-1)}(u)T_{2m+1}^{(r-1)}(u)$$
$$+ T_{2m}^{(r-2)}(u)T_m^{(r)}\Big(u - \frac{1}{2}\Big)T_m^{(r)}\Big(u + \frac{1}{2}\Big),$$

$$T_{2m+1}^{(r-1)}\Big(u - \frac{1}{2}\Big)T_{2m+1}^{(r-1)}\Big(u + \frac{1}{2}\Big) = T_{2m}^{(r-1)}(u)T_{2m+2}^{(r-1)}(u)$$
$$+ T_{2m+1}^{(r-2)}(u)T_m^{(r)}(u)T_{m+1}^{(r)}(u),$$

$$T_m^{(r)}(u - 1)T_m^{(r)}(u + 1) = T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_{2m}^{(r-1)}(u).$$

For $X_r = F_4$,

$$(2.5) \quad T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u),$$

$$T_m^{(2)}(u - 1)T_m^{(2)}(u + 1) = T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) + T_m^{(1)}(u)T_{2m}^{(3)}(u),$$

$$T_{2m}^{(3)}\Big(u - \frac{1}{2}\Big)T_{2m}^{(3)}\Big(u + \frac{1}{2}\Big) = T_{2m-1}^{(3)}(u)T_{2m+1}^{(3)}(u)$$
$$+ T_m^{(2)}\Big(u - \frac{1}{2}\Big)T_m^{(2)}\Big(u + \frac{1}{2}\Big)T_{2m}^{(4)}(u),$$

$$T_{2m+1}^{(3)}\Big(u - \frac{1}{2}\Big)T_{2m+1}^{(3)}\Big(u + \frac{1}{2}\Big) = T_{2m}^{(3)}(u)T_{2m+2}^{(3)}(u)$$
$$+ T_m^{(2)}(u)T_{m+1}^{(2)}(u)T_{2m+1}^{(4)}(u),$$

$$T_m^{(4)}\Big(u - \frac{1}{2}\Big)T_m^{(4)}\Big(u + \frac{1}{2}\Big) = T_{m-1}^{(4)}(u)T_{m+1}^{(4)}(u) + T_m^{(3)}(u).$$

For $X_r = G_2$,

$$(2.6) \quad T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_{3m}^{(2)}(u),$$

$$T_{3m}^{(2)}\Big(u - \frac{1}{3}\Big)T_{3m}^{(2)}\Big(u + \frac{1}{3}\Big) = T_{3m-1}^{(2)}(u)T_{3m+1}^{(2)}(u)$$
$$+ T_m^{(1)}\Big(u - \frac{2}{3}\Big)T_m^{(1)}(u)T_m^{(1)}\Big(u + \frac{2}{3}\Big),$$

$$T_{3m+1}^{(2)}\Big(u - \frac{1}{3}\Big)T_{3m+1}^{(2)}\Big(u + \frac{1}{3}\Big) = T_{3m}^{(2)}(u)T_{3m+2}^{(2)}(u)$$

$$+ T_m^{(1)}\Big(u - \frac{1}{3}\Big)T_m^{(1)}\Big(u + \frac{1}{3}\Big)T_{m+1}^{(1)}(u),$$

$$T_{3m+2}^{(2)}\Big(u - \frac{1}{3}\Big)T_{3m+2}^{(2)}\Big(u + \frac{1}{3}\Big) = T_{3m+1}^{(2)}(u)T_{3m+3}^{(2)}(u)$$

$$+ T_m^{(1)}(u)T_{m+1}^{(1)}\Big(u - \frac{1}{3}\Big)T_{m+1}^{(1)}\Big(u + \frac{1}{3}\Big).$$

The choice of the domain $U = \mathbb{C}_\xi$ of the parameter u effectively imposes an additional periodic condition:

$$(2.7) \qquad\qquad T_m^{(a)}(u) = T_m^{(a)}\Big(u + \frac{2\pi\sqrt{-1}}{\xi}\Big).$$

By the assumption, we have $2\pi\sqrt{-1}/\xi \notin \mathbb{Q}$ so that it is compatible with the relations $\mathbb{T}(X_r)$.

REMARK 2.2. Originally, the system $\mathbb{T}(X_r)$ was introduced in [KNS1] as a family of relations in the ring of commuting *transfer matrices* of the solvable lattice models. For example, for $X_r = A_r$, the relations in (2.2) are the Jacobi identities among the Jacobi-Trudi–type determinantal expression of the transfer matrices in [BR]. The T-system is a natural affinization of the *Q-system* of [Ki], [KR] (see Appendix A.1), and the idea behind the both systems was the existence of a conjectured family of exact sequences among the *Kirillov-Reshetikhin modules* [KR], [CP1], [KNS1] of the Yangian $Y(\mathfrak{g})$ and/or the untwisted quantum affine algebra $U_q(\hat{\mathfrak{g}})$ associated with the complex simple Lie algebra $\mathfrak{g}$ of type X_r [D1], [D2], [J]. The choice $U = \mathbb{C}$ corresponds to the $Y(\mathfrak{g})$ case, while the choice $U = \mathbb{C}_\xi$ corresponds to the $U_q(\hat{\mathfrak{g}})$ case, as explained below. For $U_q(\hat{\mathfrak{sl}}_2)$, the existence of such exact sequences was known by [CP1]. Later this conjecture was proved for $U_q(\hat{\mathfrak{g}})$ by [N3], [Her1] (see Theorem 2.8(2)).

It is useful to introduce the rings associated with $\mathbb{T}(X_r)$.

DEFINITION 2.3. The *unrestricted T-algebra* $\mathcal{T}(X_r)$ *of type* X_r is the ring with generators $T_m^{(a)}(u)^{\pm 1}$ $(a \in I, m \in \mathbb{N}, u \in U)$ and the relations $\mathbb{T}(X_r)$. (Here we also assume the relation $T_m^{(a)}(u)T_m^{(a)}(u)^{-1} = 1$ implicitly. We do not repeat this remark in the forthcoming similar definitions.) Also, we define the ring $\mathcal{T}^\circ(X_r)$ as the subring of $\mathcal{T}(X_r)$ generated by $T_m^{(a)}(u)$ $(a \in I, m \in \mathbb{N}, u \in U)$.

We write all the relations in $\mathbb{T}(X_r)$ in a unified manner as follows:

$$(2.8) \qquad T_m^{(a)}\Big(u - \frac{1}{t_a}\Big)T_m^{(a)}\Big(u + \frac{1}{t_a}\Big) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + M_m^{(a)}(u),$$

where $M_m^{(a)}(u)$ is the second term of the right-hand side of each relation, and t_a is the number in (2.1). Then, define $S_{amu}(T) \in \mathbb{Z}[T]$ $(a \in I, m \in \mathbb{N}, u \in U)$ by

$$(2.9) \quad S_{amu}(T) = T_m^{(a)}\Big(u - \frac{1}{t_a}\Big)T_m^{(a)}\Big(u + \frac{1}{t_a}\Big) - T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) - M_m^{(a)}(u),$$

so that all the relations in $\mathbb{T}(X_r)$ are written in the form $S_{amu}(T) = 0$. Let $I(\mathbb{T}(X_r))$ denote the ideal of $\mathbb{Z}[T]$ generated by $S_{amu}(T)$'s. We consider the natural embedding $\mathbb{Z}[T] \subset \mathbb{Z}[T^{\pm 1}]$.

We use the following description of the ring $\mathcal{T}^{\circ}(X_r)$.

LEMMA 2.4. (1) *There is a ring isomorphism:*

$$(2.10) \qquad \mathcal{T}^{\circ}(X_r) \simeq \mathbb{Z}[T]/\big(\mathbb{Z}[T^{\pm 1}]I(\mathbb{T}(X_r)) \cap \mathbb{Z}[T]\big).$$

(2) *For* $P(T) \in \mathbb{Z}[T]$, *the following conditions are equivalent:*
(i) $P(T) \in \mathbb{Z}[T^{\pm 1}]I(\mathbb{T}(X_r))$.
(ii) *There is a nonzero monomial* $M(T) \in \mathbb{Z}[T]$ *such that* $M(T)P(T) \in I(\mathbb{T}(X_r))$.

Let us clarify the relation between the ring $\mathcal{T}^{\circ}(X_r)$ and the Grothendieck ring $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$ of the category of the type 1 finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules [CP2].

Choose $\hbar \in \mathbb{C} \setminus 2\pi\sqrt{-1}\mathbb{Q}$ arbitrarily. We set the deformation parameter q of $U_q(\hat{\mathfrak{g}})$ as $q = e^{\hbar} \in \mathbb{C}^{\times}$, so that q is *not a root of unity*.

Let

$$(2.11) \qquad \chi_q : \operatorname{Rep} U_q(\hat{\mathfrak{g}}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^{\times}}$$

be the *q-character map* of $U_q(\hat{\mathfrak{g}})$ in [FR], [FM], which is an injective ring homomorphism as shown in [FR]. From now on, we employ the parameterization of the variables $Y_{a,q^{tu}}$ $(a \in I, u \in \mathbb{C}_{t\hbar})$ instead of $Y_{i,a}$ $(i \in I, a \in \mathbb{C}^{\times})$ in [FR], [FM], where t is the number in (2.1).

The *q-character ring* $\operatorname{Ch} U_q(\hat{\mathfrak{g}})$ of $U_q(\hat{\mathfrak{g}})$ is defined to be $\operatorname{Im} \chi_q$. Thus, $\operatorname{Ch} U_q(\hat{\mathfrak{g}})$ is an integral domain and isomorphic to $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$.

DEFINITION 2.5. A *Kirillov-Reshetikhin module* $W_m^{(a)}(u)$ $(a \in I, m \in \mathbb{N}, u \in \mathbb{C}_{t\hbar})$ of $U_q(\hat{\mathfrak{g}})$ is the irreducible finite dimensional $U_q(\hat{\mathfrak{g}})$-module with highest weight monomial

$$(2.12) \qquad m_+ = \prod_{j=1}^{m} Y_{a,q^{tu}q_a^{m+1-2j}},$$

where $q_a = q^{t/t_a}$. Especially, $W_1^{(a)}(u)$ $(a \in I, u \in \mathbb{C}_{t\hbar})$ is called a *fundamental module*.

REMARK 2.6. The above $W_m^{(a)}(u)$ corresponds to $W_{m,q^{tu}q_a^{-m+1}}^{(a)}$ in [N3], [Her1].

The following fact is well known.

THEOREM 2.7. (Frenkel and Reshetikhin [FR, Corollary 2]) *The ring* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ *is freely generated by the fundamental characters* $\chi_q(W_1^{(a)}(u))$ $(a \in I, u \in \mathbb{C}_{t\hbar})$.

Correspondingly, we choose the domain U of the parameter u for the T-system $\mathbb{T}(X_r)$ as $U = \mathbb{C}_{t\hbar}$. Here is an alternative description of $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ by the q-characters of the Kirillov-Reshetikhin modules and the T-system $\mathbb{T}(X_r)$.

THEOREM 2.8. *Let* $\widetilde{T} = \{\widetilde{T}_m^{(a)}(u) := \chi_q(W_m^{(a)}(u)) \mid a \in I, m \in \mathbb{N}, u \in \mathbb{C}_{t\hbar}\}$ *be the family of the q-characters of the Kirillov-Reshetikhin modules of* $U_q(\hat{\mathfrak{g}})$. *Then, we have the following.*

(1) *The family $\widetilde{T}$ generates the ring* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$.

(2) *(Nakajima [N3], Hernandez [Her1]) The family $\widetilde{T}$ satisfies the T-system $\mathbb{T}(X_r)$ in* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ *(by replacing $T_m^{(a)}(u)$ in $\mathbb{T}(X_r)$ with $\widetilde{T}_m^{(a)}(u)$).*

(3) *For any $P(T) \in \mathbb{Z}[T]$, the relation $P(\widetilde{T}) = 0$ holds in* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ *if and only if there is a nonzero monomial $M(T) \in \mathbb{Z}[T]$ such that $M(T)P(T) \in I(\mathbb{T}(X_r))$.*

Proof. (1) This is a corollary of Theorem 2.7.

(2) This was proved by [N3, Theorem 1.1] (for simply laced case) and by [Her1, Theorem 3.4] (including nonsimply laced case).

(3) The "if" part follows from (2) and from the fact that $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ is an integral domain. Let us show the "only if" part. To begin with, we introduce

the *height* of $T_m^{(a)}(u)$, ht $T_m^{(a)}(u)$, by

$$\text{for simply laced } X_r, \quad \text{ht } T_m^{(a)}(u) = m,$$

$$\text{for } X_r = B_r, \qquad \text{ht } T_m^{(a)}(u) = \begin{cases} 2m-1 & a = 1, \dots, r-1, \\ m & a = r, \end{cases}$$

$$(2.13) \quad \text{for } X_r = C_r, \qquad \text{ht } T_m^{(a)}(u) = \begin{cases} m & a = 1, \dots, r-1, \\ 2m-1 & a = r, \end{cases}$$

$$\text{for } X_r = F_4, \qquad \text{ht } T_m^{(a)}(u) = \begin{cases} 2m-1 & a = 1, 2, \\ m & a = 3, 4, \end{cases}$$

$$\text{for } X_r = G_2, \qquad \text{ht } T_m^{(a)}(u) = \begin{cases} 3m-2 & a = 1, \\ m & a = 2. \end{cases}$$

Then, the following facts can be easily checked by inspection:

(a) ht $T_m^{(a)}(u) = 1$ if and only if $m = 1$;

(b) the variable $T_m^{(a)}(u)$ ($m \geq 2$) occurs in $S_{a,m-1,u}(T)$, and ht $T_m^{(a)}(u)$ is greater than the heights of the other variables occurring in $S_{a,m-1,u}(T)$.

For a polynomial $P(T) \in \mathbb{Z}[T]$, we define ht $P(T)$ by the greatest height of all the generators $T_m^{(a)}(u)$ occurring in $P(T)$.

Now suppose that there is a nontrivial relation $P(\widetilde{T}) = 0$ in $\text{Ch } U_q(\hat{\mathfrak{g}})$ for $P(T) \in \mathbb{Z}[T]$, and that $h := \text{ht } P(T) \geq 2$. Let S be the set of all the triplets (a, m, u) such that $T_m^{(a)}(u)$ is of height h and occurs in $P(T)$. Let $M_h(T) = \prod_{(a,m,u)\in S} T_{m-2}^{(a)}(u)$. Then, thanks to (a) and (b), there is some $Q(T) \in \mathbb{Z}[T]$ with ht $Q(T) < h$ such that $Q(T) \equiv M_h(T)P(T) \bmod I(\mathbb{T}(X_r))$. Furthermore, by (2), $Q(\widetilde{T}) = 0$ in $\text{Ch } U_q(\hat{\mathfrak{g}})$. Repeat it until the relation reduces to the form $Q(\widetilde{T}) = 0$ with ht $Q(T) = 1$ or 0. However, the former does not occur, since it contradicts Theorem 2.7. Therefore, we have ht $Q(T) = 0$, that is, $Q(T) = 0$, which proves the claim.   □

COROLLARY 2.9. *The ring* $\mathcal{T}^\circ(X_r)$ *with* $U = \mathbb{C}_{t\hbar}$ *is isomorphic to* $\text{Rep } U_q(\hat{\mathfrak{g}})$ *by the correspondence* $T_m^{(a)}(u) \mapsto W_m^{(a)}(u)$.

Proof. It follows from Theorem 2.8 and Lemma 2.4 that

$$(2.14) \quad \text{Rep } U_q(\hat{\mathfrak{g}}) \simeq \text{Ch } U_q(\hat{\mathfrak{g}}) \simeq \mathbb{Z}[T]/\big(\mathbb{Z}[T^{\pm 1}]I(\mathbb{T}(X_r)) \cap \mathbb{Z}[T]\big) \simeq \mathcal{T}^\circ(X_r).$$

□

It should be proved in the same way, using the character by [Kn], that $\mathcal{T}^\circ(X_r)$ with $U = \mathbb{C}$ is isomorphic to $\operatorname{Rep} Y(\mathfrak{g})$. In Appendix A.1 we give parallel results for the ring associated with the Q-system and $\operatorname{Rep} U_q(\mathfrak{g})$.

2.2. Unrestricted Y-systems

DEFINITION 2.10. The *unrestricted Y-system* $\mathbb{Y}(X_r)$ *of type* X_r is the following system of relations for a family of variables $Y = \{Y_m^{(a)}(u) \mid a \in I,\ m \in \mathbb{N}, u \in U\}$, where $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = 0$ if they occur in the right-hand sides in the following relations.

For simply laced X_r,

$$(2.15) \qquad Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{\prod_{b \in I : C_{ab}=-1}(1 + Y_m^{(b)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}.$$

For $X_r = B_r$,

$$(2.16)\quad Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1 + Y_m^{(a-1)}(u))(1 + Y_m^{(a+1)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}$$
$$(1 \le a \le r-2),$$

$$Y_m^{(r-1)}(u-1)Y_m^{(r-1)}(u+1) = \frac{\begin{array}{c}(1 + Y_m^{(r-2)}(u))\\ \times\, (1 + Y_{2m-1}^{(r)}(u))(1 + Y_{2m+1}^{(r)}(u))\\ \times\, \left(1 + Y_{2m}^{(r)}\left(u - \tfrac{1}{2}\right)\right)\left(1 + Y_{2m}^{(r)}\left(u + \tfrac{1}{2}\right)\right)\end{array}}{(1 + Y_{m-1}^{(r-1)}(u)^{-1})(1 + Y_{m+1}^{(r-1)}(u)^{-1})},$$

$$Y_{2m}^{(r)}\left(u - \tfrac{1}{2}\right)Y_{2m}^{(r)}\left(u + \tfrac{1}{2}\right) = \frac{1 + Y_m^{(r-1)}(u)}{(1 + Y_{2m-1}^{(r)}(u)^{-1})(1 + Y_{2m+1}^{(r)}(u)^{-1})},$$

$$Y_{2m+1}^{(r)}\left(u - \tfrac{1}{2}\right)Y_{2m+1}^{(r)}\left(u + \tfrac{1}{2}\right) = \frac{1}{(1 + Y_{2m}^{(r)}(u)^{-1})(1 + Y_{2m+2}^{(r)}(u)^{-1})}.$$

For $X_r = C_r$,

$$(2.17)\quad Y_m^{(a)}\left(u - \tfrac{1}{2}\right)Y_m^{(a)}\left(u + \tfrac{1}{2}\right) = \frac{(1 + Y_m^{(a-1)}(u))(1 + Y_m^{(a+1)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}$$
$$(1 \le a \le r-2),$$

$$Y_{2m}^{(r-1)}\left(u - \tfrac{1}{2}\right)Y_{2m}^{(r-1)}\left(u + \tfrac{1}{2}\right) = \frac{(1 + Y_{2m}^{(r-2)}(u))(1 + Y_m^{(r)}(u))}{(1 + Y_{2m-1}^{(r-1)}(u)^{-1})(1 + Y_{2m+1}^{(r-1)}(u)^{-1})},$$

$$Y_{2m+1}^{(r-1)}\Big(u-\frac{1}{2}\Big)Y_{2m+1}^{(r-1)}\Big(u+\frac{1}{2}\Big)=\frac{1+Y_{2m+1}^{(r-2)}(u)}{(1+Y_{2m}^{(r-1)}(u)^{-1})(1+Y_{2m+2}^{(r-1)}(u)^{-1})},$$

$$Y_m^{(r)}(u-1)Y_m^{(r)}(u+1)=\frac{(1+Y_{2m+1}^{(r-1)}(u))(1+Y_{2m-1}^{(r-1)}(u))}{\times\big(1+Y_{2m}^{(r-1)}\big(u-\frac{1}{2}\big)\big)\big(1+Y_{2m}^{(r-1)}\big(u+\frac{1}{2}\big)\big)}{(1+Y_{m-1}^{(r)}(u)^{-1})(1+Y_{m+1}^{(r)}(u)^{-1})}.$$

For $X_r=F_4$,

$$(2.18)\quad Y_m^{(1)}(u-1)Y_m^{(1)}(u+1)=\frac{1+Y_m^{(2)}(u)}{(1+Y_{m-1}^{(1)}(u)^{-1})(1+Y_{m+1}^{(1)}(u)^{-1})},$$

$$Y_m^{(2)}(u-1)Y_m^{(2)}(u+1)=\frac{(1+Y_m^{(1)}(u))(1+Y_{2m-1}^{(3)}(u))}{\times(1+Y_{2m+1}^{(3)}(u))\big(1+Y_{2m}^{(3)}\big(u-\frac{1}{2}\big)\big)}{\times\big(1+Y_{2m}^{(3)}\big(u+\frac{1}{2}\big)\big)}{(1+Y_{m-1}^{(2)}(u)^{-1})(1+Y_{m+1}^{(2)}(u)^{-1})},$$

$$Y_{2m}^{(3)}\Big(u-\frac{1}{2}\Big)Y_{2m}^{(3)}\Big(u+\frac{1}{2}\Big)=\frac{(1+Y_m^{(2)}(u))(1+Y_{2m}^{(4)}(u))}{(1+Y_{2m-1}^{(3)}(u)^{-1})(1+Y_{2m+1}^{(3)}(u)^{-1})},$$

$$Y_{2m+1}^{(3)}\Big(u-\frac{1}{2}\Big)Y_{2m+1}^{(3)}\Big(u+\frac{1}{2}\Big)=\frac{1+Y_{2m+1}^{(4)}(u)}{(1+Y_{2m}^{(3)}(u)^{-1})(1+Y_{2m+2}^{(3)}(u)^{-1})},$$

$$Y_m^{(4)}\Big(u-\frac{1}{2}\Big)Y_m^{(4)}\Big(u+\frac{1}{2}\Big)=\frac{1+Y_m^{(3)}(u)}{(1+Y_{m-1}^{(4)}(u)^{-1})(1+Y_{m+1}^{(4)}(u)^{-1})}.$$

For $X_r=G_2$,

$$(2.19)\ Y_m^{(1)}(u-1)Y_m^{(1)}(u+1)=\frac{\begin{array}{c}(1+Y_{3m-2}^{(2)}(u))(1+Y_{3m+2}^{(2)}(u))\\\times\big(1+Y_{3m-1}^{(2)}\big(u-\frac{1}{3}\big)\big)\\\times\big(1+Y_{3m-1}^{(2)}\big(u+\frac{1}{3}\big)\big)\\\times\big(1+Y_{3m+1}^{(2)}\big(u-\frac{1}{3}\big)\big)\\\times\big(1+Y_{3m+1}^{(2)}\big(u+\frac{1}{3}\big)\big)\\\times\big(1+Y_{3m}^{(2)}\big(u-\frac{2}{3}\big)\big)\big(1+Y_{3m}^{(2)}\big(u+\frac{2}{3}\big)\big)\\\times(1+Y_{3m}^{(2)}(u))\end{array}}{(1+Y_{m-1}^{(1)}(u)^{-1})(1+Y_{m+1}^{(1)}(u)^{-1})},$$

$$Y_{3m}^{(2)}\Big(u-\frac{1}{3}\Big)Y_{3m}^{(2)}\Big(u+\frac{1}{3}\Big)=\frac{1+Y_m^{(1)}(u)}{(1+Y_{3m-1}^{(2)}(u)^{-1})(1+Y_{3m+1}^{(2)}(u)^{-1})},$$

$$Y_{3m+1}^{(2)}\Big(u - \frac{1}{3}\Big)Y_{3m+1}^{(2)}\Big(u + \frac{1}{3}\Big) = \frac{1}{(1 + Y_{3m}^{(2)}(u)^{-1})(1 + Y_{3m+2}^{(2)}(u)^{-1})},$$

$$Y_{3m+2}^{(2)}\Big(u - \frac{1}{3}\Big)Y_{3m+2}^{(2)}\Big(u + \frac{1}{3}\Big) = \frac{1}{(1 + Y_{3m+1}^{(2)}(u)^{-1})(1 + Y_{3m+3}^{(2)}(u)^{-1})}.$$

The choice of the domain $U = \mathbb{C}_\xi$ of the parameter u effectively imposes an additional periodic condition:

$$(2.20) \qquad\qquad Y_m^{(a)}(u) = Y_m^{(a)}\Big(u + \frac{2\pi\sqrt{-1}}{\xi}\Big).$$

DEFINITION 2.11. The *unrestricted Y-algebra* $\mathcal{Y}(X_r)$ *of type* X_r is the ring with generators $Y_m^{(a)}(u)^{\pm 1}$, $(1 + Y_m^{(a)}(u))^{-1}$ $(a \in I, m \in \mathbb{N}, u \in U)$, and the relations $\mathbb{Y}(X_r)$.

The system $\mathbb{Y}(X_r)$ is introduced by [KN] (see also Remark 3.7 for the origin of the Y-systems).

Though the T-systems and Y-systems arose in different contexts with different motivations, there is a simple and remarkable connection between them, as described below. Recall that $M_m^{(a)}(u)$ is defined in (2.8).

THEOREM 2.12. (1) *There is a ring homomorphism*

$$(2.21) \qquad\qquad \varphi : \mathcal{Y}(X_r) \to \mathcal{T}(X_r)$$

defined by

$$(2.22) \qquad\qquad Y_m^{(a)}(u) \mapsto \frac{M_m^{(a)}(u)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)},$$

or, equivalently, by either of

$$(2.23) \qquad\qquad 1 + Y_m^{(a)}(u) \mapsto \frac{T_m^{(a)}\big(u - \frac{1}{t_a}\big)T_m^{(a)}\big(u + \frac{1}{t_a}\big)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)},$$

$$(2.24) \qquad\qquad 1 + Y_m^{(a)}(u)^{-1} \mapsto \frac{T_m^{(a)}\big(u - \frac{1}{t_a}\big)T_m^{(a)}\big(u + \frac{1}{t_a}\big)}{M_m^{(a)}(u)},$$

where $T_0^{(a)}(u) = 1$.

(2) *There is a ring homomorphism*

$$(2.25) \qquad\qquad \psi : \mathcal{T}(X_r) \to \mathcal{Y}(X_r)$$

such that $\psi \circ \varphi = \mathrm{id}_{\mathcal{Y}(X_r)}$.

(3) $\mathcal{Y}(X_r)$ *is isomorphic to a subring and a quotient ring of* $\mathcal{T}(X_r)$.

The homomorphism φ is canonical, while ψ is neither unique nor canonical.

Proof. Statement (3) is a corollary of (1) and (2); we prove (1) and (2). Here, we concentrate on the case $U = \mathbb{C}$. The modification of the proof for the case $U = \mathbb{C}_\xi$ is straightforward.

(i) *The case* X_r *is simply laced.* (1) For simplicity, let us write the image $\varphi(Y_m^{(a)}(u))$ as $Y_m^{(a)}(u)$. Then, the relation (2.15) is shown as follows:

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1)$$

$$= \frac{\prod_{b:C_{ab}=-1} T_m^{(b)}(u-1)T_m^{(b)}(u+1)}{T_{m-1}^{(a)}(u-1)T_{m+1}^{(a)}(u-1)T_{m-1}^{(a)}(u+1)T_{m+1}^{(a)}(u+1)}$$

$$(2.26) \qquad = \frac{\prod_{b:C_{ab}=-1}\bigl(T_{m-1}^{(b)}(u)T_{m+1}^{(b)}(u) + \prod_{c:C_{bc}=-1} T_m^{(c)}(u)\bigr)}{T_{m-2}^{(a)}(u)T_m^{(a)}(u) + \prod_{b:C_{ab}=-1} T_{m-1}^{(b)}(u)}$$

$$\times \frac{1}{T_m^{(a)}(u)T_{m+2}^{(a)}(u) + \prod_{b:C_{ab}=-1} T_{m+1}^{(b)}(u)}$$

$$= \frac{\prod_{b:C_{ab}=-1}(1 + Y_m^{(b)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}.$$

We remark that the above calculation is valid also at $m = 1$ by formally setting $T_{-1}^{(0)}(u) = 0$.

(2) Below we define the image $\psi(T_m^{(a)}(u))$ $(a \in I, m \in \mathbb{N}, u \in \mathbb{C})$ in three steps and then show that they satisfy $\mathbb{T}(X_r)$. For simplicity, we write the image $\psi(T_m^{(a)}(u))$ as $T_m^{(a)}(u)$.

Step 1. We arbitrarily choose $T_1^{(a)}(u) \in \mathcal{Y}(X_r)^\times$ $(a \in I)$ for each $u \in \mathbb{C}$ in the region $-1 \le \mathrm{Re}\, u < 1$.

Step 2. We define $T_1^{(a)}(u)$ $(a \in I)$ for the rest of the region $-2 \le \mathrm{Re}\, u < 2$ by

$$(2.27) \qquad T_1^{(a)}(u \pm 1) = (1 + Y_1^{(a)}(u)^{-1})\frac{M_1^{(a)}(u)}{T_1^{(a)}(u \mp 1)}.$$

We repeat it to define $T_1^{(a)}(u)$ $(a \in I)$ for all $u \in \mathbb{C}$.

Step 3. For each a, we recursively define $T_m^{(a)}(u)$ $(m \geq 2, u \in \mathbb{C})$ by

$$(2.28) \qquad T_{m+1}^{(a)}(u) = \frac{1}{1 + Y_m^{(a)}(u)} \frac{T_m^{(a)}(u-1)T_m^{(a)}(u+1)}{T_{m-1}^{(a)}(u)},$$

where $T_0^{(a)}(u) = 1$.

CLAIM. *The family T defined above satisfies the following relations in $\mathcal{Y}(X_r)$:*

$$(2.29) \qquad 1 + Y_m^{(a)}(u) = \frac{T_m^{(a)}(u-1)T_m^{(a)}(u+1)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)},$$

$$(2.30) \qquad 1 + Y_m^{(a)}(u)^{-1} = \frac{T_m^{(a)}(u-1)T_m^{(a)}(u+1)}{M_m^{(a)}(u)}.$$

The relation (2.29) clearly holds by (2.28). The relation (2.30) is shown by the induction on m, where the $m = 1$ case is true by (2.27).

Now, taking the inverse sum of (2.29) and (2.30), we obtain (2.2). Therefore, ψ is a ring homomorphism. Furthermore, taking the ratio of (2.29) and (2.30), we obtain $Y_m^{(a)}(u) = M_m^{(a)}(u)/(T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u))$. This proves $\psi \circ \varphi = \mathrm{id}_{\mathcal{Y}(X_r)}$.

(ii) *The case X_r is nonsimply laced.* (1) This can be proved one by one with similar calculations to (2.26), though they are slightly more complicated.

(2) Below we define the image $\psi(T_m^{(a)}(u))$ $(a \in I, m \in \mathbb{N}, u \in \mathbb{C})$ in three steps and then show that they satisfy $\mathbb{T}(X_r)$. For simplicity, we again write the image $\psi(T_m^{(a)}(u))$ as $T_m^{(a)}(u)$.

Step 1. First, we arbitrarily choose $T_1^{(a)}(u) \in \mathcal{Y}(X_r)^\times$ $(a \in I)$ for each $u \in \mathbb{C}$ in the region $-1/t_a \leq \mathrm{Re}\, u < 1/t_a$. Next, for each a with $t_a = 2$ (resp. $t_a = 3$, which occurs only for $X_r = G_2$ and $a = 2$), we define $T_2^{(a)}(u)$ (resp. $T_2^{(a)}(u)$ and $T_3^{(a)}(u)$) in the region $-1/t_a \leq \mathrm{Re}\, u < 1/t_a$ by

$$(2.31) \qquad T_{m+1}^{(a)}(u) = \frac{M_m^{(a)}(u)}{Y_m^{(a)}(u)T_{m-1}^{(a)}(u)}$$

or, more explicitly,

$$(2.32) \quad T_2^{(a)}(u) = \frac{T_1^{(a-1)}(u)T_1^{(a+1)}(u)}{Y_1^{(a)}(u)}, \qquad T_3^{(2)}(u) = \frac{T_1^{(1)}\left(u - \frac{1}{3}\right)T_1^{(1)}\left(u + \frac{1}{3}\right)}{Y_2^{(2)}(u)T_1^{(2)}(u)},$$

where $T_1^{(0)}(u) = T_1^{(r+1)}(u) = 1$.

Step 2. Let t be the number in (2.1). First, we define $T_1^{(a)}(u)$ $(a \in I)$ for the rest of the region $-(1/t_a) - (1/t) \leq \operatorname{Re} u < (1/t_a) + (1/t)$ by

$$(2.33) \qquad T_1^{(a)}\Big(u \pm \frac{1}{t_a}\Big) = (1 + Y_1^{(a)}(u)^{-1})\frac{M_1^{(a)}(u)}{T_1^{(a)}\big(u \mp \frac{1}{t_a}\big)}.$$

Next, we define $T_m^{(a)}(u)$ $(t_a = 2, 3;\ m = 2, \ldots, t_a)$ for the rest of the region $-(1/t_a) - (1/t) \leq \operatorname{Re} u < (1/t_a) + (1/t)$ by (2.31). We repeat it to define $T_m^{(a)}(u)$ $(a \in I;\ m = 1, \ldots, t_a)$ for all $u \in \mathbb{C}$.

Step 3. For each a, we recursively define $T_m^{(a)}(u)$ $(m > t_a,\ u \in \mathbb{C})$ by

$$(2.34) \qquad T_{m+1}^{(a)}(u) = \frac{1}{1 + Y_m^{(a)}(u)}\frac{T_m^{(a)}\big(u - \frac{1}{t_a}\big)T_m^{(a)}\big(u + \frac{1}{t_a}\big)}{T_{m-1}^{(a)}(u)},$$

where $T_0^{(a)}(u) = 1$.

CLAIM 1. *The family T defined above satisfies the following relations in $\mathcal{Y}(X_r)$:*

$$(2.35) \qquad 1 + Y_m^{(a)}(u) = \frac{T_m^{(a)}\big(u - \frac{1}{t_a}\big)T_m^{(a)}\big(u + \frac{1}{t_a}\big)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)}$$

$$(t_a = 2, 3; m = 1, \ldots, t_a - 1),$$

$$(2.36) \qquad 1 + Y_m^{(a)}(u)^{-1} = \frac{T_m^{(a)}\big(u - \frac{1}{t_a}\big)T_m^{(a)}\big(u + \frac{1}{t_a}\big)}{M_m^{(a)}(u)}$$

$$(t_a = 2, 3; m = 2, \ldots, t_a).$$

The relation (2.35) for $m = 1$ is an immediate consequence of (2.31) and (2.33). The relation (2.36) for $m = 2$ is verified one by one. For $t_a = 3$, (2.35) for $m = 2$ is an immediate consequence of (2.31) and (2.36) for $m = 2$, and (2.36) for $m = 3$ is verified by (2.31) and (2.33).

CLAIM 2. *The family T defined above satisfies the following relations in $\mathcal{Y}(X_r)$ for any (a, m, u):*

$$(2.37) \qquad 1 + Y_m^{(a)}(u) = \frac{T_m^{(a)}\big(u - \frac{1}{t_a}\big)T_m^{(a)}\big(u + \frac{1}{t_a}\big)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)},$$

$$(2.38) \qquad 1 + Y_m^{(a)}(u)^{-1} = \frac{T_m^{(a)}\left(u - \frac{1}{t_a}\right)T_m^{(a)}\left(u + \frac{1}{t_a}\right)}{M_m^{(a)}(u)}.$$

The relation (2.37) holds for any (a, m, u) because of (2.34) and (2.35). The relation (2.38) holds for $m = 1, \dots, t_a$ because of (2.33) and (2.36). Then, one can verify (2.38) by the induction on m one by one.

The rest of the argument is the same as for the simply laced case. $\qquad \square$

REMARK 2.13. The transformation (2.22) first appeared in [KP] for the simplest case $X_r = A_1$ and was generalized in [KNS1] for general X_r. The analogous transformation plays an important role also in the approach by cluster algebras with coefficients [FZ4].

2.3. Regular solutions of T- and Y-systems

In application, we usually consider solutions of $\mathbb{T}(X_r)$ and $\mathbb{Y}(X_r)$ in a particular ring.

DEFINITION 2.14. Let R be a ring.

(i) A family $T = \{T_m^{(a)}(u) \in R \mid a \in I, \ m \in \mathbb{N}, u \in U\}$ satisfying $\mathbb{T}(X_r)$ is called a *solution of the T-system $\mathbb{T}(X_r)$ in R*. We say a solution T of $\mathbb{T}(X_r)$ in R is *regular* if $T_m^{(a)}(u) \in R^\times$ for any (a, m, u).

(ii) A family $Y = \{Y_m^{(a)}(u) \in R \mid a \in I, \ m \in \mathbb{N}, u \in U\}$ satisfying $\mathbb{Y}(X_r)$ is called a *solution of the Y-system $\mathbb{Y}(X_r)$ in R*. We say a solution Y of $\mathbb{Y}(X_r)$ in R is *regular* if $Y_m^{(a)}(u), \ 1 + Y_m^{(a)}(u) \in R^\times$ for any (a, m, u).

REMARK 2.15. Actually, any solution of $\mathbb{Y}(X_r)$ is regular because, for any $Y_m^{(a)}(u)$, there is a relation among $\mathbb{Y}(X_r)$ such that $(1 + Y_m^{(a)}(u)^{-1})^{-1}$ appears in the right-hand side. However, this is not always true for the restricted Y-system we shall discuss in Section 3. Therefore, it is convenient to introduce the above definition so that the unrestricted/restricted T-/Y-systems can be treated in a unified manner.

Clearly, there is a one-to-one correspondence between the regular solutions of $\mathbb{T}(X_r)$ (resp. $\mathbb{Y}(X_r)$) in R and the ring homomorphisms $f : \mathcal{T}(X_r) \to R$ (resp. $f : \mathcal{Y}(X_r) \to R$).

As a corollary of Theorem 2.12(2), we obtain the following.

COROLLARY 2.16. *For any ring R, the map*

$$(2.39) \qquad \varphi^* : \mathrm{Hom}\big(\mathcal{T}(X_r), R\big) \to \mathrm{Hom}\big(\mathcal{Y}(X_r), R\big),$$

induced from the homomorphism φ in (2.21), is surjective. Namely, for any regular solution Y of $\mathbb{Y}(X_r)$ in R, there exists some regular solution T of $\mathbb{T}(X_r)$ in R such that Y is expressed by T as

$$(2.40) \qquad Y_m^{(a)}(u) = \frac{M_m^{(a)}(u)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)}.$$

§3. Restricted T- and Y-systems and their periodicities

In this section we state the main claims of the paper. We first introduce the *restricted* T- and Y-systems together with the associated algebras. Then, the conjectures and the results concerning their periodicity property are presented.

3.1. Restricted T- and Y-systems

Let t_a $(a \in I)$ be the numbers in (2.1).

DEFINITION 3.1. Fix an integer $\ell \geq 2$. The *level ℓ restricted T-system* $\mathbb{T}_\ell(X_r)$ *of type X_r (with the unit boundary condition)* is the system of relations (2.2)–(2.6) naturally restricted to a family of variables $T = \{T_m^{(a)}(u) \mid a \in I; m = 1, \ldots, t_a\ell - 1; u \in U\}$, where $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$, and furthermore, $T_{t_a\ell}^{(a)}(u) = 1$ (the *unit boundary condition*) if they occur in the right-hand sides in the relations.

DEFINITION 3.2. The *level ℓ restricted T-algebra* $\mathcal{T}_\ell(X_r)$ *of type X_r* is the ring with generators $T_m^{(a)}(u)^{\pm 1}$ $(a \in I; m = 1, \ldots, t_a\ell - 1; u \in U)$ and the relations $\mathbb{T}_\ell(X_r)$. Also, we define the ring $\mathcal{T}_\ell^\circ(X_r)$ as the subring of $\mathcal{T}_\ell(X_r)$ generated by $T_m^{(a)}(u)$ $(a \in I; m = 1, \ldots, t_a\ell - 1; u \in U)$.

REMARK 3.3. The notion of the *level ℓ restriction* originates from a class of solvable lattice model, called the *level ℓ restricted solid-on-solid* (RSOS) model associated with the R-matrix of $U_q(\hat{\mathfrak{g}})$ at a $2t(h^\vee + \ell)$th root of unity [ABF], [JMO], [Pas], [BR]. The *level ℓ restricted T-system* was introduced in [KNS1], where, instead of the condition $T_{t_a\ell}^{(a)}(u) = 1$ above, a slightly weaker condition $T_{t_a\ell+1}^{(a)}(u) = 0$ was imposed. We hope that no serious confusion occurs by referring to $\mathbb{T}_\ell(X_r)$ also as *level ℓ restricted T-system* for simplicity. We impose the unit boundary condition here to ensure the periodicity property we are going to discuss. (Actually, this is not the only choice of the boundary condition showing the periodicity, but we do not discuss this point in the paper.)

PROPOSITION 3.4. *The ring $\mathfrak{T}^\circ_\ell(X_r)$ is isomorphic to a quotient of $\mathfrak{T}^\circ(X_r)$.*

Proof. First we note that the ring $\mathfrak{T}^\circ(X_r)$ is freely generated by $T_1^{(a)}(u)$ ($a \in I, u \in U$). This is true for $U = \mathbb{C}_{t\hbar}$ by Theorem 2.7, and so it is true for any choice of U, since nontrivial relations exist only among $T_{m_i}^{(a_i)}(u_i)$ with $u_i - u_j \in \mathbb{R}$. So we have a ring homomorphism

$$\pi_\ell : \mathfrak{T}^\circ(X_r) \to \mathfrak{T}^\circ_\ell(X_r) \tag{3.1}$$

uniquely determined by the condition $\pi_\ell(T_1^{(a)}(u)) = T_1^{(a)}(u)$ ($a \in I, u \in U$). We claim that $\pi_\ell(T_m^{(a)}(u)) = T_m^{(a)}(u)$ for any $m = 1, \dots, t_a\ell - 1$, from which the surjectivity of π_ℓ follows. The claim can be shown by the induction on the height of $T_m^{(a)}(u)$ in (2.13). Namely, suppose that the claim holds for any $T_k^{(b)}(v)$ such that $\operatorname{ht} T_k^{(b)}(v)$ is smaller than $\operatorname{ht} T_m^{(a)}(u)$. Let $S_{amu}(T)$ be the one in (2.9). Then, $S_{a,m-1,u}(T) = 0$ in $\mathfrak{T}^\circ(X_r)$ and $\mathfrak{T}^\circ_\ell(X_r)$; hence, $\pi_\ell(S_{a,m-1,u}(T)) = S_{a,m-1,u}(T)$ in $\mathfrak{T}^\circ_\ell(X_r)$. The claim follows from this and the induction hypothesis. □

Similarly, we define the following.

DEFINITION 3.5. Fix an integer $\ell \geq 2$. The *level ℓ restricted Y-system* $\mathbb{Y}_\ell(X_r)$ *of type X_r* is the system of relations (2.15)–(2.19) naturally restricted to a family of variables $Y = \{Y_m^{(a)}(u) \mid a \in I; m = 1, \dots, t_a\ell - 1; u \in U\}$, where $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = 0$, and furthermore, $Y_{t_a\ell}^{(a)}(u)^{-1} = 0$ if they occur in the right-hand sides in the relations.

DEFINITION 3.6. The *level ℓ restricted Y-algebra* $\mathcal{Y}_\ell(X_r)$ *of type X_r* is the ring with generators $Y_m^{(a)}(u)^{\pm 1}$, $(1 + Y_m^{(a)}(u))^{-1}$ ($a \in I; m = 1, \dots, t_a\ell - 1; u \in U$) and the relations $\mathbb{Y}_\ell(X_r)$.

REMARK 3.7. The system $\mathbb{Y}_\ell(X_r)$ was introduced by [Z] for simply laced X_r and $\ell = 2$ to characterize the solutions of the thermodynamic Bethe ansatz equations for the factorizable scattering theories. Then, it was extended to the general case by [KN] based on the thermodynamic treatment of [Ku] (see also [KNS1, Appendix B]). For simply laced X_r, it was also given by [RTV] independently.

For any ring R, one can define the *regular solutions of $\mathbb{T}_\ell(X_r)$ and $\mathbb{Y}_\ell(X_r)$ in R* in the same way as Definition 2.14. Again, they are identified with the elements in $\operatorname{Hom}(\mathfrak{T}_\ell(X_r), R)$ and $\operatorname{Hom}(\mathcal{Y}_\ell(X_r), R)$.

The restrictions of T-systems and Y-systems are *partly* compatible in view of Theorem 2.12. Namely, we have the following.

PROPOSITION 3.8. *The correspondence (2.22), with* $T_0^{(a)}(u) = T_{t_a\ell}^{(a)}(u) = 1$, *defines a ring homomorphism*

$$(3.2) \qquad\qquad \varphi_\ell : \mathcal{Y}_\ell(X_r) \to \mathcal{T}_\ell(X_r).$$

Proof. Due to Theorem 2.12, we have only to check the compatibility between the boundary conditions, $T_{t_a\ell}^{(a)}(u) = 1$ and $Y_{t_a\ell}^{(a)}(u)^{-1} = 0$. For simply laced X_r, this can be seen by formally setting $T_{\ell+1}^{(a)}(u) = 0$ at $m = \ell - 1$ in (2.26). The nonsimply laced case is similar. □

Unfortunately, the properties (2) and (3) in Theorem 2.12 do not necessarily hold for general X_r and ℓ.

EXAMPLE 3.9. (1) *The case* $X_r = A_2$ *and* $\ell = 2$. Two systems,

$$(3.3) \qquad \mathbb{T}_2(A_2) : T_1^{(1)}(u-1)T_1^{(1)}(u+1) = 1 + T_1^{(2)}(u),$$

$$T_1^{(2)}(u-1)T_1^{(2)}(u+1) = 1 + T_1^{(1)}(u),$$

$$(3.4) \qquad \mathbb{Y}_2(A_2) : Y_1^{(1)}(u-1)Y_1^{(1)}(u+1) = 1 + Y_1^{(2)}(u),$$

$$Y_1^{(2)}(u-1)Y_1^{(2)}(u+1) = 1 + Y_1^{(1)}(u),$$

are identical; moreover, we have $\varphi_2 : Y_1^{(1)}(u) \mapsto T_1^{(2)}(u)$, $Y_1^{(2)}(u) \mapsto T_1^{(1)}(u)$. Thus, φ_2 is bijective.

(2) *The case* $X_r = A_3$ *and* $\ell = 2$. We have $\varphi_2 : Y_1^{(1)}(u) \mapsto T_1^{(2)}(u)$, $Y_1^{(3)}(u) \mapsto T_1^{(2)}(u)$. Thus, φ_2 is *not* injective.

(3) *The case* $X_r = C_2$ *and* $\ell = 2$. We have $\varphi_2 : Y_1^{(1)}(u) \mapsto T_1^{(2)}(u)/T_2^{(1)}(u)$, $Y_3^{(1)}(u) \mapsto T_1^{(2)}(u)/T_2^{(1)}(u)$. Thus, φ_2 is *not* injective.

However, at least for A_r, one can resolve this incompatibility by modifying the boundary condition of $\mathbb{T}_\ell(X_r)$ while keeping the periodicity (Proposition 5.9).

There are some isomorphisms among the restricted T-algebras or Y-algebras.

EXAMPLE 3.10. (Level-rank duality) The rings $\mathcal{T}_\ell(A_{r-1})$ and $\mathcal{T}_r(A_{\ell-1})$ are isomorphic under the correspondence $T_m^{(a)}(u) \leftrightarrow T_a^{(m)}(u)$. The rings $\mathcal{Y}_\ell(A_{r-1})$ and $\mathcal{Y}_r(A_{\ell-1})$ are isomorphic under the correspondence $Y_m^{(a)}(u) \leftrightarrow Y_a^{(m)}(u)^{-1}$.

3.2. T- and Y-systems with discrete spectral parameter

So far, we assume that the spectral parameter u takes values in $U = \mathbb{C}$ or $\mathbb{C}_\xi$. In the original context of T- and Y-systems, the analyticities of $T_m^{(a)}(u)$ and $Y_m^{(a)}(u)$ with respect to u are of fundamental importance [Z], [KP], [KN], [RTV], [KNS2].

However, from the algebraic point of view, it is possible to discretize the parameter u by choosing $U = (1/t)\mathbb{Z}$, where t is the number in (2.1). There are at least two reasons why we are interested in such a discretization.

First, by regarding u as "discretized time," the T- and Y-systems have their own interests as *discrete dynamical systems*. For example, $\mathbb{T}(A_r)$ is a discrete analogue of the Toda field equation and a particular case of the Hirota's bilinear difference equation [Hi1], [Hi2], [KOS], [KLWZ] (see [KLWZ] for more information).

Second, the periodicities of the restricted T- and Y-systems, which are the subjects of this paper, concern only the algebraic aspect of the T- and Y-systems; therefore, it is adequate to discuss the periodicities in discretized systems.

From here to the end of Section 8, we assume $U = (1/t)\mathbb{Z}$ for all the T-systems and Y-systems.

3.3. Periodicity conjecture for restricted T- and Y-systems

For X_r, let $h^\vee$ be the dual Coxeter number of X_r as listed below:

(3.5)

X_r	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$h^\vee$	$r+1$	$2r-1$	$r+1$	$2r-2$	12	18	30	9	4

For simply laced X_r, $h^\vee$ equals the Coxeter number h of X_r.

Let ω be the involution on the set I such that $\omega(a) = a$ *except for the following cases (in our enumeration)*:

(3.6)
$$\omega(a) = r + 1 - a \ (a \in I) \qquad\qquad X_r = A_r,$$
$$\omega(r-1) = r, \ \omega(r) = r - 1 \qquad\qquad X_r = D_r \ (r\text{: odd}),$$
$$\omega(1) = 6, \ \omega(2) = 5, \ \omega(5) = 2, \ \omega(6) = 1 \quad X_r = E_6.$$

(Caution: for $X_r = D_r$ (r: even), $\omega(a) = a$ ($a \in I$).) The involution ω is related to the *longest element* ω_0 in the Weyl group of type X_r by $\omega_0(\alpha_a) = -\alpha_{\omega(a)}$ [B] (cf. [FZ3, Proposition 2.5]).

Now let us give the main claim of the paper.

CONJECTURE 3.11. *The following relations hold in $\mathcal{T}_\ell(X_r)$:*

(1) *Half-periodicity:* $T_m^{(a)}(u + h^\vee + \ell) = T_{t_a\ell-m}^{(\omega(a))}(u)$.
(2) *Periodicity:* $T_m^{(a)}(u + 2(h^\vee + \ell)) = T_m^{(a)}(u)$.

We may sometimes refer to (2) also as *full-periodicity* in contrast to (1). Of course, the full-periodicity follows from the half-periodicity.

This is the counterpart of the (already conjectured and partially proved) periodicity property for the restricted Y-systems in various contexts; here we present it in a form parallel to Conjecture 3.11.

CONJECTURE 3.12. *The following relations hold in $\mathcal{Y}_\ell(X_r)$.*

(1) *Half-periodicity:* $Y_m^{(a)}(u + h^\vee + \ell) = Y_{t_a\ell-m}^{(\omega(a))}(u)$.
(2) *Periodicity:* $Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u)$.

REMARK 3.13. One can rephrase these periodicity properties as those of the regular solutions of the corresponding T- and Y-systems in an arbitrary ring R. For example, suppose that Conjecture 3.11(1) is true. Then, for any regular solution T of $\mathbb{T}_\ell(X_r)$ in R, the equality $T_m^{(a)}(u + h^\vee + \ell) = T_{t_a\ell-m}^{(\omega(a))}(u)$ holds in R. The converse is also true by setting $R = \mathcal{T}_\ell(X_r)$. This remark will be applicable to any periodicity statement in the rest of the paper as well.

Let us summarize the known and/or related results on Conjectures 3.11 and 3.12 so far.

(i) Conjecture 3.12 was initially given by [Z] for simply laced X_r and $\ell = 2$ and then generalized by [RTV] for simply laced X_r and $\ell \geq 2$ (including half-periodicity) and by [KNS1, Appendix B] for general X_r and $\ell \geq 2$ (full-periodicity). The points established so far are as follows:

(a) It was proved for $X_r = A_r$ and $\ell = 2$ by Gliozzi and Tateo [GT] via three-dimensional geometry. The same case was also proved by Frenkel and Szenes [FS] with the explicit solution given.
(b) It was proved for simply laced X_r and $\ell = 2$ by Fomin and Zelevinsky [FZ3] via the cluster algebra method.
(c) It was proved for $X_r = A_r$ and $\ell \geq 2$ by Volkov [V] via the determinant method.
(d) The full-periodicity was proved for simply laced X_r and $\ell > 2$ by Keller [Kel2] via the cluster algebra/category method.

We emphasize that the nonsimply laced Y-systems treated in [FZ3], [Kel2] are *different* from ours, and their nonsimply laced Y-systems are identified with certain reductions of our Y-systems associated with the *twisted* quantum affine algebras (see Remark 9.22). In particular, there has been no systematic result on Conjecture 3.12 for the nonsimply laced case so far. The same remark applies to the T-systems as well.

(ii) Conjecture 3.11 appeared in [CGT] for simply laced X_r, while the one for nonsimply laced X_r seems new in the literature. The following related results are already known:

(a) For simply laced X_r, we will see that the ring $\mathcal{T}_\ell^\circ(X_r)$ is isomorphic to (a subring of) a certain cluster algebra. The periodicity property of the corresponding cluster algebra is known for $\ell = 2$ (including half-periodicity) by Fomin and Zelevinsky [FZ2], [FZ3] and for $\ell > 2$ (full-periodicity only) by [Kel2]. A more precise account will be given in Section 4.

(b) For $X_r = A_r$ and $\ell \geq 2$, Conjecture 3.11 follows from a more general theorem by Henriques [Hen] proved by the graph theoretical method. The same case was also proved essentially by [V] while proving Conjecture 3.12. We will give a detailed account of the latter method in Section 5.

(iii) Though Conjectures 3.11 and 3.12 are tightly connected to each other in view of the map φ_ℓ in Proposition 3.8, one is not the consequence of the other, in general. However, at least for simply laced X_r, they are unified as the periodicity property of the *F-polynomials* of the corresponding cluster algebra *with coefficients* [FZ4].

REMARK 3.14. Recall that the choice $U = \mathbb{C}_\xi$ for the domain of the parameter u of the unrestricted T-algebra $\mathcal{T}(X_r)$ imposes the period $2\pi\sqrt{-1}/\xi$ in (2.7), where ξ is taken from $\mathbb{C} \setminus 2\pi\sqrt{-1}\mathbb{Q}$ to avoid the incompatibility with the relations $\mathbb{T}(X_r)$. The level ℓ restricted T-algebra $\mathcal{T}_\ell(X_r)$ has an additional period $2(h^\vee + \ell)$. This means the choice $\xi = \pi\sqrt{-1}/(h^\vee + \ell)$ is *compatible* with the relations $\mathbb{T}_\ell(X_r)$. In the context of the q-character, we made the identification $\xi = t\hbar$, where $q = e^\hbar$. Then, the above choice corresponds to $q = \exp(\pi\sqrt{-1}/t(h^\vee + \ell))$; namely, *q is a primitive $2t(h^\vee + \ell)$th root of unity.* This is natural in view of the origin of the level ℓ restriction in Remark 3.3. We make a further remark on the implication of the periodicity of $\mathcal{T}_\ell(X_r)$ for the q-character in Section 10.

3.4. Summary of methods and results

In the following, we will study and partially prove Conjecture 3.11 by three independent methods. This is a good point to outline the methods and the results.

1. *Cluster algebra/category method applied to $\mathcal{T}_\ell(X_r)$ with simply laced X_r* (Section 4)

This is actually more than a method to prove Conjecture 3.11, since it includes the identification of the ring $\mathcal{T}_\ell^\circ(X_r)$ as a (subring of) cluster algebra.

In the simplest case $\ell = 2$, the ring $\mathcal{T}_2^\circ(X_r)$ is isomorphic to the tensor square of the cluster algebra $\mathcal{A}_Q$ of type X_r (Proposition 4.3). The ring $\mathcal{A}_Q$ is a cluster algebra of *finite type* and is particularly well studied. In particular, the periodicity property of $\mathcal{A}_Q$ is proved in [FZ2], [FZ3] by making use of the piecewise-linear modification of the simple reflections acting on the set of the almost positive roots $\Phi_{\geq -1}$ of type X_r. The periodicity of $\mathcal{T}_2(X_r)$ is its immediate corollary (Corollary 4.5).

For the case $\ell > 2$, the ring $\mathcal{T}_\ell^\circ(X_r)$ is isomorphic to the tensor square of a subring of the cluster algebra $\mathcal{A}_{Q \square Q'}$, where $Q \square Q'$ is the *square product* of quivers recently introduced by [Kel2], [HL] (Proposition 4.24). The cluster algebra $\mathcal{A}_{Q \square Q'}$ is *not* of finite type; nevertheless, it still admits the periodicity along the *bipartite belt* of [FZ4]. The periodicity of $\mathcal{A}_{Q \square Q'}$ is studied in [Kel2], in a more general situation *with coefficients*, using the categorification by the 2-Calabi-Yau category associated with the tensor product of the path algebras of quivers Q and Q'. The full-periodicity of $\mathcal{T}_\ell(X_r)$ is its immediate corollary. Furthermore, this cluster categorical approach can be adapted for the half-periodicity. Thus, we obtain the desired periodicity for $\mathcal{T}_\ell(X_r)$ (Corollary 4.29).

2. *Determinant method applied to $\mathcal{T}_\ell(A_r)$ and $\mathcal{T}_\ell(C_r)$* (Sections 5 and 6)

The method seeks a manifestly periodic expression of $T_m^{(a)}(u)$ as a minor of a matrix M over $\mathcal{T}_\ell(X_r)$ of infinitely many finite columns *with periodicity*. It was introduced by [V] to prove the periodicity of the regular solutions of $\mathbb{Y}_\ell(A_r)$ in $\mathbb{C}$.

Such a determinant expression (without periodicity) is known for the unrestricted T-system $\mathbb{T}(A_r)$ by [KLWZ, (2.25)], where the relation (2.2) of $\mathbb{T}(A_r)$ is regarded as the Hirota's bilinear difference equation. Then, the existence of such a determinant expression is viewed as a discrete analogue of the well-known relation between the Hirota's bilinear equation and the Grassmannians [S]. Remarkably, the restriction of the T-system to $\mathbb{T}_\ell(A_r)$

is compatible with this determinant expression by imposing the *periodicity* on the matrix M (Proposition 5.8). This forces the desired periodicity for $\mathcal{T}_\ell(A_r)$ (Theorem 5.3).

Since the method takes advantage of the bilinearity of the relation of $\mathbb{T}(A_r)$, it does not seem applicable to X_r other than A_r. A pleasant surprise is that it is still applicable for C_r through the relation between $\mathcal{T}(C_r)$ and a certain variant of $\mathcal{T}(A_{2r+1})$ [KOSY]. (Note that this is different from the usual "folding" relation between C_r and A_{2r-1}.) This relation is compatible with the restriction and induces the relation between $\mathcal{T}_\ell(C_r)$ and $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$, where $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$ is a variant of $\mathcal{T}_{2\ell}(A_{2r+1})$ (Proposition 6.8). Since $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$ admits the determinant expression, the desired periodicity for $\mathcal{T}_\ell(C_r)$ is obtained (Corollary 6.4). This is the first main result concerning Conjecture 3.11 for the nonsimply laced case.

At this moment the method is applicable only for these two cases, since a similar relation between $\mathcal{T}(X_r)$ and a certain variant of $\mathcal{T}(A_{r'})$ is not known for the other types X_r.

3. *Direct method applied to* $\mathcal{T}_2(A_r)$, $\mathcal{T}_2(D_r)$, *and* $\mathcal{T}_2(B_r)$ (Section 7)

The method seeks a manifestly periodic Laurent polynomial expression of $T_m^{(a)}(u)$ in terms of the "initial variables" by considering the T-system as a discrete dynamical system. At least for the above three cases, we can directly find such an expression with the aid of a computer and verify that it indeed satisfies the T-system.

The problem to express the cluster variables in terms of the initial cluster is a much-studied subject (e.g., [CC], [FZ4], [YZ]). The first two cases, $\mathcal{T}_2(A_r)$ and $\mathcal{T}_2(D_r)$, should be obtained as the specialization of those more general expressions. Our goal here is to prove the periodicity for $\mathcal{T}_2(B_r)$, which is the first nontrivial result for B_r. (Let us repeat that this is different from the tensor square of the cluster algebra of type B_r.)

§4. Cluster algebra/category method: $\mathcal{T}_\ell(X_r)$ with simply laced X_r

In this section, we study the periodicity of $\mathcal{T}_\ell(X_r)$ for simply laced X_r. We establish the relation between the ring $\mathcal{T}_\ell^\circ(X_r)$ and cluster algebras [FZ1], [FZ2]. Then, the periodicity of $\mathcal{T}_\ell(X_r)$ reduces to that of the corresponding cluster algebra. For $\ell = 2$, the periodicity of the corresponding cluster algebra is known by [FZ2], [FZ3]. For $\ell > 2$, the full-periodicity of the corresponding cluster algebra is recently shown by [Kel2] using the cluster categorical method. We prove the half-periodicity for $\ell > 2$ as well by adapting

this categorical method (see [Kel2] for a comprehensive review of cluster algebras and cluster categories).

4.1. Cluster algebra

For a finite quiver Q without loops or 2-cycles with vertex set, say, $I = \{1, \ldots, n\}$ and an I-tuple of variables $x = \{x_1, \ldots, x_n\}$, we define a *cluster algebra (with trivial coefficients)* $\mathcal{A}_Q$ [FZ1], [FZ2], which is a $\mathbb{Z}$-subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$, as follows.

(1) We start from the pair ("initial seed") (Q, x), where Q and x are as above.

(2) For each $k = 1, \ldots, n$, we define another pair ("seed") $(R, y) = \mu_k(Q, x)$ of a quiver R without loops or 2-cycles with vertex set I and an I-tuple $y = \{y_1, \ldots, y_n\}$, $y_i \in \mathbb{Q}(x_1, \ldots, x_n)$, called the *mutation of (Q, x) at k*, where y is given by the *exchange relation*,

$$(4.1) \qquad y_i = \begin{cases} x_i & i \neq k, \\ \frac{1}{x_k}\left(\prod_{\text{arrows } j \to k \text{ of } Q} x_j + \prod_{\text{arrows } k \to j \text{ of } Q} x_j\right) & i = k, \end{cases}$$

while $R = \mu_k(Q)$ is obtained from Q by the following *mutation rule*.

(i) For each $i \to k \to j$ of Q, create a new arrow $i \to j$.
(ii) Replace each $i \to k$ of Q with $k \to i$, and $k \to j$ of Q with $j \to k$, respectively.
(iii) Remove a maximal disjoint collection of 2-cycles of the resulting quiver after (i) and (ii).

(3) Iterate the mutation for every new seed at every k and collect all the (possibly infinite number of) seeds. For any seed (R, y), y is called a *cluster*, and each element y_i of y is called a *cluster variable*.

(4) The cluster algebra $\mathcal{A}_Q$ is the $\mathbb{Z}$-subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.

Due to the *Laurent phenomenon* [FZ1], $\mathcal{A}_Q$ is a subring of $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

4.2. Level 2 case

Here we study the periodicity of $\mathcal{T}_2(X_r)$ for simply laced X_r. Since the case $X_r = A_1$ is trivial, we assume $X_r \neq A_1$.

Let $X_r \,(\neq A_1)$ be a simply laced Dynkin diagram, and let $I = I_+ \sqcup I_-$ be a bipartite decomposition of the vertex set I of X_r; namely, $C_{ab} = 0$ for any $a, b \in I_\pm$ with $a \neq b$. We set $\varepsilon(a) = \pm$ for $a \in I_\pm$.

Recall that the ring $\mathcal{T}_2(X_r)$ has the generators $T = \{T_1^{(a)}(u)^{\pm 1} \mid a \in I, u \in \mathbb{Z}\}$ and the relations $\mathbb{T}_2(X_r)$:

$$(4.2) \qquad T_1^{(a)}(u-1)T_1^{(a)}(u+1) = 1 + \prod_{b \in I: C_{ab}=-1} T_1^{(b)}(u).$$

Let $\mathcal{T}_2^\circ(X_r)_\pm$ be the subring of $\mathcal{T}_2^\circ(X_r)$ generated by $T_1^{(a)}(u)$ $(a \in I, u \in \mathbb{Z})$ such that $\varepsilon(a)(-1)^u = \pm$, where we identify $+$ and $-$ with 1 and -1, respectively. Since the relation (4.2) closes among those $T_1^{(a)}(u)$ with fixed parity $\varepsilon(a)(-1)^u$, we have

$$(4.3) \qquad \mathcal{T}_2^\circ(X_r) \simeq \mathcal{T}_2^\circ(X_r)_+ \otimes_\mathbb{Z} \mathcal{T}_2^\circ(X_r)_-, \qquad \mathcal{T}_2^\circ(X_r)_+ \simeq \mathcal{T}_2^\circ(X_r)_-.$$

Let $Q = Q(X_r)$ be the *alternating quiver* such that X_r is the underlying graph, $a \in I_+$ is a source, and $a \in I_-$ is a sink of Q. We introduce an I-tuple of variables $x = \{x_a\}_{a \in I}$, and we define $\mathcal{A}_Q$ to be the cluster algebra with initial seed (Q, x).

Following [FZ2], [FZ3], we introduce composed mutations $\mu_\pm = \prod_{a \in I_\pm} \mu_a$ and $\mu = \mu_- \mu_+$ for $\mathcal{A}_Q$. We set $x = x(0)$, and we define clusters $x(u) = \{x_a(u)\}_{a \in I}$ $(u \in \mathbb{Z})$ of $\mathcal{A}_Q$ by the following sequence of the mutations:

$$(4.4) \qquad \cdots \xleftrightarrow{\mu_+} \left(Q^{\mathrm{op}}, x(-1)\right) \xleftrightarrow{\mu_-} \left(Q, x(0)\right) \xleftrightarrow{\mu_+} \left(Q^{\mathrm{op}}, x(1)\right)$$
$$\xleftrightarrow{\mu_-} \left(Q, x(2)\right) \xleftrightarrow{\mu_+} \cdots,$$

where Q^{op} is the *opposite quiver* of Q, that is, the quiver obtained from Q by reversing all the arrows. In particular,

$$(4.5) \qquad x_a(u+1) = x_a(u) \qquad \text{if } \varepsilon(a)(-1)^u = -,$$

$$(4.6) \qquad \begin{aligned} \left(Q, x(2k)\right) &= \mu^k\left(Q, x(0)\right) & (k \in \mathbb{Z}), \\ \left(Q^{\mathrm{op}}, x(2k+1)\right) &= \mu_+ \mu^k\left(Q, x(0)\right) & (k \in \mathbb{Z}). \end{aligned}$$

Furthermore, any cluster variable of $\mathcal{A}_Q$ occurs in $x(u)$ for some $u \in \mathbb{Z}$, due to [FZ2, Theorems 1.9 and 3.1]. (This is not true for a general finite quiver Q.)

LEMMA 4.1 ([FZ4, (8.12)]). *The family* $\{x_a(u) \mid a \in I, u \in \mathbb{Z}\}$ *satisfies the T-system* $\mathbb{T}_2(X_r)$ *in* $\mathcal{A}_Q$; *namely,*

$$(4.7) \qquad x_a(u-1)x_a(u+1) = 1 + \prod_{b \in I: C_{ab}=-1} x_b(u).$$

Proof. For example, suppose that $a \in I_+$ and that u is odd. Then,

$$(4.8) \qquad x_a(u-1) = \mu_+\big(x_a(u)\big) = \frac{1}{x_a(u)}\Big(1 + \prod_{b \in I : C_{ab} = -1} x_b(u)\Big)$$

by (4.1), and $x_a(u) = x_a(u+1)$ by (4.5). The other cases are similar. $\qquad\square$

Now let us describe the relation between the rings, $\mathfrak{T}_2(X_r)$ and $\mathcal{A}_Q$. Define a ring homomorphism $f : \mathcal{A}_Q \to \mathfrak{T}_2(X_r)$ as the restriction of the ring homomorphism $\mathbb{Z}[x_a^{\pm 1}]_{a \in I} \to \mathfrak{T}_2(X_r)$ given by

$$(4.9) \qquad f : x_a^{\pm 1} \mapsto \begin{cases} T_1^{(a)}(0)^{\pm 1} & a \in I_+, \\ T_1^{(a)}(1)^{\pm 1} & a \in I_-. \end{cases}$$

Then, we have the following (see Figure 2).

LEMMA 4.2. *For the above homomorphism* $f : \mathcal{A}_Q \to \mathfrak{T}_2(X_r)$,

$$(4.10) \qquad f : x_a(u) \mapsto \begin{cases} T_1^{(a)}(u) & \varepsilon(a)(-1)^u = +, \\ T_1^{(a)}(u+1) & \varepsilon(a)(-1)^u = -. \end{cases}$$

Proof. For $u = 0$, (4.10) holds by the definition of f. Then, one can prove (4.10) by the induction on $\pm u$ with (4.2), (4.5), and (4.7). $\qquad\square$

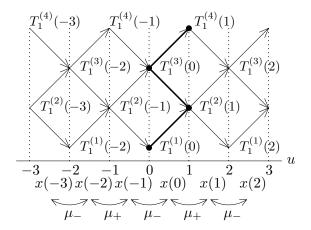

Figure 2: Relation of $x(u)$ and $T_1^{(a)}(u)$ for $X_r = A_4$. The thick quiver corresponds to the initial seed $(Q, x(0))$ of the cluster algebra $\mathcal{A}_Q$, where we take $I_+ = \{1, 3\}$, $I_- = \{2, 4\}$.

PROPOSITION 4.3. *The ring $\mathcal{T}_2^\circ(X_r)$ is isomorphic to $\mathcal{A}_Q \otimes_{\mathbb{Z}} \mathcal{A}_Q$.*

Proof. It follows from Lemma 4.2 that the image $f(\mathcal{A}_Q)$ is $\mathcal{T}_2^\circ(X_r)_+$. Furthermore, the inverse correspondence $g : \mathcal{T}_2^\circ(X_r)_+ \to \mathcal{A}_Q$, $T_1^{(a)}(u) \mapsto x_a(u)$ defines a homomorphism by Lemma 4.1 and the fact that $\mathcal{A}_Q$ is an integral domain. Therefore,

$$\text{(4.11)} \qquad \mathcal{T}_2^\circ(X_r)_+ \simeq \mathcal{A}_Q,$$

and we obtain the assertion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Thanks to the isomorphism, the periodicity of $\mathcal{T}_2(X_r)$ is reduced to the known periodicity of $\mathcal{A}_Q$. By the correspondence (4.10), it is easy to check that the periodicity of $\mathcal{T}_2(X_r)$ is translated as

$$\text{(4.12)} \qquad \begin{aligned} &\text{half-periodicity:} \quad x_a(u + h + 2) = x_{\omega(a)}(u), \\ &\text{periodicity:} \qquad x_a\big(u + 2(h+2)\big) = x_a(u). \end{aligned}$$

Recall that the Coxeter number $h = h^\vee$ of X_r is odd if and only if $X_r = A_r$ (r: even); furthermore, the involution $\omega : I \to I$ induces a quiver isomorphism $\omega : Q \to Q^{\mathrm{op}}$ if $X_r = A_r$ (r: even) and $\omega : Q \to Q$ otherwise. For a pair of seeds (R, y) and (R', y'), we write $(R, y) \overset{\nu}{=} (R', y')$ if $\nu : I \to I$ is a bijection that induces a quiver isomorphism $R \to R'$ and $y'_{\nu(a)} = y_a$ for any $a \in R$. The following periodicity of $\mathcal{A}_Q$ is due to [FZ2, Theorems 1.9 and 3.1] and [FZ3, Propositions 2.5 and 2.6].

THEOREM 4.4 (Fomin and Zelevinsky [FZ2], [FZ3]). *The following equalities hold for $\mathcal{A}_Q$ (u: even).*

(1) *Half-periodicity:*

(i) *for X_r other than A_r (r: even), where h is even,*

$$\text{(4.13)} \qquad \mu^{(h+2)/2}\big(Q, x(u)\big) \overset{\omega}{=} \big(Q, x(u)\big);$$

(ii) *for $X_r = A_r$ (r: even), where h is odd,*

$$\text{(4.14)} \qquad \mu_+ \mu^{(h+1)/2}\big(Q, x(u)\big) \overset{\omega}{=} \big(Q, x(u)\big).$$

(2) *Periodicity: For any X_r,*

$$\text{(4.15)} \qquad \mu^{h+2}\big(Q, x(u)\big) \overset{\mathrm{id}}{=} \big(Q, x(u)\big).$$

COROLLARY 4.5. *The following relations hold in $\mathfrak{T}_2(X_r)$ for any simply laced X_r.*

(1) *Half-periodicity:* $T_1^{(a)}(u + h + 2) = T_1^{(\omega(a))}(u)$.
(2) *Periodicity:* $T_1^{(a)}(u + 2(h + 2)) = T_1^{(a)}(u)$.

Proof. The relations in (4.12) immediately follow from Theorem 4.4 and (4.6). □

4.3. Alternative proof of Theorem 4.4 by cluster category

Here we present an alternative proof of Theorem 4.4 based on the categorification of $\mathcal{A}_Q$ by the cluster category $\mathcal{C}_Q$, in the spirit of [Kel2]. The definitions and results here will be also used to prove the periodicity for the levels greater than two in Section 4.5.

Let Q be the alternating quiver whose underlying graph is simply laced X_r other than A_1 as in Section 4.2. Let K be an algebraically closed field, and let KQ be the path algebra of Q [ARS], [ASS]. We denote by $\mathcal{D}_Q = \mathcal{D}^{\mathrm{b}}(\operatorname{mod} KQ)$ the bounded derived category of finite dimensional KQ-modules. Then $\mathcal{D}_Q$ forms a K-linear triangulated category with the suspension functor [1]. We denote by D the K-dual. The autoequivalence

$$(4.16) \qquad \tau := D(KQ)[-1] \overset{\mathbf{L}}{\otimes}_{KQ} - : \mathcal{D}_Q \to \mathcal{D}_Q$$

is called the *Auslander-Reiten translation* and plays an important role in representation theory of KQ [ARS], [ASS], [Ha].

Now we define another autoequivalence of $\mathcal{D}_Q$ by $F := \tau^{-1} \circ [1]$. Then the *cluster category of Q* [BMRRT] is defined as the orbit category

$$(4.17) \qquad \mathcal{C}_Q := \mathcal{D}_Q/F,$$

which means that $\mathcal{C}_Q$ has the same objects with $\mathcal{D}_Q$, and the morphism space is given by

$$(4.18) \qquad \operatorname{Hom}_{\mathcal{C}_Q}(X, Y) := \bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}_{\mathcal{D}_Q}\big(X, F^i(Y)\big)$$

for any $X, Y \in \mathcal{C}_Q$. Then $\mathcal{C}_Q$ forms a triangulated category with the suspension functor [1], and the natural functor $\mathcal{D}_Q \to \mathcal{C}_Q$ is a triangle functor [Kel1].

For $a \in I$, we denote by e_a the path of length zero in Q. Define KQ-modules by

$$(4.19) \qquad P_a := (KQ)e_a, \qquad P := KQ = \bigoplus_{a \in I} P_a.$$

The following description of indecomposable objects in $\mathcal{C}_Q$ follows from Gabriel's theorem [ASS] and from Fomin and Zelevinsky's [FZ2] description of finite type cluster algebras.

THEOREM 4.6. *There exists a bijection*

(4.20)
$$X : \{\textit{indecomposable objects in } \mathcal{C}_Q\}/\simeq \rightarrow \{\textit{cluster variables in } \mathcal{A}_Q\}$$

satisfying $X_{P_a} = x_a(0)$ *for any* $a \in I$.

We say that an object $T = \bigoplus_{a \in I} T_a \in \mathcal{C}_Q$ is *cluster tilting* if
(1) each T_a is indecomposable and mutually nonisomorphic, and
(2) $\mathrm{Hom}_{\mathcal{C}_Q}(T_a, T_b[1]) = 0$ holds for any $a, b \in I$.
For a cluster tilting object $T \in \mathcal{C}_Q$, we denote by Q_T the quiver of the endomorphism ring $\mathrm{End}_{\mathcal{C}_Q}(T)$ [ARS], [ASS].
We give two important examples of cluster tilting objects.

EXAMPLE 4.7. (1) The KQ-module

(4.21)
$$P = \bigoplus_{a \in I} P_a$$

gives a cluster tilting object. We have $Q_P = Q$, since $\mathrm{End}_{\mathcal{C}_Q}(P) = KQ$.
(2) We define KQ-modules by

(4.22)
$$U_a := \begin{cases} \tau^{-1}(P_a) & a \in I_+, \\ P_a & a \in I_-, \end{cases} \quad U := \bigoplus_{a \in I} U_a.$$

Then U is a tilting KQ-module, so it gives a cluster tilting object in $\mathcal{C}_Q$. We have $Q_U = Q^{\mathrm{op}}$, since $\mathrm{End}_{\mathcal{D}_Q}(U) \simeq KQ^{\mathrm{op}}$.

The mutation of cluster tilting objects is introduced in [BMRRT, Theorem 5.1].

THEOREM 4.8. *Let* $T = \bigoplus_{a \in I} T_a \in \mathcal{C}_Q$ *be a cluster tilting object. For any* $a \in I$, *there exists a unique indecomposable object* $T_a^* \in \mathcal{C}_Q$ *which is not isomorphic to* T_a *such that* $(T/T_a) \oplus T_a^*$ *is a cluster tilting object.*

We call the above $(T/T_a) \oplus T_a^*$ the *cluster tilting mutation of T at a*, and we denote it by $\mu_a(T)$. We have the following key observation [BMR, Theorem 6.1].

THEOREM 4.9. (1) *We have a bijection*

(4.23) $\widetilde{X}: \{$ *cluster tilting objects in* $\mathcal{C}_Q\}/\simeq \to \{$ *seeds in* $\mathcal{A}_Q\}$

defined by

(4.24) $$T = \bigoplus_{a \in I} T_a \mapsto (Q_T, \{X_{T_a}\}_{a \in I}).$$

(2) *We have* $\widetilde{X} \circ \mu_a = \mu_a \circ \widetilde{X}$ *for any* $a \in I$.

We have

(4.25) $$\widetilde{X}_P = \big(Q, x(0)\big).$$

We number elements of I_+ and I_- as $\{a_1, \ldots, a_s\}$ and $\{a_{s+1}, \ldots, a_r\}$, respectively. Define composed cluster tilting mutations by

(4.26) $\mu_+ := \mu_{a_s} \cdots \mu_{a_1}, \qquad \mu_- := \mu_{a_r} \cdots \mu_{a_{s+1}}, \qquad \mu := \mu_- \mu_+.$

For a cluster tilting object $T = \bigoplus_{a \in I} T_a$, let $[T]_a$ denote T_a; similarly, for a seed (R, y) of $\mathcal{A}_Q$, let $[(R, y)]_a$ denote y_a. By Theorem 4.9(2), we have the following relationship between the seed mutation and the cluster tilting mutation:

(4.27) $[\mu^k(\widetilde{X}_T)]_a = X_{[\mu^k(T)]_a}, \qquad [\mu_+ \mu^k(\widetilde{X}_T)]_a = X_{[\mu_+ \mu^k(T)]_a},$

for any cluster tilting object $T \in \mathcal{C}_Q$, $k \in \mathbb{Z}$, and $a \in I$. The following observation is a key result.

PROPOSITION 4.10. *For any* $k \in \mathbb{Z}$ *and* $a \in I$, *the following assertions hold.*
  (1) $[\mu^k(P)]_a \simeq \tau^{-k}(P_a)$ *in* $\mathcal{C}_Q$.
  (2) $[\mu_+ \mu^k(P)]_a \simeq \tau^{-k}(U_a)$ *in* $\mathcal{C}_Q$.

Proof. Since cluster tilting mutation commutes with any autoequivalence of $\mathcal{C}_Q$, we have only to show the assertion for $k = 1$ for (1) and $k = 0$ for (2).
  (2) We put $I^i := \{a_1, \ldots, a_i\}$ and

$$T^i := \Big(\bigoplus_{a \in I^i} \tau^{-1}(P_a)\Big) \oplus \Big(\bigoplus_{a \in I \setminus I^i} P_a\Big).$$

Then T^i is a tilting KQ-module by [ASS]. Thus it is a cluster tilting object by [BMRRT]. Since T^{i-1} and T^i have the same indecomposable direct summands except P_{a_i}, we have $\mu_{a_i}(T^{i-1}) = T^i$. In particular, we have $\mu_+(P) = U$.

(1) By a similar argument to (2), we have $\mu(P) = \mu_-(U) = \tau^{-1}(P)$. $\square$

We use the following classical periodicity result.

PROPOSITION 4.11. (1) *We have that $\tau^{-h}(X)$ is isomorphic to $X[2]$ for any $X \in \mathcal{D}_Q$.*
(2) *We have that $\tau^{-h-2}(X)$ is isomorphic to X for any $X \in \mathcal{C}_Q$.*

Proof. (1) This follows from the structure of the Auslander-Reiten quiver of $\mathcal{D}_Q$ [G], [Ha] (see Figure 3).
(2) By (1), we have $\tau^{-h-2}(X) \simeq \tau^{-2}(X[2]) \simeq F^2(X) \simeq X$. $\square$
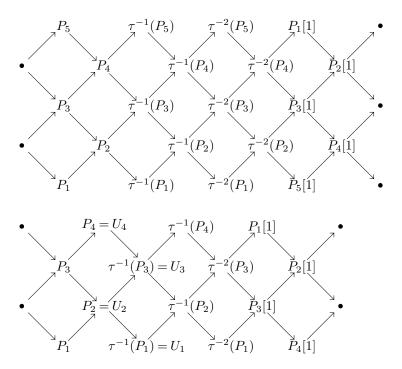
Now we are ready to prove Theorem 4.4(2) (full-periodicity).

Figure 3: The Auslander-Reiten quivers of $\mathcal{D}_Q$ for $X_r = A_5$ (the above) and A_4 (the below)

THEOREM 4.12 (Theorem 4.4(2)). $\left[\mu^{h+2}(Q,x(0))\right]_a = x_a(0)$.

Proof. We have $[\mu^{h+2}(P)]_a \overset{\text{Prop.4.10(1)}}{=} \tau^{-h-2}(P_a) \overset{\text{Prop.4.11(2)}}{=} P_a$ for any $a \in I$. Applying $\widetilde{X}$, we have

$$\left[\mu^{h+2}\big(Q,x(0)\big)\right]_a \overset{(4.25)}{=} [\mu^{h+2}(\widetilde{X}_P)]_a \overset{(4.27)}{=} X_{[\mu^{h+2}(P)]_a} = X_{P_a} = x_a(0).$$

$\square$

Next we prove Theorem 4.4(1) (half-periodicity). We divide the proof into two cases.

Case 1: h is even.

In this case, the map $\omega : I \to I$ induces the quiver automorphism $\omega : Q \to Q$. Thus ω induces an automorphism $\omega : KQ \to KQ$ of our K-algebra KQ, and we have an autoequivalence

$$(4.28) \qquad\qquad \omega : \mathcal{D}_Q \to \mathcal{D}_Q$$

of categories. We have $\omega(P_a) = P_{\omega(a)}$ for any $a \in I$.

We have the following periodicity.

PROPOSITION 4.13. (1) $\tau^{-h/2}(X) \simeq \omega(X[1])$ *for any* $X \in \mathcal{D}_Q$;
(2) $\tau^{-(h+2)/2}(X) \simeq \omega(X)$ *for any* $X \in \mathcal{C}_Q$.

Proof. (1) This follows from the structure of the Auslander-Reiten quiver of $\mathcal{D}_Q$ [G], [Ha] (see Figure 3).
(2) By (1), we have $\tau^{-(h+2)/2}(X) \simeq \tau^{-1}\big(\omega(X[1])\big) \simeq F(\omega(X)) \simeq \omega(X)$. $\square$

We prove the first part of Theorem 4.4(1):

THEOREM 4.14 (Theorem 4.4(1-i)). $\left[\mu^{(h+2)/2}(Q,x(0))\right]_a = x_{\omega(a)}(0)$.

Proof. We have $[\mu^{(h+2)/2}(P)]_a \overset{\text{Prop.4.10(1)}}{=} \tau^{-(h+2)/2}(P_a) \overset{\text{Prop.4.13(2)}}{=} P_{\omega(a)}$. Applying $\widetilde{X}$, we have $\left[\mu^{(h+2)/2}(Q,x(0))\right]_a \overset{(4.25)}{=} [\mu^{(h+2)/2}(\widetilde{X}_P)]_a \overset{(4.27)}{=} X_{[\mu^{(h+2)/2}(P)]_a} = X_{P_{\omega(a)}} = x_{\omega(a)}(0)$. $\square$

Case 2: h is odd.

In this case, the map $\omega : I \to I$ induces the quiver isomorphism $\omega : Q \to Q^{\text{op}}$. Thus ω induces an isomorphism $\omega : KQ \to KQ^{\text{op}}$ of K-algebras, and we have an equivalence

$$(4.29) \qquad\qquad \omega : \mathcal{D}_Q \to \mathcal{D}_{Q^{\text{op}}}$$

of categories. On the other hand, the tilting KQ-module U in Example 4.7(2) induces an equivalence

$$(4.30) \qquad U \overset{\mathbf{L}}{\otimes}_{KQ^{\mathrm{op}}} - : \mathcal{D}_{Q^{\mathrm{op}}} \to \mathcal{D}_Q.$$

Composing them, we have an autoequivalence

$$(4.31) \qquad r : \mathcal{D}_Q \xrightarrow{\omega} \mathcal{D}_{Q^{\mathrm{op}}} \xrightarrow{U \overset{\mathbf{L}}{\otimes}_{KQ^{\mathrm{op}}} -} \mathcal{D}_Q.$$

We have $r(P_a) = U_{\omega(a)}$ and $r(U_a) = \tau^{-1}(P_{\omega(a)})$ for any $a \in I$ (see Figure 3); hence, $r^2(X) \simeq \tau^{-1}(X)$ for any $X \in \mathcal{D}_Q$.

We have the following periodicity.

PROPOSITION 4.15. (1) $r\tau^{-(h-1)/2}(X) \simeq X[1]$ for any $X \in \mathcal{D}_Q$;
(2) $\tau^{-(h+1)/2}(X) \simeq r^{-1}(X)$ for any $X \in \mathcal{C}_Q$.

Proof. (1) This follows from the structure of the Auslander-Reiten quiver of $\mathcal{D}_Q$ [G], [Ha] (see Figure 3).

(2) By (1), we have $r\tau^{-(h+1)/2}(X) \simeq \tau^{-1}(X[1]) \simeq X$, from which the claim follows. □

We prove the second part of Theorem 4.4(1).

THEOREM 4.16 (Theorem 4.4(1-ii)). $\left[\mu_+\mu^{(h+1)/2}(Q, x(0))\right]_a = x_{\omega(a)}(0)$.

Proof. We have $\left[\mu_+\mu^{(h+1)/2}(P)\right]_a \overset{\text{Prop. }4.10(2)}{=} \tau^{-(h+1)/2}(U_a) \overset{\text{Prop. }4.15(2)}{=}$ $r^{-1}(U_a) = P_{\omega(a)}$. Applying $\widetilde{X}$, we have $\left[\mu_+\mu^{(h+1)/2}(Q, x(0))\right]_a \overset{(4.25)}{=}$ $[\mu_+\mu^{(h+1)/2}(\widetilde{X}_P)]_a \overset{(4.27)}{=} X_{[\mu_+\mu^{(h+1)/2}(P)]_a} = X_{P_{\omega(a)}} = x_{\omega(a)}(0)$. □

4.4. Level greater than two case

Here we study the periodicity of $\mathcal{T}_\ell(X_r)$ for simply laced X_r and $\ell > 2$. Since the case $X_r = A_1$ reduces to $\mathcal{T}_2(A_{\ell-1})$ by the level-rank duality (Example 3.10), we continue to assume $X_r \neq A_1$.

First, let us establish a connection between the ring $\mathcal{T}_\ell^\circ(X_r)$ and the cluster algebra $\mathcal{A}_{Q \square Q'}$ considered in [Kel2], [HL]. In doing that, we slightly generalize the problem and consider a pair of simply laced Dynkin diagrams $X_r, X'_{r'} (\neq A_1)$. For X_r (resp. $X'_{r'}$), let $C, h, I, I_\pm, \varepsilon, Q, \omega$ (resp. $C', h',$ $I', I'_\pm, \varepsilon', Q', \omega'$) be the same as Section 4.2.

DEFINITION 4.17. The *T-system* $\mathbb{T}(X_r, X'_{r'})$ *of type* $(X_r, X'_{r'})$ is the following system of relations for a family of variables $T = \{T_{a,b}(u) \mid a \in I, b \in I', u \in \mathbb{Z}\}$:

$$(4.32) \quad T_{a,b}(u-1)T_{a,b}(u+1) = \prod_{k \in I': C'_{bk} = -1} T_{a,k}(u) + \prod_{k \in I: C_{ak} = -1} T_{k,b}(u).$$

DEFINITION 4.18. The *T-algebra* $\mathcal{T}(X_r, X'_{r'})$ *of type* $(X_r, X'_{r'})$ is the ring with generators $T_{a,b}(u)^{\pm 1}$ $(a \in I, b \in I', u \in \mathbb{Z})$ and the relations $\mathbb{T}(X_r, X'_{r'})$. Also, we define the ring $\mathcal{T}^\circ(X_r, X'_{r'})$ as the subring of $\mathcal{T}(X_r, X'_{r'})$ generated by $T_{a,b}(u)$ $(a \in I, b \in I', u \in \mathbb{Z})$.

The system $\mathbb{T}(X_r, X'_{r'})$ is the counterpart of the Y-system of [RTV] studied in [Kel2], and $\mathcal{T}_\ell(X_r) = \mathcal{T}(X_r, A_{\ell-1})$ by identifying $T_m^{(a)}(u)$ with $T_{a,m}(u)$. We are going to show the following periodicity of $\mathcal{T}(X_r, X'_{r'})$:

$$(4.33) \quad \begin{aligned} &\text{half-periodicity:} \quad T_{a,b}(u+h+h') = T_{\omega(a),\omega'(b)}(u), \\ &\text{periodicity:} \quad T_{a,b}\big(u+2(h+h')\big) = T_{a,b}(u). \end{aligned}$$

Let $\mathcal{T}^\circ(X_r, X'_{r'})_\pm$ be the subring of $\mathcal{T}^\circ(X_r, X'_{r'})$ generated by $T_{a,b}(u)$ $(a \in I, b \in I', u \in \mathbb{Z})$ such that $\varepsilon(a)\varepsilon'(b)(-1)^u = \pm$. Then, we have

$$(4.34) \quad \mathcal{T}^\circ(X_r, X'_{r'}) \simeq \mathcal{T}^\circ(X_r, X'_{r'})_+ \otimes_{\mathbb{Z}} \mathcal{T}^\circ(X_r, X'_{r'})_-,$$

$$\mathcal{T}^\circ(X_r, X'_{r'})_+ \simeq \mathcal{T}^\circ(X_r, X'_{r'})_-.$$

To describe the corresponding cluster algebra to $\mathcal{T}^\circ(X_r, X'_{r'})$, we introduce two kinds of quivers, $Q \square Q'$ and $Q \otimes Q'$.

DEFINITION 4.19 ([Kel2]). (i) The *square product* $Q \square Q'$ of Q and Q' is the quiver obtained from the product $Q \times Q'$ by reversing all the arrows in the full subquivers $\{a\} \times Q'$ (a: sink of Q) and $Q \times \{b\}$ (b: source of Q').

(ii) The *tensor product* $Q \otimes Q'$ of Q and Q' is the quiver obtained from the product $Q \times Q'$ by adding an arrow $(a_2, b_2) \to (a_1, b_1)$ for each pair of arrows $a_1 \to a_2$ of Q and $b_1 \to b_2$ of Q'.

EXAMPLE 4.20. Since $a \in I_+$ is a source of Q, in our convention, the ordinary product $Q \times Q'$ consists of the following type of squares:

$$(4.35) \quad Q \times Q': \quad \begin{array}{ccc} (+-) & \to & (--) \\ \uparrow & & \uparrow \\ (++) & \to & (-+) \end{array},$$

where $(+-)$, for example, represents a vertex (a,b) of $Q \times Q'$ with $a \in I_+$, $b \in I'_-$. Correspondingly, $Q \square Q'$ and $Q \otimes Q'$ consist of the following types of squares:

$$(4.36) \quad Q \square Q': \quad \begin{array}{ccc} (+-) & \rightarrow & (--) \\ \uparrow & & \downarrow \\ (++) & \leftarrow & (-+) \end{array}, \quad Q \otimes Q': \quad \begin{array}{ccc} (+-) & \rightarrow & (--) \\ \uparrow & \swarrow & \uparrow \\ (++) & \rightarrow & (-+) \end{array}.$$

Using these diagrams, one can easily check that

$$(4.37) \quad \begin{aligned} Q \square Q' &= Q^{\mathrm{op}} \square Q'^{\mathrm{op}}, \\ (Q \square Q')^{\mathrm{op}} &= Q^{\mathrm{op}} \square Q' = Q \square Q'^{\mathrm{op}}, \\ (Q \otimes Q')^{\mathrm{op}} &= Q^{\mathrm{op}} \otimes Q'^{\mathrm{op}}. \end{aligned}$$

We define composed mutations,

$$(4.38) \quad \mu_{\pm\pm} = \prod_{(a,b) \in I_\pm \times I'_\pm} \mu_{a,b}, \qquad \mu_{\pm\mp} = \prod_{(a,b) \in I_\pm \times I'_\mp} \mu_{a,b},$$

where $\mu_{a,b}$ is the mutation at (a,b). Then, the following cycle of mutations of quivers occurs (the "eyeglass diagram"):

$$(4.39) \quad \begin{array}{ccccccc} & Q \otimes Q'^{\mathrm{op}} & & & & Q \otimes Q' & \\ {}^{\mu_{++}}\nearrow & & \searrow^{\mu_{--}} & {}^{\mu_{+-}}\nearrow & & \searrow^{\mu_{-+}} & \\ Q \square Q' & & (Q \square Q')^{\mathrm{op}} & & & Q \square Q'. \\ {}_{\mu_{--}}\searrow & & \nearrow_{\mu_{++}} & {}_{\mu_{-+}}\searrow & & \nearrow_{\mu_{+-}} & \\ & Q^{\mathrm{op}} \otimes Q' & & & Q^{\mathrm{op}} \otimes Q'^{\mathrm{op}} & \end{array}$$

We further define composed mutations [Kel2]

$$(4.40) \quad \begin{aligned} \mu_- &= \mu_{+-}\mu_{--}, \qquad \mu_+ = \mu_{++}\mu_{-+}, \\ \mu_\otimes &= \mu_-\mu_+ = \mu_{+-}\mu_{--}\mu_{++}\mu_{-+}. \end{aligned}$$

In particular, $\mu_\otimes$ preserves $Q \otimes Q'$.

REMARK 4.21. (1) The mutations $\mu_{\pm\pm}$, $\mu_{\pm\mp}$, and $\mu_\otimes$ here correspond to $\mu_{\mp,\mp}$, $\mu_{\mp,\pm}$, and the inverse of $\mu_\otimes$ in [Kel2]. This is due to our convention of the assignment of $+/-$ for the sources/sinks of Q and Q' and is not essential at all.

(2) Instead of (4.40), one may set $\mu_+ = \mu_{+-}\mu_{++}$, $\mu_- = \mu_{--}\mu_{-+}$, and $\mu_\otimes = \mu_+\mu_-$. This is again a matter of choice.

We introduce the $(I \times I')$-tuple of variables $x = \{x_{a,b} \mid a \in I, b \in I'\}$, and we define $\mathcal{A}_{Q \square Q'}$ to be the cluster algebras with initial seeds $(Q \square Q', x)$. We set $x = x(0)$, and we define clusters $x(u)$, $z(u)$, $\overline{z}(u)$ $(u \in \mathbb{Z})$ of $\mathcal{A}_{Q \square Q'}$ by the following sequence of mutations (u: even):

$$
\begin{array}{ccccc}
& \big(Q \otimes Q'^{\mathrm{op}}, z(u+1)\big) & & \big(Q \otimes Q', z(u+2)\big) & \\
& {\scriptstyle \mu_{++}} \nearrow \quad \searrow {\scriptstyle \mu_{--}} & {\scriptstyle \mu_{+-}} \nearrow \quad \searrow {\scriptstyle \mu_{-+}} & \\
(4.41) \quad \big(Q \square Q', x(u)\big) & & \big((Q \square Q')^{\mathrm{op}}, x(u+1)\big) & & \big(Q \square Q', x(u+2)\big). \\
& {\scriptstyle \mu_{--}} \searrow \quad \nearrow {\scriptstyle \mu_{++}} & {\scriptstyle \mu_{-+}} \searrow \quad \nearrow {\scriptstyle \mu_{+-}} & \\
& \big(Q^{\mathrm{op}} \otimes Q', \overline{z}(u+1)\big) & & \big(Q^{\mathrm{op}} \otimes Q'^{\mathrm{op}}, \overline{z}(u+2)\big) &
\end{array}
$$

In particular,

$$(4.42) \qquad x_{a,b}(u+1) = x_{a,b}(u) \quad \text{if } \varepsilon(a)\varepsilon'(b)(-1)^u = -,$$

$$
z_{a,b}(u) = \begin{cases} x_{a,b}(u) & a \in I_+, \\ x_{a,b}(u-1) & a \in I_-, \end{cases}
$$

(4.43)

$$
\overline{z}_{a,b}(u) = \begin{cases} x_{a,b}(u-1) & a \in I_+, \\ x_{a,b}(u) & a \in I_-, \end{cases}
$$

$$
\begin{aligned}
(4.44) \qquad & \big(Q \otimes Q', z(2k)\big) = \mu_{\otimes}^k\big(Q \otimes Q', z(0)\big) \quad (k \in \mathbb{Z}), \\
& \big(Q \otimes Q'^{\mathrm{op}}, z(2k+1)\big) = \mu_+ \mu_{\otimes}^k\big(Q \otimes Q', z(0)\big) \quad (k \in \mathbb{Z}).
\end{aligned}
$$

LEMMA 4.22. *The family $\{x_{a,b}(u) \mid a \in I, b \in I', u \in \mathbb{Z}\}$ satisfies the T-system $\mathbb{T}(X_r, X'_{r'})$ in $\mathcal{A}_{Q \square Q'}$; namely,*

$$(4.45) \quad x_{a,b}(u-1)x_{a,b}(u+1) = \prod_{k \in I' : C'_{bk} = -1} x_{a,k}(u) + \prod_{k \in I : C_{ak} = -1} x_{k,b}(u).$$

Proof. The proof is the same as Lemma 4.1 by replacing μ_+ (resp. μ_-) therein with $\mu_{--}\mu_{++}$ (resp. $\mu_{-+}\mu_{+-}$). $\qquad\square$

Define a ring homomorphism $f : \mathcal{A}_{Q \square Q'} \to \mathcal{T}(X_r, X'_{r'})$ as the restriction of the ring homomorphism $\mathbb{Z}[x_{a,b}^{\pm 1}]_{(a,b) \in I \times I'} \to \mathcal{T}(X_r, X'_{r'})$ given by

$$(4.46) \qquad\qquad f : x_{a,b}^{\pm 1} \mapsto \begin{cases} T_{a,b}(0)^{\pm 1} & \varepsilon(a)\varepsilon'(b) = +, \\ T_{a,b}(1)^{\pm 1} & \varepsilon(a)\varepsilon'(b) = -. \end{cases}$$

Then, as Lemma 4.2, we have the following.

LEMMA 4.23. *For the above homomorphism* $f : \mathcal{A}_{Q \square Q'} \to \mathfrak{T}(X_r, X'_{r'})$,

$$(4.47) \qquad f : x_{a,b}(u) \mapsto \begin{cases} T_{a,b}(u) & \varepsilon(a)\varepsilon'(b)(-1)^u = +, \\ T_{a,b}(u+1) & \varepsilon(a)\varepsilon'(b)(-1)^u = -. \end{cases}$$

Let $\mu_\square := \mu_{-+}\mu_{+-}\mu_{--}\mu_{++}$. We define $\mathcal{A}^{\mathrm{t}}_{Q \square Q'}$ to be the subring of $\mathcal{A}_{Q \square Q'}$ generated by $x_{a,b}(u)$ $(a \in I, b \in I', u \in 2\mathbb{Z})$, that is, the cluster variables belonging to the seeds $\mu^k_\square(Q \square Q', x(0))$ $(k \in \mathbb{Z})$. We call $\mathcal{A}^{\mathrm{t}}_{Q \square Q'}$ the *translation subalgebra of* $\mathcal{A}_{Q \square Q'}$ *with respect to* $\mu_\square$. The ring $\mathcal{A}^{\mathrm{t}}_{Q \square Q'}$ is no longer a cluster algebra. We remark that, by (4.42), the ring $\mathcal{A}^{\mathrm{t}}_{Q \square Q'}$ coincides with the subring of $\mathcal{A}_{Q \square Q'}$ generated by $x_{a,b}(u)$ $(a \in I, b \in I', u \in \mathbb{Z})$, which are the cluster variables belonging to the "bipartite belt" in [FZ4, Section 8].

PROPOSITION 4.24. *The ring* $\mathfrak{T}^\circ(X_r, X'_{r'})$ *is isomorphic to* $\mathcal{A}^{\mathrm{t}}_{Q \square Q'} \otimes_{\mathbb{Z}} \mathcal{A}^{\mathrm{t}}_{Q \square Q'}$.

Proof. By Lemma 4.23, the restriction $f : \mathcal{A}^{\mathrm{t}}_{Q \square Q'} \to \mathfrak{T}^\circ(X_r, X'_{r'})_+$ is surjective. Furthermore, the inverse correspondence $g : \mathfrak{T}^\circ(X_r, X'_{r'})_+ \to \mathcal{A}^{\mathrm{t}}_{Q \square Q'}$, $T_{a,b}(u) \mapsto x_{a,b}(u)$ defines a homomorphism by Lemma 4.22 and the fact that $\mathcal{A}^{\mathrm{t}}_{Q \square Q'}$ is an integral domain. Therefore,

$$(4.48) \qquad \mathfrak{T}^\circ(X_r, X'_{r'})_+ \simeq \mathcal{A}^{\mathrm{t}}_{Q \square Q'},$$

and we obtain the assertion. □

REMARK 4.25. Hernandez and Leclerc [HL] also study the relation between the cluster algebra $\mathcal{A}_{Q \square Q'}$ with $X'_{r'} = A_{r'}$ and the T-system in view of the categorification of $\mathcal{A}_{Q \square Q'}$ by a subcategory of the category of the finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules.

Now let us turn to the periodicity problem. Let $\mathcal{A}_{Q \otimes Q'}$ be the cluster algebra with initial seed $(Q \otimes Q', z)$, $z = \{z_{a,b} \mid a \in I, b \in I'\}$. Two cluster algebras $\mathcal{A}_{Q \otimes Q'}$ and $\mathcal{A}_{Q \square Q'}$ coincide by setting $z = z(0)$ in (4.41). A crucial observation made by Keller [Kel2] is that the periodicity of $\mathcal{A}_{Q \square Q'}$ is more transparent in the "$\otimes$-picture" than in the "$\square$-picture" from the cluster categorical point of view. By (4.43) and (4.47), it is easy to check that the periodicity (4.33) of $\mathfrak{T}(X_r, X'_{r'})$ is translated as

$$(4.49) \qquad \text{half-periodicity:} \quad z_{a,b}(u + h + h') = \begin{cases} z_{\omega(a),\omega'(b)}(u) & h: \text{even}, \\ \overline{z}_{\omega(a),\omega'(b)}(u) & h: \text{odd}, \end{cases}$$

$$\text{periodicity:} \quad z_{a,b}\big(u + 2(h + h')\big) = z_{a,b}(u).$$

The following periodicity of $\mathcal{A}_{Q\otimes Q'}$ is immediately obtained from the results in [Kel2].

THEOREM 4.26 (Keller [Kel2]). *The following equality holds for $\mathcal{A}_{Q\otimes Q'}$ (u: even):*

$$(4.50) \qquad \mu_{\otimes}^{h+h'}\big(Q \otimes Q', z(u)\big) \overset{\mathrm{id}}{=} \big(Q \otimes Q', z(u)\big).$$

Proof. The F-polynomials of $\mathcal{A}_{Q\otimes Q'}$ relevant to the above periodicity are expressed in terms of the triangulated category $\mathcal{C}_{KQ\otimes KQ'}$ [Kel2, Theorem 7.13(c)]; furthermore, $\mathcal{C}_{KQ\otimes KQ'}$ has the desired periodicity [Kel2, Proposition 8.5]. Therefore, the cluster variable $z(u)$ has the same periodicity by [FZ4, Corollary 6.3, Proposition 6.9]. □

As a refinement of Theorem 4.26, we also show the half-periodicity of $\mathcal{A}_{Q\otimes Q'}$.

THEOREM 4.27. *The following equalities hold for $\mathcal{A}_{Q\otimes Q'}$ (u: even).*

(1) *For $(h, h') = (even, even)$,*

$$(4.51) \qquad \mu_{\otimes}^{(h+h')/2}\big(Q \otimes Q', z(u)\big) \overset{\omega \times \omega'}{=} \big(Q \otimes Q', z(u)\big).$$

(2) *For $(h, h') = (odd, odd)$,*

$$(4.52) \qquad \mu_{\otimes}^{(h+h')/2}\big(Q \otimes Q', z(u)\big) \overset{\omega \times \omega'}{=} \big(Q^{\mathrm{op}} \otimes Q'^{\mathrm{op}}, \overline{z}(u)\big).$$

(3) *For $(h, h') = (even, odd)$,*

$$(4.53) \qquad \mu_{+}\mu_{\otimes}^{(h+h'-1)/2}\big(Q \otimes Q', z(u)\big) \overset{\omega \times \omega'}{=} \big(Q \otimes Q', z(u)\big).$$

(4) *For $(h, h') = (odd, even)$,*

$$(4.54) \qquad \mu_{+}\mu_{\otimes}^{(h+h'-1)/2}\big(Q \otimes Q', z(u)\big) \overset{\omega \times \omega'}{=} \big(Q^{\mathrm{op}} \otimes Q'^{\mathrm{op}}, \overline{z}(u)\big).$$

The proof of Theorem 4.27 is given in Section 4.5.

COROLLARY 4.28. *The following relations hold in $\mathcal{T}(X_r, X'_{r'})$:*
(1) *Half-periodicity: $T_{a,b}(u + h + h') = T_{\omega(a),\omega'(b)}(u)$.*
(2) *Periodicity: $T_{a,b}(u + 2(h + h')) = T_{a,b}(u)$.*

Proof. The relations in (4.49) immediately follow from Theorems 4.26 and 4.27 and (4.44). □

By Corollaries 4.5 and 4.28, we obtain the main result of this section, which is as follows.

COROLLARY 4.29. *The following relations hold in $\mathcal{T}_\ell(X_r)$ for any simply laced X_r and any $\ell \geq 2$.*

(1) *Half-periodicity: $T_m^{(a)}(u + h + \ell) = T_{\ell-m}^{(\omega(a))}(u)$.*

(2) *Periodicity: $T_m^{(a)}(u + 2(h + \ell)) = T_m^{(a)}(u)$.*

4.5. Proof of Theorems 4.26 and 4.27 by cluster category

Here we prove Theorem 4.27 by adapting the method of [Kel2, Theorem 8.2] for our situation. In the course we also include a proof of Theorem 4.26 without using the F-polynomials, for the reader's convenience. We present the proof as parallel as possible to the level 2 case in Section 4.3.

Let Q and Q' continue to be the alternating quivers in Section 4.4 whose underlying graphs are simply laced X_r and $X'_{r'}$ other than A_1, respectively. For Q (resp. Q'), let KQ, $\mathcal{D}_Q$, $\tau : \mathcal{D}_Q \to \mathcal{D}_Q$ (resp. KQ', $\mathcal{D}_{Q'}$, $\tau' : \mathcal{D}_{Q'} \to \mathcal{D}_{Q'}$) be the ones in Section 4.3.

We denote the tensor product $\otimes_K$ simply by $\otimes$, and we define a finite dimensional K-algebra A by

$$(4.55) \qquad A := KQ \otimes KQ'.$$

Let $\mathcal{D}_A = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ be the bounded derived category of finite dimensional A-modules, and let

$$(4.56) \qquad \tau_\otimes := DA[-1] \overset{\mathbf{L}}{\otimes}_A - : \mathcal{D}_A \to \mathcal{D}_A$$

be the Auslander-Reiten translation for $\mathcal{D}_A$. Now we define another autoequivalence of $\mathcal{D}_A$ by $F := \tau_\otimes^{-1} \circ [1]$. *We need the following easy observation.*

LEMMA 4.30. *We have $\tau(X) \otimes \tau'(Y) \simeq F^{-1}(X \otimes Y)$ in $\mathcal{D}_A$ for any $X \in \mathcal{D}_Q$ and $Y \in \mathcal{D}_{Q'}$.*

Proof. We have $\tau(X) \otimes \tau'(Y) \simeq (D(KQ) \overset{\mathbf{L}}{\otimes}_{KQ} X[-1]) \otimes (D(KQ') \overset{\mathbf{L}}{\otimes}_{KQ'} Y[-1]) \simeq (D(KQ) \otimes D(KQ'))[-2] \overset{\mathbf{L}}{\otimes}_A (X \otimes Y) = F^{-1}(X \otimes Y)$. $\square$

The orbit category

$$(4.57) \qquad \mathcal{D}_A/F$$

has the same objects with $\mathcal{D}_A$, and the morphism space is given by

$$(4.58) \qquad \operatorname{Hom}_{(\mathcal{D}_A/F)}(X,Y) := \bigoplus_{i\in\mathbf{Z}} \operatorname{Hom}_{\mathcal{D}_A}\big(X, F^i(Y)\big)$$

for any $X, Y \in \mathcal{D}_A/F$. In contrast to (4.17), $\mathcal{D}_A/F$ is no longer a triangulated category in general. However, based on the works of Keller (see [Kel1], [Kel4]), Amiot [A, Section 4] constructed a triangulated hull $\mathcal{C}_A$ of $\mathcal{D}_A/F$, which is a 2-Calabi-Yau triangulated category with a fully faithful functor $\mathcal{D}_A/F \to \mathcal{C}_A$ satisfying a certain universal property. (Here we need the fact that the functor $H^0(F-)$ is nilpotent on $\operatorname{mod} A$, which follows from Lemma 4.30.) We call $\mathcal{C}_A$ the *(generalized) cluster category of A*.

We say that an object $T = \bigoplus_{(a,b)\in I\times I'} T_{a,b} \in \mathcal{C}_A$ is *cluster tilting* if

(1) each $T_{a,b}$ is indecomposable and mutually nonisomorphic, and
(2) $\operatorname{add} T = \big\{ X \in \mathcal{C}_A \mid \operatorname{Hom}_{\mathcal{C}_A}(T, X[1]) = 0 \big\}$.

(To simplify our proof, we assume that the index set of direct summands of T is $I \times I'$. This does not affect the definition essentially due to Example 4.31 and [DeK, Theorem 2.4].) For a cluster tilting object $T \in \mathcal{C}_A$, we denote by Q_T the quiver of the endomorphism ring $\operatorname{End}_{\mathcal{C}_A}(T)$ [ARS], [ASS].

EXAMPLE 4.31 (cf. Example 4.7). For $a \in I$ and $b \in I'$, we denote by e_a and e_b' the paths of length zero in Q and Q', respectively. As in Example 4.7, we define KQ-modules P, U and KQ'-modules P', U' by

$$(4.59) \qquad P_a := (KQ)e_a, \qquad\qquad P := KQ = \bigoplus_{a\in I} P_a,$$

$$(4.60) \qquad U_a := \begin{cases} \tau^{-1}(P_a) & a \in I_+, \\ P_a & a \in I_-, \end{cases} \quad U := \bigoplus_{a\in I} U_a,$$

$$(4.61) \qquad P_b' := (KQ')e_b', \qquad\qquad P' := KQ' = \bigoplus_{b\in I'} P_b',$$

$$(4.62) \qquad U_b' := \begin{cases} \tau'^{-1}(P_b') & b \in I_+', \\ P_b' & b \in I_-', \end{cases} \quad U' := \bigoplus_{b\in I'} U_b'.$$

Then

$$(4.63) \qquad P \otimes P' = \bigoplus_{(a,b)\in I\times I'} P_a \otimes P_b'$$

is a cluster tilting object in $\mathcal{C}_A$ with $Q_{P\otimes P'} = Q \otimes Q'$ [A, Theorem 4.10].

Again we can define the mutation of cluster tilting objects as follows [IY, Theorem 5.3].

THEOREM 4.32 (cf. Theorem 4.8). *Let $T = \bigoplus_{(a,b)\in I \times I'} T_{a,b} \in \mathcal{C}_A$ be a cluster tilting object. For any $(a,b) \in I \times I'$, there exists a unique indecomposable object $T_{a,b}^* \in \mathcal{C}_A$ that is not isomorphic to $T_{a,b}$ such that $(T/T_{a,b}) \oplus T_{a,b}^*$ is a cluster tilting object.*

We call the above $(T/T_{a,b}) \oplus T_{a,b}^*$ the *cluster tilting mutation of T at* (a,b), and we denote it by $\mu_{a,b}(T)$.

We number elements of $I_- \times I'_+$ as $\{c_1, \dots, c_s\}$ $(s := |I_- \times I'_+|)$, and we define a composed cluster tilting mutation μ_{-+} by

$$(4.64) \qquad \mu_{-+} := \mu_{c_s} \cdots \mu_{c_1}.$$

Similarly, we define

$$(4.65) \qquad \mu_{++}, \qquad \mu_{--}, \qquad \mu_{+-}$$

by using $I_+ \times I'_+$, $I_- \times I'_-$, and $I_+ \times I'_-$, respectively. We further define

$$(4.66) \qquad \mu_\otimes := \mu_{+-}\mu_{--}\mu_{++}\mu_{-+}, \qquad \mu_+ := \mu_{++}\mu_{-+}.$$

Thus we have a numbering $\{c_1, \dots, c_{rr'}\}$ of the elements of $I \times I'$ such that $\mu_\otimes = \mu_{c_{rr'}} \cdots \mu_{c_1}$.

For a cluster tilting object $T = \bigoplus_{(a,b)\in I \times I'} T_{a,b}$, let $[T]_{a,b}$ denote $T_{a,b}$. We have the following key observation.

PROPOSITION 4.33 (cf. Proposition 4.10). (1) *We have a diagram*

$$(4.67)$$

$$
\begin{array}{ccccc}
P \otimes \tau'^{-k}(P') & & P \otimes \tau'^{-k}(U') & & P \otimes \tau'^{-k-1}(P') \\
& \searrow \mu_{-+} \quad \mu_{++} \nearrow & & \searrow \mu_{--} \quad \mu_{+-} \nearrow & \\
& V^k & & W^k & \\
& \nearrow \mu_{+-} \quad \mu_{--} \searrow & & \nearrow \mu_{++} \quad \mu_{-+} \searrow & \\
U \otimes \tau'^{-k}(U') & & U \otimes \tau'^{-k-1}(P') & & U \otimes \tau'^{-k-1}(U')
\end{array}
$$

of composed cluster tilting mutations for any $k \in \mathbb{Z}$, where

$$(4.68) \quad V^k := \Big(\bigoplus_{(a,b)\in I_- \times I'_+} P_a \otimes \tau'^{-k-1}(P'_b) \Big) \oplus \Big(\bigoplus_{(a,b)\notin I_- \times I'_+} P_a \otimes \tau'^{-k}(P'_b) \Big),$$

$$(4.69) \quad W^k := \Big(\bigoplus_{(a,b)\notin I_+ \times I'_-} P_a \otimes \tau'^{-k-1}(P'_b) \Big) \oplus \Big(\bigoplus_{(a,b)\in I_+ \times I'_-} P_a \otimes \tau'^{-k}(P'_b) \Big).$$

(2) *We have*

$$[\mu_\otimes^k(P \otimes P')]_{a,b} = P_a \otimes \tau'^{-k}(P'_b),$$

(4.70)

$$[\mu_+\mu_\otimes^k(P \otimes P')]_{a,b} = P_a \otimes \tau'^{-k}(U'_b)$$

for any $k \in \mathbb{Z}$ and $(a,b) \in I \times I'$.

Proof. (2) is an immediate consequence of (1).

(1) We show only $\mu_{-+}(P \otimes P') = V^0$ since other cases are shown similarly. We put $J^\ell := \{c_i \mid 1 \le i \le \ell\}$. Define $T^\ell \in \mathcal{D}_A$ by

$$T^\ell := \Big(\bigoplus_{(a,b) \in J^\ell} P_a \otimes \tau'^{-1}(P'_b) \Big) \oplus \Big(\bigoplus_{(a,b) \in (I \times I') \setminus J^\ell} P_a \otimes P'_b \Big)$$

for any $0 \le \ell \le s = |I_- \times I'_+|$. Clearly we have $T^0 = P \otimes P'$ and $T^s = V^0$.

The following observation is crucial.

PROPOSITION 4.34. (1) T^ℓ *is a tilting A-module for any $0 \le \ell \le s$.*

(2) *The algebra $B := \mathrm{End}_A(T^\ell)$ has global dimension at most two.*

(3) *For any simple B-modules S and S', we have $\mathrm{Ext}_B^1(S,S) = 0$, and either $\mathrm{Ext}_B^1(S,S') = 0$ or $\mathrm{Ext}_B^2(S,S') = 0$ holds.*

Proof. (1) Since Q' is not of type A_1, our T^ℓ is an A-module. Clearly T^ℓ has projective dimension at most one, and the number of indecomposable direct summands of T^ℓ is rr'. Thus we have only to show $\mathrm{Ext}_A^1(T^\ell, T^\ell) = 0$. Fix $(a,b), (c,d) \in I \times I'$, and we shall show

(4.71) $$\mathrm{Hom}_{\mathcal{D}_A}(T_{a,b}^\ell, T_{c,d}^\ell[1]) = 0.$$

We have a general equality

(4.72) $$\mathrm{Hom}_{\mathcal{D}_A}\big(X \otimes X', (Y \otimes Y')[i]\big)$$

$$= \bigoplus_{j+j'=i} \mathrm{Hom}_{\mathcal{D}_Q}(X, Y[j]) \otimes \mathrm{Hom}_{\mathcal{D}_{Q'}}(X', Y'[j'])$$

for any $X, Y \in \mathcal{D}_Q$ and $X', Y' \in \mathcal{D}_{Q'}$. We also have

$$\mathrm{Hom}_{\mathcal{D}_Q}(P_a, P_c[j]) = 0$$

(4.73)

if "$j \ne 0$" or "$j = 0$, $a \ne c$, and there is no arrow $a \to c$ in Q."

We divide the proof of (4.71) into four cases.

(i) Assume $(a,b),(c,d) \notin J^\ell$. Then (4.71) follows from (4.72), (4.73), and

$$(4.74) \qquad \operatorname{Hom}_{\mathcal{D}_{Q'}}(P'_b, P'_d[j']) = 0 \quad \text{if } j' \neq 0.$$

(ii) Assume $(a,b),(c,d) \in J^\ell$. Then (4.71) follows from (4.72), (4.73), and

$$(4.75) \qquad \operatorname{Hom}_{\mathcal{D}_{Q'}}\big(\tau'^{-1}(P'_b), \tau'^{-1}(P'_d)[j']\big)$$
$$\simeq \operatorname{Hom}_{\mathcal{D}_{Q'}}(P'_b, P'_d[j']) = 0 \quad \text{if } j' \neq 0.$$

(iii) Assume $(a,b) \in J^\ell$ and $(c,d) \notin J^\ell$. We have

$$(4.76) \qquad \operatorname{Hom}_{\mathcal{D}_{Q'}}\big(\tau'^{-1}(P'_b), P'_d[j']\big) \simeq D\operatorname{Hom}_{\mathcal{D}_{Q'}}(P'_d[j'], P'_b[1]) = 0$$
$$\text{if } \text{``}j' \neq 1\text{''} \text{ or } \text{``}j' = 1,\ b \neq d, \text{ and there is no arrow } d \to b \text{ in } Q'.\text{''}$$

Since $a \in I_-$, $b \in I'_+$, and $(a,b) \neq (c,d)$ hold, either "$a \neq c$ and there is no arrow $a \to c$ in Q" or "$b \neq d$ and there is no arrow $d \to b$ in Q'." Thus (4.71) follows from (4.72) and (4.73).

(iv) Assume $(a,b) \notin J^\ell$ and $(c,d) \in J^\ell$. Since $d \in I'_+$ and Q' is not type A_1, we have

$$(4.77) \qquad \operatorname{Hom}_{\mathcal{D}_{Q'}}\big(P'_b, \tau'^{-1}(P'_d)[j']\big) = 0 \quad \text{if } j' \neq 0.$$

Thus (4.71) follows from (4.72) and (4.73).

(2) and (3) It is enough to prove the following claim.

CLAIM. *Every simple B-module S has a projective resolution*

$$(4.78) \qquad 0 \to B_2 \to B_1 \to B_0 \to S \to 0$$

such that B_1 and $B_0 \oplus B_2$ have no nonzero common direct summand.

Let $(a,b) \in I \times I'$ be such that S is the top of the indecomposable projective B-module $\operatorname{Hom}_A(T^\ell, T^\ell_{a,b})$. We divide the proof of the claim into four cases.

(i) Assume that $a \in I_+$. Then the radical of the A-module $P_a \otimes P'_b$ is $\bigoplus_{d \to b} P_a \otimes P'_d$, which belongs to add T^ℓ. Here the direct sum is taken over all arrows in Q' with target b. Thus we have a projective resolution

$$(4.79) \qquad 0 \to \operatorname{Hom}_A\Big(T^\ell, \bigoplus_{d \to b} P_a \otimes P'_d\Big) \to \operatorname{Hom}_A(T^\ell, P_a \otimes P'_b) \to S \to 0,$$

and the claim follows.

(ii) Assume that $(a, b) \in (I_- \times I'_+) \setminus J^\ell$. Then the radical of the A-module $P_a \otimes P'_b$ is $\bigoplus_{c \to a} P_c \otimes P'_b$, which belongs to $\operatorname{add} T^\ell$. Thus we have a projective resolution

$$(4.80) \qquad 0 \to \operatorname{Hom}_A\Big(T^\ell, \bigoplus_{c \to a} P_c \otimes P'_b\Big) \to \operatorname{Hom}_A(T^\ell, P_a \otimes P'_b) \to S \to 0,$$

and the claim follows.

(iii) Assume that $(a, b) \in J^\ell$. Applying $(P_a \otimes -)$ to the Auslander-Reiten sequence $0 \to P'_b \to \bigoplus_{b \to d} P'_d \to \tau'^{-1}(P'_b) \to 0$ of KQ'-modules, we have an exact sequence

$$(4.81) \qquad 0 \to P_a \otimes P'_b \to \bigoplus_{b \to d} P_a \otimes P'_d \xrightarrow{f} P_a \otimes \tau'^{-1}(P'_b) \to 0$$

of A-modules whose middle term belongs to $\operatorname{add} T^\ell$. Clearly, any morphism $T^\ell \to P_a \otimes \tau'^{-1}(P'_b)$ that is not a split epimorphism factors through f. Moreover, the left term $P_a \otimes P'_b$ does not belong to $\operatorname{add} T^\ell$, but its radical $\bigoplus_{c \to a} P_c \otimes P'_b$ belongs to $\operatorname{add} T^\ell$. Consequently, we have a projective resolution

$$(4.82) \quad \begin{aligned} 0 &\to \operatorname{Hom}_A\Big(T^\ell, \bigoplus_{c \to a} P_c \otimes P'_b\Big) \to \operatorname{Hom}_A\Big(T^\ell, \bigoplus_{b \to d} P_a \otimes P'_d\Big) \\ &\xrightarrow{f} \operatorname{Hom}_A\big(T^\ell, P_a \otimes \tau'^{-1}(P'_b)\big) \to S \to 0 \end{aligned}$$

and the claim follows.

(iv) Assume $(a, b) \in I_- \times I'_-$. Taking a tensor product of exact sequences $0 \to \bigoplus_{c \to a} P_c \to P_a$ and $0 \to \bigoplus_{d \to b, (a,d) \notin J^\ell} P'_d \to P'_b$, we have an exact sequence

$$(4.83) \quad \begin{aligned} 0 &\to \bigoplus_{\substack{c \to a \\ d \to b \\ (a,d) \notin J^\ell}} P_c \otimes P'_d \\ &\to \Big(\bigoplus_{\substack{d \to b \\ (a,d) \notin J^\ell}} P_a \otimes P'_d\Big) \oplus \Big(\bigoplus_{c \to a} P_c \otimes P'_b\Big) \xrightarrow{f} P_a \otimes P'_b \end{aligned}$$

of A-modules whose terms belong to $\operatorname{add} T^\ell$. Clearly, any morphism $T^\ell \to P_a \otimes P'_b$ which is not a split epimorphism factors through f. Thus we have

a projective resolution

$$0 \to \mathrm{Hom}_A\Big(T^\ell, \bigoplus_{\substack{c \to a \\ d \to b \\ (a,d)\notin J^\ell}} P_c \otimes P_d'\Big)$$

(4.84)
$$\to \mathrm{Hom}_A\Big(T^\ell, \Big(\bigoplus_{\substack{d \to b \\ (a,d)\notin J^\ell}} P_a \otimes P_d'\Big) \oplus \Big(\bigoplus_{c \to a} P_c \otimes P_b'\Big)\Big)$$

$$\xrightarrow{f} \mathrm{Hom}_A(T^\ell, P_a \otimes P_b') \to S \to 0$$

and the claim follows. $\qquad\qquad\square$

By Proposition 4.34(1)–(2) together with [A, Theorem 4.10], T^ℓ is a cluster tilting object in $\mathcal{C}_A$ for any $1 \le \ell \le s$. Since $T^{\ell-1}$ and T^ℓ have the same indecomposable direct summands except $T^\ell_{c_\ell}$, we have $\mu_{c_\ell}(T^{\ell-1}) = T^\ell$. Consequently, we have $\mu_{-+}(P \otimes P') = \mu_{-+}(T^0) = T^s = V^0$. $\qquad\square$

We define a set $\mathbf{T}$ of cluster tilting objects by

$$\mathbf{T} = \big\{ \mu_{c_i} \cdots \mu_{c_1} \mu_\otimes^u (P \otimes P') \mid u \in \mathbf{Z}, 0 \le i < rr' \big\}.$$

We have the following result by [Kel2, Proposition 8.3].

PROPOSITION 4.35. *The quiver Q_T has no loops and 2-cycles for any $T \in \mathbf{T}$.*

Proof. This is a consequence of Proposition 4.34(3) and [A, Proposition 4.16]. $\qquad\qquad\square$

The following result is crucial in our proof.

THEOREM 4.36 (cf. Theorems 4.6 and 4.9). *There exists a map*

(4.85)
$$X : \{indecomposable\ direct\ summands\ of\ objects\ in\ \mathbf{T}\}/\simeq$$
$$\to \{cluster\ variables\ in\ \mathcal{A}_{Q\otimes Q'}\}$$

such that we have a map

(4.86)
$$\widetilde{X} : \mathbf{T} \to \{seeds\ in\ \mathcal{A}_{Q\otimes Q'}\}$$

defined by

(4.87)
$$T = \bigoplus_{(a,b)\in I\times I'} T_{a,b} \mapsto (Q_T, \{X_{T_{a,b}}\}_{(a,b)\in I\times I'})$$

satisfying the following conditions.

(1) $\widetilde{X}_{P \otimes P'} = (Q \otimes Q', z(0))$.

(2) If T and $\mu_{a,b}(T)$ belong to $\mathbf{T}$, then $\widetilde{X}_{\mu_{a,b}(T)} = \mu_{a,b}(\widetilde{X}_T)$.

Proof. The assertion follows from Proposition 4.35 [Pal, Theorem 4] and [BIRS, Theorem I.1.6]. □

By Theorem 4.36, (4.41), and (4.67), we also obtain

$$(4.88) \qquad \begin{aligned} \widetilde{X}_{U \otimes U'} &= \left(Q^{\mathrm{op}} \otimes Q'^{\mathrm{op}}, \overline{z}(0)\right), \\ \widetilde{X}_{P \otimes U'} &= \left(Q \otimes Q'^{\mathrm{op}}, z(1)\right), \qquad \widetilde{X}_{U \otimes P'} = \left(Q^{\mathrm{op}} \otimes Q', \overline{z}(-1)\right). \end{aligned}$$

Now we are ready to prove Theorem 4.26 (full-periodicity).

PROPOSITION 4.37. *We use the following periodicity result by [Kel2, Proposition 8.5]: $X \otimes \tau'^{-h-h'}(Y) \simeq X \otimes Y$ in $\mathcal{C}_A$ for any $X \in \mathcal{D}_Q$ and $Y \in \mathcal{D}_{Q'}$.*

Proof. We have $X \otimes \tau'^{-h-h'}(Y) \simeq F^{-h}(X \otimes \tau'^{-h-h'}(Y)) \overset{\mathrm{Lem.4.30}}{\simeq} \tau^h(X) \otimes \tau'^{-h'}(Y) \overset{\mathrm{Prop.4.11(1)}}{\simeq} X[-2] \otimes Y[2] \simeq X \otimes Y$. □

For a seed (R, y) of $\mathcal{A}_{Q \otimes Q'}$, let $[(R, y)]_{a,b}$ denote $y_{a,b}$.

THEOREM 4.38 (Theorem 4.26). *Now we are ready to prove Theorem 4.26 (full-periodicity): $[\mu_{\otimes}^{h+h'}(Q \otimes Q', z(0))]_{a,b} = z_{a,b}(0)$.*

Proof. We have $[\mu_{\otimes}^{h+h'}(P \otimes P')]_{a,b} \overset{\mathrm{Prop.4.33(2)}}{=} P_a \otimes \tau'^{-h-h'}(P'_b) \overset{\mathrm{Prop.4.37}}{=} P_a \otimes P'_b$ for any $(a,b) \in I \times I'$. Applying $\widetilde{X}$, we have $[\mu_{\otimes}^{h+h'}(Q \otimes Q', z(0))]_{a,b} \overset{\mathrm{Thm.4.36(1)}}{=} [\mu_{\otimes}^{h+h'}(\widetilde{X}_{P \otimes P'})]_{a,b} \overset{\mathrm{Thm.4.36(2)}}{=} X_{[\mu_{\otimes}^{h+h'}(P \otimes P')]_{a,b}} = X_{P_a \otimes P'_b} = z_{a,b}(0)$. □

Next we prove Theorem 4.27 (half-periodicity). We divide the proof into four cases.

Recall the following facts in Section 4.3: when h is even, the map $\omega : I \to I$ induces the quiver automorphism $\omega : Q \to Q$, the K-algebra automorphism $\omega : KQ \to KQ$, and an autoequivalence $\omega : \mathcal{D}_Q \to \mathcal{D}_Q$. We have $\omega(P_a) = P_{\omega(a)}$ for any $a \in I$. When h is odd, the map $\omega : I \to I$ induces the quiver isomorphism $\omega : Q \to Q^{\mathrm{op}}$, the K-algebra isomorphism $\omega : KQ \to KQ^{\mathrm{op}}$, and an autoequivalence

$$(4.89) \qquad r : \mathcal{D}_Q \overset{\omega}{\to} \mathcal{D}_{Q^{\mathrm{op}}} \xrightarrow{U \overset{\mathbf{L}}{\otimes}_{KQ^{\mathrm{op}}} -} \mathcal{D}_Q.$$

We have $r(P_a) = U_{\omega(a)}$ for any $a \in I$, and we have $r^2(X) \simeq \tau^{-1}(X)$ for any $X \in \mathcal{D}_Q$.

We also define $\omega' : \mathcal{D}_{Q'} \to \mathcal{D}_{Q'}$ for even h', and we define $r' : \mathcal{D}_{Q'} \to \mathcal{D}_{Q'}$ for odd h' in the same way.

Case 1: Both h and h' are even.

PROPOSITION 4.39. *We have* $X \otimes \tau'^{-(h+h')/2}(Y) \simeq \omega(X) \otimes \omega'(Y)$ *in* $\mathcal{C}_A$ *for any* $X \in \mathcal{D}_Q$ *and* $Y \in \mathcal{D}_{Q'}$.

Proof. We have $X \otimes \tau'^{-(h+h')/2}(Y) \simeq F^{-h/2}\big(X \otimes \tau'^{-(h+h')/2}(Y)\big) \overset{\text{Lem.4.30}}{\simeq}$ $\tau^{h/2}(X) \otimes \tau'^{-h'/2}(Y) \overset{\text{Prop.4.13(1)}}{\simeq} \omega^{-1}(X[-1]) \otimes \omega'(Y[1]) \simeq \omega(X) \otimes \omega'(Y)$. $\quad\square$

THEOREM 4.40 (Theorem 4.27(1)). *We have the following:*
$$\big[\mu_\otimes^{(h+h')/2}\big(Q \otimes Q', z(0)\big)\big]_{a,b} = z_{\omega(a),\omega'(b)}(0).$$

Proof. We have
$$[\mu_\otimes^{(h+h')/2}(P \otimes P')]_{a,b} \overset{\text{Prop.4.33(2)}}{=} P_a \otimes \tau'^{-(h+h')/2}(P_b') \overset{\text{Prop.4.39}}{=} P_{\omega(a)} \otimes P_{\omega'(b)}'.$$

Applying $\widetilde{X}$, we have
$$\big[\mu_\otimes^{(h+h')/2}\big(Q \otimes Q', z(0)\big)\big]_{a,b} \overset{\text{Thm.4.36(1)}}{=} [\mu_\otimes^{(h+h')/2}(\widetilde{X}_{P \otimes P'})]_{a,b}$$
$$\overset{\text{Thm.4.36(2)}}{=} X_{[\mu_\otimes^{(h+h')/2}(P \otimes P')]_{a,b}}$$
$$= X_{P_{\omega(a)} \otimes P_{\omega'(b)}'} = z_{\omega(a),\omega'(b)}(0).$$
$\quad\square$

Case 2: Both h and h' are odd.

PROPOSITION 4.41. *We have* $X \otimes \tau'^{-(h+h')/2}(Y) \simeq r(X) \otimes r'(Y)$ *in* $\mathcal{C}_A$ *for any* $X \in \mathcal{D}_Q$ *and* $Y \in \mathcal{D}_{Q'}$.

Proof. We have
$$X \otimes \tau'^{-(h+h')/2}(Y) \simeq F^{-(h-1)/2}\big(X \otimes \tau'^{-(h+h')/2}(Y)\big)$$
$$\overset{\text{Lem.4.30}}{\simeq} \tau^{(h-1)/2}(X) \otimes \tau'^{-1}\tau'^{-(h'-1)/2}(Y)$$
$$\overset{\text{Prop.4.15(1)}}{\simeq} r(X[-1]) \otimes \tau'^{-1}r'^{-1}(Y[1]) \simeq r(X) \otimes r'(Y).$$
$\quad\square$

THEOREM 4.42 (Theorem 4.27(2)). *We have*

$$\left[\mu_\otimes^{(h+h')/2}\bigl(Q\otimes Q',z(0)\bigr)\right]_{a,b} = \overline{z}_{\omega(a),\omega'(b)}(0).$$

Proof. We have

$$[\mu_\otimes^{(h+h')/2}(P\otimes P')]_{a,b} \overset{\text{Prop.4.33(2)}}{=} P_a \otimes \tau'^{-(h+h')/2}(P'_b)$$

$$\overset{\text{Prop.4.41}}{=} r(P_a) \otimes r'(P'_b) = U_{\omega(a)} \otimes U'_{\omega'(b)}.$$

Applying $\widetilde{X}$, we have

$$\left[\mu_\otimes^{(h+h')/2}\bigl(Q\otimes Q',z(0)\bigr)\right]_{a,b} \overset{\text{Thm.4.36(1)}}{=} [\mu_\otimes^{(h+h')/2}(\widetilde{X}_{P\otimes P'})]_{a,b}$$

$$\overset{\text{Thm.4.36(2)}}{=} X_{[\mu_\otimes^{(h+h')/2}(P\otimes P')]_{a,b}}$$

$$= X_{U_{\omega(a)}\otimes U'_{\omega'(b)}} = \overline{z}_{\omega(a),\omega'(b)}(0).$$

$\square$

Case 3: h is even, and h' is odd.

PROPOSITION 4.43. *We have* $X \otimes \tau'^{-(h+h'-1)/2}(Y) \simeq \omega(X) \otimes r'^{-1}(Y)$ *in* $\mathcal{C}_A$ *for any* $X \in \mathcal{D}_Q$ *and* $Y \in \mathcal{D}_{Q'}$.

Proof. We have

$$X \otimes \tau'^{-(h+h'-1)/2}(Y) \simeq F^{-h/2}\bigl(X \otimes \tau'^{-(h+h'-1)/2}(Y)\bigr)$$

$$\overset{\text{Lem.4.30}}{\simeq} \tau^{h/2}(X) \otimes \tau'^{-(h'-1)/2}(Y)$$

$$\overset{\text{Prop.4.13(1),4.15(1)}}{\simeq} \omega^{-1}(X[-1]) \otimes r'^{-1}(Y[1])$$

$$\simeq \omega(X) \otimes r'^{-1}(Y).$$

$\square$

THEOREM 4.44 (Theorem 4.27(3)). *We have* $[\mu_+\mu_\otimes^{(h+h'-1)/2}(Q \otimes Q',$ $z(0))]_{a,b} = z_{\omega(a),\omega'(b)}(0)$.

Proof. We have

$$[\mu_+\mu_\otimes^{(h+h'-1)/2}(P\otimes P')]_{a,b} \overset{\text{Prop.4.33(2)}}{=} P_a \otimes \tau'^{-(h+h'-1)/2}(U'_b)$$

$$\overset{\text{Prop.4.43}}{=} \quad P_{\omega(a)} \otimes r'^{-1}(U'_b)$$

$$= \quad P_{\omega(a)} \otimes P'_{\omega'(b)}.$$

Applying $\widetilde{X}$, we have

$$\left[\mu_+\mu_\otimes^{(h+h'-1)/2}(Q \otimes Q', z(0))\right]_{a,b} \overset{\text{Thm.4.36(1)}}{=} \left[\mu_+\mu_\otimes^{(h+h'-1)/2}(\widetilde{X}_{P\otimes P'})\right]_{a,b}$$

$$\overset{\text{Thm.4.36(2)}}{=} X_{[\mu_+\mu_\otimes^{(h+h'-1)/2}(P\otimes P')]_{a,b}}$$

$$= X_{P_{\omega(a)}\otimes P'_{\omega'(b)}} = z_{\omega(a),\omega'(b)}(0).$$

$\square$

Case 4: h is odd, and h' is even.

PROPOSITION 4.45. *We have* $X \otimes \tau'^{-(h+h'-1)/2}(Y) \simeq r(X) \otimes \omega'(Y)$ *in* $\mathcal{C}_A$ *for any* $X \in \mathcal{D}_Q$ *and* $Y \in \mathcal{D}_{Q'}$.

Proof. We have

$$X \otimes \tau'^{-(h+h'-1)/2}(Y) \simeq F^{-(h-1)/2}\left(X \otimes \tau'^{-(h+h'-1)/2}(Y)\right)$$

$$\overset{\text{Lem.4.30}}{\simeq} \tau^{(h-1)/2}(X) \otimes \tau'^{-(h'/2)}(Y)$$

$$\overset{\text{Prop.4.13(1),4.15(1)}}{\simeq} r(X[-1]) \otimes \omega'(Y[1]) \simeq r(X) \otimes \omega'(Y).$$

$\square$

THEOREM 4.46 (Theorem 4.27(4)). *We have* $[\mu_+\mu_\otimes^{(h+h'-1)/2}(Q \otimes Q', z(0))]_{a,b} = \overline{z}_{\omega(a),\omega'(b)}(0)$.

Proof. We have

$$[\mu_+\mu_\otimes^{(h+h'-1)/2}(P \otimes P')]_{a,b} \overset{\text{Prop.4.33(2)}}{=} P_a \otimes \tau'^{-(h+h'-1)/2}(U'_b)$$

$$\overset{\text{Prop.4.45}}{=} r(P_a) \otimes \omega(U'_b) = U_{\omega(a)} \otimes U'_{\omega'(b)}.$$

Applying $\widetilde{X}$, we have

$$\left[\mu_+\mu_\otimes^{(h+h'-1)/2}(Q \otimes Q', z(0))\right]_{a,b} \overset{\text{Thm.4.36(1)}}{=} \left[\mu_+\mu_\otimes^{(h+h'-1)/2}(\widetilde{X}_{P\otimes P'})\right]_{a,b}$$

$$\overset{\text{Thm.4.36(2)}}{=} X_{[\mu_+\mu_\otimes^{(h+h'-1)/2}(P\otimes P')]_{a,b}}$$

$$= X_{U_{\omega(a)}\otimes U'_{\omega'(b)}} = \overline{z}_{\omega(a),\omega'(b)}(0).$$

$\square$

§5. Determinant method I: $\mathcal{T}_\ell(A_r)$

Volkov [V] proved the periodicity of the regular solutions of $\mathbb{Y}_\ell(A_r)$ in $\mathbb{C}$ for any $\ell \geq 2$ by constructing the manifestly periodic determinant expression. In the process of the proof, he essentially proved the periodicity of the regular solutions of $\mathbb{T}_\ell(A_r)$ in $\mathbb{C}$ as well. In this section, we prove the periodicities of $\mathcal{T}_\ell(A_r)$ and $\mathcal{Y}_\ell(A_r)$ for any $\ell \geq 2$ by reformulating the determinant method in our setting to avoid the projective geometrical arguments used in [V], with the application to the C_r case also in mind.

5.1. Level ℓ restricted T-system with spiral boundary condition

Following [V], we introduce the level ℓ restricted T-system of type A_r with a more general boundary condition than the unit boundary condition for $\mathbb{T}_\ell(A_r)$ in Definition 3.1; we call it the *spiral boundary condition.*

Let us set

$$H = \{(a,m,u) \mid a = 0,\ldots,r+1; m = 0,\ldots,\ell; u \in \mathbb{Z}\},$$
$$H^\circ = \{(a,m,u) \in H \mid a \neq 0, r+1; m \neq 0, \ell\},$$
$$(5.1) \qquad \partial H = H \setminus H^\circ,$$
$$H_e = \{(a,m,u) \in H \mid a + m + u \text{ is even}\},$$
$$H_o = \{(a,m,u) \in H \mid a + m + u \text{ is odd}\}.$$

We also use the combined notations $H_o^\circ = H^\circ \cap H_o$, $\partial H_e = \partial H \cap H_e$, and so forth.

DEFINITION 5.1. Fix an integer $\ell \geq 2$. The *level ℓ restricted T-system* $\widetilde{\mathbb{T}}_\ell(A_r)$ *of type A_r with the spiral boundary condition* is the following system of relations for a family of variables $T = \{T_m^{(a)}(u) \mid (a,m,u) \in H\}$, as follows.
   (1) T-system:

$$(5.2) \quad T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u)$$
$$((a,m,u) \in H^\circ),$$

(2) Spiral boundary condition:

$$\begin{aligned}
T_0^{(a)}(u+1) &= T_0^{(a-1)}(u) && (a=1,\dots,r+1), \\
T_\ell^{(a)}(u+1) &= T_\ell^{(a+1)}(u) && (a=0,\dots,r), \\
T_m^{(0)}(u+1) &= T_{m+1}^{(0)}(u) && (m=0,\dots,\ell-1), \\
T_m^{(r+1)}(u+1) &= T_{m-1}^{(r+1)}(u) && (m=1,\dots,\ell).
\end{aligned}$$

(5.3)

DEFINITION 5.2. Let $\widetilde{\mathcal{T}}_\ell(A_r)$ be the ring with generators $T_m^{(a)}(u)^{\pm 1}$ $((a,m, u) \in H)$ and the relations $\widetilde{\mathbb{T}}_\ell(A_r)$.

Condition (5.3) means that $T_m^{(a)}(u)$ is *constant along spirals* on the boundary ∂H. In particular, if we impose $T_m^{(a)}(u)=1$ for any $(a,m,u) \in \partial H$, then $\widetilde{\mathbb{T}}_\ell(A_r)$ reduces to $\mathbb{T}_\ell(A_r)$. In other words, $\mathcal{T}_\ell(A_r)$ is isomorphic to $\widetilde{\mathcal{T}}_\ell(A_r)/J$, where J is the ideal of $\widetilde{\mathcal{T}}_\ell(A_r)$ generated by $T_m^{(a)}(u) - 1$ $((a,m,u) \in \partial H)$.

Recall that the dual Coxeter number of A_r is $r+1$. We will prove the following.

THEOREM 5.3 (Henriques [Hen, Theorem 5]). *The following relations hold in $\widetilde{\mathcal{T}}_\ell(A_r)$.*

(1) *Half-periodicity:* $T_m^{(a)}(u+r+1+\ell) = T_{\ell-m}^{(r+1-a)}(u)$.
(2) *Periodicity:* $T_m^{(a)}(u+2(r+1+\ell)) = T_m^{(a)}(u)$.

By the paragraph that follows Definition 5.2, we obtain the following.

COROLLARY 5.4. *The following relations hold in $\mathcal{T}_\ell(A_r)$.*

(1) *Half-periodicity:* $T_m^{(a)}(u+r+1+\ell) = T_{\ell-m}^{(r+1-a)}(u)$.
(2) *Periodicity:* $T_m^{(a)}(u+2(r+1+\ell)) = T_m^{(a)}(u)$.

REMARK 5.5. Theorem 5.3(1) is a corollary of a more general theorem by Henriques [Hen, Theorem 5], since the spiral boundary condition ensures that $c=1$ for [Hen, (11)]. However, as we mentioned earlier, we prove Theorem 5.3 by the determinant method of [V] to have the application to the C_r case in mind.

5.2. Proof of Theorem 5.3

Let R be any ring. Let us take an arbitrary $\ell \times \infty$ matrix M over R such that $M = [x_k]_{k \in \mathbb{Z}}$, $x_k \in R^\ell$ with the following periodicity:

(5.4) $$x_{k+r+1+\ell} = (-1)^{\ell-1} x_k.$$

Let $D_M = \{D_m^{(a)}(u) \mid (a, m, u) \in H_e\}$ be a family of minors of M defined by

$$(5.5) \qquad D_m^{(a)}(u) = \det[\overbrace{x_\beta x_{\beta+1} \cdots}^{m} \overbrace{\cdots \underset{\vee}{}}^{a} \overbrace{\cdots}^{\ell-m} \underset{\vee}{} \overbrace{\cdots \underset{x_{\beta+r+\ell}}{\vee}}^{r+1-a}],$$

$$(5.6) \qquad \beta = -\frac{a+m+u}{2} \in \mathbb{Z},$$

where $\overset{\vee}{x}_k$ means the omission of x_k as usual.

PROPOSITION 5.6. *The family $D_M = \{D_m^{(a)}(u) \mid (a, m, u) \in H_e\}$ satisfies the following relations in the ring R.*

(1) *T-system:*

$$(5.7) \quad D_m^{(a)}(u-1)D_m^{(a)}(u+1) = D_{m-1}^{(a)}(u)D_{m+1}^{(a)}(u) + D_m^{(a-1)}(u)D_m^{(a+1)}(u)$$
$$((a, m, u) \in H_o^\circ),$$

(2) *Spiral boundary condition:*

$$(5.8) \quad \begin{aligned} D_0^{(a)}(u+1) &= D_0^{(a-1)}(u) & (a = 1, \ldots, r+1), \\ D_\ell^{(a)}(u+1) &= D_\ell^{(a+1)}(u) & (a = 0, \ldots, r), \\ D_m^{(0)}(u+1) &= D_{m+1}^{(0)}(u) & (m = 0, \ldots, \ell-1), \\ D_m^{(r+1)}(u+1) &= D_{m-1}^{(r+1)}(u) & (m = 1, \ldots, \ell). \end{aligned}$$

(3) *Half-periodicity:*

$$(5.9) \qquad D_m^{(a)}(u+r+1+\ell) = D_{\ell-m}^{(r+1-a)}(u).$$

Proof. (1) They are the Plücker relations among minors.

(2) The first three relations follow from the definition of $D_m^{(a)}(u)$. To show the last relation, we also use the (anti-)periodicity (5.4) of x_k.

(3) It also follows from the definition of $D_m^{(a)}(u)$ and (5.4). □

REMARK 5.7. This determinant solution of the T-system (5.7) is regarded as the restricted version of [KLWZ, (2.25)].

Observe that, if we divide the family of generators $T = \{T_m^{(a)}(u) \mid (a, m, u) \in H\}$ of $\widetilde{\mathcal{T}}_\ell(A_r)$ into two subfamilies, $T_e = \{T_m^{(a)}(u) \mid (a, m, u) \in H_e\}$ and

$T_o = \{T_m^{(a)}(u) \mid (a, m, u) \in H_o\}$, then T_e and T_o have no mutual relation in $\widetilde{\mathbb{T}}_\ell(X_r)$. Therefore, to prove Theorem 5.3, it is enough to consider the half-family T_e of T (cf. (4.34)). Then, to prove Theorem 5.3, we have only to show the following.

PROPOSITION 5.8. *There exists some $\ell \times \infty$ matrix $M = [x_k]_{k\in\mathbb{Z}}$ over $\widetilde{\mathcal{T}}_\ell(A_r)$ satisfying the condition (5.4) such that, for $D_M = \{D_m^{(a)}(u) \mid (a, m, u) \in H_e\}$, the following relation holds in $\widetilde{\mathcal{T}}_\ell(A_r)$:*

$$(5.10) \qquad T_m^{(a)}(u) = D_m^{(a)}(u) \quad ((a, m, u) \in H_e).$$

Proof. We define $x_0, \ldots, x_{r+\ell} \in (\widetilde{\mathcal{T}}_\ell(A_r))^\ell$ as follows. First, let us arbitrarily choose $x_1, \ldots, x_{\ell-1}$ such that

$$(5.11) \qquad D_0^{(0)}(0) = \det[x_0 \cdots x_{\ell-1}] = T_0^{(0)}(0) = T_0^{(1)}(1)$$

holds. We define x_ℓ by

$$(5.12) \qquad x_\ell = \frac{1}{T_0^{(1)}(1)} \sum_{m=0}^{\ell-1} (-1)^{\ell-1-m} T_m^{(1)}(-1-m) x_m.$$

Then, the following equality holds:

$$(5.13) \qquad D_m^{(1)}(-1-m) = T_m^{(1)}(-1-m) \quad (m = 0, \ldots, \ell-1).$$

For example,

$$(5.14) \qquad \begin{aligned} D_0^{(1)}(-1) &= \det[x_1 \cdots x_\ell] \\ &= \frac{(-1)^{\ell-1} T_0^{(1)}(-1)}{T_0^{(1)}(1)} \det[x_1 \cdots x_{\ell-1} x_0] = T_0^{(1)}(-1). \end{aligned}$$

Similarly, we recursively define the rest, $x_{\ell+1}, \ldots, x_{r+\ell}$, by

$$(5.15) \quad x_{\ell+j} = \frac{1}{T_0^{(1)}(1-2j)} \sum_{m=0}^{\ell-1} (-1)^{\ell-1-m} T_m^{(1)}(-1-2j-m) x_{j+m}$$

$$(j = 1, \ldots, r)$$

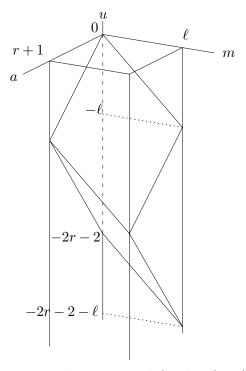so that the following equality holds (including (5.13) as $j = 0$):

Figure 4: The prism P defined in (5.17)

$$(5.16) \qquad D_m^{(1)}(-1-2j-m) = T_m^{(1)}(-1-2j-m)$$

$$(m = 0, \ldots, \ell-1; j = 0, \ldots, r).$$

Finally, we define the matrix $M = [x_k]_{k \in \mathbb{Z}}$ by extending the above $x_0, \ldots,$ $x_{r+\ell}$ with (5.4).

For D_M, we claim that the relation (5.10) holds in $\widetilde{\mathcal{T}}_\ell(A_r)$. This will be shown inductively, based on the fact that the T-system and the spiral boundary condition are satisfied by both T_e and D_M.

To proceed the induction, it is convenient to introduce a prism P, where

$$(5.17) \qquad P = \left\{ (a, m, u) \in H \mid a + m + u \leq 0, a - m - u \leq 2r + 2 \right\}$$

(see Figure 4). We use the notations $P_e = P \cap H_e$, $P_e[a = 1] = \{(a, m, u) \in P_e \mid a = 1\}$, and so forth.

First, we show that (5.10) is true for $(a, m, u) \in P_e$ by the induction on a. By (5.11), (5.16), and the spiral boundary condition, we see that (5.10)

is true for any (a, m, u) in the set

$$(5.18) \qquad P_e[a = 0] \cup P_e[a = 1] \cup P_e[m = 0] \cup P_e[m = \ell].$$

Assume that (5.10) is true up to a. By (5.2), we have

$$(5.19) \quad T_m^{(a+1)}(u) = \frac{1}{T_m^{(a-1)}(u)} \big(T_m^{(a)}(u - 1) T_m^{(a)}(u + 1) - T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) \big).$$

On the other hand, by (5.7) and the induction hypothesis, we have

$$(5.20) \quad D_m^{(a+1)}(u) = \frac{1}{T_m^{(a-1)}(u)} \big(T_m^{(a)}(u - 1) T_m^{(a)}(u + 1) - T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) \big).$$

Thus, the relation $D_m^{(a+1)}(u) = T_m^{(a+1)}(u)$ is obtained. Therefore, (5.10) is true for $(a, m, u) \in P_e$.

Next, we show that (5.10) is true for any $(a, m, u) \in H_e \setminus P_e$. We first remark that, by the spiral boundary condition, (5.10) is now true for any $(a, m, u) \in \partial H_e$. Then, using the T-system (5.2) once again as

$$(5.21) \quad T_m^{(a)}(u \pm 1) = \frac{1}{T_m^{(a)}(u \mp 1)} \big(T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u) T_m^{(a+1)}(u) \big),$$

and repeating the same argument as before, one can inductively, with respect to $\pm u$, conclude that (5.10) is true for any $(a, m, u) \in H_e \setminus P_e$. $\qquad \square$

This completes the proof of Theorem 5.3.

5.3. Periodicity of Y-system

The periodicity of $\mathcal{Y}_\ell(A_r)$ follows from that of $\widetilde{\mathcal{T}}_\ell(A_r)$. This is due to the fact that, unlike the unit boundary condition, the spiral boundary condition is *fully* compatible with the restriction of the Y-system in view of Theorem 2.12 (cf. Proposition 3.8). Namely, we have the following.

PROPOSITION 5.9. (1) *There is a ring homomorphism*

$$(5.22) \qquad\qquad \widetilde{\varphi}_\ell : \mathcal{Y}_\ell(A_r) \to \widetilde{\mathcal{T}}_\ell(A_r)$$

defined by

$$(5.23) \qquad Y_m^{(a)}(u) \mapsto \frac{T_m^{(a-1)}(u) T_m^{(a+1)}(u)}{T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u)} \quad ((a, m, u) \in H^\circ).$$

(2) *There is a ring homomorphism*

(5.24) $$\psi_\ell : \widetilde{\mathbb{T}}_\ell(A_r) \to \mathbb{Y}_\ell(A_r)$$

such that $\psi_\ell \circ \widetilde{\varphi}_\ell = \mathrm{id}_{\mathbb{Y}_\ell(X_r)}$.

Proof. (1) It is enough to check the compatibility between the boundary conditions of $\widetilde{\mathbb{T}}_\ell(A_r)$ and $\mathbb{Y}_\ell(A_r)$ as Theorem 2.12 and Proposition 3.8. For example, to see the compatibility with $Y_0^{(a)}(u)^{-1} = 0$ ($a \neq 0$), we formally extend (5.2) to $m = 0$ as

$$T_0^{(a)}(u-1)T_0^{(a)}(u+1) = T_{-1}^{(a)}(u)T_1^{(a)}(u) + T_0^{(a-1)}(u)T_0^{(a+1)}(u)$$

and use (5.3). Then we have $T_{-1}^{(a)}(u) = 0$. The other cases are checked similarly.

(2) We define the image $\psi_\ell(T_m^{(a)}(u))$ for the half-family T_e of T. The other half, T_o, is completely parallel. The construction is in four steps and similar to the one for Theorem 2.12. For simplicity, we write the image $\psi_\ell(T_m^{(a)}(u))$ as $T_m^{(a)}(u)$. Let P be the prism defined in (5.17).

Step 1. We arbitrarily choose $T_0^{(0)}(-2j) \in \mathbb{Y}_\ell(A_r)^\times$ ($j = 0, 1, \ldots, r+1$). Then, we define $T_m^{(a)}(u)$ for the rest of $(a, m, u) \in P_e[a = 0] \cup P_e[m = 0] \cup P_e[m = \ell]$ by (5.8).

Step 2. We arbitrarily choose $T_m^{(1)}(-1 - m) \in \mathbb{Y}_\ell(A_r)^\times$ ($m = 1, \ldots, \ell - 1$). Then, we recursively, with respect to u, define $T_m^{(a)}(u)$ for the rest of $(a, m, u) \in P_e[a = 1]$ by

(5.25) $$T_m^{(1)}(u-1) = \left(1 + Y_m^{(1)}(u)\right)\frac{T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u)}{T_m^{(1)}(u+1)}.$$

Step 3. We recursively, with respect to a, define $T_m^{(a)}(u)$ for the rest of $(a, m, u) \in P_e$ by

(5.26) $$T_m^{(a+1)}(u) = \frac{1}{1 + Y_m^{(a)}(u)^{-1}}\frac{T_m^{(a)}(u-1)T_m^{(a)}(u+1)}{T_m^{(a-1)}(u)}.$$

Step 4. We define $T_m^{(a)}(u)$ for the rest of $(a, m, u) \in \partial H_e$ by (5.8). Then, we recursively, with respect to $\pm u$, define $T_m^{(a)}(u)$ for the rest of $(a, m, u) \in H_e$ by

(5.27) $$T_m^{(a)}(u \pm 1) = \left(1 + Y_m^{(a)}(u)^{-1}\right)\frac{T_m^{(a-1)}(u)T_m^{(a+1)}(u)}{T_m^{(a)}(u \mp 1)}.$$

CLAIM. *The family $\{T_m^{(a)}(u) \mid (a, m, u) \in H_e\}$ defined above satisfies the following relations in $\mathcal{Y}_\ell(A_r)$:*

$$(5.28) \qquad 1 + Y_m^{(a)}(u) = \frac{T_m^{(a)}(u-1)T_m^{(a)}(u+1)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)} \quad ((a, m, u) \in H_o^\circ),$$

$$(5.29) \qquad 1 + Y_m^{(a)}(u)^{-1} = \frac{T_m^{(a)}(u-1)T_m^{(a)}(u+1)}{T_m^{(a-1)}(u)T_m^{(a+1)}(u)} \quad ((a, m, u) \in H_o^\circ).$$

The relation (5.29) clearly holds by the definition of $T_m^{(a)}(u)$ in (5.26) and (5.27). With (5.26), one can verify that (5.28) is true for any $(a, m, u) \in P_o^\circ := P \cap H_o^\circ$ by induction with respect to a. Then, in a similar way, with (5.27), one can prove that (5.28) is true for the rest of $(a, m, u) \in H_o^\circ$ by induction with respect to u. This completes the proof of the claim.

Now, taking the inverse sum of (5.28) and (5.29), we obtain (5.2). Therefore, ψ_ℓ is a ring homomorphism. Furthermore, taking the ratio of (5.28) and (5.29), we obtain $Y_m^{(a)}(u) = T_m^{(a-1)}(u)T_m^{(a+1)}(u)/(T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u))$. This proves $\psi_\ell \circ \widetilde{\varphi}_\ell = \mathrm{id}_{\mathcal{Y}_\ell(A_r)}$. $\qquad\square$

By Theorem 5.3 and Proposition 5.9, we obtain the following.

COROLLARY 5.10 (cf. Volkov [V, Theorem 1]). *The following relations hold in $\mathcal{Y}_\ell(A_r)$:*

(1) *Half-periodicity: $Y_m^{(a)}(u + r + 1 + \ell) = Y_{\ell-m}^{(r+1-a)}(u)$.*
(2) *Periodicity: $Y_m^{(a)}(u + 2(r + 1 + \ell)) = Y_m^{(a)}(u)$.*

By Proposition 5.9, we also obtain the following.

COROLLARY 5.11 (cf. Volkov [V, Proposition 1]). *For any ring R, the map*

$$(5.30) \qquad \widetilde{\varphi}_\ell^* : \mathrm{Hom}\big(\widetilde{\mathcal{T}}_\ell(A_r), R\big) \to \mathrm{Hom}\big(\mathcal{Y}_\ell(A_r), R\big),$$

induced from the homomorphism $\widetilde{\varphi}_\ell$ in (5.22), is surjective.

§6. Determinant method II: $\mathcal{T}_\ell(C_r)$

In this section we prove the periodicity of $\mathcal{T}_\ell(C_r)$ for any $\ell \geq 2$. We do it in three steps. First, we introduce a ring $\widetilde{\mathcal{T}}_\ell(C_r)$ by slightly generalizing the unit boundary condition of $\mathcal{T}_\ell(C_r)$. Second, we show $\widetilde{\mathcal{T}}_\ell(C_r)$ is isomorphic to another ring $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$, which is a variant of $\mathcal{T}_{2\ell}(A_{2r+1})$. Last, we apply the determinant method to $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$.

6.1. Level ℓ restricted T-system with quasi-unit boundary condition

For $X_r = C_r$, let t_a $(a \in I)$ be the number in (2.1), that is, $t_a = 2$ for $a = 1, \ldots, r-1$ and 1 for $a = r$.

We find that it is convenient to generalize the problem slightly as follows.

DEFINITION 6.1. Fix an integer $\ell \geq 2$. The *level ℓ restricted T-system* $\widetilde{\mathbb{T}}_\ell(C_r)$ *of type C_r with the quasi-unit boundary condition* is the following system of relations for a family of variables $T = \{T_m^{(a)}(u) \mid a = 1, \ldots, r; m = 1, \ldots, t_a \ell - 1; u \in (1/2)\mathbb{Z}\} \cup \{T_\ell^{(r)}(u) \mid u \in (1/2)\mathbb{Z}\}$.

(1) T-system (2.4):

$$(6.1) \qquad T_m^{(a)}\left(u - \frac{1}{2}\right) T_m^{(a)}\left(u + \frac{1}{2}\right) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u)$$
$$+ T_m^{(a-1)}(u) T_m^{(a+1)}(u)$$
$$(a = 1, \ldots, r-2; m = 1, \ldots, 2\ell - 1; u \in \tfrac{1}{2}\mathbb{Z}),$$

$$(6.2) \quad T_{2m}^{(r-1)}\left(u - \frac{1}{2}\right) T_{2m}^{(r-1)}\left(u + \frac{1}{2}\right) = T_{2m-1}^{(r-1)}(u) T_{2m+1}^{(r-1)}(u)$$
$$+ T_{2m}^{(r-2)}(u) T_m^{(r)}\left(u - \frac{1}{2}\right) T_m^{(r)}\left(u + \frac{1}{2}\right)$$
$$(m = 1, \ldots, \ell - 1; u \in \tfrac{1}{2}\mathbb{Z}),$$

$$(6.3) \quad T_{2m+1}^{(r-1)}\left(u - \frac{1}{2}\right) T_{2m+1}^{(r-1)}\left(u + \frac{1}{2}\right) = T_{2m}^{(r-1)}(u) T_{2m+2}^{(r-1)}(u)$$
$$+ T_{2m+1}^{(r-2)}(u) T_m^{(r)}(u) T_{m+1}^{(r)}(u)$$
$$(m = 0, \ldots, \ell - 1; u \in \tfrac{1}{2}\mathbb{Z}),$$

$$(6.4) \qquad T_m^{(r)}(u-1) T_m^{(r)}(u+1) = T_{m-1}^{(r)}(u) T_{m+1}^{(r)}(u) + T_{2m}^{(r-1)}(u)$$
$$(m = 1, \ldots, \ell - 1; u \in \tfrac{1}{2}\mathbb{Z}).$$

(2) Quasi-unit boundary condition:

$$(6.5) \qquad\qquad T_m^{(0)}(u) = T_0^{(a)}(u) = 1,$$

$$(6.6) \qquad\qquad T_{2\ell}^{(a)}(u) = 1 \quad (a = 1, \ldots, r-1),$$

if they occur in the right-hand sides of the relations, and

$$(6.7) \qquad\qquad T_\ell^{(r)}(u)^2 = 1, \qquad T_\ell^{(r)}(u+1) = T_\ell^{(r)}(u).$$

DEFINITION 6.2. Let $\widetilde{\mathfrak{T}}_\ell(C_r)$ be the ring with generators $T_m^{(a)}(u)^{\pm 1}$ ($a = 1, \ldots, r; m = 1, \ldots, t_a\ell - 1; u \in (1/2)\mathbb{Z}$), $T_\ell^{(r)}(u)$ ($u \in (1/2)\mathbb{Z}$) and the relations $\widetilde{\mathbb{T}}_\ell(C_r)$.

In (6.7), if we impose $T_\ell^{(r)}(u) = 1$ ($u \in (1/2)\mathbb{Z}$), then $\widetilde{\mathbb{T}}_\ell(C_r)$ reduces to $\mathbb{T}_\ell(C_r)$. In other words, $\mathfrak{T}_\ell(C_r)$ is isomorphic to $\widetilde{\mathfrak{T}}_\ell(C_r)/J$, where J is the ideal of $\widetilde{\mathfrak{T}}_\ell(C_r)$ generated by $T_\ell^{(r)}(u) - 1$ ($u \in (1/2)\mathbb{Z}$).

Recall that the dual Coxeter number of C_r is $r + 1$. We will prove the following.

THEOREM 6.3. *The following relations hold in $\widetilde{\mathfrak{T}}_\ell(C_r)$:*

(1) *Half-periodicity:* $T_m^{(a)}(u+r+1+\ell) = \begin{cases} T_{2\ell-m}^{(a)}(u) & a = 1, \ldots, r-1, \\ T_\ell^{(r)}(u)T_{\ell-m}^{(r)}(u) & a = r. \end{cases}$

(2) *Periodicity:* $T_m^{(a)}(u + 2(r+1+\ell)) = T_m^{(a)}(u).$

By the above remark, we obtain the following.

COROLLARY 6.4. *The following relations hold in $\mathfrak{T}_\ell(C_r)$.*

(1) *Half-periodicity:* $T_m^{(a)}(u + r + 1 + \ell) = T_{t_a\ell-m}^{(a)}(u).$

(2) *Periodicity:* $T_m^{(a)}(u + 2(r+1+\ell)) = T_m^{(a)}(u).$

6.2. System $\widehat{\mathbb{T}}_{2\ell}(A_{2r+1})$

To prove Theorem 6.3, we introduce another system of relations *that is equivalent to* $\widetilde{\mathbb{T}}_\ell(C_r)$.

Let us set

$$(6.8) \qquad \widehat{H} = \left\{ (a, m, u) \,\middle|\, a = 0, \ldots, 2r+2; m = 0, \ldots, 2\ell; u \in \frac{1}{2}\mathbb{Z} \right\},$$

$$(6.9) \qquad \widehat{H}^\circ = \left\{ (a, m, u) \in \widehat{H} \mid a \neq 0, 2r+2; m \neq 0, 2\ell \right\},$$

$$(6.10) \qquad \partial\widehat{H} = \widehat{H} \setminus \widehat{H}^\circ,$$

$$(6.11) \qquad \widehat{H}_e = \left\{ (a, m, u) \in \widehat{H} \mid a + m + 2u \text{ is even} \right\},$$

$$(6.12) \qquad \widehat{H}_o = \left\{ (a, m, u) \in \widehat{H} \mid a + m + 2u \text{ is odd} \right\}.$$

We again use the combined notations $\widehat{H}_o^\circ = \widehat{H}^\circ \cap \widehat{H}_o$, $\partial\widehat{H}_e = \partial\widehat{H} \cap \widehat{H}_e$, and so forth.

DEFINITION 6.5. Fix an integer $\ell \geq 2$. The *level 2ℓ restricted T-system* $\widehat{\mathbb{T}}_{2\ell}(A_{2r+1})$ *of type A_{2r+1} with the quasi-symmetric condition* is the following system of relations for a family of variables $S = \{S_m^{(a)}(u) \mid (a, m, u) \in \widehat{H}^\circ\}$.

(1) T-system:

$$(6.13) \quad S_m^{(a)}\left(u - \frac{1}{2}\right) S_m^{(a)}\left(u + \frac{1}{2}\right) = S_{m-1}^{(a)}(u) S_{m+1}^{(a)}(u) + S_m^{(a-1)}(u) S_m^{(a+1)}(u).$$

(2) Quasi-symmetric condition:

$$(6.14) \qquad\qquad S_m^{(a)}(u) = (-1)^m S_m^{(2r+2-a)}(u).$$

In particular, $S_m^{(r+1)}(u) = 0$ for odd m.

(3) Quasi-symmetric unit boundary condition:

$$(6.15) \qquad\qquad S_0^{(a)}(u) = S_{2\ell}^{(a)}(u) = S_m^{(0)}(u) = 1,$$

$$(6.16) \qquad\qquad S_m^{(2r+2)}(u) = (-1)^m,$$

if they occur in the right-hand side of the relations.

DEFINITION 6.6. Let $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$ be the ring with generators $S_m^{(a)}(u)^{\pm 1}$ $((a,m,u) \in \widehat{H}^\circ, (a,m) \neq (r+1, \text{odd}))$ and the relations $\widehat{\mathbb{T}}_{2\ell}(A_{2r+1})$.

REMARK 6.7. The system $\widehat{\mathbb{T}}_{2\ell}(A_{2r+2})$ and the map ρ below are the restricted versions of those considered in [KOSY, Section 4].

PROPOSITION 6.8. *There is a ring isomorphism*

$$(6.17) \qquad\qquad \rho : \widehat{\mathcal{T}}_{2\ell}(A_{2r+1}) \xrightarrow{\sim} \widetilde{\mathcal{T}}_\ell(C_r)$$

defined by

$$(6.18) \qquad S_m^{(a)}(u) \mapsto T_m^{(a)}(u) \quad (a = 1, \ldots, r-1),$$

$$(6.19) \qquad S_{2m}^{(r)}(u) \mapsto T_m^{(r)}\left(u - \frac{1}{2}\right) T_m^{(r)}\left(u + \frac{1}{2}\right),$$

$$(6.20) \qquad S_{2m+1}^{(r)}(u) \mapsto T_m^{(r)}(u) T_{m+1}^{(r)}(u),$$

$$(6.21) \qquad S_{2m}^{(r+1)}(u) \mapsto T_m^{(r)}(u)^2,$$

$$(6.22) \qquad \rho\big(S_m^{(a)}(u)\big) = (-1)^m \rho\big(S_m^{(2r+2-a)}(u)\big) \quad (a = r+2, \ldots, 2r+1),$$

where $T_0^{(r)}(u) = 1$.

Proof. It is easy to check that the map ρ is a ring homomorphism by substitution (cf. [KOSY, Proposition 4.3]). For simplicity, we write the image $\rho(S_m^{(a)}(u))$ as $S_m^{(a)}(u)$. Then, for example, to show

$$(6.23) \qquad S_{2m}^{(r)}\Big(u - \frac{1}{2}\Big) S_{2m}^{(r)}\Big(u + \frac{1}{2}\Big) = S_{2m-1}^{(r)}(u) S_{2m+1}^{(r)}(u) \\ + S_{2m}^{(r-1)}(u) S_{2m}^{(r+1)}(u),$$

multiply (6.4) by $T_m^{(r)}(u)^2$, and then use (6.19)–(6.21).

Let us consider the inverse map $\rho^{-1} : \widetilde{\mathfrak{T}}_\ell(C_r) \to \widehat{\mathfrak{T}}_{2\ell}(A_{2r+1})$. Looking at (6.18) and (6.20), it should be given by the following correspondence:

$$(6.24) \qquad T_m^{(a)}(u) \mapsto S_m^{(a)}(u) \quad (a = 1, \dots, r-1),$$

$$(6.25) \qquad T_m^{(r)}(u) \mapsto \begin{cases} \dfrac{S_{2m-1}^{(r)}(u) S_{2m-5}^{(r)}(u) \cdots S_3^{(r)}(u)}{S_{2m-3}^{(r)}(u) S_{2m-7}^{(r)}(u) \cdots S_1^{(r)}(u)} & (m: \text{even}), \\[3mm] \dfrac{S_{2m-1}^{(r)}(u) S_{2m-5}^{(r)}(u) \cdots S_1^{(r)}(u)}{S_{2m-3}^{(r)}(u) S_{2m-7}^{(r)}(u) \cdots S_3^{(r)}(u)} & (m: \text{odd}). \end{cases}$$

For simplicity, we write the image $\rho^{-1}(T_m^{(a)}(u))$ as $T_m^{(a)}(u)$.

CLAIM. *The family $T = \{T_m^{(a)}(u)\}$ above satisfies the following relations in $\widehat{\mathfrak{T}}_{2\ell}(A_{2r+1})$:*

$$(6.26) \qquad T_m^{(r)}\Big(u - \frac{1}{2}\Big) T_m^{(r)}\Big(u + \frac{1}{2}\Big) = S_{2m}^{(r)}(u) \qquad (m = 1, \dots, \ell),$$

$$(6.27) \qquad T_m^{(r)}(u) T_{m+1}^{(r)}(u) = S_{2m+1}^{(r)}(u) \quad (m = 1, \dots, \ell - 1),$$

$$(6.28) \qquad T_m^{(r)}(u)^2 = S_{2m}^{(r+1)}(u) \quad (m = 1, \dots, \ell),$$

where $S_{2\ell}^{(r)}(u) = S_{2\ell}^{(r+1)}(u) = 1$.

Indeed, (6.27) follows immediately from (6.25), while (6.26) and (6.28) are proved by the induction with respect to m.

Now, it is easy to check that ρ^{-1} is a ring homomorphism by substitution.

By comparing (6.18)–(6.21) with (6.24), (6.26)–(6.28), we see that ρ^{-1} is the inverse of ρ. $\square$

The following theorem is an analogue of Theorem 5.3.

THEOREM 6.9. *The following relations hold in $\widehat{\mathfrak{T}}_{2\ell}(A_{2r+1})$:*

(1) *Half-periodicity:* $S_m^{(a)}(u+r+1+\ell) = (-1)^m S_{2\ell-m}^{(2r+2-a)}(u) = S_{2\ell-m}^{(a)}(u)$.

(2) *Periodicity:* $S_m^{(a)}(u+2(r+1+\ell)) = S_m^{(a)}(u)$.

A proof of Theorem 6.9 by the determinant method will be given in Section 6.3.

Admitting Theorem 6.9, let us prove Theorem 6.3 first.

Proof of Theorem 6.3. It is enough to prove (1). For simplicity, we write the image $\rho(S_m^{(a)}(u))$ as $S_m^{(a)}(u)$. By (6.18), we immediately obtain the relation for $a = 1, \ldots, r-1$. Let us verify the case $a = r$. By Proposition 6.8 and Theorem 6.9, we have (e.g., for ℓ, m: even)

$$
\begin{aligned}
&T_m^{(r)}(u+r+1+\ell) \\
&= \frac{S_{2m-1}^{(r)}(u+r+1+\ell)S_{2m-5}^{(r)}(u+r+1+\ell)\cdots S_3^{(r)}(u+r+1+\ell)}{S_{2m-3}^{(r)}(u+r+1+\ell)S_{2m-7}^{(r)}(u+r+1+\ell)\cdots S_1^{(r)}(u+r+1+\ell)} \\
&= \frac{S_{2\ell-2m+1}^{(r)}(u)S_{2\ell-2m+5}^{(r)}(u)\cdots S_{2\ell-3}^{(r)}(u)}{S_{2\ell-2m+3}^{(r)}(u)S_{2\ell-2m+7}^{(r)}(u)\cdots S_{2\ell-1}^{(r)}(u)} \\
&= T_\ell^{(r)}(u)\frac{S_{2\ell-2m-1}^{(r)}(u)S_{2\ell-2m-5}^{(r)}(u)\cdots S_3^{(r)}(u)}{S_{2\ell-2m-3}^{(r)}(u)S_{2\ell-2m-7}^{(r)}(u)\cdots S_1^{(r)}(u)} \\
&= T_\ell^{(r)}(u)T_{\ell-m}^{(r)}(u).
\end{aligned}
$$

(6.29)

$\square$

6.3. Proof of Theorem 6.9

The outline of the proof is the same as for Theorem 5.3, but there are some points where extra caution is necessary due to the singularities $S_m^{(r+1)}(u) = 0$ (m: odd).

Let R be any ring. Let us take an arbitrary $2\ell \times \infty$ matrix M over R such that $M = [x_k]_{k\in\mathbb{Z}}$, $x_k \in (\widehat{\mathcal{T}}_{2\ell}(A_{2r+1}))^{2\ell}$ with the following periodicity:

(6.30)
$$x_{k+2r+2+2\ell} = x_k.$$

Let $D_M = \{D_m^{(a)}(u) \mid (a, m, u) \in \widehat{H}_e\}$ be a family of minors of M defined by

(6.31)
$$D_m^{(a)}(u) = \det[\overbrace{x_\beta x_{\beta+1} \cdots}^{m} \overset{\vee}{\cdots} \overset{\vee}{\overbrace{\cdots}^{a}} \overset{\vee}{\overbrace{\cdots}^{2\ell-m}} \overset{\vee}{\overbrace{\cdots x_{\beta+2r+2\ell+1}}^{2r+2-a}}],$$

(6.32) $$\beta = -\frac{a+m+2u}{2} \in \mathbb{Z}.$$

Then, as Proposition 5.6, we have the following.

PROPOSITION 6.10. *The family* $D_M = \{D_m^{(a)}(u) \mid (a,m,u) \in \widehat{H}_e\}$ *satisfies the following relations in* R.

(1) *T-system:*

$$(6.33) \qquad D_m^{(a)}\left(u - \frac{1}{2}\right) D_m^{(a)}\left(u + \frac{1}{2}\right) = D_{m-1}^{(a)}(u) D_{m+1}^{(a)}(u)$$
$$+ D_m^{(a-1)}(u) D_m^{(a+1)}(u)$$
$$((a,m,u) \in \widehat{H}_o^\circ),$$

(2) *Boundary condition:*

$$(6.34) \qquad \begin{aligned} D_0^{(a)}\left(u + \frac{1}{2}\right) &= D_0^{(a-1)}(u) \quad (a = 1, \ldots, 2r+2), \\ D_{2\ell}^{(a)}\left(u + \frac{1}{2}\right) &= D_{2\ell}^{(a+1)}(u) \quad (a = 0, \ldots, 2r+1), \\ D_m^{(0)}\left(u + \frac{1}{2}\right) &= D_{m+1}^{(0)}(u) \quad (m = 0, \ldots, 2\ell - 1), \\ D_m^{(2r+2)}\left(u + \frac{1}{2}\right) &= (-1) D_{m-1}^{(2r+2)}(u) \quad (m = 1, \ldots, 2\ell). \end{aligned}$$

(3) *Half-periodicity:*

$$(6.35) \qquad D_m^{(a)}(u + r + 1 + \ell) = (-1)^m D_{2\ell-m}^{(2r+2-a)}(u).$$

We consider an extended family of generators $S = \{S_m^{(a)}(u) \mid (a,m,u) \in \widehat{H}\}$ of $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$, where $S_m^{(r+1)}(u) = 0$ (*m: odd*) and $S_m^{(a)}(u)$ $((a,m,u) \in \partial\widehat{H})$ are given by (6.15) and (6.16). Again, we divide the family S into two subfamilies, $S_e = \{S_m^{(a)}(u) \mid (a,m,u) \in \widehat{H}_e\}$ and $S_o = \{S_m^{(a)}(u) \mid (a,m,u) \in \widehat{H}_o\}$, and we concentrate on the half-family S_e.

Theorem 6.9 follows from Proposition 6.10 and the following proposition.

PROPOSITION 6.11. *There exists some* $2\ell \times \infty$ *matrix* $M = \{x_k\}_{k\in\mathbb{Z}}$ *satisfying the condition* (6.30) *such that, for* $D_M = \{D_m^{(a)}(u) \mid (a,m,u) \in \widehat{H}_e\}$, *the following relation holds in* $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$:

$$(6.36) \qquad S_m^{(a)}(u) = D_m^{(a)}(u) \quad ((a,m,u) \in \widehat{H}_e).$$

Proof. We define $x_0, \dots, x_{2r+2\ell+1} \in \big(\widehat{\mathfrak{T}}_{2\ell}(A_{2r+1})\big)^{2\ell}$ as follows. First, let us arbitrarily choose $x_1, \dots, x_{2\ell-1}$ such that

$$(6.37) \qquad D_0^{(0)}(0) := \det[x_0 \cdots x_{2\ell-1}] = 1 = S_0^{(0)}(0) = S_0^{(1)}\Big(\frac{1}{2}\Big)$$

holds. Second, we recursively define the rest, $x_{2\ell}, \dots, x_{2r+2\ell+1}$, by

$$(6.38) \quad x_{2\ell+j} = \frac{1}{S_0^{(1)}\big(\frac{1}{2} - j\big)} \sum_{m=0}^{2\ell-1} (-1)^{2\ell-1-m} S_m^{(1)}\Big(-\frac{1+2j+m}{2}\Big) x_{j+m}$$

$$(j = 0, \dots, 2r+1)$$

so that the following equality holds:

$$(6.39) \qquad D_m^{(1)}\Big(-\frac{1+2j+m}{2}\Big) = S_m^{(1)}\Big(-\frac{1+2j+m}{2}\Big)$$

$$(m = 0, \dots, 2\ell-1; j = 0, \dots, 2r+1).$$

Last, we define the matrix $M = [x_k]_{k \in \mathbb{Z}}$ by extending the above $x_0, \dots, x_{2r+2\ell+1}$ with the condition (6.30).

For D_M, we claim that the relation (6.36) holds in $\widehat{\mathfrak{T}}_{2\ell}(A_{2r+1})$. This will be shown inductively, based on the fact that the T-system and the boundary condition are satisfied by both S_e and D_M.

To proceed with the induction, we introduce a prism

$$(6.40) \qquad \widehat{P} = \big\{(a, m, u) \in \widehat{H} \mid a+m+2u \le 0, a-m-2u \le 4r+4\big\}.$$

We use the notations, $\widehat{P}_e = \widehat{P} \cap \widehat{H}_e$, $\widehat{P}_e[a=1] = \{(a, m, u) \in \widehat{P}_e \mid a = 1\}$, and so on, once again.

First, we show that (6.36) is true for $(a, m, u) \in \widehat{P}_e$ by the induction on a. By (6.37), (6.39), and (6.34), we see that (6.36) is true for any (a, m, u) in the set

$$(6.41) \qquad \widehat{P}_e[a=0] \cup \widehat{P}_e[a=1] \cup \widehat{P}_e[m=0] \cup \widehat{P}_e[m=2\ell].$$

Assume that (6.36) is true up to a. By the T-system (6.13), we have

$$(6.42) \qquad S_m^{(a+1)}(u) = \frac{1}{S_m^{(a-1)}(u)} \Big(S_m^{(a)}\Big(u - \frac{1}{2}\Big) S_m^{(a)}\Big(u + \frac{1}{2}\Big)$$

$$- S_{m-1}^{(a)}(u) S_{m+1}^{(a)}(u) \Big).$$

On the other hand, by (6.33) and the induction hypothesis, we have

$$(6.43) \qquad D_m^{(a+1)}(u) = \frac{1}{S_m^{(a-1)}(u)} \left(S_m^{(a)}\left(u - \frac{1}{2}\right) S_m^{(a)}\left(u + \frac{1}{2}\right) \right.$$
$$\left. - S_{m-1}^{(a)}(u) S_{m+1}^{(a)}(u) \right).$$

Thus, the relation $S_m^{(a+1)}(u) = D_m^{(a+1)}(u)$ is obtained. The induction works up to $a = r + 2$. However, since $S_{2m+1}^{(r+1)}(u) = D_{2m+1}^{(r+1)}(u) = 0$ ($m = 0, \ldots, \ell - 1$) by the assumption, the relation $S_{2m+1}^{(r+3)}(u) = D_{2m+1}^{(r+3)}(u)$ is not trivial. To overcome the point, we have to show the following claim:

CLAIM 1. *For* $(r + 3, 2m + 1, u) \in \widehat{P}_e$, *the following relation holds in* $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$:

$$(6.44) \qquad D_{2m+1}^{(r+3)}(u) = -D_{2m+1}^{(r-1)}(u).$$

Once the claim is verified, we have

$$S_{2m+1}^{(r+3)}(u) = -S_{2m+1}^{(r-1)}(u) = -D_{2m+1}^{(r-1)}(u) = D_{2m+1}^{(r+3)}(u)$$

so that the induction continues and completes.

Let us prove Claim 1. For simplicity, we write the ring $\widehat{\mathcal{T}}_{2\ell}(A_{2r+1})$ as R. Consider a deformation $M_\varepsilon = [x_k + \varepsilon x_k']$ of the matrix M with a formal parameter ε and some $x_k' \in R^{2\ell}$ ($k \in \mathbb{Z}$, $x_{k+2r+2+2\ell}' = x_k'$). Let $D_m^{(a)}(u)_\varepsilon \in R[\varepsilon]$ be the corresponding minor for M_ε. Fix $2m + 1$ and u in (6.44). We choose M_ε so that $D_{2m+1}^{(r+1)}(u)_\varepsilon = \varepsilon D_{2m+1}^{(r+1)}(u)' + o(\varepsilon)$ with $D_{2m+1}^{(r+1)}(u)' \in R^\times$. (This is possible. For example, let x' be the $2m + 2$th column for $D_{2m+1}^{(r)}(u + \frac{1}{2})$. Then, add $\varepsilon x'$ to M at the position of the last column for $D_{2m+1}^{(r+1)}(u)$. This deformation yields $D_{2m+1}^{(r+1)}(u)' = D_{2m+1}^{(r)}(u + \frac{1}{2}) = T_{2m+1}^{(r)}(u + \frac{1}{2}) \in R^\times$.) Now, by (6.33), we have

$$(6.45) \quad \begin{aligned} D_{2m+1}^{(r-1)}(u)_\varepsilon D_{2m+1}^{(r+1)}(u)_\varepsilon &= D_{2m+1}^{(r)}\left(u - \frac{1}{2}\right)_\varepsilon D_{2m+1}^{(r)}\left(u + \frac{1}{2}\right)_\varepsilon \\ &\quad - D_{2m}^{(r)}(u)_\varepsilon D_{2m+2}^{(r)}(u)_\varepsilon, \\ D_{2m+1}^{(r+3)}(u)_\varepsilon D_{2m+1}^{(r+1)}(u)_\varepsilon &= D_{2m+1}^{(r+2)}\left(u - \frac{1}{2}\right)_\varepsilon D_{2m+1}^{(r+2)}\left(u + \frac{1}{2}\right)_\varepsilon \\ &\quad - D_{2m}^{(r+2)}(u)_\varepsilon D_{2m+2}^{(r+2)}(u)_\varepsilon. \end{aligned}$$

We take the ratio of the relations in (6.45),

$$(6.46) \qquad \frac{D_{2m+1}^{(r+3)}(u)_\varepsilon}{D_{2m+1}^{(r-1)}(u)_\varepsilon}$$

$$= \frac{D_{2m+1}^{(r+2)}\big(u - \tfrac{1}{2}\big)_\varepsilon D_{2m+1}^{(r+2)}\big(u + \tfrac{1}{2}\big)_\varepsilon - D_{2m}^{(r+2)}(u)_\varepsilon D_{2m+2}^{(r+2)}(u)_\varepsilon}{D_{2m+1}^{(r)}\big(u - \tfrac{1}{2}\big)_\varepsilon D_{2m+1}^{(r)}\big(u + \tfrac{1}{2}\big)_\varepsilon - D_{2m}^{(r)}(u)_\varepsilon D_{2m+2}^{(r)}(u)_\varepsilon},$$

which is in $R[[\varepsilon]]$. Let us calculate the right-hand side of (6.46). Let $\equiv$ mean the equality in $R[[\varepsilon]]$ modulo $\varepsilon R[[\varepsilon]]$. By (6.33) and $D_{2m+1}^{(r+1)}(u)_\varepsilon \equiv 0$, we have

$$D_{2m}^{(r+2)}(u)_\varepsilon \equiv D_{2m}^{(r+1)}\Big(u - \frac{1}{2}\Big)_\varepsilon D_{2m}^{(r+1)}\Big(u + \frac{1}{2}\Big)_\varepsilon / D_{2m}^{(r)}(u)_\varepsilon,$$

$$D_{2m+2}^{(r+2)}(u)_\varepsilon \equiv D_{2m+2}^{(r+1)}\Big(u - \frac{1}{2}\Big)_\varepsilon D_{2m+2}^{(r+1)}\Big(u + \frac{1}{2}\Big)_\varepsilon / D_{2m+2}^{(r)}(u)_\varepsilon,$$

$$(6.47)$$
$$D_{2m+1}^{(r+2)}\Big(u + \frac{1}{2}\Big)_\varepsilon \equiv -D_{2m}^{(r+1)}\Big(u + \frac{1}{2}\Big)_\varepsilon D_{2m+2}^{(r+1)}\Big(u + \frac{1}{2}\Big)_\varepsilon / D_{2m+1}^{(r)}\Big(u + \frac{1}{2}\Big)_\varepsilon,$$

$$D_{2m+1}^{(r+2)}\Big(u - \frac{1}{2}\Big)_\varepsilon \equiv -D_{2m}^{(r+1)}\Big(u - \frac{1}{2}\Big)_\varepsilon D_{2m+2}^{(r+1)}\Big(u - \frac{1}{2}\Big)_\varepsilon / D_{2m+1}^{(r)}\Big(u - \frac{1}{2}\Big)_\varepsilon.$$

Thanks to (6.47), the right-hand side of (6.46) equals, modulo $\varepsilon R[[\varepsilon]]$,

$$(6.48) \qquad (-1) \frac{D_{2m}^{(r+1)}\big(u - \tfrac{1}{2}\big) D_{2m+2}^{(r+1)}\big(u - \tfrac{1}{2}\big) D_{2m}^{(r+1)}\big(u + \tfrac{1}{2}\big) D_{2m+2}^{(r+1)}\big(u + \tfrac{1}{2}\big)}{D_{2m+1}^{(r)}\big(u - \tfrac{1}{2}\big) D_{2m+1}^{(r)}\big(u + \tfrac{1}{2}\big) D_{2m}^{(r)}(u) D_{2m+2}^{(r)}(u)}.$$

Then, using the induction hypothesis (6.36) up to $a = r + 2$ in $\widehat{P}_e$ and the relations (6.13) and (6.14), one can show that (6.48) is equal to -1. Therefore, we have $D_{2m+1}^{(r+3)}(u)_\varepsilon \equiv -D_{2m+1}^{(r-1)}(u)_\varepsilon$, which means (6.44). This ends the proof of Claim 1.

Next, we show that (6.36) is true for any $(a, m, u) \in \widehat{H}_e \setminus \widehat{P}_e$. We first remark that, by (6.34), (6.36) is now true for any $(a, m, u) \in \partial \widehat{H}_e$. Then, using the T-system (6.13) as

$$(6.49) \qquad S_m^{(a)}\Big(u \pm \frac{1}{2}\Big) = \frac{1}{S_m^{(a)}\big(u \mp \tfrac{1}{2}\big)} \big(S_{m-1}^{(a)}(u) S_{m+1}^{(a)}(u)$$
$$+ S_m^{(a-1)}(u) S_m^{(a+1)}(u)\big),$$

and repeating the same argument as before, one can inductively, with respect to $\pm u$, conclude that (6.36) is true for any $(a, m, u) \in \widehat{H}_e \setminus \widehat{P}_e$. Once again,

in the induction process we need the following claim, which can be verified by a similar deformation argument as Claim 1.

CLAIM 2. *For $(r+1, 2m+1, u) \in \widehat{H}_e \setminus \widehat{P}_e$, the following relation holds in $\widehat{\mathfrak{T}}_{2\ell}(A_{2r+1})$:*

$$(6.50) \qquad D_{2m+1}^{(r+1)}(u) = 0.$$

This ends the proof of Proposition 6.11. ∎

REMARK 6.12. The system $\widetilde{\mathbb{T}}_\ell(C_r)$ plays a role analogous to that of $\widetilde{\mathbb{T}}_\ell(A_r)$ in the determinant method. Thus, it is natural to expect that the analogue of Proposition 5.9 also holds. Unfortunately, this is not true. The simplest case, C_2 and $\ell = 2$, provides a counterexample by the similar reason to Example 3.9(3).

§7. Direct method: $\mathfrak{T}_2(A_r)$, $\mathfrak{T}_2(D_r)$, and $\mathfrak{T}_2(B_r)$

In this section we give some explicit formulae of generators $T_m^{(a)}(u)$ in terms of "initial variables" for $\mathfrak{T}_2(A_r)$, $\mathfrak{T}_2(D_r)$, and $\mathfrak{T}_2(B_r)$. Our goal here is to prove the periodicity for $\mathfrak{T}_2(B_r)$. The formulae for $\mathfrak{T}_2(A_r)$ and $\mathfrak{T}_2(D_r)$ should be obtained as a specialization of the more general formulae by Caldero and Chapoton [CC, Theorem 3.4] and/or by Yang and Zelevinsky [YZ, Theorems 1.10, 1.12]. Nevertheless, we include these formulae as well, since they are good examples showing how the periodicity actually happens in the T-systems.

7.1. Explicit formula by initial variables for $\mathfrak{T}_2(A_r)$

Throughout this subsection, $I = \{1, \ldots, r\}$ is the set enumerating the diagram A_r in Figure 1. Recall that the (dual) Coxeter number of A_r is $r + 1$.

DEFINITION 7.1. For a family of variables $X = \{x_a \mid a \in I\}$, we define $\gamma_n^{(j)}(X), \nu_n^{(j)}(X) \in \mathbb{Z}[X^{\pm 1}]$ $(1 \leq j \leq r, 0 \leq n \leq j)$ as follows:

$$(7.1) \qquad \gamma_n^{(j)}(X) = \frac{1}{x_n} + \frac{x_{n-1}}{x_n x_{n+1}} + \frac{x_{n-1}}{x_{n+1} x_{n+2}} + \cdots + \frac{x_{n-1}}{x_{j-2} x_{j-1}},$$

$$(7.2) \qquad \nu_n^{(j)}(X) = \frac{x_{n-1}}{x_{j-1}},$$

where we set $x_{-1} = 0$ and $x_0 = 1$. In particular,

$$(7.3) \qquad \gamma_0^{(j)}(X) = 1, \qquad \nu_0^{(j)}(X) = 0,$$

$$(7.4) \qquad \gamma_j^{(j)}(X) = 0, \qquad \nu_j^{(j)}(X) = 1.$$

The following lemma is easily checked.

LEMMA 7.2. *The following relations hold for any $1 \leq j \leq r$ and $1 \leq n \leq j$.*

$$(7.5) \qquad x_{n-1}\gamma_{n-1}^{(j)}(X) = 1 + x_{n-2}\gamma_n^{(j)}(X),$$

$$(7.6) \qquad x_{j-1}\gamma_{n-1}^{(j)}(X) = x_{j-1}\gamma_{n-1}^{(j-1)}(X) + \nu_{n-1}^{(j-1)}(X).$$

We define a family $\tau = \{\tau^{(a)}(u; X) \in \mathbb{Z}[X^{\pm 1}] \mid a \in I, u \in \mathbb{Z}, a + u \text{ is odd}\}$ as follows. First, we define, for $a \in I$ and $0 \leq k \leq r + 2$,

$$(7.7) \quad \tau^{(a)}(a - 1 + 2k; X) = \begin{cases} \gamma_k^{(a+k)}(X)x_{a+k} + \nu_k^{(a+k)}(X) \\ \quad (0 \leq k \leq r - a), \\ \gamma_{r+1-a}^{(r)}(X) + \nu_{r+1-a}^{(r)}(X)\frac{1+x_{r-1}}{x_r} \\ \quad (k = r - a + 1), \\ \gamma_{a+k-(r+2)}^{(k-1)}(X)x_{k-1} + \nu_{a+k-(r+2)}^{(k-1)}(X) \\ \quad (r - a + 2 \leq k \leq r + 1), \\ \gamma_a^{(r)}(X) + \nu_a^{(r)}(X)\frac{1+x_{r-1}}{x_r} \quad (k = r + 2). \end{cases}$$

Then, we extend the definition by the following periodicity:

$$(7.8) \qquad \tau^{(a)}(u + 2(r + 3); X) = \tau^{(a)}(u; X).$$

By (7.3), we have

$$(7.9) \qquad \tau^{(a)}(a - 1; X) = x_a,$$

$$(7.10) \qquad \tau^{(a)}(2r + 3 - a; X) = x_{r+1-a}.$$

See Figure 5 for the fundamental domain of $\tau^{(a)}(u; X)$ in the strip $\{(a, u) \mid a \in I, u \in \mathbb{Z}, a + u \text{ is odd}\}$, which consists of four parts, corresponding to the cases in (7.7):

$$(7.11) \quad \begin{aligned} D_1 &= \{(a, a - 1 + 2k) \mid a \in I, 0 \leq k \leq r - a\}, \\ D_2 &= \{(a, a - 1 + 2k) \mid a \in I, k = r - a + 1\}, \\ D_3 &= \{(a, a - 1 + 2k) \mid a \in I, r - a + 2 \leq k \leq r + 1\}, \\ D_4 &= \{(a, a - 1 + 2k) \mid a \in I, k = r + 2\}. \end{aligned}$$
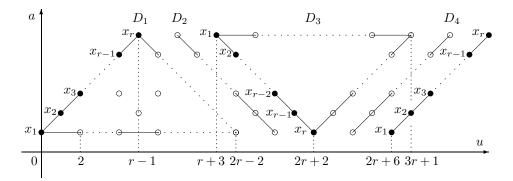
Figure 5: The fundamental domain of $\tau^{(a)}(u;X)$ for $\mathfrak{T}_2(A_r)$. The solid circles in D_1 are identified with the initial variables of $\mathfrak{T}_2(A_r)_+$.

PROPOSITION 7.3. *The family τ satisfies the following relations in $\mathbb{Z}[X^{\pm 1}]$.*

(1) *Half-periodicity:*

$$(7.12) \qquad \tau^{(a)}(u + r + 3; X) = \tau^{(r+1-a)}(u; X).$$

(2) *T-system $\mathbb{T}_2(A_r)$:*

$$(7.13) \qquad \tau^{(a)}(u-1; X)\tau^{(a)}(u+1; X) = 1 + \tau^{(a-1)}(u; X)\tau^{(a+1)}(u; X),$$

where $\tau^{(0)}(u; X) = \tau^{(r+1)}(u; X) = 1$ if they occur in the right-hand side.

Proof. For simplicity, in this proof we write $\gamma_k^{(j)} = \gamma_k^{(j)}(X)$, $\nu_k^{(j)} = \nu_k^{(j)}(X)$, and $\tau^{(a)}(u) = \tau^{(a)}(u; X)$.

(1) As an equivalent claim to (7.12), we prove

$$(7.14) \qquad \tau^{(a)}(u) = \tau^{(r+1-a)}(u + r + 3).$$

Thanks to (7.8), it is enough to prove it for $(a, u) \in D_1 \sqcup D_2$. Suppose that $(a, u) = (a, a - 1 + 2k) \in D_1$. Then we have

$$(7.15) \qquad \tau^{(a)}(a - 1 + 2k) = \gamma_k^{(a+k)} x_{a+k} + \nu_k^{(a+k)}.$$

On the other hand, $(r + 1 - a, u + r + 3) = (a', a' - 1 + 2k') \in D_3$, where $a' = r + 1 - a$ and $k' = a + k + 1$. Therefore, we have

$$
\begin{aligned}
\tau^{(r+1-a)}(u + r + 3) &= \gamma_{a'+k'-(r+2)}^{(k'-1)} x_{k'-1} + \nu_{a'+k'-(r+2)}^{(k'-1)} \\
&= \gamma_k^{(a+k)} x_{a+k} + \nu_k^{(a+k)}.
\end{aligned}
$$

(7.16)

Thus the claim follows. The remaining case $(a, u) \in D_2$ is similar and easier.

(2) Thanks to (1), it is enough to show (7.13) for $(a, u - 1) = (a, a - 1 + 2k) \in D_1 \sqcup D_2$; that is, $0 \leq k \leq r - a + 1$. It is proved by a direct calculation using Lemma 7.2. □

Like (4.3), it is enough to consider the subring $\mathcal{T}_2(A_r)_+$ of $\mathcal{T}_2(A_r)$ generated by $T_1^{(a)}(u)^{\pm 1}$ ($a \in I$, $u \in \mathbb{Z}$, $a + u$: odd). Set

$$(7.17) \qquad T^{\mathrm{init}} = \big\{ T_1^{(a)}(a - 1) \mid a \in I \big\}.$$

We call the elements of T^{init} the *initial variables* of $\mathcal{T}_2(A_r)_+$.

By Proposition 7.3(2), we have the following *explicit formula* of other variables $T_1^{(a)}(u) \in \mathcal{T}_2(A_r)_+$ by Laurent polynomials in the initial variables.

THEOREM 7.4. *The following relation holds in* $\mathcal{T}_2(A_r)_+$:

$$(7.18) \qquad T_1^{(a)}(u) = \tau^{(a)}(u; T^{\mathrm{init}}),$$

where we set $x_a = T_1^{(a)}(a - 1)$ *($a \in I$) in the right-hand side.*

Proof. In the following we simply write $\tau^{(a)}(u)$ for $\tau^{(a)}(u; T^{\mathrm{init}})$.

We rewrite (7.18) as ($n \in \mathbb{Z}$),

$$(7.19) \qquad T_1^{(a)}(a - 1 + 2n) = \tau^{(a)}(a - 1 + 2n),$$

and we show it by induction on n. Recall that $\mathbb{T}_2(A_r)$ becomes

$$(7.20) \qquad T_1^{(a)}(a - 1 + 2n)T_1^{(a)}(a + 1 + 2n)$$
$$= 1 + T_1^{(a-1)}(a + 2n)T_1^{(a+1)}(a + 2n).$$

By (7.9), the $n = 0$ case of (7.19), $\tau^{(a)}(a - 1) = T_1^{(a)}(a - 1)$, is satisfied. Assume that (7.19) holds up to n (> 0). Then (7.13) with $u = a + 2n$ becomes

$$(7.21) \qquad \begin{aligned} & T_1^{(a)}(a - 1 + 2n)\tau^{(a)}\big(a - 1 + 2(n + 1)\big) \\ & = 1 + \tau^{(a-1)}\big(a - 2 + 2(n + 1)\big)T_1^{(a+1)}(a + 2n). \end{aligned}$$

By setting $a = 1$ in (7.20) and (7.21), we obtain the relation

$$(7.22) \qquad \tau^{(1)}\big(2(n + 1)\big) = \frac{1 + T_1^{(2)}(1 + 2n)}{T_1^{(1)}(2n)} = T^{(1)}\big(2(n + 1)\big).$$

By increasing a one by one up to r, we recursively obtain the relation

$$\tag{7.23}
\begin{aligned}
\tau^{(a)}\big(a - 1 + 2(n+1)\big) &= \frac{1 + T_1^{(a-1)}(a+2n)T_1^{(a+1)}(a+2n)}{T_1^{(a)}(a-1+2n)} \\
&= T^{(a)}\big(a - 1 + 2(n+1)\big).
\end{aligned}
$$

The case of $n\ (<0)$ is similar. $\qquad\qquad\square$

In the above proof, we carefully stay inside the ring $\mathcal{T}_2(A_r)_+$ by avoiding the division by $\tau^{(a)}(u)$'s. As a corollary of Proposition 7.3(1) and Theorem 7.4, we once again obtain the periodicity of $\mathcal{T}_2(A_r)$ (Corollary 4.5 for $X_r = A_r$).

7.2. Explicit formula by initial variables for $\mathcal{T}_2(D_r)$

The method is parallel to the A_r case, so we present only the results for the most part. Throughout this subsection, $I = \{1, \ldots, r\}$ is the set enumerating the diagram D_r in Figure 1. Recall that the (dual) Coxeter number of D_r is $2r - 2$.

For a family of variables $X = \{x_a \mid a \in I\}$, we define $\gamma_n^{(j)}(X), \nu_n^{(j)}(X) \in \mathbb{Z}[X^{\pm 1}]$ $(1 \le j \le r - 1, 0 \le n \le j)$ by Definition 7.1. We set

$$\tag{7.24}
\alpha_n = \gamma_n^{(r-1)}(X), \qquad \beta_n = \nu_n^{(r-1)}(X), \qquad z = x_{r-1}x_r.
$$

Note that $\alpha_{r-1} = \beta_0 = 0$, $\alpha_0 = \beta_{r-1} = 1$, and $\beta_1 = 1/x_{r-2}$.

DEFINITION 7.5. Define $\Gamma_n^{(j)}(X), \Pi_n^{(j)}(X), \Omega_n^{(j)}(X) \in \mathbb{Z}[X^{\pm 1}]$ $(0 \le j \le r - 2, 0 \le n \le r - 1 - j)$ as

$$\tag{7.25}
\Gamma_n^{(j)}(X) = \alpha_j \alpha_{j+n},
$$

$$\tag{7.26}
\Pi_n^{(j)}(X) = \alpha_j \beta_{j+n} + \beta_j \alpha_{j+n} \frac{2 + \beta_1}{\beta_1},
$$

$$\tag{7.27}
\Omega_n^{(j)}(X) = \beta_j \beta_{j+n} \Big(\frac{1 + \beta_1}{\beta_1}\Big)^2.
$$

In particular,

$$\tag{7.28}
\Gamma_0^{(j)}(X) = \alpha_j^2, \qquad \Pi_0^{(j)}(X) = 2\alpha_j \beta_j \frac{1 + \beta_1}{\beta_1}, \qquad \Omega_0^{(j)}(X) = \beta_j \Big(\frac{1 + \beta_1}{\beta_1}\Big)^2.
$$

Note that they are independent of x_{r-1}, x_r.

LEMMA 7.6. *The following relations hold:*

$$(7.29) \qquad \alpha_n \beta_{n+1} - \alpha_{n+1}\beta_n = \beta_1 \quad (0 \le n \le r-2),$$

$$(7.30) \quad \begin{aligned} &\Gamma_n^{(j)}(X)\Omega_n^{(j+1)}(X) + \Gamma_n^{(j+1)}(X)\Omega_n^{(j)}(X) \\ &\quad - \Gamma_{n+1}^{(j)}(X)\Omega_{n-1}^{(j+1)}(X) - \Gamma_{n-1}^{(j+1)}(X)\Omega_{n+1}^{(j)}(X) = (\beta_1+1)^2 \\ &\qquad\qquad\qquad\qquad\qquad (0 \le j \le r-3, 1 \le n \le r-2-j), \end{aligned}$$

$$(7.31) \quad \Pi_n^{(j)}(X)\Pi_n^{(j+1)}(X) - \Pi_{n+1}^{(j)}(X)\Pi_{n-1}^{(j+1)}(X) = -\beta_1(2+\beta_1)$$
$$(0 \le j \le r-3, 1 \le n \le r-2-j).$$

Proof. It is easy to check (7.29). The rest are the consequences of Definition 7.5 and (7.29). $\square$

We define a family $\tau = \{\tau^{(a)}(u; X) \in \mathbb{Z}[X^{\pm 1}] \mid a \in I, u \in \mathbb{Z}, a+u \text{ is odd/even if } a \ne r/a = r\}$ as follows. First, we define, for $1 \le a \le r-2$ and $0 \le k \le r-1$,

$$(7.32) \quad \tau^{(a)}(a-1+2k; X)$$
$$= \begin{cases} \gamma_k^{(a+k)}(X)x_{a+k} + \nu_k^{(a+k)}(X) & (0 \le k \le r-a-2), \\ \Gamma_{r-a-1}^{(a+k-r+1)}(X)z + \Pi_{r-a-1}^{(a+k-r+1)}(X) + \dfrac{\Omega_{r-a-1}^{(a+k-r+1)}(X)}{z} & \\ & (r-a-1 \le k \le r-1), \end{cases}$$

and, for $a = r-1, r$ and $0 \le k \le r-1$,

$$(7.33) \qquad \tau^{(r-1)}(r-2+2k; X) = \begin{cases} \dfrac{\alpha_k z + \beta_k(1+\beta_1)/\beta_1}{x_{r-1}} & (k : \text{odd}), \\ \dfrac{\alpha_k z + \beta_k(1+\beta_1)/\beta_1}{x_r} & (k : \text{even}), \end{cases}$$

$$(7.34) \qquad \tau^{(r)}(r-2+2k; X) = \begin{cases} \dfrac{\alpha_k z + \beta_k(1+\beta_1)/\beta_1}{x_r} & (k : \text{odd}), \\ \dfrac{\alpha_k z + \beta_k(1+\beta_1)/\beta_1}{x_{r-1}} & (k : \text{even}). \end{cases}$$

Then, we extend the definition by the following half-periodicity:

$$(7.35) \qquad\qquad \tau^{(a)}(u+2r; X) = \tau^{(\omega(a))}(u; X),$$

from which the periodicity $\tau^{(a)}(u+4r; X) = \tau^{(a)}(u; X)$ also follows.

By (7.3), we have

$$(7.36) \quad \tau^{(a)}(a-1; X) = x_a \quad (a = 1, \ldots, r-1), \qquad \tau^{(r)}(r-2; X) = x_r.$$
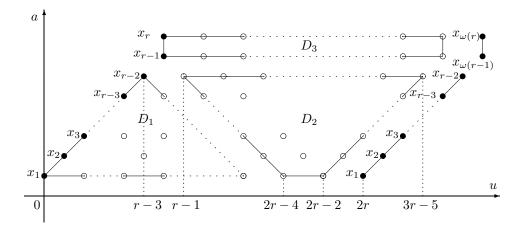
Figure 6: The half of the fundamental domain of $\tau^{(a)}(u; X)$ for $\mathcal{T}_2(D_r)$

See Figure 6 for the half of the fundamental domain of $\tau^{(a)}(u; X)$, which consists of three parts. The domains D_1 and D_2 correspond to the two cases of (7.32), while D_3 does the cases of (7.33) and (7.34):

$$D_1 = \big\{ (a, a - 1 + 2k) \mid a = 1, \ldots, r - 2; 0 \leq k \leq r - a - 2 \big\},$$

$$(7.37) \qquad D_2 = \big\{ (a, a - 1 + 2k) \mid a = 1, \ldots, r - 2; r - a - 1 \leq k \leq r - 1 \big\},$$

$$D_3 = \big\{ (r - 1, r - 2 + 2k), (r, r - 2 + 2k) \mid 0 \leq k \leq r - 1 \big\}.$$

Using Lemmas 7.2 and 7.6, one can verify, case by case, the following.

PROPOSITION 7.7. *The family τ satisfies the T-system $\mathbb{T}_2(D_r)$ in $\mathbb{Z}[X^{\pm 1}]$:*

$$(7.38) \qquad \tau^{(a)}(u - 1; X)\tau^{(a)}(u + 1; X) = 1 + \prod_{b \in I : C_{ab} = -1} \tau^{(b)}(u; X).$$

Let $\mathcal{T}_2(D_r)_+$ be the subring of $\mathcal{T}_2(D_r)$ generated by $T_1^{(a)}(u)^{\pm 1}$ ($a \in I$, $u \in \mathbb{Z}$, $a + u$ is odd/even if $a \neq r/a = r$). Set

$$(7.39) \qquad T^{\text{init}} = \big\{ T_1^{(a)}(a - 1) \ (a = 1, \ldots, r - 1), T_1^{(r)}(r - 2) \big\}.$$

We call the elements of T^{init} the *initial variables* of $\mathcal{T}_2(D_r)_+$.

THEOREM 7.8. *The following relation holds in $\mathcal{T}_2(D_r)_+$:*

$$(7.40) \qquad T_1^{(a)}(u) = \tau^{(a)}(u; T^{\text{init}}),$$

where we set $x_a = T_1^{(a)}(a-1)$ $(a \neq r)$ *and* $x_r = T_1^{(r)}(r-2)$ *in the right-hand side.*

As a corollary of Theorem 7.8 and (7.35), we once again obtain the periodicity of $\mathcal{T}_2(D_r)$ (Corollary 4.5 for $X_r = D_r$).

7.3. Explicit formula by initial variables for $\mathcal{T}_2(B_r)$

Again, the method is parallel to the former cases, so we present only the results for the most part. Throughout this subsection, $I = \{1, \ldots, r\}$ is the set enumerating the diagram B_r in Figure 1. Recall that the dual Coxeter number of B_r is $2r - 1$, and the number t_a in (2.1) is 2 (resp. 1) if $a = r$ (otherwise).

For a family of variables $X = \{x_a, \overline{x}_a \ (a = 1, \ldots, r-1), w_1, w_2, w_3\}$, we define $\gamma_n^{(j)}(X), \nu_n^{(j)}(X) \in \mathbb{Z}[X^{\pm 1}]$ $(1 \leq j \leq r-1, 0 \leq n \leq j)$ by Definition 7.1. We set

$$(7.41) \qquad \alpha_n = \gamma_n^{(r-1)}(X), \qquad \beta_n = \nu_n^{(r-1)}(X), \qquad z = w_1 w_3.$$

Note that $\alpha_{r-1} = \beta_0 = 0$ and $\alpha_0 = \beta_{r-1} = 1$.

In the following, for $p(X) \in \mathbb{Z}[X^{\pm 1}]$, we write $\overline{p}(X)$ for what obtained from $p(X)$ by replacing x_i with $\overline{x}_i$. Similarly, we write $\widehat{p}(X)$ for what obtained from $p(X)$ by swapping w_1 and w_3.

DEFINITION 7.9. We define $\mu_n(X), \delta_n(X) \in \mathbb{Z}[X^{\pm 1}]$ $(0 \leq n \leq r)$ as

$$(7.42) \qquad \mu_n(X) = \frac{x_{n-1}}{x_{r-1}}, \qquad \delta_n(X) = \begin{cases} \alpha_n + \frac{\beta_n}{x_{r-1}} & 0 \leq n \leq r-1, \\ 0 & n = r, \end{cases}$$

where we set $x_0 = \overline{x}_0 = 1, x_{-1} = \overline{x}_{-1} = 0$. In particular, we have $\mu_0(X) = 0$, $\mu_r(X) = 1$, $\delta_0(X) = 1$, and $\delta_{r-1}(X) = 1/x_{r-1}$.

Note that $\mu_n(X)$ and $\delta_n(X)$ depend on x_{r-1}, while $\gamma_n^{(j)}(X)$ and $\nu_n^{(j)}(X)$ do not.

DEFINITION 7.10. We define $P_n^{(j)}(X), Q_n^{(j)}(X), R_n^{(j)}(X), \eta_n(X), \xi_n(X) \in \mathbb{Z}[X^{\pm 1}]$ as follows. For $0 \leq j \leq r, 0 \leq k \leq r, 1 \leq j + k \leq r + 1$,

$$P_{2k}^{(j)}(X) = \frac{x_{k-1}}{z} \overline{\delta}_{j-1+k}(X),$$

$$(7.43) \qquad Q_{2k}^{(j)}(X) = x_{k-1}\overline{\mu}_{j+k-1} + \overline{\delta}_{j+k-1}(X)\overline{x}_{r-1}\left(\delta_k(X) + \frac{2x_{k-1}}{z}\right),$$

$$R_{2k}^{(j)}(X) = \overline{\delta}_{j-1+k}(X)x_{k-1}\overline{x}_{r-1}\left(\frac{\overline{x}_{r-1}}{z} + \frac{1}{x_{r-1}}\right).$$

For $0 \le j \le r, 0 \le k \le r, 0 \le j + k \le r$,

$$P_{2k+1}^{(j)}(X) = \overline{\delta}_k(X)\delta_{j+k}(X),$$

(7.44)
$$Q_{2k+1}^{(j)}(X) = z\delta_{j+k}(X)\overline{\mu}_k(X) + \overline{\delta}_k(X)\mu_{j+k}(X),$$

$$R_{2k+1}^{(j)}(X) = zx_{j-1+k}\overline{\mu}_k(X)\Big(\frac{\overline{x}_{r-1}}{z} + \frac{1}{x_{r-1}}\Big).$$

And,

$$\eta_{2k}(X) = w_1\delta_k(X) + \frac{x_{k-1}}{w_3},$$

$$\xi_{2k}(X) = w_1 x_{k-1}\Big(\frac{\overline{x}_{r-1}}{z} + \frac{1}{x_{r-1}}\Big) \quad (0 \le k \le r),$$

(7.45)
$$\eta_{2k+1}(X) = \frac{\overline{\delta}_k(X)}{w_1},$$

$$\xi_{2k+1}(X) = \frac{1}{w_1}\big(\overline{x}_{r-1}\overline{\delta}_k(X) + z\overline{\mu}_k(X)\big) \quad (0 \le k \le r-1).$$

In particular, we have $Q_1^{(0)}(X) = R_1^{(0)}(X) = P_{2r+1}^{(0)}(X) = Q_{2r+1}^{(0)}(X) = 0$ and $P_1^{(0)}(X) = 1$.

LEMMA 7.11. *The following relations hold. For $1 \le k \le r$,*

(7.46)
$$\mu_{k-1}(X)x_{k-1} - \mu_k(X)x_{k-2} = \overline{\mu}_{k-1}(X)\overline{x}_{k-1} - \overline{\mu}_k(X)\overline{x}_{k-2} = 0,$$
$$\delta_{k-1}(X)x_{k-1} - \delta_k(X)x_{k-2} = \overline{\delta}_{k-1}(X)\overline{x}_{k-1} - \overline{\delta}_k(X)\overline{x}_{k-2} = 1.$$

For $1 \le j \le r-1,\ 0 \le k \le 2r-2,\ 1 \le j + [k/2] \le r$,

$$Q_k^{(j)}(X)R_{k+2}^{(j)}(X) + Q_{k+2}^{(j)}(X)R_k^{(j)}(X)$$
$$\qquad - Q_{k+2}^{(j-1)}(X)R_k^{(j+1)}(X) - Q_k^{(j+1)}(X)R_{k+2}^{(j-1)}(X) = 0,$$

$$Q_k^{(j)}(X)P_{k+2}^{(j)}(X) + Q_{k+2}^{(j)}(X)P_k^{(j)}(X)$$
$$\qquad - Q_{k+2}^{(j-1)}(X)P_k^{(j+1)}(X) - Q_k^{(j+1)}(X)P_{k+2}^{(j-1)}(X) = 0,$$

(7.47)
$$Q_k^{(j)}(X)Q_{k+2}^{(j)}(X) - Q_{k+2}^{(j-1)}(X)Q_k^{(j+1)}(X)$$
$$\qquad = \begin{cases} -z\mu_1(X)\overline{\mu}_1(X) & k : odd, \\ 1 & k : even, \end{cases}$$

$$P_k^{(j)}(X)R_{k+2}^{(j)}(X) + P_{k+2}^{(j)}(X)R_k^{(j)}(X) - P_{k+2}^{(j-1)}(X)R_k^{(j+1)}(X)$$

$$- P_k^{(j+1)}(X)R_{k+2}^{(j-1)}(X) = \begin{cases} 1 + z\mu_1(X)\overline{\mu}_1(X) & k : odd, \\ 0 & k : even. \end{cases}$$

For $1 \le k \le 2r,$

$$Q_k^{(0)}(X)R_{k+1}^{(0)}(X) + Q_{k+1}^{(0)}(X)R_k^{(0)}(X)$$

$$- Q_k^{(1)}(X)R_{k-1}^{(1)}(X) - Q_{k-1}^{(1)}(X)R_k^{(1)}(X)$$

$$= \begin{cases} 0 & k : odd, \\ \xi_k(X)\widehat{\eta}_k(X) + \widehat{\xi}_k(X)\eta_k(X) & k : even, \end{cases}$$

$$Q_k^{(0)}(X)P_{k+1}^{(0)}(X) + Q_{k+1}^{(0)}(X)P_k^{(0)}(X)$$

$$(7.48) \qquad - Q_k^{(1)}(X)P_{k-1}^{(1)}(X) - Q_{k-1}^{(1)}(X)P_k^{(1)}(X)$$

$$= \begin{cases} \xi_k(X)\widehat{\eta}_k(X) + \widehat{\xi}_k(X)\eta_k(X) & k : odd, \\ 0 & k : even, \end{cases}$$

$$P_k^{(0)}(X)R_{k+1}^{(0)}(X) + P_{k+1}^{(0)}(X)R_k^{(0)}(X) - P_k^{(1)}(X)R_{k-1}^{(1)}(X)$$

$$- P_{k-1}^{(1)}(X)R_k^{(1)}(X) + Q_k^{(0)}(X)Q_{k+1}^{(0)}(X) - Q_{k+1}^{(1)}(X)Q_k^{(1)}(X)$$

$$= \begin{cases} \xi_k(X)\widehat{\xi}_k(X) & k : odd, \\ \eta_k(X)\widehat{\eta}_k(X) & k : even. \end{cases}$$

Proof. It is easy to check (7.46). The rest are the consequence of Definitions 7.9, 7.10, and (7.46). □

For a triplet (a, m, u) $(a \in I; m = 1, \ldots, 2t_a - 1; u \in \frac{1}{2}\mathbb{Z})$, we set the condition,

$$(7.49) \qquad \text{Condition (P):} \quad u \in \begin{cases} \mathbb{Z} + \frac{1}{2} & (a, m) = (r, 1), (r, 3), \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

We define a family $\tau = \{\tau_m^{(a)}(u; X) \in \mathbb{Z}[X^{\pm 1}] \mid a \in I; m = 1, \ldots, 2t_a - 1; u \in (1/2)\mathbb{Z}; \text{Condition (P)}\}$ as follows. First, we define $\tau_m^{(a)}(u; X)$ in the following

region:

$$(7.50) \quad \begin{cases} \tau_1^{(a)}(a-1+2k;X) = \gamma_k^{(a+k)}(X)x_{a+k} + \nu_k^{(a+k)}(X) \\ \qquad (1 \le a \le r-1,\ 0 \le k \le r-1-a), \\ \tau_1^{(a)}(a+2k;X) = \overline{\tau}_1^{(a)}(a-1+2k;X) \\ \qquad (1 \le a \le r-1,\ 0 \le k \le r-1-a), \end{cases}$$

$$(7.51) \quad \begin{cases} \tau_1^{(a)}\big(a+2(r-1-a)+k;X\big) \\ \qquad = P_k^{(r-a)}(X)w_2 + Q_k^{(r-a)}(X) + \frac{R_k^{(r-a)}(X)}{w_2} \\ \qquad (1 \le a \le r-1,\ 1 \le k \le 2a+1), \\ \tau_2^{(r)}(r-1+k;X) = P_{k+1}^{(0)}(X)w_2 + Q_{k+1}^{(0)}(X) + \frac{R_{k+1}^{(0)}(X)}{w_2} \\ \qquad (0 \le k \le 2r), \end{cases}$$

$$(7.52) \quad \begin{cases} \tau_1^{(r)}\big(r+2k-\frac{1}{2};X\big) = \eta_{2k+1}(X)w_2 + \xi_{2k+1}(X) & (0 \le k \le r-1), \\ \tau_1^{(r)}\big(r+2k-\frac{3}{2};X\big) = \eta_{2k}(X) + \frac{\xi_{2k}(X)}{w_2} & (0 \le k \le r), \\ \tau_3^{(r)}\big(r+k-\frac{3}{2};X\big) = \widehat{\tau}_1^{(r)}\big(r+k-\frac{3}{2};X\big) & (0 \le k \le 2r). \end{cases}$$

Then, we extend the definition by the following half-periodicity:

$$(7.53) \qquad\qquad \tau_m^{(a)}(u+2r+1;X) = \widehat{\tau}_m^{(a)}(u;X),$$

from which the periodicity $\tau_m^{(a)}(u+2(2r+1);X) = \tau_m^{(a)}(u;X)$ also follows.

By (7.3), we have

$$\tau_1^{(a)}(a-1;X) = x_a, \qquad \tau_1^{(a)}(a;X) = \overline{x}_a \quad (a=1,\dots,r-1),$$

$$(7.54) \qquad \tau_1^{(r)}\left(r-\frac{3}{2};X\right) = w_1, \qquad \tau_2^{(r)}(r-1;X) = w_2,$$

$$\tau_3^{(r)}\left(r-\frac{3}{2};X\right) = w_3.$$

See Figure 7 for the half of the fundamental domain of $\tau_m^{(a)}(u;X)$ which consists of three parts. The domains D_1, D_2, and D_3 correspond to the cases (7.50), (7.51), and (7.52), respectively:

$$D_1 = \big\{(a, a-1+k) \,|\, 1 \le a \le r-1; 0 \le k \le 2(r-a)-1\big\},$$

$$D_2 = \big\{(a, a-1+k) \,|\, 1 \le a \le r-1; 2(r-a) \le k \le 2r\big\}$$

$$(7.55) \qquad\qquad \sqcup \big\{(r_2, r-1+k) \,|\, 0 \le k \le 2r\big\},$$

$$D_3 = \left\{\left(r_1, r+k-\frac{3}{2}\right), \left(r_3, r+k-\frac{3}{2}\right) \,\Big|\, 0 \le k \le 2r\right\},$$
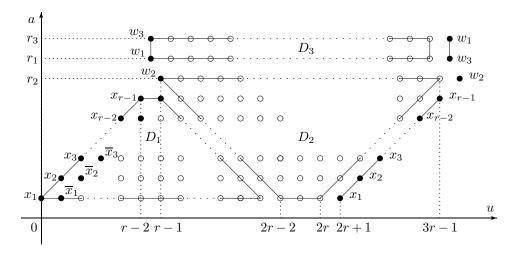
Figure 7: The half of the fundamental domain of $\tau_m^{(a)}(u; X)$ for $\mathcal{T}_2(B_r)$

where the point (r_i, u) corresponds to $\tau_i^{(r)}(u)$.

Using Lemmas 7.2 and 7.11, one can verify the following, case by case.

PROPOSITION 7.12. *The family τ satisfies the T-system $\mathbb{T}_2(B_r)$ in $\mathbb{Z}[X^{\pm 1}]$ (by replacing $T_m^{(a)}(u)$ in $\mathbb{T}_2(B_r)$ with $\tau_m^{(a)}(u; X)$).*

Let $\mathcal{T}_2(B_r)_+$ be the subring of $\mathcal{T}_2(B_r)$ generated by $T_m^{(a)}(u)^{\pm 1}$ ($a \in I; m = 1, \ldots, 2t_a - 1; u \in (1/2)\mathbb{Z}$; Condition (P)). Set

(7.56)
$$T^{\text{init}} = \left\{ T_1^{(a)}(a-1), T_1^{(a)}(a)(a = 1, \ldots, r-1), \right.$$
$$\left. T_1^{(r)}\left(r - \frac{3}{2}\right), T_2^{(r)}(r-1), T_3^{(r)}\left(r - \frac{3}{2}\right) \right\}.$$

We call the elements of T^{init} the *initial variables* of $\mathcal{T}_2(B_r)_+$.

THEOREM 7.13. *The following relation holds in $\mathcal{T}_2(B_r)_+$:*

(7.57)
$$T_m^{(a)}(u) = \tau_m^{(a)}(u; T^{\text{init}}),$$

where we set $x_a = T_1^{(a)}(a-1)$, $\overline{x}_a = T_1^{(a)}(a)$ $(a \neq r)$, $w_1 = T_1^{(r)}(r - (3/2))$, $w_2 = T_2^{(r)}(r-1)$, and $w_3 = T_3^{(r)}(r - (3/2))$ in the right-hand side.

As a corollary of Theorem 7.13 and (7.53), we obtain the following.

COROLLARY 7.14. *The following relations hold in* $\mathcal{T}_2(B_r)$*:*

(1) *Half-periodicity:* $T_m^{(a)}(u + 2r + 1) = T_{2t_a-m}^{(a)}(u)$.
(2) *Periodicity:* $T_m^{(a)}(u + 2(2r + 1)) = T_m^{(a)}(u)$.

§8. Periodicities of restricted T- and Y-systems at levels 1 and 0

So far, we have assumed that the level ℓ for the restriction is greater than or equal to 2. In this section we extend the periodicity property of the restricted T- and Y-systems at *levels* 1 *and* 0.

8.1. Periodicities of restricted T- and Y-systems at level 1

In the systems $\mathbb{T}_\ell(X_r)$ and $\mathbb{Y}_\ell(X_r)$, we treat the variables $T_m^{(a)}(u)$ and $Y_m^{(a)}(u)$ with $m = 1, \ldots, t_a \ell - 1$. Thus at $\ell = 1$, these systems are *void* for simply laced X_r; however, Definitions 3.1, 3.2, 3.5, and 3.6 still make sense for nonsimply laced X_r.

The level 1 T- and Y-systems in these cases are actually equivalent to the systems of type A. To illustrate, consider $\mathbb{T}_1(F_4)$:

$$
(8.1) \qquad
\begin{aligned}
T_1^{(3)}\Big(u - \frac{1}{2}\Big)T_1^{(3)}\Big(u + \frac{1}{2}\Big) &= 1 + T_1^{(4)}(u), \\
T_1^{(4)}\Big(u - \frac{1}{2}\Big)T_1^{(4)}\Big(u + \frac{1}{2}\Big) &= 1 + T_1^{(3)}(u),
\end{aligned}
$$

where we have omitted the first three relations in (2.5), which are void at $\ell = 1$. To be precise, let us introduce another level ℓ restricted T-system $\mathbb{T}'_\ell(A_r)$ for $T = \{T_m^{(a)}(u) \mid a = 1, \ldots, r; m = 1, \ldots, \ell - 1; u \in (1/\ell)\mathbb{Z}\}$ with the relations

$$
(8.2) \quad T_m^{(a)}\Big(u - \frac{1}{\ell}\Big)T_m^{(a)}\Big(u + \frac{1}{\ell}\Big) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u),
$$

where the left-hand side of (8.2) differs from (2.2) for $\mathbb{T}_\ell(A_r)$. Then, the relations in (8.1) are equivalent to $\mathbb{T}'_2(A_2)$. In other words, $\mathcal{T}_1(F_4) \simeq \mathcal{T}'_2(A_2)$, where $\mathcal{T}'_\ell(A_r)$ denotes the T-algebra associated with $\mathbb{T}'_\ell(A_r)$. A similar reduction of the Y-system, $\mathcal{Y}_1(F_4) \simeq \mathcal{Y}'_2(A_2)$, happens, where $\mathbb{Y}'_\ell(A_r)$ and $\mathcal{Y}'_\ell(A_r)$ are defined in the same way.

In general, $\mathcal{T}_1(X_r) \simeq \mathcal{T}'_t(A_{r'})$ and $\mathcal{Y}_1(X_r) \simeq \mathcal{Y}'_t(A_{r'})$ hold for nonsimply laced X_r, where t is the number in (2.1) and r' equals the number of the short simple roots of X_r.

Let us summarize the relevant data for the periodicities of $\mathcal{T}_1(X_r)$ and $\mathcal{Y}_1(X_r)$:

(8.3)

X_r	$A_{r'}$	t	$h^\vee + 1$	$(r'+1+t)/t$
B_r	A_1	2	$2r$	2
C_r	A_{r-1}	2	$r+2$	$(r+2)/2$
F_4	A_2	2	10	$5/2$
G_2	A_1	3	5	$5/3$

THEOREM 8.1. (i) *For any nonsimply laced X_r, the following relations hold in $\mathcal{T}_1(X_r)$:*

(1) *Half-periodicity:* $T_m^{(a)}(u + h^\vee + 1) = T_{t_a-m}^{(a)}(u)$.

(2) *Periodicity:* $T_m^{(a)}(u + 2(h^\vee + 1)) = T_m^{(a)}(u)$.

(ii) *For any nonsimply laced X_r, the following relations hold in $\mathcal{Y}_1(X_r)$:*

(1) *Half-periodicity:* $Y_m^{(a)}(u + h^\vee + 1) = Y_{t_a-m}^{(a)}(u)$.

(2) *Periodicity:* $Y_m^{(a)}(u + 2(h^\vee + 1)) = Y_m^{(a)}(u)$.

Proof. By inspecting (8.3), one can check that the half-periodicities of $\mathcal{T}_1(X_r)$ and $\mathcal{Y}_1(X_r)$ follow from the half- or full-periodicities of the corresponding $\mathcal{T}'_t(A_{r'})$ and $\mathcal{Y}'_t(A_{r'})$. $\qquad\square$

8.2. Periodicities of restricted T- and Y-systems at level 0

At level 0, one can still introduce, at least formally, a restricted T- and Y-system for any X_r, and study their periodicity.

DEFINITION 8.2. The *level 0 restricted T-system* $\mathbb{T}_0(X_r)$ *of type X_r is the following system of relations for a family of variables* $T = \{T^{(a)}(u) \mid a \in I, u \in U\}$, *where* $T^{(0)}(u) = 1$ *if they occur in the right-hand sides in the relations.*

For simply laced X_r,

$$(8.4) \qquad T^{(a)}(u-1)T^{(a)}(u+1) = \prod_{b\in I : C_{ab}=-1} T^{(b)}(u).$$

For $X_r = B_r$,

$$(8.5) \qquad T^{(a)}(u-1)T^{(a)}(u+1) = T^{(a-1)}(u)T^{(a+1)}(u) \quad (1 \le a \le r-2),$$

$$T^{(r-1)}(u-1)T^{(r-1)}(u+1) = T^{(r-2)}(u)T^{(r)}(u),$$

$$T^{(r)}\left(u - \frac{1}{2}\right)T^{(r)}\left(u + \frac{1}{2}\right) = T^{(r-1)}\left(u - \frac{1}{2}\right)T^{(r-1)}\left(u + \frac{1}{2}\right).$$

For $X_r = C_r$,

$$(8.6) \quad T^{(a)}\Big(u - \frac{1}{2}\Big)T^{(a)}\Big(u + \frac{1}{2}\Big) = T^{(a-1)}(u)T^{(a+1)}(u) \quad (1 \le a \le r - 2),$$

$$T^{(r-1)}\Big(u - \frac{1}{2}\Big)T^{(r-1)}\Big(u + \frac{1}{2}\Big) = T^{(r-2)}(u)T^{(r)}\Big(u - \frac{1}{2}\Big)T^{(r)}\Big(u + \frac{1}{2}\Big),$$

$$T^{(r)}(u - 1)T^{(r)}(u + 1) = T^{(r-1)}(u).$$

For $X_r = F_4$,

$$(8.7) \qquad T^{(1)}(u - 1)T^{(1)}(u + 1) = T^{(2)}(u),$$

$$T^{(2)}(u - 1)T^{(2)}(u + 1) = T^{(1)}(u)T^{(3)}(u),$$

$$T^{(3)}\Big(u - \frac{1}{2}\Big)T^{(3)}\Big(u + \frac{1}{2}\Big) = T^{(2)}\Big(u - \frac{1}{2}\Big)T^{(2)}\Big(u + \frac{1}{2}\Big)T^{(4)}(u),$$

$$T^{(4)}\Big(u - \frac{1}{2}\Big)T^{(4)}\Big(u + \frac{1}{2}\Big) = T^{(3)}(u).$$

For $X_r = G_2$,

$$(8.8) \qquad T^{(1)}(u - 1)T^{(1)}(u + 1) = T^{(2)}(u),$$

$$T^{(2)}\Big(u - \frac{1}{3}\Big)T^{(2)}\Big(u + \frac{1}{3}\Big) = T^{(1)}\Big(u - \frac{2}{3}\Big)T^{(1)}(u)T^{(1)}\Big(u + \frac{2}{3}\Big).$$

DEFINITION 8.3. The *level 0 restricted T-group* $\mathfrak{T}_0(X_r)$ *of type* X_r is the abelian group with generators $T^{(a)}(u)$ $(a \in I, u \in U)$ and the relations $\mathbb{T}_0(X_r)$.

REMARK 8.4. $\mathbb{T}_0(X_r)$ is obtained from the unrestricted T-system $\mathbb{T}(X_r)$ (2.2)–(2.6) by setting $T_m^{(a)}(u) = T^{(a)}(u)$ if $m = 0$ and $T_m^{(a)}(u) = 0$ otherwise. It was originally introduced in [KNS2, Section 2.2] as "bulk T-system."

Similarly, we define the following.

DEFINITION 8.5. The *level 0 restricted Y-system* $\mathbb{Y}_0(X_r)$ *of type* X_r is the following system of relations for a family of variables $Y = \{Y^{(a)}(u) \mid a \in I, u \in U\}$, where $Y^{(0)}(u) = 1$ if they occur in the right-hand sides in the relations.

For simply laced X_r,

$$(8.9) \qquad Y^{(a)}(u - 1)Y^{(a)}(u + 1) = \prod_{b \in I : C_{ab} = -1} Y^{(b)}(u).$$

For $X_r = B_r$,

$$(8.10) \quad Y^{(a)}(u-1)Y^{(a)}(u+1) = Y^{(a-1)}(u)Y^{(a+1)}(u) \quad (1 \le a \le r-2),$$

$$Y^{(r-1)}(u-1)Y^{(r-1)}(u+1) = Y^{(r-2)}(u)Y^{(r)}\left(u-\frac{1}{2}\right)Y^{(r)}\left(u+\frac{1}{2}\right),$$

$$Y^{(r)}\left(u-\frac{1}{2}\right)Y^{(r)}\left(u+\frac{1}{2}\right) = Y^{(r-1)}(u).$$

For $X_r = C_r$,

$$(8.11) \quad Y^{(a)}\left(u-\frac{1}{2}\right)Y^{(a)}\left(u+\frac{1}{2}\right) = Y^{(a-1)}(u)Y^{(a+1)}(u) \quad (1 \le a \le r-2),$$

$$Y^{(r-1)}\left(u-\frac{1}{2}\right)Y^{(r-1)}\left(u+\frac{1}{2}\right) = Y^{(r-2)}(u)Y^{(r)}(u),$$

$$Y^{(r)}(u-1)Y^{(r)}(u+1) = Y^{(r-1)}\left(u-\frac{1}{2}\right)Y^{(r-1)}\left(u+\frac{1}{2}\right).$$

For $X_r = F_4$,

$$(8.12) \quad Y^{(1)}(u-1)Y^{(1)}(u+1) = Y^{(2)}(u),$$

$$Y^{(2)}(u-1)Y^{(2)}(u+1) = Y^{(1)}(u)Y^{(3)}\left(u-\frac{1}{2}\right)Y^{(3)}\left(u+\frac{1}{2}\right),$$

$$Y^{(3)}\left(u-\frac{1}{2}\right)Y^{(3)}\left(u+\frac{1}{2}\right) = Y^{(2)}(u)Y^{(4)}(u),$$

$$Y^{(4)}\left(u-\frac{1}{2}\right)Y^{(4)}\left(u+\frac{1}{2}\right) = Y^{(3)}(u).$$

For $X_r = G_2$,

$$(8.13) \quad Y^{(1)}(u-1)Y^{(1)}(u+1) = Y^{(2)}\left(u-\frac{2}{3}\right)Y^{(2)}(u)Y^{(2)}\left(u+\frac{2}{3}\right),$$

$$Y^{(2)}\left(u-\frac{1}{3}\right)Y^{(2)}\left(u+\frac{1}{3}\right) = Y^{(1)}(u).$$

DEFINITION 8.6. The *level* 0 *restricted Y-group* $\mathcal{Y}_0(X_r)$ *of type* X_r is the abelian group with generators $Y^{(a)}(u)$ ($a \in I, u \in U$) and the relations $\mathbb{Y}_0(X_r)$.

REMARK 8.7. $\mathbb{Y}_0(X_r)$ is obtained from unrestricted Y-system $\mathbb{Y}(X_r)$ (2.15)–(2.19) by first making the replacement $(1+Y_m^{(a)}(u), 1+Y_m^{(a)}(u)^{-1}) \to (Y_m^{(a)}(u), 1)$ in the right-hand sides (i.e., taking a formal limit $Y_m^{(a)}(u) \to \infty$), and then setting $Y_m^{(a)}(u) = Y^{(a)}(u)$ if $m = 0$ and $Y_m^{(a)}(u) = 1$ otherwise.

From (8.4) and (8.9), we see that $\mathcal{T}_0(X_r) \simeq \mathcal{Y}_0(X_r)$ if X_r is simply laced.

The following periodicity property justifies that we call these systems "level 0." Notice that the half-periodicity here contains the *inverse* in the right-hand sides, in contrast with the level $\ell \geq 1$ case.

THEOREM 8.8. (i) *The following relations hold in $\mathcal{T}_0(X_r)$.*

(1) *Half-periodicity:* $T^{(a)}(u + h^\vee) = T^{(\omega(a))}(u)^{-1}$.
(2) *Periodicity:* $T^{(a)}(u + 2h^\vee) = T^{(a)}(u)$.

(ii) *The following relations hold in $\mathcal{Y}_0(X_r)$.*

(1) *Half-periodicity:* $Y^{(a)}(u + h^\vee) = Y^{(\omega(a))}(u)^{-1}$.
(2) *Periodicity:* $Y^{(a)}(u + 2h^\vee) = Y^{(a)}(u)$.

Proof. It is enough to show the half-periodicity, and it can be proved by elementary manipulations. Especially for exceptional $X_r = E_6, E_7, E_8, F_4$, and G_2, it is a matter of a direct check. As an illustration we present a proof for $\mathcal{T}_0(D_r)$ ($\simeq \mathcal{Y}_0(D_r)$) and $\mathcal{Y}_0(B_r)$ below. The cases $\mathcal{T}_0(B_r), \mathcal{T}_0(C_r), \mathcal{T}_0(F_4)$, and $\mathcal{T}_0(G_2)$ have been treated in [KNS2, Section 2.2.1 and Appendix A].

First we consider $\mathcal{T}_0(D_r)$. From (8.4) we have

$$(8.14) \qquad T^{(a)}(u) = \prod_{s=1}^{a} T^{(1)}(u - a - 1 + 2s) \quad (1 \leq a \leq r - 2),$$

$$(8.15) \qquad T^{(r-1)}(u)T^{(r)}(u) = \prod_{s=1}^{r-1} T^{(1)}(u - r + 2s),$$

$$(8.16) \quad T^{(a)}(u - 1)T^{(a)}(u + 1) = T^{(r-2)}(u) \quad (a = r - 1, r).$$

In (8.15) replace u by $u \pm 1$ and take the product. Using (8.14) with $a = r - 2$ and (8.16), one can express all the factors by $T^{(1)}$ only, leading to $T^{(1)}(u - r + 1)T^{(1)}(u + r - 1) = 1$. In view of $h^\vee = 2r - 2$ and (8.14), this verifies the claim of the theorem $T^{(a)}(u)T^{(a)}(u + h^\vee) = 1$ except for $a = r - 1$ and r.

Suppose r is even. Then for $a = r - 1, r$ we have

$$T^{(a)}(u)T^{(a)}(u + h^\vee) = \frac{\prod_{s=0}^{r-1} T^{(a)}(u + 2s)}{\prod_{s=1}^{r-2} T^{(a)}(u + 2s)} = \frac{\prod_{s=0}^{r/2-1} T^{(r-2)}(u + 1 + 4s)}{\prod_{s=1}^{r/2-1} T^{(r-2)}(u - 1 + 4s)},$$

where the second equality is due to (8.16). From (8.14) with $a = r - 2$, the ratio is expressed by $T^{(1)}$ only, which turns out to be 1 owing to

$T^{(1)}(u)T^{(1)}(u+h^\vee)=1$. If r is odd, let $\bar{r}=r-1$ and $\overline{r-1}=r$. Then we have

$$
\begin{aligned}
T^{(a)}(u)T^{(\bar{a})}(u+h^\vee) &= \frac{\left(\prod_{s=0}^{r-1}T^{(a)}(u+2s)\right)T^{(\bar{a})}(u+h^\vee)}{\prod_{s=1}^{r-1}T^{(a)}(u+2s)} \\
&= \frac{\prod_{s=0}^{(r-3)/2}T^{(r-2)}(u+1+4s)}{\prod_{s=1}^{(r-1)/2}T^{(r-2)}(u-1+4s)} \\
&\quad \times T^{(r-1)}(u+h^\vee)T^{(r)}(u+h^\vee),
\end{aligned}
$$

where the second equality is due to (8.16). Again, this can be shown to be 1 from (8.14) with $a=r-2$, (8.15), and $T^{(1)}(u)T^{(1)}(u+h^\vee)=1$.

Next we consider $\mathcal{Y}_0(B_r)$. From (8.10) we find

$$
(8.17) \qquad\qquad Y^{(a)}(u)=\prod_{s=1}^{a}Y^{(1)}(u-a-1+2s)
$$

$$
(1\le a\le r-1),
$$

$$
(8.18) \qquad Y^{(r-1)}(u-1)Y^{(r-1)}(u+1)=Y^{(r-2)}(u)Y^{(r-1)}(u),
$$

$$
(8.19) \qquad Y^{(r)}\left(u-\frac{1}{2}\right)Y^{(r)}\left(u+\frac{1}{2}\right)=Y^{(r-1)}(u).
$$

Substituting (8.17) into (8.18), we get $Y^{(r-1)}(u)=\prod_{s=1}^{r}Y^{(1)}(u-r-1+2s)$. Comparing this with expression (8.17) with $a=r-1$, we have

$$
(8.20) \qquad\qquad Y^{(r-1)}(u)=Y^{(r-1)}(u\pm 1)Y^{(1)}(u\mp r\pm 1).
$$

The two relations imply $Y^{(1)}(u-r+1)Y^{(1)}(u+r)=1$. In view of $h^\vee=2r-1$ and (8.17), this verifies the claim of the theorem $Y^{(a)}(u)Y^{(a)}(u+h^\vee)=1$ except for $a=r$. From either of the relations (8.20) and (8.17) with $a=r-1$, one can derive $\prod_{s=1}^{r}Y^{(r-1)}(u-r-1+2s)=\prod_{s=1}^{r-1}Y^{(r-1)}(u-r+2s)$. Substitution of (8.19) into this gives $Y^{(r)}(u)Y^{(r)}(u+h^\vee)=1$. □

For simply laced X_r, a more intrinsic proof of Theorem 8.8 by the *Coxeter element* of the Weyl group is available, following the remarkable idea by Fomin and Zelevinsky [FZ3] used for the proof of the periodicity of $\mathcal{Y}_2(X_r)$.

Alternative proof for simply laced X_r. Assume that X_r is simply laced. Let $I=I_+\sqcup I_-$ be the bipartite decomposition of the index set I, and define $\varepsilon(a)$ by $\varepsilon(a)=\pm$ for $a\in I_\pm$. Since $\mathbb{T}_0(X_r)$ closes among those $T^{(a)}(u)$

with fixed "parity" $\varepsilon(a)(-1)^u$, it is no problem to impose an additional relation in $\mathcal{T}_0(X_r)$,

$$(8.21) \qquad T^{(a)}(u+1) = T^{(a)}(u)^{-1} \quad \text{whenever } \varepsilon(a)(-1)^u = +,$$

in order to prove its periodicity. Let W be the Weyl group of type X_r with the simple reflections s_a $(a \in I)$, which acts on $\mathcal{T}_0(X_r)$ by

$$(8.22) \qquad s_b\big(T^{(a)}(u)\big) = T^{(a)}(u)T^{(b)}(u)^{-C_{ba}}.$$

Define $\tau_\pm = \prod_{a \in I_\pm} s_a$. Then, $\tau_\pm \tau_\mp$ is the Coxeter element of W, and τ_ε acts as

$$(8.23) \qquad \tau_\varepsilon\big(T^{(a)}(u)\big) = \begin{cases} T^{(a)}(u)^{-1} & \varepsilon(a) = \varepsilon, \\ T^{(a)}(u) \prod_{b \neq a} T^{(b)}(u)^{-C_{ba}} & \varepsilon(a) \neq \varepsilon. \end{cases}$$

By (8.21), (8.23), and $\mathbb{T}_0(X_r)$ in (8.4), we have $T^{(a)}(u+1) = \tau_{(-1)^u}(T^{(a)}(u))$. The following fact is known ([B, Chapter V, 6.2], [FZ3, Lemma 2.1]):

$$(8.24) \qquad \underbrace{\cdots \tau_- \tau_+ \tau_- \tau_+}_{h \text{ times}} = \underbrace{\cdots \tau_+ \tau_- \tau_+ \tau_-}_{h \text{ times}} = \omega_0 \quad \text{(the longest element of } W),$$

where h is the Coxeter number of X_r. Also, $\omega_0(T^{(a)}(u)) = T^{(\omega(a))}(u)^{-1}$ due to the remark after (3.6). Using these results, we obtain

$$(8.25) \qquad T^{(a)}(u+h) = \big(\underbrace{\cdots \tau_\mp \tau_\pm \tau_\mp \tau_\pm}_{h \text{ times}}\big)\big(T^{(a)}(u)\big)$$

$$= \omega_0\big(T^{(a)}(u)\big) = T^{(\omega(a))}(u)^{-1}.$$

$$\square$$

REMARK 8.9. As for the half-periodicity of $\mathcal{T}_0(X_r)$, a similar result has been obtained in [KNS2, (2.8)]. Compared with Theorem 8.8 here, the result there is weaker in that it does not cover A_r $(r \geq 2)$ or individual $T_m^{(a)}(u)$ for E_6 and E_8. Moreover, [KNS2, (2.8a)] should be corrected for $X_r = D_r$ with r odd and $a = r-1, r$.

§9. Periodicities of restricted T- and Y-systems for twisted quantum affine algebras

The T- and Y-systems considered so far are associated with the *untwisted* quantum affine algebra $U_q(\hat{\mathfrak{g}})$ (when $U = U_{t\hbar}$) as explained in Section 2. In this section we consider T- and Y-systems associated with the *twisted* quantum affine algebra $U_q(\hat{\mathfrak{g}}^\sigma)$ following [KS], [Her2]. All the basic results presented for the untwisted case can be naturally extended to the twisted case as well. Moreover, the periodicity property of the twisted case reduces to that of the untwisted case.

9.1. Dynkin diagrams of twisted affine type

Throughout this section, we let X_N exclusively denote a Dynkin diagram of type A_N ($N \geq 2$), D_N ($N \geq 4$), or E_6. We keep the enumeration of the nodes of X_N by the set $I = \{1, \ldots, N\}$ as in Figure 1. For a pair $(X_N, \kappa) = (A_N, 2)$, $(D_N, 2)$, $(E_6, 2)$, or $(D_4, 3)$, we define the diagram automorphism $\sigma : I \to I$ of X_N of order κ as $\sigma(a) = a$ *except for the following cases (in our enumeration)*:

$$
\begin{array}{lll}
(9.1) & \sigma(a) = N + 1 - a \quad (a \in I) & (X_N, \kappa) = (A_N, 2), \\
& \sigma(N-1) = N, \ \sigma(N) = N - 1 & (X_N, \kappa) = (D_N, 2), \\
& \sigma(1) = 6, \ \sigma(2) = 5, \ \sigma(5) = 2, \ \sigma(6) = 1 & (X_N, \kappa) = (E_6, 2), \\
& \sigma(1) = 3, \ \sigma(3) = 4, \ \sigma(4) = 1 & (X_N, \kappa) = (D_4, 3).
\end{array}
$$

The map σ is the same as the involution $\omega : I \to I$ in (3.6) except for $X_N = D_N$ (N: even). Let I/σ be the set of the σ-orbits of nodes of X_N. We choose, at our discretion, a complete set of representatives $I_\sigma \subset I$ of I/σ as

$$
(9.2) \qquad I_\sigma = \begin{cases} \{1, 2, \ldots, r\} & (X_N, \kappa) = (A_{2r-1}, 2), (A_{2r}, 2), (D_{r+1}, 2), \\ \{1, 2, 3, 4\} & (X_N, \kappa) = (E_6, 2), \\ \{1, 2\} & (X_N, \kappa) = (D_4, 3). \end{cases}
$$

Let $X_N^{(\kappa)} = A_{2r-1}^{(2)}$ ($r \geq 2$), $A_{2r}^{(2)}$ ($r \geq 1$), $D_{r+1}^{(2)}$ ($r \geq 3$), $E_6^{(2)}$, or $D_4^{(3)}$ be a Dynkin diagram of twisted affine type [Ka]. We enumerate the nodes of $X_N^{(\kappa)}$ with $I_\sigma \cup \{0\}$ as in Figure 8, where I_σ is the one for (X_N, κ). By this, we have established the identification of the non-zeroth nodes of the diagram $X_N^{(\kappa)}$ with the nodes of the diagram X_N belonging to the set I_σ.
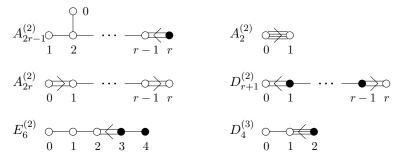
Figure 8: The Dynkin diagrams $X_N^{(\kappa)}$ of twisted affine type and their enumerations by $I_\sigma \cup \{0\}$. For a solid node a, $\sigma(a) = a$ (i.e., $\kappa_a = \kappa$) holds.

For example, for $E_6^{(2)}$, the correspondence is as follows:

$$(9.3) \qquad$$

The solid nodes 3,4 in $E_6^{(2)}$ correspond to the fixed nodes by σ in E_6. We use this identification throughout the section. (The zeroth node of $X_N^{(\kappa)}$ is irrelevant in our setting here.)

We define κ_a $(a \in I_\sigma)$ as

$$(9.4) \qquad \kappa_a = \begin{cases} 1 & \sigma(a) \neq a, \\ \kappa & \sigma(a) = a. \end{cases}$$

Note that $X_N^{(2)} = A_{2r}^{(2)}$ is the unique case in which $\kappa_a = 1$ for any $a \in I_\sigma$.

9.2. Unrestricted T-systems

Choose $\hbar \in \mathbb{C} \setminus 2\pi\sqrt{-1}\mathbb{Q}$ arbitrarily.

DEFINITION 9.1. The *unrestricted T-system* $\mathbb{T}(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the following system of relations for a family of variables $T = \{T_m^{(a)}(u) \mid a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a \hbar}\}$, where $\Omega = 2\pi\sqrt{-1}/\kappa\hbar$, and $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$ if they occur in the right-hand sides in the relations.

For $X_N^{(\kappa)} = A_{2r-1}^{(2)}$,

$$(9.5) \quad T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u)$$

$$(1 \leq a \leq r-1),$$

$$T_m^{(r)}(u-1)T_m^{(r)}(u+1) = T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u)$$
$$+ T_m^{(r-1)}(u)T_m^{(r-1)}(u+\Omega).$$

For $X_N^{(\kappa)} = A_{2r}^{(2)}$,

(9.6) $\quad T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)$
$$+ T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \leq a \leq r-1),$$
$$T_m^{(r)}(u-1)T_m^{(r)}(u+1) = T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_m^{(r-1)}(u)T_m^{(r)}(u+\Omega).$$

For $X_N^{(\kappa)} = D_{r+1}^{(2)}$,

(9.7) $\quad T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)$
$$+ T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \leq a \leq r-2),$$
$$T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) = T_{m-1}^{(r-1)}(u)T_{m+1}^{(r-1)}(u)$$
$$+ T_m^{(r-2)}(u)T_m^{(r)}(u)T_m^{(r)}(u+\Omega),$$
$$T_m^{(r)}(u-1)T_m^{(r)}(u+1) = T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_m^{(r-1)}(u).$$

For $X_N^{(\kappa)} = E_6^{(2)}$,

(9.8) $\quad T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u),$
$$T_m^{(2)}(u-1)T_m^{(2)}(u+1) = T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) + T_m^{(1)}(u)T_m^{(3)}(u),$$
$$T_m^{(3)}(u-1)T_m^{(3)}(u+1) = T_{m-1}^{(3)}(u)T_{m+1}^{(3)}(u)$$
$$+ T_m^{(2)}(u)T_m^{(2)}(u+\Omega)T_m^{(4)}(u),$$
$$T_m^{(4)}(u-1)T_m^{(4)}(u+1) = T_{m-1}^{(4)}(u)T_{m+1}^{(4)}(u) + T_m^{(3)}(u).$$

For $X_N^{(\kappa)} = D_4^{(3)}$,

(9.9) $\quad T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u),$
$$T_m^{(2)}(u-1)T_m^{(2)}(u+1) = T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u)$$
$$+ T_m^{(1)}(u)T_m^{(1)}(u-\Omega)T_m^{(1)}(u+\Omega).$$

The domain $\mathbb{C}_{\kappa_a\hbar}$ of the parameter u effectively imposes the following periodic condition:

$$(9.10) \qquad T_m^{(a)}(u) = \begin{cases} T_m^{(a)}(u + \kappa\Omega) & \sigma(a) \neq a, \\ T_m^{(a)}(u + \Omega) & \sigma(a) = a. \end{cases}$$

DEFINITION 9.2. The *unrestricted T-algebra* $\mathcal{T}(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the ring with generators $T_m^{(a)}(u)^{\pm 1}$ $(a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a\hbar})$ and the relations $\mathbb{T}(X_N^{(\kappa)})$. Also, we define the ring $\mathcal{T}^\circ(X_N^{(\kappa)})$ as the subring of $\mathcal{T}(X_N^{(\kappa)})$ generated by $T_m^{(a)}(u)$ $(a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a\hbar})$.

The following are some features of the T-system $\mathbb{T}(X_N^{(\kappa)})$ that are specific to the twisted case.

(i) The relations include the *two* basic units of the parameter u, 1 and Ω, which are $\mathbb{Z}$-linearly independent under our assumption of $\hbar \notin 2\pi\sqrt{-1}\mathbb{Q}$.

(ii) The domain $\mathbb{C}_{\kappa_a\hbar}$ of the parameter u and the resulting periodic condition (9.10) *depend on* $a \in I_\sigma$.

(iii) We do *not* consider the T-system $\mathbb{T}(X_N^{(\kappa)})$ whose domain U of the parameter u is $\mathbb{C}$. This is because the periodic condition (9.10) is now an integral part of the relations $\mathbb{T}(X_N^{(\kappa)})$ due to (i). This is also natural given that no Yangian analogue of the twisted quantum affine algebra $U_q(\hat{\mathfrak{g}}^\sigma)$ is known.

(iv) The discrete version of $\mathbb{T}(X_N^{(\kappa)})$ is available by taking the domain U of the parameter u as $U = \mathbb{Z} \times \mathbb{Z}_\kappa$, where $(a, b) \in U$ corresponds to $u = a + b\Omega$, and imposing the periodic condition (9.10).

REMARK 9.3. The T-system $\mathbb{T}(X_N^{(\kappa)})$ was introduced in [KS] as a family of relations in the ring of the commuting transfer matrices for solvable lattice models associated with the twisted quantum affine algebra $U_q(\hat{\mathfrak{g}}^\sigma)$ of type $X_N^{(\kappa)}$.

REMARK 9.4. Unifying the untwisted and twisted cases, the T-system $\mathbb{T}(X_r)$ and the Y-system $\mathbb{Y}(X_r)$ of type X_r in Section 2 are also said to be *of type* $X_r^{(1)}$ and are denoted by $\mathbb{T}(X_r^{(1)})$ and $\mathbb{Y}(X_r^{(1)})$. Strictly speaking, this should be applied only when the domain U of the parameter u is $\mathbb{C}_{t\hbar}$. However, as we have seen, such a distinction of U is not so essential in many aspects of $\mathbb{T}(X_r)$ and $\mathbb{Y}(X_r)$.

There is a simple relation between the rings $\mathbb{T}(X_N^{(\kappa)})$ and $\mathbb{T}(X_N)$. Let $\{\hat{T}_m^{(a)}(u)^{\pm 1} \mid a \in I, m \in \mathbb{N}, u \in \mathbb{C}_\hbar\}$ be the set of generators of $\mathbb{T}(X_N)$. Let $\mathcal{J}^\sigma$ be the ideal of $\mathbb{T}(X_N)$ generated by

$$(9.11) \qquad \hat{T}_m^{(a)}(u) - \hat{T}_m^{(\sigma(a))}(u+\Omega) \quad (a \in I, m \in \mathbb{N}_{\geq 1}, u \in \mathbb{C}_\hbar).$$

Then one can choose a generating set of $\mathbb{T}(X_N)/\mathcal{J}^\sigma$ as $\{\hat{T}_m^{(a)}(u)^{\pm 1} \mid a \in I_\sigma, m \in \mathbb{Z}_{\geq 1}, u \in \mathbb{C}_{\kappa_a \hbar}\}$.

PROPOSITION 9.5. *There is a ring isomorphism*

$$(9.12) \qquad \begin{aligned} \mathbb{T}(X_N)/\mathcal{J}^\sigma &\to \mathbb{T}(X_N^{(\kappa)}), \\ \hat{T}_m^{(a)}(u) &\mapsto T_m^{(a)}(u) \quad (a \in I_\sigma). \end{aligned}$$

Proof. It is easy to check that the relations of the both rings are identical under the correspondence. □

The T-system $\mathbb{T}(X_N^{(\kappa)})$ plays the same role in the Grothendieck ring $\mathrm{Rep}\, U_q(\hat{\mathfrak{g}}^\sigma)$ of the category of type 1 finite-dimensional $U_q(\hat{\mathfrak{g}}^\sigma)$-modules for the twisted quantum affine algebra $U_q(\hat{\mathfrak{g}}^\sigma)$ of type $X_N^{(\kappa)}$ as does the untwisted case.

For arbitrarily chosen $\hbar \in \mathbb{C} \setminus 2\pi\sqrt{-1}\mathbb{Q}$, we set the deformation parameter q of the twisted quantum affine algebras $U_q(\hat{\mathfrak{g}})$ [J], [D1], [D2] as $q = e^\hbar \in \mathbb{C}^\times$, so that q is *not a root of unity*.

The q-character map χ_q^σ of $U_q(\hat{\mathfrak{g}}^\sigma)$ is defined by Hernandez [Her2] as an injective ring homomorphism

$$(9.13) \qquad \chi_q^\sigma : \mathrm{Rep}\, U_q(\hat{\mathfrak{g}}^\sigma) \to \mathbb{Z}[Z_{i,a}^{\pm 1}]_{i \in I_\sigma, a \in \mathbb{C}^\times}.$$

Consult [Her2] for more information on $U_q(\hat{\mathfrak{g}}^\sigma)$ and χ_q^σ. The enumeration of I_σ in [Her2] is the same as the present one except for $A_{2r}^{(2)}$, where $1, 2, \ldots, r$ here correspond to $r-1, \ldots, 1, 0$ in [Her2]. To make the description uniform, for $a \in I_\sigma = \{1, \ldots, r\}$ we set $\overline{a} = r - a$ for $X_N^{(\kappa)} = A_{2r}^{(2)}$ and $\overline{a} = a$ otherwise. (This notation $\overline{a}$ will be used only in the rest of this section.) From now on, we employ the parameterization of the variables $Z_{\overline{a}, q^{\kappa_a u}}$ ($a \in I_\sigma$, $u \in \mathbb{C}_{\kappa_a \hbar}$) instead of $Z_{i,a}$ ($i \in I_\sigma$, $a \in \mathbb{C}^\times$) in [Her2]. The *q-character ring* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}}^\sigma)$ of $U_q(\hat{\mathfrak{g}}^\sigma)$ is defined to be $\mathrm{Im}\, \chi_q^\sigma$.

DEFINITION 9.6. A *Kirillov-Reshetikhin module* $W_m^{(a)}(u)$ $(a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a \hbar})$ of $U_q(\hat{\mathfrak{g}}^\sigma)$ of type $X_N^{(\kappa)}$ is the irreducible finite dimensional $U_q(\hat{\mathfrak{g}}^\sigma)$-module with highest weight monomial

$$(9.14) \qquad \prod_{j=1}^m Z_{\overline{a}, q^{\kappa_a(u+m+1-2j)}}.$$

REMARK 9.7. The above $W_m^{(a)}(u)$ corresponds to $W_{m, q^{\kappa_a(u-m+1)}}^{(\overline{a})}$ in [Her2]. The T-system $\mathbb{T}(X_N^{(\kappa)})$ in Definition 9.1 agrees with the one in [Her2, Section 4.3] under the identification $T_m^{(a)}(u) = X_{m, q^{\kappa_a(u-m+1)}}^{(\overline{a})}$.

In the same way as (2.9), we define $S_{amu}(T) \in \mathbb{Z}[T]$ $(a \in I, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a \hbar})$, so that all the relations in $\mathbb{T}(X_N^{(\kappa)})$ are written in the form $S_{amu}(T) = 0$. Let $I(\mathbb{T}(X_N^{(\kappa)}))$ be the ideal of $\mathbb{Z}[T]$ generated by $S_{amu}(T)$.

THEOREM 9.8. *Let* $\widetilde{T} = \{\widetilde{T}_m^{(a)}(u) := \chi_q^\sigma(W_m^{(a)}(u)) \mid a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a \hbar}\}$ *be the family of the q-characters of the Kirillov-Reshetikhin modules of* $U_q(\hat{\mathfrak{g}}^\sigma)$ *of type* $X_N^{(\kappa)}$. *Then,*

(1) *the family* $\widetilde{T}$ *generates the ring* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}}^\sigma)$;

(2) *(Hernandez [Her2]) the family* $\widetilde{T}$ *satisfies the T-system* $\mathbb{T}(X_N^{(\kappa)})$ *in* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}}^\sigma)$ *(by replacing* $T_m^{(a)}(u)$ *in* $\mathbb{T}(X_N^{(\kappa)})$ *with* $\widetilde{T}_m^{(a)}(u)$);

(3) *for any* $P(T) \in \mathbb{Z}[T]$, *the relation* $P(\widetilde{T}) = 0$ *holds in* $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}}^\sigma)$ *if and only if there is a nonzero monomial* $M(T) \in \mathbb{Z}[T]$ *such that* $M(T)P(T) \in I(\mathbb{T}(X_N^{(\kappa)}))$.

Proof. (1) The fundamental character $\chi_q^\sigma(W_1^{(a)}(u))$ has the form

$$(9.15) \qquad \chi_q^\sigma\big(W_1^{(a)}(u)\big) = Z_{\overline{a}, q^{\kappa_a u}} + (\text{lower term}),$$

where "lower" means lower in the weight lattice for the subalgebra $U_q(\mathfrak{g}^\sigma)$ of $U_q(\hat{\mathfrak{g}}^\sigma)$ [Her2]. Thus, if there is a nontrivial relation among the fundamental characters, then it causes some nontrivial relation among $Z_{\overline{a}, q^{\kappa_a u}}$'s. This is a contradiction.

(2) This was proved by [Her2, Theorem 4.2].

(3) The proof is completely parallel with Theorem 2.8(3) by setting the height as $\mathrm{ht}\, T_m^{(a)}(u) = m$. $\qquad \square$

COROLLARY 9.9. *The ring* $\mathcal{T}^\circ(X_N^{(\kappa)})$ *is isomorphic to* $\mathrm{Rep}\, U_q(\hat{\mathfrak{g}}^\sigma)$ *by the correspondence* $T_m^{(a)}(u) \mapsto W_m^{(a)}(u)$.

In Appendix A.2 we give parallel results for the ring associated with the Q-system and $\operatorname{Rep} U_q(\mathfrak{g}^\sigma)$.

9.3. Unrestricted Y-systems

DEFINITION 9.10. The *unrestricted Y-system* $\mathbb{T}(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the following system of relations for a family of variables $Y = \{Y_m^{(a)}(u) \mid a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a \hbar}\}$, where $\Omega = 2\pi\sqrt{-1}/\kappa\hbar$, and $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = 0$ if they occur in the right-hand sides in the following relations.

For $X_N^{(\kappa)} = A_{2r-1}^{(2)}$,

$$(9.16) \quad Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}$$
$$(1 \le a \le r-1),$$

$$Y_m^{(r)}(u-1)Y_m^{(r)}(u+1) = \frac{(1+Y_m^{(r-1)}(u))(1+Y_m^{(r-1)}(u+\Omega))}{(1+Y_{m-1}^{(r)}(u)^{-1})(1+Y_{m+1}^{(r)}(u)^{-1})}.$$

For $X_N^{(\kappa)} = A_{2r}^{(2)}$,

$$(9.17) \quad Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}$$
$$(1 \le a \le r-1),$$

$$Y_m^{(r)}(u-1)Y_m^{(r)}(u+1) = \frac{(1+Y_m^{(r-1)}(u))(1+Y_m^{(r)}(u+\Omega))}{(1+Y_{m-1}^{(r)}(u)^{-1})(1+Y_{m+1}^{(r)}(u)^{-1})}.$$

For $X_N^{(\kappa)} = D_{r+1}^{(2)}$,

$$(9.18) \quad Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}$$
$$(1 \le a \le r-2),$$

$$Y_m^{(r-1)}(u-1)Y_m^{(r-1)}(u+1) = \frac{\begin{array}{c}(1+Y_m^{(r-2)}(u))(1+Y_m^{(r)}(u)) \\ \times (1+Y_m^{(r)}(u+\Omega))\end{array}}{(1+Y_{m-1}^{(r-1)}(u)^{-1})(1+Y_{m+1}^{(r-1)}(u)^{-1})},$$

$$Y_m^{(r)}(u-1)Y_m^{(r)}(u+1) = \frac{1+Y_m^{(r-1)}(u)}{(1+Y_{m-1}^{(r)}(u)^{-1})(1+Y_{m+1}^{(r)}(u)^{-1})}.$$

For $X_N^{(\kappa)} = E_6^{(2)}$,

$$(9.19) \qquad Y_m^{(1)}(u-1)Y_m^{(1)}(u+1) = \frac{1 + Y_m^{(2)}(u)}{(1 + Y_{m-1}^{(1)}(u)^{-1})(1 + Y_{m+1}^{(1)}(u)^{-1})},$$

$$Y_m^{(2)}(u-1)Y_m^{(2)}(u+1) = \frac{(1 + Y_m^{(1)}(u))(1 + Y_m^{(3)}(u))}{(1 + Y_{m-1}^{(2)}(u)^{-1})(1 + Y_{m+1}^{(2)}(u)^{-1})},$$

$$Y_m^{(3)}(u-1)Y_m^{(3)}(u+1) = \frac{\begin{array}{c}(1 + Y_m^{(2)}(u))(1 + Y_m^{(2)}(u+\Omega)) \\ \times (1 + Y_m^{(4)}(u))\end{array}}{(1 + Y_{m-1}^{(3)}(u)^{-1})(1 + Y_{m+1}^{(3)}(u)^{-1})},$$

$$Y_m^{(4)}(u-1)Y_m^{(4)}(u+1) = \frac{1 + Y_m^{(3)}(u)}{(1 + Y_{m-1}^{(4)}(u)^{-1})(1 + Y_{m+1}^{(4)}(u)^{-1})}.$$

For $X_N^{(\kappa)} = D_4^{(3)}$,

$$(9.20) \qquad Y_m^{(1)}(u-1)Y_m^{(1)}(u+1) = \frac{1 + Y_m^{(2)}(u)}{(1 + Y_{m-1}^{(1)}(u)^{-1})(1 + Y_{m+1}^{(1)}(u)^{-1})},$$

$$Y_m^{(2)}(u-1)Y_m^{(2)}(u+1) = \frac{\begin{array}{c}(1 + Y_m^{(1)}(u))(1 + Y_m^{(1)}(u-\Omega)) \\ \times (1 + Y_m^{(1)}(u+\Omega))\end{array}}{(1 + Y_{m-1}^{(2)}(u)^{-1})(1 + Y_{m+1}^{(2)}(u)^{-1})}.$$

The domain $\mathbb{C}_{\kappa_a \hbar}$ of the parameter u effectively imposes the following periodic condition:

$$(9.21) \qquad Y_m^{(a)}(u) = \begin{cases} Y_m^{(a)}(u + \kappa\Omega) & \sigma(a) \neq a, \\ Y_m^{(a)}(u + \Omega) & \sigma(a) = a. \end{cases}$$

DEFINITION 9.11. The *unrestricted Y-algebra* $\mathcal{Y}(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the ring with generators $Y_m^{(a)}(u)^{\pm 1}$, $(1 + Y_m^{(a)}(u))^{-1}$ $(a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a \hbar})$ and the relations $\mathbb{Y}(X_N^{(\kappa)})$.

Let $\{\hat{Y}_m^{(a)}(u)^{\pm 1}, (1 + \hat{Y}_m^{(a)}(u))^{-1} \mid a \in I, m \in \mathbb{N}, u \in \mathbb{C}_\hbar\}$ be the set of generators of $\mathcal{Y}(X_N)$. Let $\mathcal{I}^\sigma$ be the ideal of $\mathcal{Y}(X_N)$ generated by

$$(9.22) \qquad \hat{Y}_m^{(a)}(u) - \hat{Y}_m^{(\sigma(a))}(u + \Omega) \quad (a \in I, m \in \mathbb{N}, u \in \mathbb{C}_\hbar).$$

Then one can choose a generating set of $\mathcal{Y}(X_N)/\mathcal{I}^\sigma$ as $\{\hat{Y}_m^{(a)}(u)^{\pm 1}, (1 + \hat{Y}_m^{(a)}(u))^{-1} \mid a \in I_\sigma, m \in \mathbb{N}, u \in \mathbb{C}_{\kappa_a \hbar}\}$.

As Proposition 9.5, we have the following.

PROPOSITION 9.12. *There is a ring isomorphism*

$$(9.23) \qquad \begin{aligned} \mathcal{Y}(X_N)/\mathcal{I}^\sigma &\to \mathcal{Y}(X_N^{(\kappa)}), \\ \hat{Y}_m^{(a)}(u) &\mapsto Y_m^{(a)}(u) \quad (a \in I_\sigma). \end{aligned}$$

The following theorem is an analogue of Theorem 2.12.

THEOREM 9.13. (1) *There is a ring homomorphism*

$$(9.24) \qquad \varphi : \mathcal{Y}(X_N^{(\kappa)}) \to \mathcal{T}(X_N^{(\kappa)})$$

defined by

$$(9.25) \qquad Y_m^{(a)}(u) \mapsto \frac{M_m^{(a)}(u)}{T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u)},$$

where $T_0^{(a)}(u) = 1$.

(2) *There is a ring homomorphism*

$$(9.26) \qquad \psi : \mathcal{T}(X_N^{(\kappa)}) \to \mathcal{Y}(X_N^{(\kappa)})$$

such that $\psi \circ \varphi = \mathrm{id}_{\mathcal{Y}(X_N^{(\kappa)})}$.

Proof. We derive the theorem from the results of Theorem 2.12 and Propositions 9.5 and 9.12.

(1) Let $\hat{\varphi} : \mathcal{Y}(X_N) \to \mathcal{T}(X_N)$ be the homomorphism in (2.21). Let $\mathcal{J}^\sigma$ and $\mathcal{I}^\sigma$ be the ideals of $\mathcal{T}(X_N)$ and $\mathcal{Y}(X_N)$ in Propositions 9.5 and 9.12, respectively. We claim that $\hat{\varphi}(\mathcal{I}^\sigma) \subset \mathcal{J}^\sigma$. In fact,

$$
\begin{aligned}
\hat{\varphi}\big(\hat{Y}_m^{(a)}(u)\big) &= \frac{\hat{M}_m^{(a)}(u)}{\hat{T}_{m-1}^{(a)}(u)\hat{T}_{m+1}^{(a)}(u)} \\
(9.27) \qquad &\equiv \frac{\hat{M}_m^{(\sigma(a))}(u+\Omega)}{\hat{T}_{m-1}^{(\sigma(a))}(u+\Omega)\hat{T}_{m+1}^{(\sigma(a))}(u+\Omega)} \quad \mathrm{mod}\, \mathcal{J}^\sigma \\
&= \hat{\varphi}\big(\hat{Y}_m^{(\sigma(a))}(u+\Omega)\big),
\end{aligned}
$$

where we also used the invariance of $\mathbb{T}(X_N)$ by σ in the second equality. Then, the induced homomorphism

$$(9.28) \qquad \hat{\varphi}^\sigma : \mathcal{Y}(X_N)/\mathcal{I}^\sigma \to \mathcal{T}(X_N)/\mathcal{J}^\sigma$$

gives the desired homomorphism φ under the isomorphisms in Propositions 9.5 and 9.12.

(2) Let $\hat{\psi} : \mathcal{T}(X_N) \to \mathcal{Y}(X_N)$ be the homomorphism in (2.25), where we modify Step 1 of the construction of $\hat{\psi}$ in the proof of Theorem 2.12 with the following. (For simplicity, we write $\hat{\psi}(T_1^{(a)}(u))$ as $T_1^{(a)}(u)$.)

Step 1. We arbitrarily choose $T_1^{(a)}(u) \in \mathcal{Y}(X_N)^\times$ $(a \in I)$ for each $u \in \mathbb{C}_\hbar$ in the region $-1 \leq \operatorname{Re} u < 1$ *such that*

$$(9.29) \qquad T_1^{(a)}(u) \equiv T_1^{(\sigma(a))}(u+\Omega) \mod \mathcal{J}^\sigma.$$

(For example, just take $T_1^{(a)}(u) = 1$.)

Then, one can easily show that $T_m^{(a)}(u) \equiv T_m^{(\sigma(a))}(u+\Omega) \mod \mathcal{J}^\sigma$ for any $T_m^{(a)}(u)$ constructed in Steps 2 and 3 of Theorem 2.12, again by the invariance of $\mathbb{T}(X_N)$ by σ. Then, the induced isomorphism

$$(9.30) \qquad \hat{\psi}^\sigma : \mathcal{T}(X_N)/\mathcal{J}^\sigma \to \mathcal{Y}(X_N)/\mathcal{I}^\sigma$$

gives the desired homomorphism ψ under the isomorphisms in Propositions 9.5 and 9.12. The property $\psi \circ \varphi = \operatorname{id}_{\mathcal{Y}(X_N^{(\kappa)})}$ follows from $\hat{\psi}^\sigma \circ \hat{\varphi}^\sigma = \operatorname{id}_{\mathcal{Y}(X_N)/\mathcal{I}^\sigma}$. $\qquad\square$

9.4. Restricted T- and Y-systems

DEFINITION 9.14. Fix an integer $\ell \geq 2$. The *level ℓ restricted T-system* $\mathbb{T}_\ell(X_N^{(\kappa)})$ *of type $X_N^{(\kappa)}$ with the unit boundary condition* is the system of relations (9.5)–(9.9) naturally restricted to a family of variables $T = \{T_m^{(a)}(u) \mid a \in I_\sigma; m = 1, \dots, \ell-1; u \in \mathbb{C}_{\kappa_a\hbar}\}$, where $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$, and furthermore, $T_\ell^{(a)}(u) = 1$ (the *unit boundary condition*) if they occur in the right-hand sides in the relations.

DEFINITION 9.15. The *level ℓ restricted T-algebra* $\mathcal{T}_\ell(X_N^{(\kappa)})$ *of type $X_N^{(\kappa)}$* is the ring with generators $T_m^{(a)}(u)^{\pm 1}$ $(a \in I_\sigma; m = 1, \dots, \ell-1; u \in \mathbb{C}_{\kappa_a\hbar})$ and the relations $\mathbb{T}_\ell(X_N^{(\kappa)})$. Also, we define the ring $\mathcal{T}_\ell^\circ(X_N^{(\kappa)})$ as the subring of $\mathcal{T}_\ell(X_N^{(\kappa)})$ generated by $T_m^{(a)}(u)$ $(a \in I_\sigma; m = 1, \dots, \ell-1; u \in \mathbb{C}_{\kappa_a\hbar})$.

DEFINITION 9.16. Fix an integer $\ell \geq 2$. The *level ℓ restricted Y-system* $\mathbb{Y}_\ell(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the system of relations (9.16)–(9.20) naturally restricted to a family of variables $Y = \{Y_m^{(a)}(u) \mid a \in I_\sigma; m = 1, \ldots, \ell - 1; u \in \mathbb{C}_{\kappa_a \hbar}\}$, where $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = 0$, and furthermore, $Y_\ell^{(a)}(u)^{-1} = 0$ if they occur in the right-hand sides in the relations.

DEFINITION 9.17. The *level ℓ restricted Y-algebra* $\mathcal{Y}_\ell(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the ring with generators $Y_m^{(a)}(u)^{\pm 1}$, $(1 + Y_m^{(a)}(u))^{-1}$ ($a \in I_\sigma; m = 1, \ldots, \ell - 1; u \in \mathbb{C}_{\kappa_a \hbar}$) and the relations $\mathbb{Y}_\ell(X_N^{(\kappa)})$.

REMARK 9.18. The level restrictions of the T- and Y-systems for the twisted case are introduced here for the first time. The former is defined such that dropping the parameter u, namely, the formal replacement $T_m^{(a)}(u) \to Q_m^{(a)}$, coincides with the level ℓ restricted Q-system introduced in [HKOTT, (6.2)].

Propositions 9.5 and 9.12 have natural counterparts in the level restricted situation. Let $\{\hat{T}_m^{(a)}(u)^{\pm 1} \mid a \in I, m = 1, \ldots, \ell - 1, u \in \mathbb{C}_\hbar\}$ be the set of generators of $\mathcal{T}_\ell(X_N)$. Let $\mathcal{J}_\ell^\sigma$ be the ideal of $\mathcal{T}_\ell(X_N)$ generated by

$$(9.31) \qquad \hat{T}_m^{(a)}(u) - \hat{T}_m^{(\sigma(a))}(u + \Omega) \quad (a \in I; m = 1, \ldots, l - 1; u \in \mathbb{C}_\hbar).$$

Then one can choose a generating set of $\mathcal{T}_\ell(X_N)/\mathcal{J}_\ell^\sigma$ as $\{\hat{T}_m^{(a)}(u)^{\pm 1} \mid a \in I_\sigma; m = 1, \ldots, \ell - 1; u \in \mathbb{C}_{\kappa_a \hbar}\}$.

Similarly, let $\{\hat{Y}_m^{(a)}(u)^{\pm 1}, (1 + \hat{Y}_m^{(a)}(u))^{-1} \mid a \in I; m = 1, \ldots, \ell - 1; u \in \mathbb{C}_\hbar\}$ be the set of generators of $\mathcal{Y}_\ell(X_N)$. Let $\mathcal{I}_\ell^\sigma$ be the ideal of $\mathcal{Y}_\ell(X_N)$ generated by

$$(9.32) \qquad \hat{Y}_m^{(a)}(u) - \hat{Y}_m^{(\sigma(a))}(u + \Omega) \quad (a \in I; m = 1, \ldots, \ell - 1; u \in \mathbb{C}_\hbar).$$

Then one can choose a generating set of $\mathcal{Y}_\ell(X_N)/\mathcal{I}_\ell^\sigma$ as $\{\hat{Y}_m^{(a)}(u)^{\pm 1}, (1 + \hat{Y}_m^{(a)}(u))^{-1} \mid a \in I_\sigma; m = 1, \ldots, \ell - 1; u \in \mathbb{C}_{\kappa_a \hbar}\}$.

PROPOSITION 9.19. *There is a ring isomorphism*

$$(9.33) \qquad \begin{aligned} \mathcal{T}_\ell(X_N)/\mathcal{J}_\ell^\sigma &\to \mathcal{T}_\ell(X_N^{(\kappa)}), \\ \hat{T}_m^{(a)}(u) &\mapsto T_m^{(a)}(u) \quad (a \in I_\sigma). \end{aligned}$$

Similarly, there is a ring isomorphism

$$(9.34) \qquad \begin{aligned} \mathcal{Y}_\ell(X_N)/\mathcal{I}_\ell^\sigma &\to \mathcal{Y}_\ell(X_N^{(\kappa)}), \\ \hat{Y}_m^{(a)}(u) &\mapsto Y_m^{(a)}(u) \quad (a \in I_\sigma). \end{aligned}$$

Proof. It is compatible to set $\hat{T}_\ell^{(a)}(u) = T_\ell^{(a)}(u) = 1$ in Proposition 9.5. It is also compatible to set $\hat{Y}_\ell^{(a)}(u)^{-1} = Y_\ell^{(a)}(u)^{-1} = 0$ in Proposition 9.12. $\square$

9.5. Periodicities of restricted T- and Y-systems

Let $h^\vee$ be the dual Coxeter number of $X_N^{(\kappa)}$ [Ka]. It is the same with the (dual) Coxeter number of X_N and the following:

$$(9.35) \qquad \begin{array}{c|ccccc} X_N^{(\kappa)} & A_{2r-1}^{(2)} & A_{2r}^{(2)} & D_{r+1}^{(2)} & E_6^{(2)} & D_4^{(3)} \\ \hline h^\vee & 2r & 2r+1 & 2r & 12 & 6 \end{array}$$

The periodicity of $\mathcal{T}_\ell(X_N^{(\kappa)})$ reduces to that of $\mathcal{T}_\ell(X_N)$ proved in Corollary 4.29.

THEOREM 9.20. *The following relations hold in* $\mathcal{T}_\ell(X_N^{(\kappa)})$:

(1) *Half-periodicity:*

$$T_m^{(a)}(u + h^\vee + \ell) = \begin{cases} T_{\ell-m}^{(a)}(u) & \text{if } X_N^{(\kappa)} = D_{r+1}^{(2)} \text{ (r+1: even) or } D_4^{(3)}, \\ T_{\ell-m}^{(a)}(u + \Omega) & \text{otherwise.} \end{cases}$$

(2) *Periodicity:* $T_m^{(a)}(u + 2(h^\vee + \ell)) = T_m^{(a)}(u)$.

Proof. It suffices to prove the half-periodicity (1). First consider the case $X_N^{(\kappa)} \neq D_{r+1}^{(2)}$ (r+1: even) and $D_4^{(3)}$. Then we have $\sigma = \omega$ and $\kappa = 2$. Thanks to the first half of Proposition 9.19, it is equivalent to showing that $\hat{T}_m^{(a)}(u + h^\vee + \ell) = \hat{T}_{\ell-m}^{(a)}(u + \Omega)$ for $a \in I_\sigma$ in $\mathcal{T}_\ell(X_N)/\mathcal{J}_\ell^\sigma$. Both the left and the right sides coincide with $\hat{T}_{\ell-m}^{(\sigma(a))}(u)$ due to the half-periodicity of X_N and (9.11), respectively.

Next consider the remaining case $X_N^{(\kappa)} = D_{r+1}^{(2)}$ (r+1: even) or $D_4^{(3)}$. Then we have $\omega = \mathrm{id}$. By the same reason as before, we show that $\hat{T}_m^{(a)}(u + h^\vee + \ell) = \hat{T}_{\ell-m}^{(a)}(u)$ for $a \in I_\sigma$ in $\mathcal{T}_\ell(X_N)/\mathcal{J}_\ell^\sigma$. Again, this is guaranteed by the half-periodicity of X_N. $\square$

Similarly, the periodicity of $\mathcal{Y}_\ell(X_N^{(\kappa)})$ reduces to that of $\mathcal{Y}_\ell(X_N)$. Recall that Conjecture 3.12 has been proved for $\mathcal{Y}_\ell(X_N)$, except for the half-periodicity for D_N and E_6.

THEOREM 9.21. *Suppose that Conjecture 3.12(1) is also true for* D_N *and* E_6. *Then, the following relations hold in* $\mathcal{Y}_\ell(X_N^{(\kappa)})$:

(1) *Half-periodicity:*

$$Y_m^{(a)}(u + h^\vee + \ell) = \begin{cases} Y_{\ell-m}^{(a)}(u) & \text{if } X_N^{(\kappa)} = D_{r+1}^{(2)} \ (r+1\text{: even) or } D_4^{(3)}, \\ Y_{\ell-m}^{(a)}(u + \Omega) & otherwise. \end{cases}$$

(2) *Periodicity:* $Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u)$.

REMARK 9.22. By formally setting $\Omega = 0$ (i.e., $\hbar \to \infty$; $q \to 0, \infty$) or by imposing a further relation $Y_m^{(a)}(u) = Y_m^{(a)}(u + \Omega)$ in $\mathcal{Y}_\ell(X_N^{(\kappa)})$ for $X_N^{(\kappa)} = A_{2r-1}^{(2)}$ (resp. $D_{r+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$), one gets the Y-system of the form

$$(9.36) \qquad Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{\prod_{b \in I}(1 + Y_m^{(b)}(u))^{2\delta_{ab} - C_{ab}}}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}.$$

Here, $C_{ab} = 2(\alpha_a, \alpha_b)/(\alpha_a, \alpha_a)$ is the Cartan matrix for B_r (resp. C_r, F_4, G_2) with the enumeration I in Figure 1. This is the Y-system for nonsimply laced X_r considered in [FZ3], [Kel2]. It was proved, by [FZ3] for $\ell = 2$ and by [Kel2] for any $\ell \geq 2$, that the system (9.36) has the full-period $2(h + \ell)$, where h is the *Coxeter number* of B_r (resp. C_r, F_4, G_2). This completely agrees with Theorem 9.21, since the dual Coxeter number $h^\vee$ of $A_{2r-1}^{(2)}$ (resp. $D_{r+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$) equals the Coxeter number h of B_r (resp. C_r, F_4, G_2). By the same token, one can obtain from $\mathcal{T}_\ell(X_N^{(\kappa)})$ the T-system of the form

$$(9.37) \quad T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{b \in I} T_m^{(b)}(u)^{2\delta_{ab} - C_{ab}},$$

whose periodicity is the same as (9.36).

9.6. Periodicities of restricted T- and Y-systems at level 0

Here we introduce the level 0 restricted T-system $\mathbb{T}_0(X_N^{(\kappa)})$ and T-group $\mathcal{T}_0(X_N^{(\kappa)})$ of type $X_N^{(\kappa)}$ in a manner similar to Section 3. The analogous construction of the Y-system and Y-group leads to exactly the same objects. Thus one should understand $\mathcal{T}_0(X_N^{(\kappa)}) \simeq \mathcal{Y}_0(X_N^{(\kappa)})$ by $T^{(a)}(u) \leftrightarrow Y^{(a)}(u)$ in the sequel. Such a coincidence has been already encountered in the untwisted case between $\mathbb{T}_0(X_r)$ (8.4) and $\mathbb{Y}_0(X_r)$ (8.9) for simply laced X_r.

DEFINITION 9.23. The *level 0 restricted T-system* $\mathbb{T}_0(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the following system of relations for a family of variables $T = \{T^{(a)}(u) \mid a \in I_\sigma, u \in \mathbb{C}_{\kappa_a \hbar}\}$, where $\Omega = 2\pi\sqrt{-1}/\kappa\hbar$, and $T^{(0)}(u) = 1$ if they occur in the right-hand sides in the relations.

For $X_N^{(\kappa)} = A_{2r-1}^{(2)}$,

$$(9.38) \quad T^{(a)}(u-1)T^{(a)}(u+1) = T^{(a-1)}(u)T^{(a+1)}(u) \quad (1 \le a \le r-1),$$

$$T^{(r)}(u-1)T^{(r)}(u+1) = T^{(r-1)}(u)T^{(r-1)}(u+\Omega).$$

For $X_N^{(\kappa)} = A_{2r}^{(2)}$,

$$(9.39) \quad T^{(a)}(u-1)T^{(a)}(u+1) = T^{(a-1)}(u)T^{(a+1)}(u) \quad (1 \le a \le r-1),$$

$$T^{(r)}(u-1)T^{(r)}(u+1) = T^{(r-1)}(u)T^{(r)}(u+\Omega).$$

For $X_N^{(\kappa)} = D_{r+1}^{(2)}$,

$$(9.40) \quad T^{(a)}(u-1)T^{(a)}(u+1) = T^{(a-1)}(u)T^{(a+1)}(u) \quad (1 \le a \le r-2),$$

$$T^{(r-1)}(u-1)T^{(r-1)}(u+1) = T^{(r-2)}(u)T^{(r)}(u)T^{(r)}(u+\Omega),$$

$$T^{(r)}(u-1)T^{(r)}(u+1) = T^{(r-1)}(u).$$

For $X_N^{(\kappa)} = E_6^{(2)}$,

$$(9.41) \quad T^{(1)}(u-1)T^{(1)}(u+1) = T^{(2)}(u),$$

$$T^{(2)}(u-1)T^{(2)}(u+1) = T^{(1)}(u)T^{(3)}(u),$$

$$T^{(3)}(u-1)T^{(3)}(u+1) = T^{(2)}(u)T^{(2)}(u+\Omega)T^{(4)}(u),$$

$$T^{(4)}(u-1)T^{(4)}(u+1) = T^{(3)}(u).$$

For $X_N^{(\kappa)} = D_4^{(3)}$,

$$(9.42) \quad T^{(1)}(u-1)T^{(1)}(u+1) = T^{(2)}(u),$$

$$T^{(2)}(u-1)T^{(2)}(u+1) = T^{(1)}(u)T^{(1)}(u-\Omega)T^{(1)}(u+\Omega).$$

DEFINITION 9.24. The *level 0 restricted T-group* $\mathcal{T}_0(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the abelian group with generators $T^{(a)}(u)$ ($a \in I_\sigma, u \in \mathbb{C}_{\kappa_a \hbar}$) and the relations $\mathbb{T}_0(X_N^{(\kappa)})$.

REMARK 9.25. $\mathbb{T}_0(X_N^{(\kappa)})$ is obtained from the unrestricted T-system $\mathbb{T}(X_N^{(\kappa)})$ (9.5)–(9.9) by setting $T_m^{(a)}(u) = T^{(a)}(u)$ if $m = 0$ and $T_m^{(a)}(u) = 0$ otherwise.

Let $\{\hat{T}^{(a)}(u) \mid a \in I, u \in \mathbb{C}_\hbar\}$ be the set of generators of $\mathcal{T}_0(X_N)$. Let $\mathcal{J}_0^\sigma$ be the subgroup of $\mathcal{T}_0(X_N)$ generated by

$$(9.43) \qquad \hat{T}^{(a)}(u)\hat{T}^{(\sigma(a))}(u+\Omega)^{-1} \quad (a \in I, u \in \mathbb{C}_\hbar).$$

Then one can choose a generating set of $\mathcal{T}_0(X_N)/\mathcal{J}_0^\sigma$ as $\{\hat{T}^{(a)}(u)^{\pm 1} \mid a \in I_\sigma, u \in \mathbb{C}_{\kappa_a \hbar}\}$.

PROPOSITION 9.26. *There is a group isomorphism*

$$(9.44) \qquad \begin{aligned} \mathcal{T}_0(X_N)/\mathcal{J}_0^\sigma &\to \mathcal{T}_0(X_N^{(\kappa)}), \\ \hat{T}^{(a)}(u) &\mapsto T^{(a)}(u) \quad (a \in I_\sigma). \end{aligned}$$

Proof. It is easy to check that the relations of the both rings are identical under the correspondence. ∎

THEOREM 9.27. *The following relations hold in $\mathcal{T}_0(X_N^{(\kappa)})$:*
(1) *Half-periodicity:*

$$T^{(a)}(u+h^\vee) = \begin{cases} T^{(a)}(u)^{-1} & \text{if } X_N^{(\kappa)} = D_{r+1}^{(2)} \ (r+1\text{: even}) \text{ or } D_4^{(3)}, \\ T^{(a)}(u+\Omega)^{-1} & \text{otherwise.} \end{cases}$$

(2) *Periodicity:* $T^{(a)}(u+2h^\vee) = T^{(a)}(u)$.

Proof. It suffices to prove the half-periodicity (1). First consider the case $X_N^{(\kappa)} \neq D_{r+1}^{(2)}$ ($r+1$: even) and $D_4^{(3)}$. Then we have $\sigma = \omega$ and $\kappa = 2$. Thanks to Proposition 9.26, it is equivalent to showing $\hat{T}^{(a)}(u+h^\vee) = \hat{T}^{(a)}(u+\Omega)^{-1}$ for $a \in I_\sigma$ in $\mathcal{T}_0(X_N)/\mathcal{J}_0^\sigma$. This equality is verified by Theorem 8.8 and (9.43).

Next consider the remaining case $X_N^{(\kappa)} = D_{r+1}^{(2)}$ ($r+1$: even) or $D_4^{(3)}$. Then we have $\omega = \text{id}$. By the same reason as before, we show that $\hat{T}^{(a)}(u+h^\vee) = \hat{T}^{(a)}(u)$ for $a \in I_\sigma$ in $\mathcal{T}_0(X_N)/\mathcal{J}_0^\sigma$. Again, this is guaranteed by Theorem 8.8. ∎

§10. Remark on the periodicity of q-characters

We conclude the paper with a remark on a formal correspondence between the periodicity of the T-system and the q-characters of $U_q(\hat{\mathfrak{g}})$ at roots of unity.

Recall that $\mathcal{T}^\circ(X_r) \simeq \text{Ch}\, U_q(\hat{\mathfrak{g}})$ by Corollary 2.9. Let

$$(10.1) \qquad \pi_\ell : \mathcal{T}^\circ(X_r) \to \mathcal{T}_\ell^\circ(X_r)$$

be the surjective ring homomorphism in (3.1). One can easily see that $T^{(a)}_{t_a\ell}(u) - 1$ and $T^{(a)}_{t_a\ell+1}(u)$ are in $\operatorname{Ker}\pi_\ell$. Correspondingly, we define the *level ℓ restricted q-character ring* $\operatorname{Ch}_\ell U_q(\hat{\mathfrak{g}})$ of $U_q(\hat{\mathfrak{g}})$ by

$$\text{(10.2)} \qquad \operatorname{Ch}_\ell U_q(\hat{\mathfrak{g}}) = \operatorname{Ch} U_q(\hat{\mathfrak{g}})/I_\ell,$$

where I_ℓ is the ideal of $\operatorname{Ch} U_q(\hat{\mathfrak{g}})$ corresponding to $\operatorname{Ker}\pi_\ell$ under the isomorphism $\mathcal{T}^\circ(X_r) \simeq \operatorname{Ch} U_q(\hat{\mathfrak{g}})$. By construction, we have an isomorphism,

$$\text{(10.3)} \qquad \begin{aligned} &\operatorname{Ch}_\ell U_q(\hat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{T}^\circ_\ell(X_r), \\ &\chi_q\big(W^{(a)}_m(u)\big) \mapsto T^{(a)}_m(u) \quad (1 \le m \le t_a\ell - 1), \end{aligned}$$

and the periodicity of $\mathcal{T}_\ell(X_r)$ in Conjecture 3.11 is rephrased as the following periodicity of the q-characters. For $1 \le m \le t_a\ell - 1$,

$$\text{(10.4)} \qquad \chi_q\big(W^{(a)}_m(u + 2(h^\vee + \ell))\big) \equiv \chi_q\big(W^{(a)}_m(u)\big) \mod I_\ell.$$

In view of (2.12), this implies (but does not directly require) $q^{2t(h^\vee+\ell)} = 1$, which is natural, as mentioned in the end of Section 3.3. Let us make this implication, still formal, but more manifest in the form of a conjecture. Recall that $\operatorname{Ch} U_q(\hat{\mathfrak{g}})$ is a subring of the ring $\mathbb{Z}[Y^{\pm 1}_{a,q^{tu}}]_{a\in I, u\in\mathbb{C}_{t\hbar}}$. Let I'_ℓ be the ideal of $\mathbb{Z}[Y^{\pm 1}_{a,q^{tu}}]_{a\in I, u\in\mathbb{C}_{t\hbar}}$ generated by I_ℓ.

CONJECTURE 10.1. *The following equality holds in* $\mathbb{Z}[Y^{\pm 1}_{a,q^{tu}}]_{a\in I, u\in\mathbb{C}_{t\hbar}}$:

$$\text{(10.5)} \qquad Y_{a,q^{tu+2t(h^\vee+\ell)}} \equiv Y_{a,q^{tu}} \mod I'_\ell.$$

EXAMPLE 10.2. (1) $X_r = A_1$. We set $W_m(u) = W^{(1)}_m(u)$ and $Y_{q^u} = Y_{1,q^u}$, for simplicity. Recall that [FR, Section 4.1]

$$\text{(10.6)} \qquad \chi_q\big(W_m(u)\big) = \Big(\prod_{j=1}^m Y_{q^{u+m+1-2j}}\Big) \sum_{i=0}^m \prod_{j=1}^i A^{-1}_{q^{u+m+2-2j}},$$

$$A_{q^u} := Y_{q^{u-1}} Y_{q^{u+1}}.$$

Thus, we have

$$\text{(10.7)} \qquad \begin{aligned} \chi_q\big(W_{\ell+1}(u)\big) = &Y_{q^{u-\ell}} \chi_q\big(W_\ell(u+1)\big) \\ &+ Y^{-1}_{q^{u-\ell+2}} Y^{-1}_{q^{u-\ell+4}} \cdots Y^{-1}_{q^{u+\ell+2}}. \end{aligned}$$

Meanwhile, $\chi_q(W_\ell(u)) \equiv 1$ and $\chi_q(W_{\ell+1}(u)) \equiv 0$. Therefore,

$$(10.8) \qquad Y_{q^{u-\ell}} Y_{q^{u-\ell+2}} \cdots Y_{q^{u+\ell+2}} \equiv -1,$$

from which $Y_{q^{u+2(2+\ell)}} \equiv Y_{q^u}$ follows.

(2) $X_r = A_r$ $(r \geq 2)$. We remark that, in addition to $T_\ell^{(a)}(u) - 1$ and $T_{\ell+1}^{(a)}(u), T_{\ell+2}^{(1)}(u), \ldots, T_{\ell+r}^{(1)}(u)$ are also in $\operatorname{Ker} \pi_\ell$. Then, generalizing the argument of (1), one can show that

$$(10.9) \qquad Y_{a,q^u} Y_{a,q^{u+2}} \cdots Y_{a,q^{u+2(r+\ell)}} \equiv (-1)^{ar} \mod I'_\ell,$$

from which $Y_{a,q^{u+2(r+1+\ell)}} \equiv Y_{a,q^u}$ follows. More detail is given in Appendix B.

It is important to establish a precise relation between the ring $\operatorname{Ch}_\ell U_q(\hat{\mathfrak{g}})$ and the ring $\operatorname{Rep} U_\epsilon^{\operatorname{res}}(\hat{\mathfrak{g}})$ of [FM2] for a primitive $2t(h^\vee + \ell)$th root of unity ϵ. A similar remark is applicable to the twisted quantum affine algebras as well.

Appendix A. Q-systems and $\operatorname{Rep} U_q(\mathfrak{g})$

A.1. Q-systems for untwisted case

Here we present results parallel to Theorem 2.8 and Corollary 2.9 for the Q-system and $\operatorname{Rep} U_q(\mathfrak{g})$. For a Dynkin diagram X_r, let I be as in Section 2.

The following system was introduced by [Ki], [KR].

DEFINITION A.1. The *unrestricted Q-system* $\mathbb{Q}(X_r)$ *of type* X_r is the following system of relations for a family of variables $Q = \{Q_m^{(a)} \mid a \in I, m \in \mathbb{N}\}$, where $Q_m^{(0)}(u) = Q_0^{(a)}(u) = 1$ if they occur in the right-hand sides in the following relations.

For simply laced X_r,

$$(A.1) \qquad (Q_m^{(a)})^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + \prod_{b \in I : C_{ab} = -1} Q_m^{(b)}.$$

For $X_r = B_r$,

$$(A.2) \qquad (Q_m^{(a)})^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + Q_m^{(a-1)} Q_m^{(a+1)} \quad (1 \leq a \leq r-2),$$

$$(Q_m^{(r-1)})^2 = Q_{m-1}^{(r-1)} Q_{m+1}^{(r-1)} + Q_m^{(r-2)} Q_{2m}^{(r)},$$

$$(Q_{2m}^{(r)})^2 = Q_{2m-1}^{(r)}(u) Q_{2m+1}^{(r)}(u) + (Q_m^{(r-1)})^2,$$

$$(Q_{2m+1}^{(r)})^2 = Q_{2m}^{(r)} Q_{2m+2}^{(r)} + Q_m^{(r-1)} Q_{m+1}^{(r-1)}.$$

For $X_r = C_r$,

$$(\text{A.3}) \qquad (Q_m^{(a)})^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + Q_m^{(a-1)} Q_m^{(a+1)} \quad (1 \le a \le r-2),$$

$$(Q_{2m}^{(r-1)})^2 = Q_{2m-1}^{(r-1)} Q_{2m+1}^{(r-1)} + Q_{2m}^{(r-2)} (Q_m^{(r)})^2,$$

$$(Q_{2m+1}^{(r-1)})^2 = Q_{2m}^{(r-1)} Q_{2m+2}^{(r-1)} + Q_{2m+1}^{(r-2)} Q_m^{(r)} Q_{m+1}^{(r)},$$

$$(Q_m^{(r)})^2 = Q_{m-1}^{(r)} Q_{m+1}^{(r)} + Q_{2m}^{(r-1)}.$$

For $X_r = F_4$,

$$(\text{A.4}) \qquad (Q_m^{(1)})^2 = Q_{m-1}^{(1)} Q_{m+1}^{(1)} + Q_m^{(2)},$$

$$(Q_m^{(2)})^2 = Q_{m-1}^{(2)} Q_{m+1}^{(2)} + Q_m^{(1)} Q_{2m}^{(3)},$$

$$(Q_{2m}^{(3)})^2 = Q_{2m-1}^{(3)} Q_{2m+1}^{(3)} + (Q_m^{(2)})^2 Q_{2m}^{(4)},$$

$$(Q_{2m+1}^{(3)})^2 = Q_{2m}^{(3)} Q_{2m+2}^{(3)} + Q_m^{(2)} Q_{m+1}^{(2)} Q_{2m+1}^{(4)},$$

$$(Q_m^{(4)})^2 = Q_{m-1}^{(4)} Q_{m+1}^{(4)} + Q_m^{(3)}.$$

For $X_r = G_2$,

$$(\text{A.5}) \qquad (Q_m^{(1)})^2 = Q_{m-1}^{(1)} Q_{m+1}^{(1)} + Q_{3m}^{(2)},$$

$$(Q_{3m}^{(2)})^2 = Q_{3m-1}^{(2)} Q_{3m+1}^{(2)} + (Q_m^{(1)})^3,$$

$$(Q_{3m+1}^{(2)})^2 = Q_{3m}^{(2)} Q_{3m+2}^{(2)} + (Q_m^{(1)})^2 Q_{m+1}^{(1)},$$

$$(Q_{3m+2}^{(2)})^2 = Q_{3m+1}^{(2)} Q_{3m+3}^{(2)} + Q_m^{(1)}(u)(Q_{m+1}^{(1)})^2.$$

DEFINITION A.2. The *unrestricted Q-algebra* $\mathfrak{Q}(X_r)$ *of type* X_r is the ring with generators $Q_m^{(a)\pm 1}$ ($a \in I, m \in \mathbb{N}$) and the relations $\mathbb{Q}(X_r)$. Also, we define the ring $\mathfrak{Q}^\circ(X_r)$ as the subring of $\mathfrak{Q}(X_r)$ generated by $Q_m^{(a)}$ ($a \in I, m \in \mathbb{N}$).

The system $\mathbb{Q}(X_r)$ is obtained by $\mathbb{T}(X_r)$ by the reduction of the spectral parameter u. One can also define the level ℓ *restricted Q-system* $\mathbb{Q}_\ell(X_r)$ by the reduction of $\mathbb{T}_\ell(X_r)$ [KNS1]. The system $\mathbb{Q}_\ell(X_r)$ plays the central role in the dilogarithm identities for the central charges of conformal field theories (e.g., [Ki], [Ku], [KN], [KNS1], [KNS2], [RTV], [FS], [GT], etc.).

Let $\mathfrak{g}$ be the complex simple Lie algebra of type X_r, and let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. Then, $U_q(\mathfrak{g})$ is a subalgebra of

the untwisted quantum affine algebra $U_q(\hat{\mathfrak{g}})$. Let χ be the character map of $U_q(\mathfrak{g})$; it is an injective ring homomorphism $\chi \colon \operatorname{Rep} U_q(\mathfrak{g}) \to \mathbb{Z}[y_a^{\pm 1}]_{a \in I}$, where we follow the notation of [FR]. The *character ring* $\operatorname{Ch} U_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ is defined to be $\operatorname{Im} \chi$. Thus, $\operatorname{Ch} U_q(\mathfrak{g})$ is an integral domain and isomorphic to $\operatorname{Rep} U_q(\mathfrak{g})$. Let $\operatorname{Res} \colon \operatorname{Rep} U_q(\hat{\mathfrak{g}}) \to \operatorname{Rep} U_q(\mathfrak{g})$ be the restriction homomorphism induced from the inclusion $U_q(\mathfrak{g}) \to U_q(\hat{\mathfrak{g}})$.

A *Kirillov-Reshetikhin module* $W_m^{(a)}$ ($a \in I$, $m \in \mathbb{N}$) of $U_q(\mathfrak{g})$ is the (not necessarily irreducible) $U_q(\mathfrak{g})$-module defined by $W_m^{(a)} := \operatorname{Res}(W_m^{(a)}(u))$, where $W_m^{(a)}(u)$ is a Kirillov-Reshetikhin module of $U_q(\hat{\mathfrak{g}})$. We remark that $W_m^{(a)}$ is independent of u.

In the same way as (2.9), we define $S_{am}(Q) \in \mathbb{Z}[Q]$ ($a \in I, m \in \mathbb{N}$), so that all the relations in $\mathbb{Q}(X_r)$ are written in the form $S_{am}(Q) = 0$. Let $I(\mathbb{Q}(X_r))$ be the ideal of $\mathbb{Z}[Q]$ generated by $S_{am}(Q)$.

From Theorem 2.8, we obtain the following.

THEOREM A.3. *Let* $\widetilde{Q} = \{\widetilde{Q}_m^{(a)} := \chi(W_m^{(a)}) \mid a \in I, m \in \mathbb{N}\}$ *be the family of the characters of the Kirillov-Reshetikhin modules of* $U_q(\mathfrak{g})$. *Then, we have the following.*

(1) *The family* $\widetilde{Q}$ *generates the ring* $\operatorname{Ch} U_q(\mathfrak{g})$.

(2) *([N3], [Her1]) The family* $\widetilde{Q}$ *satisfies the Q-system* $\mathbb{Q}(X_r)$ *in* $\operatorname{Ch} U_q(\mathfrak{g})$ *(by replacing* $Q_m^{(a)}$ *in* $\mathbb{Q}(X_r)$ *with* $\widetilde{Q}_m^{(a)}$*).*

(3) *For any* $P(Q) \in \mathbb{Z}[Q]$, *the relation* $P(\widetilde{Q}) = 0$ *holds in* $\operatorname{Ch} U_q(\mathfrak{g})$ *if and only if there is a nonzero monomial* $M(Q) \in \mathbb{Z}[Q]$ *such that* $M(Q)P(Q) \in I(\mathbb{Q}(X_r))$.

COROLLARY A.4. *The ring* $\mathbb{Q}^\circ(X_r)$ *is isomorphic to* $\operatorname{Rep} U_q(\mathfrak{g})$ *by the correspondence* $Q_m^{(a)} \mapsto W_m^{(a)}$.

By taking $q \to 1$, one can also obtain analogous results for $\operatorname{Rep} \mathfrak{g}$.

A.2. Q-systems for twisted case

Here we present results parallel to Theorem 9.8 and Corollary 9.9 for the Q-system and $\operatorname{Rep} U_q(\mathfrak{g}^\sigma)$. For a pair $(X_N, \kappa) = (A_N, 2)$, $(D_N, 2)$, $(E_6, 2)$, or $(D_4, 3)$, let I_σ be as in Section 9.

The following system was introduced by [HKOTT].

DEFINITION A.5. The *unrestricted Q-system* $\mathbb{Q}(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the following system of relations for a family of variables $Q = \{Q_m^{(a)} \mid a \in I_\sigma, m \in$

$\mathbb{N}\}$, where $Q_m^{(0)}(u) = Q_0^{(a)}(u) = 1$ if they occur in the right-hand sides in the relations.

For $X_N^{(\kappa)} = A_{2r-1}^{(2)}$,

$$(A.6) \qquad (Q_m^{(a)})^2 = Q_{m-1}^{(a)}Q_{m+1}^{(a)} + Q_m^{(a-1)}Q_m^{(a+1)} \quad (1 \le a \le r-1),$$

$$(Q_m^{(r)})^2 = Q_{m-1}^{(r)}Q_{m+1}^{(r)} + (Q_m^{(r-1)})^2.$$

For $X_N^{(\kappa)} = A_{2r}^{(2)}$,

$$(A.7) \qquad (Q_m^{(a)})^2 = Q_{m-1}^{(a)}Q_{m+1}^{(a)} + Q_m^{(a-1)}Q_m^{(a+1)} \quad (1 \le a \le r-1),$$

$$(Q_m^{(r)})^2 = Q_{m-1}^{(r)}Q_{m+1}^{(r)} + Q_m^{(r-1)}Q_m^{(r)}.$$

For $X_N^{(\kappa)} = D_{r+1}^{(2)}$,

$$(A.8) \qquad (Q_m^{(a)})^2 = Q_{m-1}^{(a)}Q_{m+1}^{(a)} + Q_m^{(a-1)}Q_m^{(a+1)} \quad (1 \le a \le r-2),$$

$$(Q_m^{(r-1)})^2 = Q_{m-1}^{(r-1)}Q_{m+1}^{(r-1)} + Q_m^{(r-2)}(Q_m^{(r)})^2,$$

$$(Q_m^{(r)})^2 = Q_{m-1}^{(r)}Q_{m+1}^{(r)} + Q_m^{(r-1)}.$$

For $X_N^{(\kappa)} = E_6^{(2)}$,

$$(A.9) \qquad (Q_m^{(1)})^2 = Q_{m-1}^{(1)}Q_{m+1}^{(1)} + Q_m^{(2)},$$

$$(Q_m^{(2)})^2 = Q_{m-1}^{(2)}Q_{m+1}^{(2)} + Q_m^{(1)}Q_m^{(3)},$$

$$(Q_m^{(3)})^2 = Q_{m-1}^{(3)}Q_{m+1}^{(3)} + (Q_m^{(2)})^2 Q_m^{(4)},$$

$$(Q_m^{(4)})^2 = Q_{m-1}^{(4)}Q_{m+1}^{(4)} + Q_m^{(3)}.$$

For $X_N^{(\kappa)} = D_4^{(3)}$,

$$(A.10) \qquad (Q_m^{(1)})^2 = Q_{m-1}^{(1)}Q_{m+1}^{(1)} + Q_m^{(2)},$$

$$(Q_m^{(2)})^2 = Q_{m-1}^{(2)}Q_{m+1}^{(2)} + (Q_m^{(1)})^3.$$

DEFINITION A.6. The *unrestricted Q-algebra* $\mathcal{Q}(X_N^{(\kappa)})$ *of type* $X_N^{(\kappa)}$ is the ring with generators $Q_m^{(a)\pm 1}$ ($a \in I_\sigma, m \in \mathbb{N}$) and the relations $\mathbb{Q}(X_N^{(\kappa)})$. Also, we define the ring $\mathcal{Q}^\circ(X_N^{(\kappa)})$ as the subring of $\mathcal{Q}(X_N^{(\kappa)})$ generated by $Q_m^{(a)}$ ($a \in I_\sigma, m \in \mathbb{N}$).

The system $\mathbb{Q}(X_N^{(\kappa)})$ is obtained by $\mathbb{T}(X_N^{(\kappa)})$ by the reduction of the spectral parameter u. One can also define the level ℓ *restricted Q-system* $\mathbb{Q}_\ell(X_N^{(\kappa)})$ by the reduction of $\mathbb{T}_\ell(X_N^{(\kappa)})$.

Let X_N^σ be the subdiagram of $X_N^{(\kappa)}$ obtained by removing the zeroth node. Explicitly,

$$
(A.11) \qquad
\begin{array}{c|cccccc}
X_N^{(\kappa)} & A_{2r-1}^{(2)} & A_2^{(2)} & A_{2r}^{(2)} & D_{r+1}^{(2)} & E_6^{(2)} & D_4^{(3)} \\
\hline
X_N^\sigma & C_r & A_1 & B_r & B_r & F_4 & G_2
\end{array}
$$

Let $\mathfrak{g}^\sigma$ be the complex simple Lie algebra of type X_N^σ, and let $U_q(\mathfrak{g}^\sigma)$ be the quantized universal enveloping algebra of $\mathfrak{g}^\sigma$. Then, $U_q(\mathfrak{g}^\sigma)$ is a subalgebra of the twisted quantum affine algebra $U_q(\hat{\mathfrak{g}}^\sigma)$ of type $X_N^{(\kappa)}$ [Her2]. Let $\mathrm{Res}^\sigma : \mathrm{Rep}\, U_q(\hat{\mathfrak{g}}^\sigma) \to \mathrm{Rep}\, U_q(\mathfrak{g}^\sigma)$ be the restriction homomorphism induced from the inclusion $U_q(\mathfrak{g}^\sigma) \to U_q(\hat{\mathfrak{g}}^\sigma)$.

We define the (not necessarily irreducible) $U_q(\mathfrak{g}^\sigma)$-module $\dot{W}_m^{(a)}$ ($a \in I_\sigma$, $m \in \mathbb{N}$) by $\dot{W}_m^{(a)} := \mathrm{Res}^\sigma(W_m^{(a)}(u))$, where $W_m^{(a)}(u)$ is a Kirillov-Reshetikhin module of $U_q(\hat{\mathfrak{g}}^\sigma)$. We remark that $\dot{W}_m^{(a)}$ is independent of u.

In the same way as (2.9), we define $S_{am}(Q) \in \mathbb{Z}[Q]$ ($a \in I, m \in \mathbb{N}$), so that all the relations in $\mathbb{Q}(X_N^{(\kappa)})$ are written in the form $S_{am}(Q) = 0$. Let $I(\mathbb{Q}(X_N^{(\kappa)}))$ be the ideal of $\mathbb{Z}[Q]$ generated by $S_{am}(Q)$'s.

As Theorem 9.8, we obtain the following.

THEOREM A.7. *Let* $\widetilde{Q} = \{\widetilde{Q}_m^{(a)} := \chi(\dot{W}_m^{(a)}) \mid a \in I_\sigma, m \in \mathbb{N}\}$ *be the family of the characters of* $\dot{W}_m^{(a)}(u)$*'s. Then, we have the following:*

(1) *The family* $\widetilde{Q}$ *generates the ring* $\mathrm{Ch}\, U_q(\mathfrak{g}^\sigma)$.

(2) *([Her2]) The family* $\widetilde{Q}$ *satisfies the Q-system* $\mathbb{Q}(X_N^{(\kappa)})$ *in* $\mathrm{Ch}\, U_q(\mathfrak{g}^\sigma)$ *(by replacing* $Q_m^{(a)}$ *in* $\mathbb{Q}(X_N^{(\kappa)})$ *with* $\widetilde{Q}_m^{(a)}$*).*

(3) *For any* $P(Q) \in \mathbb{Z}[Q]$*, the relation* $P(\widetilde{Q}) = 0$ *holds in* $\mathrm{Ch}\, U_q(\mathfrak{g}^\sigma)$ *if and only if there is a nonzero monomial* $M(Q) \in \mathbb{Z}[Q]$ *such that* $M(Q)P(Q) \in I(\mathbb{Q}(X_N^{(\kappa)}))$.

COROLLARY A.8. *The ring* $\mathcal{Q}^\circ(X_N^{(\kappa)})$ *is isomorphic to* $\mathrm{Rep}\, U_q(\mathfrak{g}^\sigma)$ *by the correspondence* $Q_m^{(a)} \mapsto \dot{W}_m^{(a)}$.

By taking $q \to 1$, one can also obtain analogous results for $\mathrm{Rep}\,\mathfrak{g}^\sigma$.

Appendix B. Proof of (10.9) in Example 10.2(2)

First we show that $T_{\ell+2}^{(1)}(u),\ldots,T_{\ell+r}^{(1)}(u)$ are in $\operatorname{Ker}\pi_\ell$. Let $S_m^{(a)}(u) := \pi_\ell(T_m^{(a)}(u))$. Then, we have

$$\text{(B.1)} \qquad \sum_{a=0}^{r+1}(-1)^a T_1^{(a)}(u+a)T_{m-a}^{(1)}(u+m+a) = \delta_{m,0},$$

$$\text{(B.2)} \qquad \sum_{j=0}^{m}(-1)^j S_{\ell-1}^{(m-j)}(u+j)S_{\ell-j}^{(1)}(u+j-m) = \delta_{m,0},$$

$$\text{(B.3)} \qquad \sum_{a=0}^{k-1}(-1)^a S_1^{(a)}(u+a)S_{\ell+k-a}^{(1)}(u+\ell+k+a) = 0 \quad (1 \le k \le r),$$

where $T_m^{(0)}(u) = T_m^{(r+1)}(u) = S_m^{(0)}(u) = S_m^{(r+1)}(u) = 1$ $(m \ge 0)$ and $T_m^{(a)}(u) = S_m^{(a)}(u) = 0$ $(m < 0)$. We obtain (B.1) from the Jacobi-Trudi–type determinant formula in [KNS1, (2.21)], (B.2) from a similar determinant formula in $\mathcal{T}_\ell^\circ(A_r)$, and (B.3) from (B.1), (B.2), and the half-periodicity $S_1^{(a)}(u+\ell+r+1) = S_{\ell-1}^{(r+1-a)}(u)$ in $\mathcal{T}_\ell^\circ(A_r)$. It follows from (B.3) that $S_{\ell+2}^{(1)}(u) = \cdots = S_{\ell+r}^{(1)}(u) = 0$.

Next we prove the following statement in Example 10.2(2).

PROPOSITION B.1. *The following relations* $\mathrm{mod}\ I_\ell'$ *hold in* $\mathbb{Z}[Y_{a,q^u}^{\pm 1}]_{a\in I, u\in\mathbb{C}_h}$:
(1) $Y_{a,q^u}Y_{a,q^{u+2}}\cdots Y_{a,q^{u+2\ell+2r}} \equiv (-1)^{ar}$,
(2) $Y_{a,q^u} \equiv Y_{a,q^{u+2(r+1+\ell)}}$.

Let

$$\text{(B.4)} \qquad \boxed{a}_u = \frac{Y_{a,q^{u+a-1}}}{Y_{a-1,q^{u+a}}} \quad (1 \le a \le r+1),$$

where $Y_{0,q^u} = Y_{r+1,q^u} = 1$. We introduce the notation

$$\text{(B.5)} \qquad {}_u^a\overbrace{\boxed{}}^{m}{}_v^b = \sum \boxed{a_1}_u\boxed{a_2}_{u+2}\cdots\boxed{a_m}_v,$$

where $v = u + 2m - 2$, and the sum extends over all the integers $a_1,\ldots,a_m$ such that $a \le a_1 \le \cdots \le a_m \le b$. The array of boxes in the right-hand side is to be understood as the product of the monomials (B.4). By the definition,

we have the following identities:

$$
\text{(B.6)} \qquad \underset{u}{a}\overbrace{}^{m}{}_{v}^{b} = \boxed{a}_u \; \underset{u+2}{a}\overbrace{}^{m-1}{}_{v}^{b} + \underset{u}{a+1}\overbrace{}^{m}{}_{v}^{b},
$$

$$
\text{(B.7)} \qquad \underset{u}{a}\overbrace{}^{m}{}_{v}^{b} = \underset{u}{a}\overbrace{}^{m-1}{}_{v-2}^{b}\boxed{b}_v + \underset{u}{a}\overbrace{}^{m}{}_{v}^{b-1}.
$$

It is well known that the q-character of $W_m^{(1)}(u)$ is given by

$$
\text{(B.8)} \qquad \chi_q\big(W_m^{(1)}(u)\big) = \underset{u-m+1}{1}\overbrace{}^{m}{}_{u+m-1}^{r+1}.
$$

LEMMA B.2. *Let $1 \le s \le r$ and $1 \le a \le b \le r+1$. Then, the following relation $\mathrm{mod}\, I'_\ell$ holds in $\mathbb{Z}[Y_{a,q^u}^{\pm 1}]_{a\in I, u\in\mathbb{C}_\hbar}$:*

$$
\text{(B.9)} \quad \underset{u}{a}\overbrace{}^{\ell+s}{}_{v}^{b}
$$

$$
\equiv \begin{cases} (-1)^s \,{}_u\boxed{a-1,\ldots,2,1}\,\boxed{r+1,r,\ldots,b+1}_v & if\, s = r - b + a, \\ 0 & if\, s > r - b + a. \end{cases}
$$

Here $v = u + 2(\ell + s - 1)$ and the right-hand side of the first case stands for

$$
(-1)^s \prod_{\alpha=1}^{a-1} \boxed{\alpha}_{u+2a-2-2\alpha} \prod_{\beta=b+1}^{r+1} \boxed{\beta}_{v+2b+2-2\beta}.
$$

Proof. We employ the induction on s. Suppose $s = 1$. Then $s > r - b + a$ happens only if $(a,b) = (1, r+1)$; therefore, (B.9) is just $\chi_q(W_{\ell+1}^{(1)}(u')) \equiv 0$ for some u'. On the other hand, $1 = s = r - b + a$ is satisfied for $(a,b) = (2, r+1)$ and $(1, r)$. Thus we show that

$$
\underset{u}{2}\overbrace{}^{\ell+1}{}_{v}^{r+1} \equiv -\boxed{1}_u, \qquad \underset{u}{1}\overbrace{}^{\ell+1}{}_{v}^{r} \equiv -\boxed{r+1}_v.
$$

These relations follow from (B.6) and (B.7) by setting $(a,b) = (1, r+1), m = \ell+1$ and using $\chi_q(W_\ell^{(1)}(u)) \equiv 1$ and $\chi_q(W_{\ell+1}^{(1)}(u)) \equiv 0$.

Now suppose that (B.9) is valid up to $s - 1$. First we consider the case $b - a > r - s$ in (B.9). Setting $u' = u + \ell + s - 1$ and using (B.6) repeatedly,

we have

$$\text{(B.10)} \qquad 0 \equiv \chi_q\big(W^{(1)}_{\ell+s}(u')\big) = \sum_{\alpha=1}^{a-1} \boxed{\alpha}_u \, {}^{\alpha}_{u+2}\overbrace{\boxed{}}^{\ell+s-1}{}^{r+1}_v + {}^{a}_u\overbrace{\boxed{}}^{\ell+s}{}^{r+1}_v .$$

Each term in the α sum is zero $\mathrm{mod}\, I'_\ell$ due to the induction assumption. Applying (B.7) similarly to the resulting relation, we find

$$\text{(B.11)} \qquad 0 \equiv {}^{a}_u\overbrace{\boxed{}}^{\ell+s}{}^{r+1}_v = \sum_{\beta=b+1}^{r+1} {}^{a}_u\overbrace{\boxed{}}^{\ell+s-1}{}^{\beta}_{v-2}\boxed{\beta}_v + {}^{a}_u\overbrace{\boxed{}}^{\ell+s}{}^{b}_v .$$

Again, each summand in the β sum vanishes $\mathrm{mod}\, I'_\ell$ by the induction assumption, proving the latter case of (B.9).

Next we treat the former case of (B.9); namely, assume that $b - a = r - s$. If $b \leq r$, the same argument (B.10)–(B.11) as above goes through except that the $\beta = b+1$ term in (B.11) is nonvanishing, leading to

$$0 \equiv {}^{a}_u\boxed{}^{\ell+s-1}{}^{b+1}_{v-2}\boxed{b+1}_v + {}^{a}_u\overbrace{\boxed{}}^{\ell+s}{}^{b}_v .$$

Applying the induction assumption to the first term, we obtain the sought expression for the second term. If $b = r+1$, then $a = b - r + s = s + 1$ and the α sum in (B.10) contains nonzero summand at $\alpha = a - 1$, leading to

$$0 \equiv \boxed{a-1}_u \, {}^{a-1}_{u+2}\overbrace{\boxed{}}^{\ell+s-1}{}^{b}_v + {}^{a}_u\overbrace{\boxed{}}^{\ell+s}{}^{b}_v .$$

Again, rewriting the first term by using the induction assumption yields the sought expression for the second term. $\qquad\square$

Proof of Proposition B.1. Item (2) is a corollary of (1). To show (1), set $a = b$ and $s = r$ in (B.9). Substituting (B.4) into the resulting relation, we find

$$\frac{Y_{a,q^{u+1}} Y_{a,q^{u+3}} \cdots Y_{a,q^{u+2\ell+2r+1}}}{Y_{a-1,q^u} Y_{a-1,q^{u+2}} \cdots Y_{a-1,q^{u+2\ell+2r}}} \equiv (-1)^r \quad (1 \leq a \leq r+1)$$

for any $u \in \mathbb{Z}$. From $Y_{0,q^u} = 1$, the assertion follows. $\qquad\square$

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