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LEFSCHETZ OPERATOR AND LOCAL LANGLANDS MOD ℓ: THE REGULAR CASE

JEAN-FRANÇOIS DAT

To the memory of Professor Hiroshi Saito

Abstract. Let p and ℓ be two distinct primes. The aim of this paper is to show how, under a certain congruence hypothesis, the mod ℓ cohomology complex of the Lubin-Tate tower, together with a natural Lefschetz operator, provides a geometric interpretation of Vignéras's local Langlands correspondence modulo ℓ for unipotent representations.

CONTENTS

§1. Main theorem

Let K be a local nonarchimedean field with ring of integers $\mathcal{O}$ and residue field $k \simeq \mathbb{F}_q$, with q a power of a prime p. Let ℓ be another prime number, and let d be an integer. As in [10], we consider the cohomology complex

$$R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell) \in D^b\big(\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty,c}(G \times D^\times \times W_K)\big)$$

of the height d Lubin-Tate tower of K. Here $G = \mathrm{GL}_d(K)$, D is the division algebra which is central over K with invariant $1/d$, and W_K is the Weil group of K. The category $\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty,c}$ consists of $\mathbb{Z}_\ell$-representations of the triple product which are smooth for G and $D^\times$ and continuous for W_K. In [9], we

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defined a Lefschetz operator

$$L : R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell) \longrightarrow R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)[2](1)$$

as the cup-product by the Chern class of the tautological invertible sheaf on the associated Gross-Hopkins period domain.

To an irreducible $\overline{\mathbb{F}}_\ell$-representation π of G, we associate its derived π-coisotypical part

$$R_\pi := R\operatorname{Hom}_{\mathbb{Z}_\ell G}\big(R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell), \pi\big) \in D^b\big(\mathrm{Rep}^\infty_{\overline{\mathbb{F}}_\ell}(D^\times \times W_K)\big),$$

which inherits a morphism $L_\pi : R_\pi \longrightarrow R_\pi[2](1)$. We also denote by R^*_π the total cohomology of R_π, a smooth graded $\overline{\mathbb{F}}_\ell$-representation of $W_K \times D^\times$, and by $L^*_\pi : R^*_\pi \longrightarrow R^*_\pi[2](1)$ the corresponding morphism. Our aim here is to prove the following theorem, where we forget the grading.

THEOREM. *Assume that the multiplicative order of q mod ℓ is d. Then for any unipotent irreducible representation π of $\mathrm{GL}_d(K)$, there is an isomorphism*

$$(R^*_\pi, L^*_\pi)^{\mathrm{ss}} \simeq |\,\mathrm{LJ}(\pi)| \otimes \big(\sigma^{\mathrm{ss}}(\pi), L(\pi)\big).$$

The congruence condition on q modulo ℓ will be called the *Coxeter congruence relation*, by analogy with the modular Deligne-Lusztig theory where this condition arises in the context of Broué's conjecture (see, e.g., [15]). The term *unipotent* was introduced by Vignéras to denote representations that belong to the same block as the trivial representation. Finite group theorists would rather call them *principal block* representations.

Let us explain the notation of the theorem. The symbol $\mathrm{LJ}(\pi)$ stands for the Langlands-Jacquet transfer of [11]. In general, it is a virtual $\overline{\mathbb{F}}_\ell$-representation of $D^\times$, but under the congruence hypothesis it is known to be effective up to sign (see [11, (3.2.5)]), so we can put $\mathrm{LJ}(\pi) = \pm|\,\mathrm{LJ}(\pi)|$ for some semisimple $\overline{\mathbb{F}}_\ell$-representation of $D^\times$. The symbol $(\sigma^{\mathrm{ss}}(\pi), L(\pi))$ denotes the (transposed) Weil-Deligne $\overline{\mathbb{F}}_\ell$-representation associated to π by the Vignéras correspondence of [27, Théorème 1.8.2]. This is the Zelevinski-like normalization of the local Langlands correspondence mod ℓ. Therefore, to put it in simple English, the above theorem offers a geometric interpretation of the nilpotent part[†] of this Vignéras correspondence, at least for those unipotent representations such that $\mathrm{LJ}(\pi) \neq 0$.

[†]Note that, in contrast to the ℓ-adic setting, this nilpotent part has no obvious arithmetic interpretation, in the sense that it cannot be related to any infinitesimal action of the ℓ-inertia of W_K.

Let us say a few words about the proof of the theorem. Note first that, since $\mathrm{LJ}(\pi)$ is most often zero, we are soon reduced to the case when π is a subquotient of the smooth representation $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ induced from the trivial representation of some Borel subgroup. In Section 2 we classify these subquotients in a suitable way, thereby making explicit the corresponding block of the decomposition matrix of G, and we compute the associated Weil-Deligne and $D^\times$ representations. In Section 3, we study the unipotent summand of the cohomology complex. In particular, thanks to our congruence hypothesis, we may split it in a nontrivial way according to weights. Note that, in principle, all of this study can be carried out in a purely local way, using Yoshida's model of the tame Lubin-Tate space. However, for reference convenience, we invoke at some point Boyer's [3] description of the cohomology of the whole tower, the proof of which uses global arguments. An alternative argument uses the Faltings-Fargues [16] isomorphism. Then in Section 4 we prove the theorem by some fairly explicit computations.

One crucial ingredient is that we easily, and without any computation, get a complete description of (R^*_π, L^*_π) for π the trivial representation, thanks to the properties of the Gross-Hopkins period map. The theorem above is expected to hold true for *any* smooth irreducible $\overline{\mathbb{F}}_\ell$-representation π under the congruence hypothesis, but we are still missing some control on the pair (R^*_π, L^*_π) when π is a general Speh representation.

Remark. The theorem is also true under the wider assumption that q has order greater than or equal to d. This wider assumption is what we have called the *regular case* in the title, a terminology which comes from the fact that in this case, the modulus character (mod ℓ) of the Borel subgroup is indeed regular with respect to the action of the Weyl group. This regular case splits into the *Coxeter congruence* case explained above and the *banal* case. The latter is not treated in this paper because the overall strategy and all the necessary computations work exactly the same as in the ℓ-adic case. Note, however, that even in the banal case we are not able to go beyond unipotent representations regarding the computation of the Lefschetz operator.

§2. Elliptic principal series

By definition, an irreducible smooth $\overline{\mathbb{F}}_\ell$-representation is called *elliptic* if it is not a linear combination of proper parabolically induced representations. Note that by [11, Théorème 3.1.4], this is equivalent to $\mathrm{LJ}(\pi) \neq 0$.

According to [11, Corollaire 3.2.2] (where unfortunately the term *elliptic* has a slightly different meaning), any elliptic principal series is an unramified twist of a subquotient of the induced representation $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ for some Borel subgroup B. The converse is not true in general, but it is true under the Coxeter congruence relation, as we will see below.

2.1. Parameterization and decomposition matrix

2.1.1. A reminder on the ℓ-adic case. We denote by B the subgroup of upper triangular matrices in G and by S the set of simple roots of the diagonal torus T in $\mathrm{Lie}(B)$. The power set $\mathcal{P}(S)$ of S is in 1-to-1 correspondence with the set of parabolic subgroups containing B. Namely, to each subset $I \subset S$ is associated the unique parabolic subgroup P_I with $\mathrm{Lie}(P_I) = \mathrm{Lie}(B) + \sum_{\alpha\in\mathbb{Z}\langle I\rangle} \mathrm{Lie}(G)_\alpha$. In particular, we have $P_\emptyset = B$ and $P_S = G$.

Definition 2.1.1. For any ring R, we put $i_I(R) := \mathrm{Ind}_{P_I}^G(R)$, and we put

$$v_I(R) := i_I(R) \Big/ \sum_{J\supset I} i_J(R).$$

Let δ_B denote the R-valued modulus character of B. We assume that R contains a square root of q in R, and we choose such a root in order to define $\delta := \delta_B^{-1/2}$ as well as the normalized Jacquet functor r_B along B. Write $X := X_*(T) \otimes \mathbb{R}$, so that S is naturally a subset of the dual $\mathbb{R}$-vector space of X. Following [5, section 2.2.3], we associate to each subset $I \in \mathcal{P}(S)$ a connected component

$$X_I := \left\{ x \in X, \forall\alpha \in S, \varepsilon_I(\alpha)\langle x, \alpha\rangle > 0 \right\}$$

of the complement of the union of simple root hyperplanes in X. Here, ε_I is the sign function on S which takes α to -1 if and only if $\alpha \in I$. In particular, $X_\emptyset$ is the Weyl chamber associated to B, and X_S is that associated to the opposite Borel subgroup.

Fact 2.1.1 ([5], Lemme 2.3.3). If R is a field of characteristic prime to $\prod_{i=1}^d (q^i - 1)$, then the following hold.

(i) For each $I \subseteq S$, the R-representation $v_I(R)$ is irreducible, and we have

$$r_B\big(v_I(R)\big) = \bigoplus_{w(X_S)\subseteq X_I} w^{-1}(\delta).$$

(ii) The multiset $\mathrm{JH}(\mathrm{Ind}_B^G(R))$ of Jordan-Hölder factors of $\mathrm{Ind}_B^G(R)$ has multiplicity 1, hence is a set, and the map

$$I \in \mathcal{P}(S) \mapsto v_I(R) \in \mathrm{JH}\big(\mathrm{Ind}_B^G(R)\big)$$

is a bijection.

Let us label $t_0, t_1, \dots, t_{d-1}$ the diagonal entries of an element $t \in T$ (starting from the upper left corner). We get a labeling $S = \{\alpha_1, \dots, \alpha_{d-1}\}$, where $\alpha_i(t) := t_{i-1}t_i^{-1}$, and we get an identification of the Weyl group W of T with the symmetric group $\mathfrak{S}_d$ of the set $\{0, \dots, d-1\}$. Then we see that the condition $w(X_S) \subseteq X_I$ appearing in the summation of point (i) above is equivalent to the condition

$$I = \big\{\alpha_i \in S, w(i-1) < w(i)\big\}.$$

2.1.2. Classification under the Coxeter congruence relation. Here the coefficient field is $\mathbb{F}_\ell$ or $\overline{\mathbb{F}}_\ell$, and we assume that the multiplicative order of q modulo ℓ is exactly d. Denote by ν_G the unramified character $g \mapsto q^{-\mathrm{val}(\det)(g)}$. Observe that ν_G is trivial on the center of G and generates a cyclic subgroup $\langle \nu_G \rangle$ of order d of the group of $\overline{\mathbb{F}}_\ell$-valued characters of G.

We put $\widetilde{S} := S \cup \{\alpha_0\}$, where α_0 denotes the opposite of the longest root of T in $\mathrm{Lie}(B)$. Thus, if the diagonal entries of $t \in T$ are $t_0, t_1, \dots, t_{d-1}$ as above, then $\alpha_0(t) = t_{d-1}t_0^{-1}$. Note that $\widetilde{S}$ is stable under the action of the Coxeter element c of $W = \mathfrak{S}_d$, which takes $i < d-1$ to $i+1$ and $d-1$ to 0. In fact, $\widetilde{S}$ is a principal homogeneous set under the cyclic subgroup $\langle c \rangle$ of order d generated by c. Therefore, it is convenient to identify $\{0, \dots, d-1\}$ with $\mathbb{Z}/d\mathbb{Z}$ through the canonical bijection, so that we simply have

$$c(\alpha_i) = \alpha_{i+1}, \quad \forall i \in \mathbb{Z}/d\mathbb{Z}.$$

We denote by $\mathcal{P}'(\widetilde{S})$ the set of strict subsets of $\widetilde{S}$. For any $I \in \mathcal{P}'(\widetilde{S})$, we can thus choose an $i \in \widetilde{S} \setminus I$. The translated subset $c^{-i}(I)$ is then contained in S, so we can consider the representation $i_{c^{-i}I}(\overline{\mathbb{F}}_\ell) \otimes \nu_G^i$.

LEMMA 2.1.2. *Up to semisimplification, the representation $i_{c^{-i}I}(\overline{\mathbb{F}}_\ell) \otimes \nu_G^i$ is independent of the choice of i in $\widetilde{S} \setminus I$. We denote by $[i_I]$ its class in the Grothendieck group $\mathcal{R}(G, \overline{\mathbb{F}}_\ell)$.*

Proof. Up to translation by a power of c, we may assume that $I \subseteq S$, so that we have to compare $i_I(\overline{\mathbb{F}}_\ell)$ with $i_{c^{-i}I}(\overline{\mathbb{F}}_\ell) \otimes \nu_G^i$ (assuming that $i \notin I$).

Note that the parabolic subgroups P_I and $P_{c^{-i}I}$ are associated. More precisely, any element of the normalizer of T in G which projects to c^i will conjugate the Levi component $M_{c^{-i}I}$ to M_I. Therefore, [7, lemme 4.13] shows that in the Grothendieck group $\mathcal{R}(G,\overline{\mathbb{F}}_\ell)$ we have the equality $[i_{c^{-i}I}(1)] = [i_I(\gamma)]$ with $\gamma = (\delta_{P_I} c^i(\delta^{-1}_{P_{c^{-i}I}}))^{1/2}$. Thus, we have to prove that $\gamma = \nu_G^{-i}{}_{|M_I}$. Since restriction to T is injective on characters of M_I, we may restrict both sides to T. Using that $\delta_{P_I|T} = \delta_B\delta^{-1}_{B\cap M_I}$, we get that $\gamma_{|T} = (\delta_B \cdot c^i(\delta_B^{-1}))^{1/2}$.

For $k = 0,\dots,d-1$, consider the smooth character of T defined by $\varepsilon_k(t) = q^{-\mathrm{val}(t_k)}$, where $t_0,\dots,t_{d-1}$ are the diagonal entries of $t \in T$. Then we have

$$\gamma_{|T} = \Big(\prod_{k<l}\varepsilon_k\varepsilon_l^{-1}\prod_{k<l}\varepsilon_{k+i}^{-1}\varepsilon_{l+i}\Big)^{1/2} = \prod_{k<i\leq l}\varepsilon_k\varepsilon_l^{-1} = \prod_{k<i}\varepsilon_k^{d-i}\prod_{i\leq l}\varepsilon_l^{-i}.$$

Of course, in the first equality, the indices $k+i$ and $l+i$ should be read modulo d. To get the second equality, we observe that for $0 \leq k < l < d$, we have $k+i > l+i (\mathrm{mod}\ d) \Leftrightarrow k < d-i \leq l \Leftrightarrow l+i < i \leq k+i$. Now using the fact that $\varepsilon_k^d = 1$, we get $\gamma_{|T} = \prod_k \varepsilon_k^{-i} = (\nu_G^{-i})_{|T}$ as desired. ∎

Remark. In the particular case $I = \emptyset$, $i = 1$ of the above lemma tells us that $[\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)] = [\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)\otimes \nu_G]$. So the twisting action of the cyclic group $\langle \nu_G\rangle$ on the set of classes of irreducible representations preserves the multiset $\mathrm{JH}(\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell))$.

We now want to isolate a certain irreducible constituent of $[i_I]$. We follow the Zelevinski approach via degenerate Whittaker models, as in Vignéras's work. First we associate a partition λ_I of d to I in the following way. As in the previous proof, I determines a conjugacy class of Levi subgroups, namely, that of $M_{c^{-i}I}$ for any $i \in \widetilde{S}\setminus\{i\}$. This conjugacy class corresponds to a partition μ_I of d, and we let λ_I be the transpose of μ_I. For example, $\lambda_\emptyset = (d)$, and $\lambda_I = (1,\dots,1)$ whenever $|I| = d-1$.

We refer to [23, section III.1] for the basics on the theory of derivatives and to [26, paragraph V.5] for the notion of degenerate Whittaker models.

Fact 2.1.2. For $I \in \mathcal{P}'(\widetilde{S})$, the representation $[i_I]$ has a unique irreducible constituent π_I admitting a λ_I-degenerate Whittaker model. Moreover, any other irreducible constituent has λ-degenerate Whittaker models only for $\lambda < \lambda_I$.

Remark. By Lemma 2.1.2, for any $I \in \mathcal{P}'(\widetilde{S})$ we have $\pi_{c^iI} \simeq \pi_I \otimes \nu_G^i$. In particular, for any $i \in \widetilde{S}$, we have $\pi_{\widetilde{S}\setminus\{i\}} = \nu_G^i$. On the other hand, $\pi_\emptyset$ is the only generic constituent of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$.

Before proceeding, we introduce some more notation. Similarly to the banal case, we consider the complement in X of the union of all hyperplanes attached to the roots in $\widetilde{S}$. Its connected components are labeled by *proper* subsets of $\widetilde{S}$ and given by

$$\widetilde{X}_I := \{x \in X, \forall \alpha \in \widetilde{S}, \varepsilon_I(\alpha)\langle x, \alpha\rangle > 0\},$$

where ε_I is the sign function attached to I as before. Note that $\widetilde{X}_{\widetilde{S}} = \widetilde{X}_\emptyset = \emptyset$ and that $\widetilde{X}_S = X_S$ is again the opposite Weyl chamber to B. However, for I a strict subset of S, we have $\widetilde{X}_I \neq X_I$.

PROPOSITION 2.1.2. *We have the following.*

(i) *The multiset* $\mathrm{JH}(\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell))$ *is a set (multiplicity* 1*).*

(ii) *The map* $I \in \mathcal{P}'(\widetilde{S}) \mapsto \pi_I \in \mathrm{JH}(\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell))$ *is a bijection.*

(iii) *For all* $I \in \mathcal{P}'(\widetilde{S})$*, the following equality holds in* $\mathcal{R}(G, \overline{\mathbb{F}}_\ell)$*:*

$$[i_I] = \sum_{J \supseteq I} [\pi_J].$$

(iv) *If* $\pi_\emptyset$ *is a cuspidal representation, and if* $I \neq \emptyset$*, then*

$$r_B(\pi_I) = \bigoplus_{w(\widetilde{X}_S) \subset \widetilde{X}_I} w^{-1}(\delta).$$

Proof.

(i) Suppose that π is a noncuspidal irreducible subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$. Let $P = M_P U_P$ be a parabolic subgroup such that $\pi_{U_P} \neq 0$. Since ℓ is banal for M_P, the Mackey formula (or *geometric lemma*) shows that in fact $\pi_{U_B} \neq 0$. But the congruence relation and the Mackey formula imply that $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)_{U_B}$ has the multiplicity 1 property, as a representation of T. More precisely, we have $r_B(\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)) = \bigoplus_{w \in W} w(\delta)$, and $\delta = \nu_G^{(1-d)/2} \prod_{i=0}^{d-1} \varepsilon_i^i$ is a W-regular character since q has order d. Hence, π occurs with multiplicity 1 in $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$. Now any cuspidal representation is generic, so there is at most one cuspidal subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$.

(ii) This follows from the proof of [11, proposition 3.2.4]. However, the latter reference rests on Vignéras's classification [26, Theorem V.12], so in particular on a difficult result of Ariki's on the classification of simple modules of Hecke-Iwahori algebras at roots of unity. In fact, in our context the latter can be avoided and replaced by the more elementary partial classification of [24, section 2.17]. Nevertheless, for the convenience of the reader, we sketch a complete and more direct proof.

Let us first show the injectivity of the map. Let π be some irreducible subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$, and let $\lambda = \lambda_\pi$ be the partition of d obtained from π by taking successive higher derivatives. Hence, λ_1 is the order of the highest nonzero derivative of π, λ_2 is that of the derivative $\pi^{(\lambda_1)}$, and so forth. The partition λ is the greatest element in the set of all partitions λ' such that $r_{\lambda'}(\pi)$ admits a generic subquotient. Here $r_{\lambda'}$ denotes the normalized Jacquet functor associated to the standard parabolic subgroup $P_{\lambda'} = U_{\lambda'}M_{\lambda'}$ associated to λ'. Let τ denote a generic subquotient of $r_\lambda(\pi)$. We can write $\tau = \gamma(A_1) \otimes \cdots \otimes \gamma(A_{|\lambda|})$, where $A_1 \sqcup \cdots \sqcup A_{|\lambda|} = \{0, \ldots, d-1\}$ is a set-theoretical partition with $|A_i| = \lambda_i$, and for any subset $A \subset \{0, \ldots, d-1\}$, $\gamma(A)$ denotes the unique generic subquotient of the normalized induction $\bigtimes_{a \in A} \nu^{a-(d-1)/2}$ (so this is a representation of $\mathrm{GL}_{|A|}(K)$). If $\pi = \pi_I$ for some I, then $\lambda = \lambda_I$, and a computation shows that for $k = 1, \ldots, |\lambda|$,

$$\begin{aligned} A_k &= \big\{a \in \{0, \ldots, d-1\}, \{\alpha_a, c^{-1}\alpha_a, \ldots, c^{2-k}\alpha_a\} \subset I\big\} \\ &= \big\{a \in \mathbb{Z}/d\mathbb{Z}, \{\alpha_a, \alpha_{a-1}, \ldots, \alpha_{a-k+2}\} \subset I\big\}. \end{aligned}$$

In particular, the following hold:

(a) for all $k = 1, \ldots, |\lambda| - 1$, we have $A_{k+1} \subset c(A_k) = A_k + 1$;

(b) $I = \{\alpha_i \in \widetilde{S}, i \notin A_1\}$.

Hence, we see that π_I determines I, so that the map in point (ii) is injective.

In order to prove the surjectivity, it is enough to prove that $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ has at most $2^d - 2$ irreducible noncuspidal constituents. If π is such a constituent, there is a Borel subgroup B' such that π is the unique irreducible quotient of the normalized induced representation $i_{B'}^G(\delta)$. However, the same argument as in [5, section 2.5.4] shows that if both the chambers $C(B')$ and $C(B'')$ are contained in a component $\widetilde{X}_I$, then the canonical intertwining operator $i_{B'}^G(\delta) \longrightarrow i_{B''}^G(\delta)$ is an isomorphism. Indeed, we may assume as in [5, section 2.5.4] that B' and B'' are adjacent, with wall associated to some root r. Then the representation theory for GL_2 (note that ℓ is banal with respect to $\mathrm{GL}_2(K)$) tells us that the canonical intertwining operator is invertible unless $q^{l(r)} = q^{\pm 1}$ in $\overline{\mathbb{F}}_\ell$, where $l(r)$ is the height of the root. With our congruence hypothesis and the general inequality $l(r) \leq n-1$, this implies that $l(r)$ is 1, -1, or $n-1$, which is equivalent to $\pm r \in \widetilde{S}$. This gives the desired bound.

(iii) By (ii), we only have to show that if J is any other strict subset of $\widetilde{S}$, then π_J occurs in $[i_I]$ if and only if $J \supseteq I$. Start with $J \supseteq I$, and choose $i \in \widetilde{S} \setminus J$. Then we have $i_{c^{-i}J}(\mathbb{F}_\ell) \otimes \nu_G^i \subset i_{c^{-i}I}(\mathbb{F}_\ell) \otimes \nu_G^i$, so π_J

occurs in $[i_I]$. Conversely, suppose that π_J occurs in $[i_I]$. Assume first that $J\cup I\neq\widetilde{S}$, and choose $i\in\widetilde{S}\setminus(J\cup I)$. Then we see that $[\pi_J\otimes\nu_G^{-i}]$ occurs in $i_{c^{-i}I}(\mathbb{F}_\ell)\cap i_{c^{-i}J}(\mathbb{F}_\ell)=i_{c^{-i}(I\cup J)}(\mathbb{F}_\ell)$. Hence, $\lambda_J\leq\lambda_{I\cup J}$, so $\lambda_J=\lambda_{I\cup J}$, and finally $I\cup J=J$, as desired. Assume now that $J\cup I=\widetilde{S}$; choose $j\in J\setminus I$, and set $J^*:=J\setminus\{j\}$. We get that $[\pi_J\otimes\nu_G^{-j}]$ occurs in $i_{c^{-j}I}(\mathbb{F}_\ell)\cap i_{c^{-j}J^*}(\mathbb{F}_\ell)=i_{c^{-j}(I\cup J^*)}(\mathbb{F}_\ell)=i_{\widetilde{S}\setminus\{0\}}=\pi_{\widetilde{S}\setminus\{0\}}$. Hence, $\pi_J=\pi_{\widetilde{S}\setminus\{j\}}$, so $J=\widetilde{S}\setminus\{j\}$, which is impossible by definition of j.

(iv) We proved in point (ii) that $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ has exactly 2^d-2 noncuspidal irreducible subquotients. But we constructed 2^d-1 constituents, so $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ has exactly one cuspidal subquotient. We know that it is generic, so it is, by definition, $\pi_\emptyset$. Now, fix some proper subset I of $\widetilde{S}$. Again by the proof of the surjectivity in point (ii), there is a unique proper subset J such that

$$r_B(\pi_I)=\bigoplus_{w(\widetilde{X}_S)\subset\widetilde{X}_J}w^{-1}(\delta).$$

We still have to prove that $I=J$. Note that the condition $w(\widetilde{X}_S)\subset\widetilde{X}_J$ is equivalent to

$$J=\left\{\alpha_j\in\widetilde{S},wc^{-1}(j)<w(j)\right\}.$$

Now, items (a) and (b) in the proof of the injectivity in (ii) show that $I\subseteq J$. Since the map $I\mapsto J$ is a bijection, it has to be the identity. ∎

2.1.3. The decomposition matrix for elliptic representations. Recall that an admissible smooth $\overline{\mathbb{Q}}_\ell$-representation π of G is called *ℓ-integral* if it contains a G-stable $\overline{\mathbb{Z}}_\ell$-lattice. Then it is known that the reduction to $\overline{\mathbb{F}}_\ell$ of such a lattice depends on π only up to semisimplification (see [23, section II.5.11.b]). We denote by $r_\ell(\pi)$ the semisimple $\overline{\mathbb{F}}_\ell$-representation thus obtained.

PROPOSITION 2.1.3. *Let $I\subseteq S$. Then we have*

$$r_\ell\big(v_I(\overline{\mathbb{Q}}_\ell)\big)=[v_I(\overline{\mathbb{F}}_\ell)]=\begin{cases}[\pi_I]+[\pi_{I\cup\{0\}}] & \text{if } I\neq S,\\ [\pi_S] & \text{if } I=S.\end{cases}$$

Proof. Since parabolic induction commutes with inductive limits, we have $i_I(R)\simeq i_I(\mathbb{Z})\otimes R$ for any ring R. By its definition as a quotient $v_I(R)=i_I(R)/\sum_{J\supset I}i_J(R)$, we also have $v_I(R)=v_I(\mathbb{Z})\otimes R$. Now, by [21, Corollary 4.5], we know that $v_I(\mathbb{Z})$ is free over $\mathbb{Z}$. The first equality follows.

If $I=S$, we have $v_S(\overline{\mathbb{F}}_\ell)=\pi_S=\overline{\mathbb{F}}_\ell$ (trivial representation), so the second equality is clear in this case. Assume that $I\neq S$. By [21, Proposition 6.13], the following simplicial complex is exact:
(2.1.3.1)

$$0\longrightarrow i_S(\mathbb{Z})\longrightarrow\cdots\longrightarrow\bigoplus_{J\supset I,|J|=|I|+1} i_J(\mathbb{Z})\longrightarrow i_I(\mathbb{Z})\longrightarrow v_I(\mathbb{Z})\longrightarrow 0.$$

Since it consists of free $\mathbb{Z}$-modules, it remains exact after base change to $\overline{\mathbb{F}}_\ell$. Thus, we get the equality

$$[v_I(\overline{\mathbb{F}}_\ell)]=\sum_{S\supseteq J\supseteq I}(-1)^{|J\setminus I|}[i_J(\overline{\mathbb{F}}_\ell)]$$

in $\mathcal{R}(G,\overline{\mathbb{F}}_\ell)$. On the other hand, Proposition 2.1.2(iii) provides us with the equality

$$[\pi_I]=\sum_{\widetilde{S}\supset J\supseteq I}(-1)^{|J\setminus I|}[i_J]. \tag{2.1.3.2}$$

Thus, we get

$$[\pi_I]-[v_I(\overline{\mathbb{F}}_\ell)]=\sum_{\widetilde{S}\supset J\supseteq I\cup\{0\}}(-1)^{|J\setminus I|}[i_J]=-[\pi_{I\cup\{0\}}].$$

Alternatively, one could have used item (iv) in Proposition 2.1.2 and the easy fact that for any $I\subset S$, we have $X_I=\widetilde{X}_I\cup\widetilde{X}_{I\cup\{0\}}$. ∎

2.2. Corresponding representations

2.2.1. Langlands-Jacquet transfer. We refer to [11] for the definition of the Langlands-Jacquet transfer map $\mathrm{LJ}_{\overline{\mathbb{F}}_\ell}:\mathcal{R}(G,\overline{\mathbb{F}}_\ell)\longrightarrow\mathcal{R}(D^\times,\overline{\mathbb{F}}_\ell)$, which is induced by carrying Brauer characters through the usual bijection between regular elliptic conjugacy classes of G and $D^\times$. We will need the $\overline{\mathbb{F}}_\ell$-valued unramified character $\nu_D: d\mapsto q^{-\mathrm{val}\circ\mathrm{Nrd}(d)}$ of $D^\times$.

Proposition 2.2.1. *For any strict subset $I\subset\widetilde{S}$, we have*

$$\mathrm{LJ}_{\overline{\mathbb{F}}_\ell}(\pi_I)=(-1)^{|I|}\sum_{j\in\widetilde{S}\setminus I}[\nu_D^j].$$

Proof. Since the map $\mathrm{LJ}_{\overline{\mathbb{F}}_\ell}$ kills all parabolically induced representations (see [11, Théorème 3.1.4]), equality (2.1.3.2) shows that

$$\mathrm{LJ}_{\overline{\mathbb{F}}_\ell}(\pi_I)=(-1)^{|\widetilde{S}\setminus I|+1}\sum_{j\in\widetilde{S}\setminus I}\mathrm{LJ}_{\overline{\mathbb{F}}_\ell}(\pi_{\widetilde{S}\setminus\{j\}}).$$

On the other hand, $\pi_{\widetilde{S}\setminus\{j\}} = \nu_G^j = r_\ell(v_S(\overline{\mathbb{Q}}_\ell)) \otimes \nu_G^j$. By compatibility of the LJ maps with reduction modulo ℓ (see [11, Théorème 1.2.3]) and with torsion by characters, we get $\mathrm{LJ}_{\overline{\mathbb{F}}_\ell}(\pi_{\widetilde{S}\setminus\{j\}}) = (-1)^{|S|}[\nu_D^j]$. ∎

2.2.2. Different operations on Weil-Deligne representations. Before we proceed to a description of the Galois-type representations attached to the π_Is, we need to make precise some formal properties of Weil-Deligne representations.

It is convenient to work in a fairly general setting, so let $\mathcal{C}$ be an essentially small, Artinian, Noetherian, abelian category, and let $\mathcal{C}^{\mathrm{ss}}$ be the full subcategory of semisimple objects. The Jordan-Hölder theorem yields a map

$$\mathrm{Ob}(\mathcal{C})_{/\sim} \longrightarrow K_+(\mathcal{C}), \qquad V \mapsto [V]$$

from the set of isomorphism classes of objects to the free monoid on simple objects. This map induces a bijection $\mathrm{Ob}(\mathcal{C}^{\mathrm{ss}})_{/\sim} \xrightarrow{\sim} K_+(\mathcal{C})$.

Assume further that $\mathcal{C}$ is endowed with an automorphism $V \mapsto V(1)$, and denote by $V \mapsto V(n)$ its nth iteration. Consider the category $\mathcal{N}(\mathcal{C})$ with objects all pairs (V,N) with $V \in \mathrm{Ob}(\mathcal{C})$ and $N : V \longrightarrow V(-1)$ a nilpotent morphism. With the obvious notion of morphisms, $\mathcal{N}(\mathcal{C})$ is an Artinian, Noetherian, abelian category. The formalism of Deligne's filtration [12, (1.6)] yields a map

$$\mathrm{Ob}\big(\mathcal{N}(\mathcal{C})\big)_{/\sim} \longrightarrow K_+(\mathcal{C})^{(\mathbb{N})}, \qquad (V,N) \mapsto [V,N],$$

where the right-hand side is the set of almost zero sequences of elements in $K_+(\mathcal{C})$. Namely, we put $[V,N] := ([P^N_{-n}(V)])_{n\in\mathbb{N}}$, where P^N_i is the primitive part of the i-graduate of Deligne's filtration attached to N. We leave the reader to check the following fact.

LEMMA 2.2.2. *The map $(V,N) \mapsto [V,N]$ induces a bijection $\mathrm{Ob}(\mathcal{N}(\mathcal{C}^{\mathrm{ss}}))_{/\sim} \xrightarrow{\sim} K_+(\mathcal{C})^{(\mathbb{N})}$.*

As a consequence, one gets

- a *semisimplification* process $\mathrm{Ob}(\mathcal{N}(\mathcal{C}))_{/\sim} \longrightarrow \mathrm{Ob}(\mathcal{N}(\mathcal{C}^{\mathrm{ss}}))/\sim$;
- a *transposition* process $\mathrm{Ob}(\mathcal{N}(\mathcal{C}^{\mathrm{ss}}))_{/\sim} \xrightarrow{\sim} \mathrm{Ob}(\mathcal{L}(\mathcal{C}^{\mathrm{ss}}))_{/\sim}$, where $\mathcal{L}(\mathcal{C})$ denotes the category of pairs (V,L) with $L : V \longrightarrow V(1)$ nilpotent;
- a map $\mathrm{Ob}(\mathcal{N}(\mathcal{C}'^{\mathrm{ss}}))_{/\sim} \longrightarrow \mathrm{Ob}(\mathcal{N}(\mathcal{C}^{\mathrm{ss}}))_{/\sim}$ for any map $K_+(\mathcal{C}') \longrightarrow K_+(\mathcal{C})$.

As an example of application, let $\mathcal{C} = \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(W_K)$ (resp., $\mathcal{C}' = \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_K)$) be the category of finite-dimensional representations of W_K with $\overline{\mathbb{F}}_\ell$ (resp., $\overline{\mathbb{Q}}_\ell$) coefficients. In this paper, a *Weil-Deligne* $\overline{\mathbb{F}}_\ell$*-representation* is an object of $\mathcal{N}(\mathcal{C}^{\mathrm{ss}})$. (So our convention is that the Weil part of a Weil-Deligne representation is semisimple.) Applying the last item to the decomposition map $r_\ell : K_+(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_K)) \longrightarrow K_+(\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(W_K))$, we get a reduction process

$$(\sigma^{\mathrm{ss}}, N) \mapsto r_\ell(\sigma^{\mathrm{ss}}, N) = (r_\ell \sigma^{\mathrm{ss}}, N)$$

for Weil-Deligne representations.

2.2.3. The Zelevinski-Vignéras correspondence. According to [27, Théorème 1.6], there is a unique map

$$\pi \mapsto \sigma^{\mathrm{ss}}(\pi), \qquad \mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(G) \longrightarrow \{d\text{-dimensional semisimple } \overline{\mathbb{F}}_\ell\text{-reps of } W_K\}_{/\sim}$$

which is compatible with the ℓ-adic semisimple Langlands correspondence via reduction modulo ℓ in the following sense: *if π is a constituent of $r_\ell(\widetilde{\pi})$ for $\widetilde{\pi} \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$, then $\sigma^{\mathrm{ss}}(\pi) = r_\ell(\sigma^{\mathrm{ss}}(\widetilde{\pi}))$.* In [10], we gave a geometric realization of this map, as well as another proof of its existence.

Using her classification à la Zelevinski, Vignéras [27, section 1.8] explained that the above semisimple Langlands correspondence extends uniquely to a *bijection*,

$$\begin{aligned} &\mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(G) \to \{d\text{-dimensional Weil-Deligne } \overline{\mathbb{F}}_\ell\text{-reps of } W_K\}_{/\sim}, \\ &\qquad \pi \mapsto \sigma^Z(\pi) = \big(\sigma^{\mathrm{ss}}(\pi), N^Z(\pi)\big), \end{aligned}$$

such that the following compatibility with the ℓ-adic Langlands correspondence via reduction modulo ℓ holds: *if π is a constituent of $r_\ell(\widetilde{\pi})$ for $\widetilde{\pi} \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$, and if $\lambda_\pi = \lambda_{\widetilde{\pi}}$, then $\sigma^Z(\pi) = r_\ell\big(\sigma(Z(\widetilde{\pi}))\big)$.*

Here, Z denotes the Zelevinski involution for $\overline{\mathbb{Q}}_\ell$-representations, and the precise meaning of r_ℓ in the context of Weil-Deligne representations was explained in the preceding paragraph. Further, λ_π is the partition of d attached to π by taking successively higher nonzero derivatives, as in the proof of Proposition 2.1.2(ii). Note that the mere existence of a $\widetilde{\pi}$ fulfilling the conditions above is highly nontrivial in general and rests on Ariki's work on cyclotomic Hecke algebras.

Our aim in this paper is to provide a (partial) geometric interpretation of this enhanced correspondence, by means of a Lefschetz operator. Therefore, we focus on the *transposed* Weil-Deligne representation, as defined in the preceding paragraph:

$$\big(\sigma^{\rm ss}(\pi), L(\pi)\big) := {}^t\big(\sigma^{\rm ss}(\pi), N^Z(\pi)\big).$$

We now want to compute explicitly these transposed Weil-Deligne representations for the elliptic principal series. This will involve the $\overline{\mathbb{F}}_\ell$-character $\nu_W : w \mapsto q^{-\mathrm{val}(\mathrm{Art}_K^{-1}(w))}$, where Art_K is the local class field homomorphism which takes a uniformizer to a geometric Frobenius. For simplicity, we will use the Hecke normalization of Langland's correspondence.

PROPOSITION 2.2.3. *For any strict subset $I \subset \widetilde{S}$, we have $\sigma^{\rm ss}(\pi_I) \simeq \bigoplus_{i=0}^{d-1} \nu_W^i$, and in a good eigenbasis, $L(\pi_I)$ is given by the matrix $\sum_{\alpha_i \in I} E_{i-1,i}$*

Proof. The correspondence is compatible with twisting in the sense that $\sigma^Z(\pi \otimes \nu_G) = \sigma^Z(\pi) \otimes \nu_W$. Since our proposed solution is also compatible with twisting, we may assume that $I \subset S$. In this case we know that π_I appears in $r_\ell(v_I(\overline{\mathbb{Q}}_\ell))$. We also know that $\lambda_{\pi_I} = \lambda_{v_I(\overline{\mathbb{Q}}_\ell)} = \lambda_I$. Therefore, we have $(\sigma^{\rm ss}(\pi_I), L(\pi_I)) = r_\ell\big(\sigma^{\rm ss}(v_I(\overline{\mathbb{Q}}_\ell)), L(v_I(\overline{\mathbb{Q}}_\ell))\big)$. But the latter was computed in [9, proposition 3.2.4]. ∎

2.3. Computation of some Ext groups

This section is rather technical in nature and should be skipped at first reading. We first check that some computations of Ext groups between the v_J and the i_I performed by Orlik in [20] remain valid in our present context, although Orlik's hypotheses are not satisfied. Then we proceed to compute Ext groups between the values π_J and i_I.

2.3.1. Context and notation. We fix a uniformizer ϖ of K, and we will consider Yoneda extensions in the category $\mathrm{Rep}^\infty_{\overline{\mathbb{F}}_\ell}(G/\varpi^{\mathbb{Z}})$ of smooth $\overline{\mathbb{F}}_\ell$-representations of $G/\varpi^{\mathbb{Z}}$. Recall that a subset $I \subseteq S$ determines a standard parabolic subgroup P_I, the standard Levi component of which is denoted by M_I. We also denote by W_I the Weyl group of T in M_I, which is also the subgroup of W generated by reflections associated to roots in I. We define an $\overline{\mathbb{F}}_\ell$-vector space

$$Y_I := X^*\big(M_I/Z(G)\big) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell,$$

where X^* denotes the group of K-rational characters and Z means *center*.

Symbols r_P and i_P will stand for normalized parabolic functors along the parabolic subgroup P, and δ_P will denote the modulus character of P.

With this notation we have, for example, $i_I(\overline{\mathbb{F}}_\ell) = i_{P_I}(\delta_{P_I}^{-1/2})$. We will also put $\delta = \delta_B^{-1/2}$. Finally, the symbol $\mathcal{E}\mathrm{xp}(T,\sigma)$ denotes the set of characters of T occurring as subquotients of the admissible $\overline{\mathbb{F}}_\ell T$-representation σ.

LEMMA 2.3.1. *Let I be a strict subset of S.*

(i) *If π, π' are two principal series of M_I, then*

$$\big(W_I \cdot \mathcal{E}\mathrm{xp}(T, r_{B\cap M_I}(\pi)) \cap W_I \cdot \mathcal{E}\mathrm{xp}(T, r_{B\cap M_I}(\pi')) = \emptyset\big)$$
$$\Rightarrow \mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}(\pi,\pi') = 0.$$

(ii) $\mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}(\overline{\mathbb{F}}_\ell,\overline{\mathbb{F}}_\ell) = \bigwedge^* Y_I$.

Proof.

(i) The assumption means that π and π' have disjoint cuspidal supports. Since ℓ is banal for M_I, the vanishing of Ext follows from [25, Theorem 6.1].

(ii) The argument in [20, Proposition 9] shows that $\mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}(\overline{\mathbb{F}}_\ell,\overline{\mathbb{F}}_\ell) = \mathrm{Ext}^*_{M_I/M_I^0\varpi^{\mathbb{Z}}}(\overline{\mathbb{F}}_\ell,\overline{\mathbb{F}}_\ell)$, where M_I^0 is the subgroup of M_I generated by compact elements. (Note that ℓ is prime to the proindex $[M_I^0 : [M_I,M_I]]$.) Since ℓ is also prime to the torsion in the abelian group $M_I/M_I^0\varpi^{\mathbb{Z}}$, we know that $\mathrm{Ext}^*_{M_I/M_I^0\varpi^{\mathbb{Z}}}(\overline{\mathbb{F}}_\ell,\overline{\mathbb{F}}_\ell) = \bigwedge^*(\mathrm{Hom}_{\mathrm{gps}}(M_I/M_I^0\varpi^{\mathbb{Z}},\overline{\mathbb{F}}_\ell)) = \bigwedge^*(\mathrm{Hom}_{\mathrm{gps}}(M_I/M_I^0\varpi^{\mathbb{Z}},\mathbb{Z})\otimes_{\mathbb{Z}}\overline{\mathbb{F}}_\ell)$. Finally, the usual map $\chi \mapsto \mathrm{val}_K \circ \chi$ yields an isomorphism $X^*(M_I/Z(G)) \longrightarrow \mathrm{Hom}_{\mathrm{gps}}(M_I/M_I^0\varpi^{\mathbb{Z}},\mathbb{Z})$. ∎

REMARK. A consequence of item (ii) of Lemma 2.3.1 and Frobenius reciprocity is that for any representation π of $G/\varpi^{\mathbb{Z}}$, the graded space $\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}(\pi, i_I(\overline{\mathbb{F}}_\ell)) \simeq \mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}((\pi)_{U_{P_I}},\overline{\mathbb{F}}_\ell)$ is naturally a graded right module over the graded algebra $\bigwedge^* Y_I$. In particular, there is a canonical graded map

$$\mathrm{Hom}_{G/\varpi^{\mathbb{Z}}}\big(\pi, i_I(\overline{\mathbb{F}}_\ell)\big) \otimes_{\overline{\mathbb{F}}_\ell} \bigwedge^* Y_I \longrightarrow \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\pi, i_I(\overline{\mathbb{F}}_\ell)\big).$$

This map is clearly functorial in π. It is also functorial in I in the sense that if $J \subset I$, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{G/\varpi^{\mathbb{Z}}}\big(\pi, i_I(\overline{\mathbb{F}}_\ell)\big) \otimes_{\overline{\mathbb{F}}_\ell} \bigwedge^* Y_I & \longrightarrow & \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\pi, i_I(\overline{\mathbb{F}}_\ell)\big) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{G/\varpi^{\mathbb{Z}}}\big(\pi, i_J(\overline{\mathbb{F}}_\ell)\big) \otimes_{\overline{\mathbb{F}}_\ell} \bigwedge^* Y_J & \longrightarrow & \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\pi, i_J(\overline{\mathbb{F}}_\ell)\big)
\end{array}$$

where the vertical maps are induced by the inclusion $i_I(\overline{\mathbb{F}}_\ell) \hookrightarrow i_J(\overline{\mathbb{F}}_\ell)$ and the restriction map $Y_I \longrightarrow Y_J$.

2.3.2.

PROPOSITION 2.3.2. *Let I, J be two subsets of S, with I a* strict *subset. Then the canonical map*

$$\mathrm{Hom}_{G/\varpi^{\mathbb{Z}}}\big(i_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big) \otimes_{\overline{\mathbb{F}}_\ell} \bigwedge^{*} Y_I \longrightarrow \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(i_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big)$$

is an isomorphism. In other words, we have

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(i_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big) \simeq \begin{cases} \bigwedge^* Y_I & \text{if } J \supseteq I, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the natural map $\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}(i_K(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)) \longrightarrow \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}(i_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell))$ is an isomorphism for any $J \supseteq K \supseteq I$.

Proof. We follow [20, Proposition 15], but we avoid [20, Lemma 16], which might fail to be true in our context. By Frobenius reciprocity, we have

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(i_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big) = \mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}\big(r_{P_I} \circ i_{P_J}(\delta_{P_J}^{-1/2}), \delta_{P_I}^{-1/2}\big),$$

and by the geometric Mackey formula, $r_{P_I} \circ i_{P_J}(\delta_{P_J}^{-1/2})$ has a filtration with graded pieces of the form $Q_w := i_{M_I \cap w(P_J)}(w(\delta_{P_{J\cap w^{-1}(I)}}^{-1/2}))$, where w runs over all elements in W such that $w(J) \subset \Phi^+$ and $w^{-1}(I) \subset \Phi^+$. (This is a complete set of representatives of double cosets in $W_I \backslash W / W_J$.) Using again the geometric Mackey formula, we get

$$\begin{aligned} W_I \cdot \mathcal{E}\mathrm{xp}\big(T, r_{B\cap M_I}(Q_w)\big) &= W_I \cdot \mathcal{E}\mathrm{xp}\big(T, r_{B\cap M_{I\cap w(J)}}(w(\delta_{P_{J\cap w^{-1}(I)}}^{-1/2}))\big) \\ &= W_I \cdot \big\{w(\delta)\big\}. \end{aligned}$$

On the other hand, we have

$$W_I \cdot \mathcal{E}\mathrm{xp}\big(T, r_{B\cap M_I}(\delta_{P_I}^{-1/2})\big) = W_I \cdot \{\delta\}.$$

Since δ is W-regular, item (i) of Lemma 2.3.1 tells us that $\mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}(Q_w, \delta_{P_I}^{-1/2}) = 0$ unless $w \in W_I$. In this case, we must have $w = 1$ so that $Q_w =$

$Q_1 = i_{M_I\cap P_J}(\delta_{P_{I\cap J}}^{-1/2})$ is the top quotient of the geometric Mackey filtration and the canonical map

$$\mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}\big(i_{M_I\cap P_J}(\delta_{P_{I\cap J}}^{-1/2}), \delta_{P_I}^{-1/2}\big) \longrightarrow \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(i_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big)$$

is an isomorphism. Using Casselman's reciprocity, the left-hand side identifies with

$$\mathrm{Ext}^*_{M_{I\cap J}/\varpi^{\mathbb{Z}}}\big(\delta_{P_{I\cap J}}^{-1/2}, r_{M_I\cap\overline{P_J}}(\delta_{P_I}^{-1/2})\big) = \mathrm{Ext}^*_{M_{I\cap J}/\varpi^{\mathbb{Z}}}\big(\delta_{P_{I\cap J}}^{-1/2}, \delta_{P'_{I\cap J}}^{-1/2}\big),$$

where $\overline{P_J}$ is the opposite parabolic subgroup to P_J with respect to M_J, and $P'_{I\cap J}$ is the semistandard parabolic subgroup with Levi component $M_{I\cap J}$ and unipotent radical $U_I(\overline{U_J}\cap M_I)$. Let B' be the Borel subgroup with unipotent radical $U_I(\overline{U_J}\cap M_I)(U_\emptyset\cap M_{I\cap J})$. Lemma 2.3.1(i) tells us that the right-hand side of the last displayed formula vanishes unless there is $w\in W_{I\cap J}$ such that $w(B)=B'$. But then $w(B)\cap M_{I\cap J} = B'\cap M_{I\cap J}$; hence, $w=1$. Thus, $P'_{I\cap J} = P_{I\cap J}$, which is possible only if $J\supseteq I$.

We have proved the desired vanishing when J does not contain I, and we have proved that if $J\supseteq I$, the canonical map

$$\mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}(\delta_{P_I}^{-1/2}, \delta_{P_I}^{-1/2}) \longrightarrow \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(i_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big)$$

is an isomorphism. We conclude the computation by using item (ii) of Lemma 2.3.1. The last assertion follows from the functorial nature of the above map. ∎

2.3.3. The complex (2.1.3.1) yields a spectral sequence

$$E_1^{pq} = \bigoplus_{K\supseteq J, |K\setminus J|=p} \mathrm{Ext}^q_{G/\varpi^{\mathbb{Z}}}\big(i_K(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big) \Rightarrow \mathrm{Ext}^{p+q}_{G/\varpi^{\mathbb{Z}}}\big(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big),$$

which in particular yields an edge map

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\overline{\mathbb{F}}_\ell, i_I(\overline{\mathbb{F}}_\ell)\big) \longrightarrow \mathrm{Ext}^{|S\setminus J|+*}_{G/\varpi^{\mathbb{Z}}}\big(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big). \tag{2.3.3.1}$$

Thanks to Proposition 2.3.2, the same argument as [20, Proposition 17] gives the following expression.

COROLLARY 2.3.3. *Let I, J be subsets of S with I a strict subset.*

(i) *If $I\cup J\neq S$, then* $\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)) = 0$.

(ii) *If $I \cup J = S$, then the map (2.3.3.1) is an isomorphism, so we get an isomorphism*

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big) \simeq \bigwedge^{*-|S\setminus J|} Y_I.$$

Moreover, if I' is another strict subset of S which contains I, then the natural map $\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}(v_J(\overline{\mathbb{F}}_\ell), i_{I'}(\overline{\mathbb{F}}_\ell)) \longrightarrow \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell))$ is induced by the natural restriction map $Y_{I'} \longrightarrow Y_I$.

Remark. We may recast the foregoing corollary by stating that the canonical map

$$\mathrm{Ext}^{|S\setminus J|}_{G/\varpi^{\mathbb{Z}}}\big(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big) \otimes_{\overline{\mathbb{F}}_\ell} \bigwedge^* Y_I \longrightarrow \mathrm{Ext}^{*+|S\setminus J|}_{G/\varpi^{\mathbb{Z}}}\big(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big)$$

is an isomorphism and that $\mathrm{Ext}^{|S\setminus J|}_{G/\varpi^{\mathbb{Z}}}(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)) \simeq \overline{\mathbb{F}}_\ell$ if $J \cup I = S$ and is zero otherwise.

Next we turn to extensions between values π_J and i_I.

Proposition 2.3.3. *Let J be a strict subset of $\widetilde{S}$, and let I be a strict subset of S.*

(i) *If $0 \in J$, then $\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}(\pi_J, i_I(\overline{\mathbb{F}}_\ell)) = 0$.*

(ii) *Otherwise, the natural map is an isomorphism*

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(v_J(\overline{\mathbb{F}}_\ell), i_I(\overline{\mathbb{F}}_\ell)\big) \xrightarrow{\sim} \mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\pi_J, i_I(\overline{\mathbb{F}}_\ell)\big).$$

Proof. Note first that (ii) follows from (i) since $[v_J(\overline{\mathbb{F}}_\ell)] = [\pi_J] + [\pi_{J\cup\{0\}}]$. Now, in order to prove (i), we first use Frobenius reciprocity to get

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\pi_J, i_I(\overline{\mathbb{F}}_\ell)\big) = \mathrm{Ext}^*_{M_I/\varpi^{\mathbb{Z}}}\big(r_{P_I}(\pi_J), \delta_{P_I}^{-1/2}\big).$$

By Proposition 2.1.2(iv), we have $\mathcal{E}\mathrm{xp}\big(T, r_{B\cap M_I}(r_{P_I}(\pi_J))\big) = \{w^{-1}(\delta), w(\widetilde{X}_S) \subset \widetilde{X}_J\}$. Since $\mathcal{E}\mathrm{xp}(T, r_{B\cap M_I}(\delta_{P_I}^{-1/2})) = \{\delta\}$, and since δ is W-regular, Lemma 2.3.1 shows that we are left to prove that $\{w \in W, w(\widetilde{X}_S) \subset \widetilde{X}_J\} \cap W_I = \emptyset$. Now, identifying W with $\mathfrak{S}_d$ as in Section 2.1.1, the condition $w(\widetilde{X}_S) \subset \widetilde{X}_J$ is equivalent to the condition $J = \{\alpha_j \in \widetilde{S}, wc^{-1}(j) < w(j)\}$, so in particular it implies the property $w(n-1) < w(0)$. However, since I is proper, this property is never satisfied by some $w \in W_I$. ∎

§3. The cohomology complex

In this section, we focus on the useful part of the cohomology complex, namely, on that which pertains to the unipotent block of the category of smooth $\overline{\mathbb{Z}}_\ell$-representations.

3.1. The unipotent block

According to Vignéras [26, section IV.6.2], the category $\mathrm{Rep}^\infty_{\overline{\mathbb{F}}_\ell}(G)$ is a product of indecomposable Serre subcategories called *blocks*. This product of blocks corresponds to the partition of the set of irreducible $\overline{\mathbb{F}}_\ell$-representations according to the inertia class of supercuspidal support. Among them, the *unipotent block* is by definition the one which contains the trivial representation. In representation theory of finite groups, this would instead be called the *principal block*. Here we want to lift this block to $\overline{\mathbb{Z}}_\ell$-representations. Note that the usual way of lifting idempotents via Hensel's lemma is not adapted to the p-adic case, since Hecke algebras are not finitely generated modules over $\overline{\mathbb{Z}}_\ell$. Therefore, we will exhibit a progenerator of the desired block. In this section, no congruence assumption on the pair (q, ℓ) is required.

3.1.1. Unipotent blocks for a finite GL_n*.* For a finite group of Lie type $\bar{G}$, we will denote by $\mathbf{b}_{\bar{G}}$ the central idempotent in the group algebra $\mathbb{Z}_\ell[\bar{G}]$ which cuts out the direct sum of all blocks which contain a unipotent $\overline{\mathbb{Q}}_\ell$-representation (in the sense of Deligne and Luzstig).

Lemma 3.1.1. *Let $\bar{P} = \bar{M}\bar{U}$ be a parabolic subgroup of $\bar{G}$, and let $e_{\bar{U}}$ be the idempotent associated to the p-group $\bar{U}$. Then we have $e_{\bar{U}}\mathbf{b}_{\bar{G}} = e_{\bar{U}}\mathbf{b}_{\bar{M}} = \mathbf{b}_{\bar{M}}e_{\bar{U}}$.*

Proof. According to [4], an irreducible $\overline{\mathbb{Q}}_\ell$-representation π satisfies $\mathbf{b}_{\bar{G}}\pi \neq \{0\}$ if and only if it belongs to the Deligne-Lusztig series associated to some semisimple conjugacy class in the dual group $\bar{G}^*$ which consists of ℓ-elements. We call such a representation *ℓ-unipotent*. In this case, all irreducible subquotients of π_U are ℓ-unipotent representations of M. Indeed, this follows by adjunction from the "dual" statement that if σ is an ℓ-unipotent representation of M, then all irreducible subquotients of $\mathrm{Ind}_P^G(\sigma)$ are ℓ-unipotent (see [17, Corollary 6]). This shows that, denoting by $\mathbf{b}'_{\bar{G}} := 1 - \mathbf{b}_{\bar{G}}$ the complementary idempotent, we have $\mathbf{b}'_{\bar{M}}e_{\bar{U}}\mathbf{b}_{\bar{G}} = 0$ and $\mathbf{b}_{\bar{M}}e_{\bar{U}}\mathbf{b}'_{\bar{G}} = 0$. Then we get $e_{\bar{U}}\mathbf{b}_{\bar{G}} = (\mathbf{b}'_{\bar{M}} + \mathbf{b}_{\bar{M}})e_{\bar{U}}\mathbf{b}_{\bar{G}} = \mathbf{b}_{\bar{M}}e_{\bar{U}}\mathbf{b}_{\bar{G}} = \mathbf{b}_{\bar{M}}e_{\bar{U}}(1 - \mathbf{b}'_{\bar{G}}) = \mathbf{b}_{\bar{M}}e_{\bar{U}}$. ∎

FACT 3.1.1. Assume that $\bar{G} = \mathrm{GL}_n(\mathbb{F}_q)$. Then an irreducible $\overline{\mathbb{F}}_\ell$-representation $\bar{\pi}$ of $\bar{G}$ satisfies $\mathbf{b}_{\bar{G}}\bar{\pi} \neq 0$ if and only if it is a subquotient of $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\overline{\mathbb{F}}_\ell)$.

Proof. Any irreducible subquotient of $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\overline{\mathbb{F}}_\ell)$ occurs in the reduction of a unipotent irreducible $\overline{\mathbb{Q}}_\ell$-representation, and hence belongs to the category cut out by $\mathbf{b}_{\bar{G}}$. Conversely, fix $\bar{\pi}$ such that $\mathbf{b}_{\bar{G}}\bar{\pi} \neq 0$. We may assume that $\bar{\pi}$ is cuspidal, since for $\bar{P} = \bar{M}\bar{U}$ a parabolic subgroup such that $\bar{\pi}_{\bar{U}} \neq 0$ we also have $\mathbf{b}_{\bar{M}}(\bar{\pi}_{\bar{U}}) \neq 0$ (as in the previous proof). But then in terms of the Dipper-James classification, $\bar{\pi}$ is of the form $D(s,1)$ for some elliptic semisimple ℓ-element of $\bar{G}^* = \bar{G}$ (see [13, Corollary 5.23]). Thus, in terms of the James-Dipper classification, it is also of the form $D(1,(n))$ (see [14, Theorem 5.1]), which means that $\bar{\pi}$ is the only nondegenerate subquotient in $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\overline{\mathbb{F}}_\ell)$. ∎

3.1.2. Construction of the block. Here we put $\bar{G} = \mathrm{GL}_d(\mathbb{F}_q)$. We may view $\mathbf{b}_{\bar{G}}$ as a central idempotent of the $\mathbb{Z}_\ell$-algebra $\mathcal{H}_{\mathbb{Z}_\ell}(\mathrm{GL}_d(\mathcal{O}))$ of locally constant distributions on $\mathrm{GL}_d(\mathcal{O})$. Then we put

$$P_{\mathbf{b}} := \mathrm{ind}_{\mathrm{GL}_d(\mathcal{O})}^{G}\big(\mathbf{b}_{\bar{G}}\mathcal{H}_{\mathbb{Z}_\ell}(\mathrm{GL}_d(\mathcal{O}))\big),$$

and we define $\mathrm{Rep}_{\mathbf{b}}^{\infty}(G)$ as the full subcategory of $\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty}(G)$ consisting of all objects V that are generated by $\mathbf{b}_{\bar{G}}V$ over $\mathbb{Z}_\ell G$.

We will use similar notation to denote somewhat more familiar objects; letting $\mathbf{e}_{\bar{G}}$ be the idempotent attached to the pro-p-radical of $\mathrm{GL}_d(\mathcal{O})$, we also put

$$P_{\mathbf{e}} := \mathrm{ind}_{\mathrm{GL}_d(\mathcal{O})}^{G}\big(\mathbf{e}_{\bar{G}}\mathcal{H}_{\mathbb{Z}_\ell}(\mathrm{GL}_d(\mathcal{O}))\big),$$

and we define the category $\mathrm{Rep}_{\mathbf{e}}^{\infty}(G)$ as above. We recall the following result, which is a special case of level decomposition (see, e.g., [7, Appendice A]).

FACT 3.1.2. The category $\mathrm{Rep}_{\mathbf{e}}^{\infty}(G)$ is a direct factor of $\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty}(G)$ and is progenerated by $P_{\mathbf{e}}$. In particular, there is an idempotent $\mathbf{e}$ of the center of the category $\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty}(G)$ such that for any object V we have $\mathbf{e}V = \sum_{g\in G/\mathrm{GL}_d(\mathcal{O})} g\mathbf{e}_{\bar{G}}V$.

Now we can state the main result of this section.

PROPOSITION 3.1.2. *The category* $\mathrm{Rep}_{\mathbf{b}}^{\infty}(G)$ *is a direct factor of* $\mathrm{Rep}_{\mathbf{e}}^{\infty}(G)$ *and is progenerated by* $P_{\mathbf{b}}$*. It consists of all objects* V*, all irreducible subquotients of which are not annihilated by* $\mathbf{b}_{\bar{G}}$*.*

Proof. From its definition, $P_{\mathbf{b}}$ clearly is a generator of the category $\mathrm{Rep}^{\infty}_{\mathbf{b}}(G)$ and is a finitely generated projective object of $\mathrm{Rep}^{\infty}_{\overline{\mathbb{Z}}_\ell}(G)$.

Let us prove that $\mathrm{Rep}^{\infty}_{\mathbf{b}}(G)$ is a Serre subcategory. For this, we will apply the general result of [18, Theorem 3.1]. For any vertex x of the semisimple building of G, we denote by G_x its stabilizer, G_x^+ the pro-p-radical of its stabilizer, and $\bar{G}_x := G_x/G_x^+$ the reductive quotient. We thus get an idempotent $\mathbf{b}_x \in \mathcal{H}_{\overline{\mathbb{Z}}_\ell}(G_x)$ by inflation from $\mathbf{b}_{\bar{G}_x}$. If $g \in G$, we clearly have $\mathbf{b}_{gx} = g\mathbf{b}_x g^{-1}$. Therefore, to apply [18, Theorem 3.1], we are left to check the two following properties (see [18, Definition 2.1]):

(i) $\mathbf{b}_x\mathbf{b}_y = \mathbf{b}_y\mathbf{b}_x$ for any adjacent vertices x, y;
(ii) $\mathbf{b}_x\mathbf{b}_z\mathbf{b}_y = \mathbf{b}_x\mathbf{b}_y$ whenever z is adjacent to x and belongs to the convex simplicial hull of $\{x, y\}$.

Note first that the definition of $\mathbf{b}_x$ extends to any facet F of the building. Further, let $\mathbf{e}_F := e_{G_F^+}$ denote the idempotent associated to the pro-p-group G_F^+. We know that properties (i) and (ii) are satisfied by the system $(\mathbf{e}_x)_x$, and more precisely, we have $\mathbf{e}_x\mathbf{e}_y = \mathbf{e}_{[x,y]}$ whenever x and y are adjacent vertices. Therefore, Lemma 3.1.1 shows that $\mathbf{b}_x\mathbf{e}_y = \mathbf{b}_x\mathbf{e}_{[x,y]} = \mathbf{b}_{[x,y]}$, and thus $\mathbf{b}_x\mathbf{b}_y = \mathbf{b}_{[x,y]} = \mathbf{b}_y\mathbf{b}_x$. As for property (ii), starting from $\mathbf{e}_x\mathbf{e}_y = \mathbf{e}_x\mathbf{e}_z\mathbf{e}_y$, we get $\mathbf{b}_x\mathbf{b}_y = \mathbf{b}_x\mathbf{e}_z\mathbf{b}_y = \mathbf{b}_x\mathbf{b}_z\mathbf{b}_y$, as desired.

We now know that $\mathrm{Rep}^{\infty}_{\mathbf{b}}(G)$ is a Serre subcategory of the Serre subcategory $\mathrm{Rep}^{\infty}_{\mathbf{e}}(G)$ cut out by the system $(\mathbf{e}_x)_x$. For a vertex x, define $\mathbf{b}'_x := \mathbf{e}_x - \mathbf{b}_x$, which is lifted from the idempotent $1 - \mathbf{b}_{\bar{G}_x}$ of $\overline{\mathbb{Z}}_\ell[\bar{G}_x]$. The same argument as above shows that the system $(\mathbf{b}'_x)_x$ satisfies properties (i) and (ii) and therefore cuts out a Serre subcategory of $\mathrm{Rep}^{\infty}_{\mathbf{e}}(G)$, which is easily seen to be a complement to $\mathrm{Rep}^{\infty}_{\mathbf{b}}(G)$. Therefore, the latter is a direct factor in $\mathrm{Rep}^{\infty}_{\mathbf{e}}(G)$. The last statement of the proposition is clear. ∎

Notation. We will denote by $\mathbf{b}$ the idempotent of the center of the category $\mathrm{Rep}^{\infty}_{\overline{\mathbb{Z}}_\ell}(G)$ which projects a representation V to its largest subobject $\mathbf{b}V$ in $\mathrm{Rep}^{\infty}_{\mathbf{b}}(G)$. Concretely, we have $\mathbf{b}V = \sum_{g \in \mathrm{GL}_d(K)/\mathrm{GL}_d(\mathcal{O})} g \cdot \mathbf{b}_{\bar{G}} V$.

Proposition 3.1.3. *A representation $\pi \in \mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(G)$ belongs to $\mathrm{Rep}^{\infty}_{\mathbf{b}}(G)$ if and only if it is an irreducible subquotient of some $\mathrm{Ind}_B^G(\chi)$ with χ an unramified character of B. In particular, $\mathrm{Rep}^{\infty}_{\mathbf{b}}(G) \cap \mathrm{Rep}^{\infty}_{\overline{\mathbb{F}}_\ell}(G)$ is Vignéras's unipotent block [26, section IV.6.3].*

Proof. Let $\mathbf{e}_{\bar{G}} \in \mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathrm{GL}_d(\mathcal{O}))$ be the idempotent associated to the kernel of the reduction map $\mathrm{GL}_d(\mathcal{O}) \longrightarrow \mathrm{GL}_d(\mathbb{F}_q)$. By the Mackey formula, the residual representation of $\bar{G}$ on $\mathbf{e}_{\bar{G}}\operatorname{Ind}_B^G(\chi)$ is isomorphic to $\operatorname{Ind}_{\bar{B}}^{\bar{G}}(\overline{\mathbb{F}}_\ell)$ with obvious notation. Since $\operatorname{Ind}_B^G(\chi)$ belongs to the level zero subcategory $\operatorname{Rep}_{\mathbf{e}}^\infty(G)$, so does each of its irreducible subquotients. Hence, for such a subquotient π, $\mathbf{e}_{\bar{G}}\pi$ is a nonzero subquotient of $\operatorname{Ind}_{\bar{B}}^{\bar{G}}(\overline{\mathbb{F}}_\ell)$, so that $\mathbf{b}_{\bar{G}}\pi \neq 0$ and π belongs to $\operatorname{Rep}_{\mathbf{b}}^\infty(G)$.

Conversely, let π be an irreducible $\overline{\mathbb{F}}_\ell$-representation such that $\mathbf{b}\pi \neq 0$. Choose a parabolic subgroup $P = MU$ and a supercuspidal $\overline{\mathbb{F}}_\ell$-representation σ of M such that π occurs as a subquotient of $\operatorname{Ind}_P^G(\sigma)$. As above, the Mackey formula tells us that $\mathbf{e}_{\bar{G}}\operatorname{Ind}_P^G(\sigma) \simeq \operatorname{Ind}_{\bar{P}}^{\bar{G}}(\mathbf{e}_{\bar{M}}\sigma)$, with obvious notation. So by Lemma 3.1.1 we get $\mathbf{b}_{\bar{G}}\operatorname{Ind}_P^G(\sigma) \simeq \operatorname{Ind}_{\bar{P}}^{\bar{G}}(\mathbf{b}_{\bar{M}}\sigma)$, and finally $\mathbf{b}_{\bar{M}}\sigma \neq 0$. By [23, section III.3.15], we know that σ is of the form $\operatorname{ind}_{M\cap \mathrm{GL}_n(\mathcal{O})}^M(\bar{\sigma})$ for some *super*cuspidal $\overline{\mathbb{F}}_\ell$-representation $\bar{\sigma}$ of the Levi subgroup $\bar{M}$ of $\bar{G}$, image of $M \cap \mathrm{GL}_n(\mathcal{O})$ by the projection to $\bar{G}$. Here, supercuspidal is equivalent to the fact that the semisimple elliptic class s associated to $\bar{\sigma}$ consists of ℓ'-elements. However, an easy computation in [23, lemme III.3.14] shows that $\mathbf{b}_{\bar{M}}\bar{\sigma} = \mathbf{b}_{\bar{M}}\sigma$. Therefore, $\mathbf{b}_{\bar{M}}\bar{\sigma}$ is nonzero, and s consists of ℓ-elements by definition. Hence, $s = 1$, or equivalently, M is a torus, and $\bar{\sigma}$ is the trivial representation of $\bar{M}$. ∎

Remark. In terms of the Langlands correspondence, the irreducible $\overline{\mathbb{F}}_\ell$-representations π of the principal/unipotent block are those such that $\sigma(\pi)^{\mathrm{ss}}$ is a sum of unramified characters. This formulation might extend to other p-adic groups, as suggested by the finite field picture.

3.2. The complex

In the first two paragraphs of this subsection, no congruence hypothesis on the pair (q, ℓ) is required. From Section 3.2.3 on, we will work under the Coxeter congruence relation.

3.2.1. The tower and its cohomology complexes. We refer to [22] or [6, section 3.1] for the definition of the Lubin-Tate space $\mathcal{M}_{\mathrm{LT},n}$ of height d and level n, which we see as a $\breve{K}$-analytic space, endowed with a continuous action of $D^\times$, an action of $\mathrm{GL}_d(\mathcal{O}/\varpi^n\mathcal{O})$, and a Weil descent datum to K. Although in this paper we will be mainly interested in the tame level $\mathcal{M}_{\mathrm{LT},1}$, the formalism used to define the complex requires the whole tower $(\mathcal{M}_{\mathrm{LT},n})_{n\in\mathbb{N}}$ and, in particular, the action of $G = \mathrm{GL}_d(K)$ which can be defined on this tower. Maybe the most precise way to describe this action is to introduce the category $\mathbb{N}(G)$ with set of objects $\mathbb{N}$ and arrows given by $\operatorname{Hom}(n, m) := \{g \in G, gM_d(\mathcal{O})g^{-1} \subset \varpi^{m-n}M_d(\mathcal{O})\}$ and to note that the

$\mathcal{M}_{\mathrm{LT},n}$ are the image of a functor from $\mathbb{N}(G)$ to the category whose objects are $\breve{K}$-analytic spaces with continuous action of $D^\times$ and Weil descent datum to K, and morphisms are finite étale equivariant morphisms. This allows one to define the complex

$$R\Gamma_c := R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell) \in D^b\big(\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty,c}(G \times D^\times \times W_K)\big)$$

as in [6, section 3.3.3]. Let us note that the diagonal subgroup $K^\times$ of $G \times D^\times$ acts trivially on the tower, hence also on the cohomology.

It is technically important to recall that the tower is induced from a subtower denoted $(\mathcal{M}_{\mathrm{LT},n}^{(0)})_{n\in\mathbb{N}}$ which is stable under the subgroup

$$(GDW)^0 := \big\{(g,\delta,w) \in G \times D^\times \times W_K, |\det(g)|^{-1}|\mathrm{Nr}(\delta)||\mathrm{Art}_K(w)| = 1\big\}.$$

So we have a complex $R\Gamma_c^{(0)} := R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca},(0)}, \mathbb{Z}_\ell) \in D^b(\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty,c}(GDW)^0)$ together with an isomorphism [6, (3.5.2)]

$$R\Gamma_c \simeq \mathrm{Irr}_{(GDW)^0}^{GDW} R\Gamma_c^{(0)}.$$

An important consequence of this is the following compatibility with twisting. For any smooth character χ of $K^\times$ and any representation π of G, we have

$$\text{(3.2.1.1)} \quad R_{(\chi\circ\det)\otimes\pi} \simeq \big(\chi \circ (\mathrm{Nr}\cdot\mathrm{Art}_K)\big) \otimes R_\pi \text{ in } D^b\big(\mathrm{Rep}_{\mathbb{Z}_\ell}(D^\times \times W_K)\big).$$

We need yet another variant. Let us fix a uniformizer ϖ of K and see it as a central element of G. Its action on the tower is free (it permutes the connected components), so we may consider the quotient tower $(\mathcal{M}_{\mathrm{LT},n}/\varpi^{\mathbb{Z}})_{n\in\mathbb{N}}$ and its cohomology complex

$$R\Gamma_{c,\varpi} := R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}/\varpi^{\mathbb{Z}}, \mathbb{Z}_\ell) \in D^b\big(\mathrm{Rep}_{\mathbb{Z}_\ell}^{\infty,c}(G/\varpi^{\mathbb{Z}} \times D^\times \times W)\big).$$

We then have isomorphisms (see [6, section 3.5.3])

$$R\Gamma_{c,\varpi} \simeq R\Gamma_c \otimes_{\mathbb{Z}_\ell[\varpi^{\mathbb{Z}}]}^L \mathbb{Z}_\ell \simeq \mathrm{Irr}_{(GDW)^0\varpi^{\mathbb{Z}}}^{GDW} R\Gamma_c^{(0)}.$$

Because of the first isomorphism, if π is a representation on which ϖ acts trivially, then $R_\pi \simeq R\mathrm{Hom}_{G/\varpi^{\mathbb{Z}}}(R\Gamma_{c,\varpi}, \pi)$. Since any irreducible representation may be twisted to achieve this condition $\pi(\varpi) = 1$, we see that we do not lose any generality in restricting attention to $R\Gamma_{c,\varpi}$.

3.2.2. The tame part. We take up the notation $\mathbf{e}$, $\mathbf{e}_{\bar{G}}$ of the previous section and denote by $\mathcal{H}_{\mathbf{e}}$ the commuting algebra $End_{\mathbb{Z}_\ell G}(P_{\mathbf{e}})$, which identifies with the Hecke algebra of compactly supported $\mathbb{Z}_\ell$-valued $(1+\varpi M_d(\mathcal{O}))$-bi-invariant measures on G.

The complex $\mathbf{e}_{\bar{G}} R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)$ is naturally an object of $D^b(\mathrm{Rep}^\infty_{\mathcal{H}_{\mathbf{e}}}(D^\times \times W_K))$, and we recover the direct summand $\mathbf{e}R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)$ via the usual equivalence of categories. Namely, we have, as in [6, Lemme 3.5.9],

$$\mathbf{e}R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell) \simeq P_{\mathbf{e}} \otimes^L_{\mathcal{H}_{\mathbf{e}}} \mathbf{e}_{\bar{G}} R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell).$$

Now, if we restrict the action to $\mathrm{GL}_d(\mathcal{O})$, we have by construction an isomorphism in $D^b(\mathrm{Rep}^{\infty,c}_{\mathbb{Z}_\ell}(\bar{G} \times D^\times \times W_K))$:

$$\mathbf{e}_{\bar{G}} R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell) \xrightarrow{\sim} R\Gamma_c(\mathcal{M}_{\mathrm{LT},1}^{\mathrm{ca}}, \mathbb{Z}_\ell) \simeq \mathrm{Irr}_{\bar{G}\times (DW)^0}^{\bar{G}\times D^\times \times W_K} R\Gamma_c(\mathcal{M}_{\mathrm{LT},1}^{\mathrm{ca},(0)}, \mathbb{Z}_\ell).$$

The tame Lubin-Tate space $\mathcal{M}_{\mathrm{LT},1}^{(0)}$ was studied by Yoshida [28]. He exhibited in particular a certain affinoid subset $\mathcal{N}$ of $\mathcal{M}_{\mathrm{LT},1}^{0}$ which acquires good reduction over $\breve{K}[\varpi^{1/(q^d-1)}]$, with special fiber equivariantly isomorphic to the Deligne-Lusztig covering $Y(c)$ associated to the Coxeter element of $\bar{G}$. In [8], we showed that the restriction map induces an isomorphism $R\Gamma(\mathcal{M}_{\mathrm{LT},1}^{\mathrm{ca},(0)}, \mathbb{Z}_\ell) \xrightarrow{\sim} R\Gamma(\mathcal{N}^{\mathrm{ca}}, \mathbb{Z}_\ell)$. Taking duals, we thus get an isomorphism in $D^b(\mathrm{Rep}_{\mathbb{Z}_\ell}(\bar{G}))$,

$$R\Gamma_c(Y(c), \mathbb{Z}_\ell) \xrightarrow{\sim} R\Gamma_c(\mathcal{M}_{\mathrm{LT},1}^{\mathrm{ca},(0)}, \mathbb{Z}_\ell). \tag{3.2.2.1}$$

In particular, we get the following important property.

PROPOSITION 3.2.2. *The cohomology spaces of both the complexes* $\mathbf{e}R\Gamma_c$ *and* $\mathbf{e}R\Gamma_{c,\varpi}$ *are torsion-free.*

Indeed, the torsion-freeness for $Y(c)$ follows from [1, Lemma 3.9, Corollary 4.3]. We emphasize the fact that no hypothesis on the pair (q,ℓ) is required here.

3.2.3. The unipotent part: ℓ-adic cohomology. From this paragraph on, we assume that the order of q in $\mathbb{F}_\ell^\times$ is d. We take up the notation of the previous section, and we consider the direct summand $\mathbf{b}R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)$ or, rather, its variant $\mathbf{b}R\Gamma_{c,\varpi}$. There is a fairly explicit description of the $\mathbb{Q}_\ell$-cohomology of this complex. We first recall a classical construction. Let $\theta : \mathbb{F}_{q^d}^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character which is Frob_q-regular. Define

- a representation $\rho(\theta) := \mathrm{Irr}_{\mathcal{O}_D^\times \varpi^{\mathbb{Z}}}^{D^\times}(\theta)$ of $D^\times$, where $\mathcal{O}_D^\times \varpi^{\mathbb{Z}}$ acts via the reduction map $\mathcal{O}_D^\times \longrightarrow \mathbb{F}_{q^d}^\times$;
- a representation $\sigma(\theta) := \mathrm{ind}_{I_K \varphi^{d\mathbb{Z}}}^{W_K}(\theta)$, where $I_K \varphi^{d\mathbb{Z}}$ acts via the tame inertia map $I_K \longrightarrow \mu_{q^d-1} \simeq \mathbb{F}_{q^d-1}$;
- a representation $\pi(\theta) := \mathrm{ind}_{\mathrm{GL}_d(\mathcal{O})\varpi^{\mathbb{Z}}}^{G}(\pi_\theta^0)$, where π_θ^0 is the cuspidal representation of $\mathrm{GL}_d(\mathbb{F}_q)$ associated to θ by the Green (or Deligne-Lusztig) correspondence.

All these representations are irreducible and depend only on the Frob_q-conjugacy class of θ. Moreover, they are associate by the Langlands and Jacquet-Langlands correspondences.

FACT 3.2.3. Let $I_\varpi := \mathbb{Z}_\ell[G \times D^\times \times W_K/(GDW)^0\varpi^{\mathbb{Z}}]$.

(i) For $i = 1, \ldots, d-1$, there is an isomorphism

$$\mathcal{H}^{d-1+i}(\mathbf{b}R\Gamma_{c,\varpi}) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} v_{\{1,\ldots,i\}}(\mathbb{Q}_\ell)(-i) \otimes I_\varpi.$$

(ii) For $i = 0$, there is a (split) exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{H}^{d-1}(\mathbf{b}R\Gamma_{c,\varpi}) \otimes \mathbb{Q}_\ell \longrightarrow v_\emptyset(\mathbb{Q}_\ell) \otimes I_\varpi \longrightarrow 0$$

and an isomorphism $\mathcal{K} \otimes \overline{\mathbb{Q}}_\ell \simeq \bigoplus_\theta \pi(\theta) \otimes \rho(\theta)^\vee \otimes \sigma(\theta)^\vee$, where θ runs over Frob_q-conjugacy classes of Frob_q-regular characters $\mathbb{F}_{q^d}^\times \longrightarrow \overline{\mathbb{Z}}_\ell^\times$ which are ℓ-congruent to the trivial character.

Proof. The shortest argument here is to invoke Boyer's description of the $\overline{\mathbb{Q}}_\ell$-cohomology of the whole Lubin-Tate tower in [3] (see [6, théorème 4.1.2] for an account featuring a notation consistent with that of the present paper), together with the characterization of irreducible objects of the unipotent block in Proposition 3.1.3. We note that the maps $\mathcal{H}^{d-1+i}(\mathbf{b}R\Gamma_{c,\varpi}) \otimes \mathbb{Q}_\ell \longrightarrow v_{\{1,\ldots,i\}}(\mathbb{Q}_\ell)(-i)$ are induced by the canonical morphism

$$R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca},(0)}, \mathbb{Z}_\ell) \longrightarrow R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca},(0)}, \mathbb{Z}_\ell) \otimes^L_{\mathbb{Z}_\ell[\mathcal{O}_D^\times]} \mathbb{Z}_\ell. \tag{3.2.3.1}$$

Alternatively, if one wants to avoid Boyer's machinery, it is possible to derive almost everything from Yoshida's construction in [28] via isomorphism (3.2.2.1). More precisely, put $w_i := \mathcal{H}^{d-1+i}(\mathbf{b}R\Gamma_{c,\varpi} \otimes^L_{\mathbb{Z}_\ell[\mathcal{O}_D^\times/\varpi^{\mathbb{Z}}]} \mathbb{Z}_\ell)$. Then, by using a similar feature of Deligne-Lusztig varieties, one can show that the above morphism of complexes induces isomorphisms

$$\mathcal{H}^{d-1+i}(\mathbf{b}R\Gamma_{c,\varpi}) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} w_i \otimes \mathbb{Q}_\ell$$

for $i>0$, as well as an exact sequence

$$\mathcal{K} \hookrightarrow \mathcal{H}^{d-1}(\mathbf{b}R\Gamma_{c,\varpi}) \otimes \mathbb{Q}_\ell \twoheadrightarrow w_0 \otimes \mathbb{Q}_\ell.$$

Further, one finds an isomorphism $\mathbf{e}_{\bar{G}} w_i \simeq \mathbf{e}_{\bar{G}}(v_{\{1,\ldots,i\}}(\mathbb{Z}_\ell) \otimes I_\varpi)$. However, what is a priori missing is enough information on Hecke operators acting on $\mathbf{e}_{\bar{G}} w_i$ to recognize w_i as isomorphic to $v_{\{1,\ldots,i\}}(\mathbb{Z}_\ell) \otimes I_\varpi$. One highly nontrivial way to get around this problem is to invoke the Faltings-Fargues isomorphism of [16] (see [6, section 3.4] for a brief description) to move to the so-called *Drinfeld tower* (see [6, section 3.2] for an overview on this tower). Then the morphism of complexes (3.2.3.1) is carried to

$$R\Gamma_c(\mathcal{M}^{\mathrm{ca},(0)}_{Dr}, \mathbb{Z}_\ell) \longrightarrow R\Gamma_c(\mathcal{M}^{\mathrm{ca},(0)}_{Dr,0}, \mathbb{Z}_\ell), \tag{3.2.3.2}$$

and the right-hand side is the Drinfeld upper half-space whose cohomology is computed by combinatorics and shown by Schneider and Stuhler [21] to be isomorphic to $v_{\{1,\ldots,i\}}(\mathbb{Z}_\ell)$. ∎

We let Π be a uniformizer of D such that $\Pi^d = \varpi$, and we fix a geometric Frobenius element φ in W_K. We are going to decompose the complex $\mathbf{b}R\Gamma_{c,\varpi}$ in the category $D^b(\mathrm{Rep}_{\mathbb{Z}_\ell}(G/\varpi^{\mathbb{Z}}))$ according to the action of Π and φ. Since $K^\times_{\mathrm{diag}}$ acts trivially on the tower, the action of Π on $R\Gamma_{c,\varpi}$ is obviously killed by the polynomial X^d-1. Further, as a corollary to the description above and to the torsion-freeness result of Proposition 3.2.2, we get the following.

Corollary 3.2.3. *For any integer $0 \le i \le d-1$, the action of φ on $\mathcal{H}^{d-1+i}(\mathbf{b}R\Gamma_{c,\varpi})$ is killed by the polynomial $X^d - q^{id}$.*

3.2.4. The unipotent part: splitting. Put $P_\varphi(X) := \prod_{i=0}^{d-1}(X^d - q^{id})$. By Corollary 3.2.3 and [6, lemme A.1.4(i)], $P_\varphi(\varphi)$ acts by zero on the whole complex $\mathbf{b}R\Gamma_{c,\varpi}$. The ring $A_\varphi := \mathbb{Z}_\ell[X]/P_\varphi(X)$ is a semilocal ring, and hence decomposes as a product $A_\varphi = \prod_{\mathfrak{m}} A_{\varphi\mathfrak{m}}$ of its localizations at maximal ideals. Since q is a primitive d-root of unity in $\mathbb{F}_\ell$ by our congruence hypothesis, the maximal ideals of this ring are $\mathfrak{m}_i := (\ell, X - q^i)$, $i = 0, \ldots, d-1$, and we denote $A_\varphi = \prod_{i=0}^{d-1} A_{\varphi,i}$ the associated decomposition. Accordingly, we get a decomposition [19, Proposition 1.6.8]

$$\mathbf{b}R\Gamma_{c,\varpi} \simeq \bigoplus_{i=0}^{d-1} (\mathbf{b}R\Gamma_{c,\varpi})_i \quad \text{in } D^b\big(\mathrm{Rep}^\infty_{\mathbb{Z}_\ell}(G/\varpi^{\mathbb{Z}})\big).$$

Similarly, the ring $A_\Pi := \mathbb{Z}_\ell[X]/(X^d - 1)$ is semilocal with maximal ideals $(\ell, X - q^j)$, $j = 0, \ldots, d-1$, and we get a sharper decomposition

$$\mathbf{b}R\Gamma_{c,\varpi} \simeq \bigoplus_{i,j=0}^{d-1} (\mathbf{b}R\Gamma_{c,\varpi})_{i,j} \quad \text{in } D^b\big(\mathrm{Rep}^\infty_{\mathbb{Z}_\ell}(G/\varpi^{\mathbb{Z}})\big).$$

Note that each of these direct summands is preserved by the action of φ and Π, but not necessarily by that of I_K and $\mathcal{O}_D^\times$. Let ζ denote the Teichmüller lift of q, that is, the only primitive d-root of unity in $\mathbb{Z}_\ell$ which is ℓ-congruent to q. By construction, the action of Π on $(\mathbf{b}R\Gamma_{c,\varpi})_{i,j}$ is by multiplication by ζ^j, while that of φ is killed by the polynomial $\prod_k (X - q^{i-k}\zeta^k)$.

Moreover, these direct summands satisfy the following properties:

(3.2.4.1)
$$(\mathbf{b}R\Gamma_{c,\varpi})_{i,j} \simeq \zeta^{j\,\mathrm{val}_K \circ \det^{-1}} (\mathbf{b}R\Gamma_{c,\varpi})_{i-j,0}$$
with action of φ and Π twisted by ζ^j.

This follows indeed from (3.2.1.1).

(3.2.4.2) There is a distinguished triangle
$$c_i[0] \longrightarrow (\mathbf{b}R\Gamma_{c,\varpi})_{i,0}[d-1] \longrightarrow h_i(-i)[-i] \xrightarrow{+1}$$
with c_i a cuspidal ℓ-torsion free representation and h_i a G-invariant lattice in $v_{\{1,\ldots,i\}}(\mathbb{Q}_\ell)$.

This follows from Fact 3.2.3 and Proposition 3.2.2. Note that by convention we set $\{1, \ldots, i\} = \emptyset$ if $i = 0$.

Let us put $\bar{h}_i := h_i \otimes \overline{\mathbb{F}}_\ell$. By Proposition 2.1.3, we have the following equality in the Grothendieck group:

(3.2.4.3)
$$[\bar{h}_i] = [v_{\{1,\ldots,i\}}(\overline{\mathbb{F}}_\ell)] = [\pi_{\{1,\ldots,i\}}] + [\pi_{\{0,\ldots,i\}}].$$

Remark. For $i \neq d-1$, it can be shown that $\bar{h}_i$ is not isomorphic to $v_{\{1,\ldots,i\}}(\overline{\mathbb{F}}_\ell)$. More precisely, $v_{\{1,\ldots,i\}}(\overline{\mathbb{F}}_\ell)$ is a nonsplit extension of $\pi_{\{0,\ldots,i\}}$ by $\pi_{\{1,\ldots,i\}}$, while $\bar{h}_i$ is a nontrivial extension going the other way. The same phenomenon appears for the Deligne-Lusztig variety (see in particular [15, Theorem 4.1], which provides a description of the finite field analogue of $\bar{h}_i$). In the present context, let us simply mention without proof that the morphism (3.2.3.2) induces a map

(3.2.4.4)
$$H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca}}_{Dr,1}, \overline{\mathbb{F}}_\ell) = \bar{h}_i \longrightarrow H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca}}_{Dr,0}, \overline{\mathbb{F}}_\ell) = v_{\{1,\ldots,i\}}(\overline{\mathbb{F}}_\ell)$$

which is nonzero, with kernel and cokernel both isomorphic to $\pi_{\{0,\ldots,i\}}$. Of course, this map is also induced by the morphism (3.2.3.1).

§4. Proof of the main theorem

Let π be an $\overline{\mathbb{F}}_\ell$-representation of G. Recall the definition of the graded vector space R^*_π from the introduction. For convenience, we will shift this definition by $[1-d]$; that is, we consider now

$$R^*_\pi := \mathcal{H}^*\big(R\operatorname{Hom}_{\mathbb{Z}_\ell G}(R\Gamma_c[d-1],\pi)\big).$$

This is a graded smooth $\overline{\mathbb{F}}_\ell$-representation of $D^\times \times W_K$, whose grading is supported in the range $[1-d, d-1]$ by [10, Proposition 2.1.3].

In this entire section, we work under the Coxeter congruence hypothesis; that is, we assume that the order of q in $\mathbb{F}_\ell^\times$ is d.

4.1. Computation of R^*_π for π an elliptic principal series

4.1.1. Preliminaries. Assume now that π belongs to the unipotent block and that its central character is trivial on ϖ. Then we have $R^*_\pi = \mathcal{H}^*(R\operatorname{Hom}_{\mathbb{Z}_\ell(G/\varpi^\mathbb{Z})}(R\Gamma_{c,\varpi}[1-d],\pi))$, and according to Section 3.2.4, we may decompose it as

$$R^*_\pi = \bigoplus_{i,j=0}^{d-1} (R^*_\pi)_{i,j}, \quad \text{where } (R^*_\pi)_{i,j} := \mathcal{H}^*\big(R\operatorname{Hom}_{\mathbb{Z}_\ell(G/\varpi^\mathbb{Z})}((\mathbf{b}R\Gamma_{c,\varpi})_{i,j},\pi)\big).$$

Concretely, $(R^*_\pi)_{i,j}$ is the intersection of the generalized q^{-i}-eigenspace of φ with the generalized q^{-j}-eigenspace of Π. As already mentioned, these summands need not be stable under the action of I_K and $\mathcal{O}_D^\times$. However, the description of the ℓ-adic cohomology of $\mathbf{b}R\Gamma_{c,\varpi}$ in Section 3.2.3, together with the ℓ-torsion freeness of its integral cohomology, shows that both I_K and $\mathcal{O}_D^\times$ act trivially on the $D^\times \times W_K$ semisimplifications $\mathcal{H}^k(\mathbf{b}R\Gamma_{c,\varpi}\otimes^L_{\mathbb{Z}_\ell}\overline{\mathbb{F}}_\ell)^{\mathrm{ss}}$, $k\in\mathbb{N}$. Therefore, the same is true for $R^{*,\mathrm{ss}}_\pi$. As a consequence, letting I_K and $\mathcal{O}_D^\times$ act trivially on each $(R^*_\pi)^{\mathrm{ss}}_{i,j}$, we get the following equality in $\mathcal{R}(D^\times\times W_K,\overline{\mathbb{F}}_\ell)$:

$$R^{*,\mathrm{ss}}_\pi \simeq \bigoplus_{i,j=0}^{d-1}(R^*_\pi)^{\mathrm{ss}}_{i,j} = \bigoplus_{i,j=0}^{d-1}(\nu_D^j\otimes\nu_W^i)^{\dim_{\overline{\mathbb{F}}_\ell}(R^*_\pi)_{i,j}}. \tag{4.1.1.1}$$

Recall also from property (3.2.1.1) that we have $R^*_{\nu_G\pi}\simeq(\nu_D\otimes\nu_W)\otimes R^*_\pi$. Therefore, we get isomorphisms

$$(R^*_{\nu_G\pi})_{i,j}\simeq(R^*_\pi)_{i-1,j-1}. \tag{4.1.1.2}$$

The aim of this section is to prove Theorem 4.1.3 below, which describes explicitly each $(R^*_{\pi_I})_{i,j}$, that is, which computes the dimension of each $(R^*_{\pi_I})_{i,j}$. We first introduce some notation.

4.1.2. For an integer k between 0 and $d-1$ and a subset I of S, we put

$$\partial_I(k) := k - \delta(k,I), \quad \text{where } \delta(k,I) := |I \cup \{1,\dots,k\}| - |I \cap \{1,\dots,k\}|.$$

These functions already appear in [5] (see, esp., [5, Lemma 4.4.1]). The following property is elementary.

FACT 4.1.2. The map $k \in \{0,\dots,d-1\} \mapsto \partial_I(k) \in \mathbb{Z}$ is nondecreasing, with image $\{-|I|, -|I|+2, \dots, |I|-2, |I|\}$. More precisely, writing $I = \{i_1,\dots,i_{|I|}\}$ and putting $i_0 := 0$ and $i_{|I|+1} := d$, we have $\partial_I^{-1}(-|I|+2j) = \{i_j,\dots,i_{j+1}-1\}$.

In the next statement, we extend the function ∂_I to $\mathbb{Z}$ by making it d-periodic.

4.1.3.

THEOREM 4.1.3. *Let I be a strict subset of $\widetilde{S}$, and let i,j be integers between 0 and $d-1$. We have*

$$(R^*_{\pi_I})_{i,j} \simeq \begin{cases} \overline{\mathbb{F}}_\ell[\partial_{c^{-j}I}(i-j)] & \text{if } j \notin I, \\ 0 & \text{if } j \in I. \end{cases}$$

Since $\pi_I \simeq \nu_G^j \pi_{c^{-j}I}$, we see that the statement above is compatible with the twisting property (4.1.1.2). Therefore, we only have to prove it when $j=0$. We will treat separately the vanishing statement (when $0 \in I$) and the nonzero cases (when $0 \notin I$), and we start with a special case.

4.1.4. The case $|I| = d-1$. Here we prove Theorem 4.1.3 for characters, that is, for $|I| = d-1$. By the above remark on the effect of twisting by ν_G, we may assume that $I = S$, so that $\pi_I = \overline{\mathbb{F}}_\ell$ is the trivial representation of G. In this case, we have

$$R^*_{\overline{\mathbb{F}}_\ell} = \mathcal{H}^*\big(R\operatorname{Hom}_{\overline{\mathbb{F}}_\ell}(\overline{\mathbb{F}}_\ell \otimes^L_{\overline{\mathbb{F}}_\ell G} R\Gamma_c, \overline{\mathbb{F}}_\ell)[1-d]\big).$$

By [10, paragraph A.1.1, second lemma], we have

$$\mathcal{H}^*(\overline{\mathbb{F}}_\ell \otimes^L_{\overline{\mathbb{F}}_\ell G} R\Gamma_c) \simeq H^*(\mathbb{P}^{d-1,\mathrm{ca}}, \overline{\mathbb{F}}_\ell) = \bigoplus_{i=0}^{d-1} \overline{\mathbb{F}}_\ell[-2i](-i),$$

where the action of $D^\times$ is trivial and that of W_K is described by the Tate twists. Forgetting about technicalities, this merely expresses the fact that G acts freely on the tower $(\mathcal{M}_{\mathrm{LT},n})_{n\in\mathbb{N}}$ and that the quotient is the so-called Gross-Hopkins period space, which is isomorphic to the projective space $\mathbb{P}^{d-1}$ over $\widehat{K^{nr}}$. It follows that

$$R^*_{\overline{\mathbb{F}}_\ell} = \bigoplus_{i=0}^{d-1} (\nu_D^0 \otimes \nu_W^i)[1-d+2i].$$

Since $\partial_S(i) = 1-d+2i$, we have proved Theorem 4.1.3 for $I = S$ and thus for any $I \subset \widetilde{S}$ of cardinality $d-1$.

4.1.5. Vanishing when $j \in I$. As already mentioned, we may assume that $j = 0$. Fix a strict subset I of $\widetilde{S}$ which contains 0. We will prove in this section that

(4.1.5.1) $$\text{for all } i = 0, \dots, d-1, \quad \text{we have } (R^*_{\pi_I})_{i,0} = 0.$$

We argue by decreasing induction on $|I|$. The case $|I| = d-1$ was treated in Section 4.1.4, so let us assume that $|I| < d-1$. Recall from Lemma 2.1.2 and Proposition 2.1.2(iii) that for any $k \in \widetilde{S} \setminus I$, we have $c^{-k}I \subset S$ and

$$[\nu_G^k \otimes i_{c^{-k}I}(\overline{\mathbb{F}}_\ell)] = [i_I] = \sum_{J \supseteq I} [\pi_J].$$

Therefore, using the induction hypothesis, it is enough to find a $k \in \widetilde{S} \setminus I$ such that

$$R\operatorname{Hom}_{\mathbb{Z}_\ell(G/\varpi^{\mathbb{Z}})}\big((\mathbf{b}R\Gamma_{c,\varpi})_{i,0}, \nu_G^k \otimes i_{c^{-k}I}(\overline{\mathbb{F}}_\ell)\big) = 0.$$

Let us start with a random k in $\widetilde{S} \setminus I$. By (3.2.4.2) and Frobenius reciprocity, we have an isomorphism

$$\begin{aligned}\mathcal{H}^*\big(R\operatorname{Hom}_{\mathbb{Z}_\ell(G/\varpi^{\mathbb{Z}})}((\mathbf{b}R\Gamma_{c,\varpi})_{i,0}, \nu_G^k \otimes i_{c^{-k}I}(\overline{\mathbb{F}}_\ell))\big) \\ \simeq \operatorname{Ext}^{*+i}_{\overline{\mathbb{F}}_\ell(G/\varpi^{\mathbb{Z}})}\big(\nu_G^{-k} \otimes \bar{h}_i, i_{c^{-k}I}(\overline{\mathbb{F}}_\ell)\big).\end{aligned}$$

Further, by (3.2.4.3) we have $[\nu_G^{-k} \otimes \bar{h}_i] = [\pi_{c^{-k}\{0,\dots,i\}}] + [\pi_{c^{-k}\{1,\dots,i\}}]$. Therefore, applying Proposition 2.3.4(i), we get

(4.1.5.2) $$\operatorname{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\nu_G^{-k} \otimes \bar{h}_i, i_{c^{-k}I}(\overline{\mathbb{F}}_\ell)\big) = 0 \quad \text{whenever } k \in \{1, \dots, i\}.$$

In other words, if $k \in \{1,\dots,i\}$, we are done. Let us thus assume that $k \notin \{1,\dots,i\}$. In this case, Proposition 2.3.4(ii) and Corollary 2.3.3(i) tell us that

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\nu_G^{-k} \otimes \bar{h}_i, i_{c^{-k}I}(\overline{\mathbb{F}}_\ell)\big) = 0 \quad \text{whenever } c^{-k}\{1,\dots,i\} \cup c^{-k}I \neq S.$$

This means that if $I \cup \{1,\dots,i\} \neq \widetilde{S} \setminus \{k\}$, we are done. In particular, if $i = 0$ (in which case $\{1,\dots,i\} = \emptyset$ by convention), we are done, because $|I| < d-1$. Now let us assume the contrary, that is, that $I \cup \{1,\dots,i\} = \widetilde{S} \setminus \{k\}$ (and therefore $i \geq 1$). Again, because of $|I| < d-1$, this means that $\{1,\dots,i\}$ contains an element k' which does not belong to I. Applying (4.1.5.2) to this k', we get

$$\mathrm{Ext}^*_{G/\varpi^{\mathbb{Z}}}\big(\nu_G^{-k'} \otimes \bar{h}_i, i_{c^{-k'}I}(\overline{\mathbb{F}}_\ell)\big) = 0,$$

and this finishes the proof of (4.1.5.1).

4.1.6. Computation when $j \notin I$. Again we may assume that $j = 0$ and hence that $I \subset S$. The vanishing property of Section 4.1.5 shows that the map $\pi_I \hookrightarrow v_I := v_I(\overline{\mathbb{F}}_\ell)$ induces isomorphisms

$$(R^*_{\pi_I})_{i,0} \xrightarrow{\sim} (R^*_{v_I})_{i,0} \quad \text{for } i = 0,\dots,d-1 \tag{4.1.6.1}$$

because the cokernel v_I/π_I is isomorphic to $\pi_{I\cup\{0\}}$.

Now, we will use the exact sequence (2.1.3.1) in order to compute $(R^*_{v_I})_{i,0}$. It provides us with a spectral sequence

$$E_1^{pq} = \bigoplus_{\substack{S\supseteq J\supseteq I\\ |J\setminus I|=-p}} (R^q_{i_J})_{i,0} \Rightarrow (R^{p+q}_{v_I})_{i,0},$$

where we have abbreviated $i_J := i_J(\overline{\mathbb{F}}_\ell)$. A priori, this spectral sequence vanishes outside the range $-|S \setminus I| \leq p \leq 0$ and $q \geq -i$. Its differential d_1 has degree $(1,0)$ and is given by the natural maps $(R^*_{i_{J'}})_{i,0} \longrightarrow (R^*_{i_J})_{i,0}$ with signs associated to the simplicial set of subsets of $S \setminus I$.

The graded space $R^*_{i_J}$ is already known for $J = S$ by Section 4.1.4: we have $R^*_{i_S} = \overline{\mathbb{F}}_\ell[1-d+2i]$. Let us thus fix $J \subsetneq S$. Using Frobenius reciprocity and (3.2.4.2), we then have isomorphisms

$$\mathrm{Ext}^{*+i}_{G/\varpi^{\mathbb{Z}}}(\bar{h}_i, i_J) \xrightarrow{\sim} (R^*_{i_J})_{i,0}$$

for $i \in \{0, \dots, d-1\}$. Further, using equality (3.2.4.3), Proposition 2.3.4, and Corollary 2.3.3(ii), we get isomorphisms

$$\begin{aligned}(R^*_{i_J})_{i,0} &\\ &\simeq \mathrm{Ext}^{*+i}_{G/\varpi^{\mathbb{Z}}}(\bar{h}_i, i_J) \simeq \mathrm{Ext}^{*+i}_{G/\varpi^{\mathbb{Z}}}(\pi_{\{1,\dots,i\}}, i_J) \simeq \mathrm{Ext}^{*+i}_{G/\varpi^{\mathbb{Z}}}(v_{\{1,\dots,i\}}, i_J) \\ &\simeq \begin{cases} \mathrm{Ext}^{*+i-|S\setminus\{1,\dots,i\}|}_{G/\varpi^{\mathbb{Z}}}(\overline{\mathbb{F}}_\ell, i_J) \simeq \bigwedge^{*+2i+1-d} Y_J & \text{if } \{i+1,\dots,d-1\} \subseteq J, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Observe that the smallest J which contributes is $J(i,I) := I \cup \{i+1, \dots, d-1\}$. In particular, the E_1 page of the spectral sequence is supported in the vertical strip defined by

$$-|S \setminus I| \leq p \leq -|J(i,I) \setminus I|.$$

Moreover, since $\dim Y_J = d-1-|J|$, we see that for each p in the above range, the column E_1^{p*} is supported in the range

$$d-1-2i \leq q \leq 2d-2-2i-p-|I|.$$

In other words, the E_1 page is supported in the half square with left corner

$$(-|S \setminus I|, d-1-2i)$$

and right corners

$$\bigl(-|J(i,I) \setminus I|, d-1-2i\bigr) \qquad \text{and} \qquad \bigl(-|J(i,I) \setminus I|, 2d-2-2i-|J(i,I)|\bigr).$$

Now, we observe that the $E_1^{*\bullet}$ of our spectral sequence is the same, up to some shifts, as that which occurs in [20, proof of Theorem 1] and [2, Chapter X, Proposition 4.7]. We still have to compare the differential d_1 with that of these two references. Using Proposition 2.3.4(ii) again, we see that for $J' \supseteq J$ with J' a strict subset of S, the nonzero map $(R^*_{i_{J'}})_{i,0} \longrightarrow (R^*_{i_J})_{i,0}$ is induced by the natural map $Y_{J'} \longrightarrow Y_J$. It follows that for $p > -|S/I|$ (i.e., everywhere except maybe on the first nonzero column), the differential d_1^{pq} is the same as that of the two references cited above. In fact, the only possible difference concerns $d_1^{-|S/I|,d-1-2i}$, for which we have no control yet in our setting, except in the trivial case where $i=0$, because in this case, the E_1 is supported on one point. For $i>0$, in order to ensure that $d_1^{-|S/I|,d-1-2i}$

is the same as in the two references cited above, we have to prove that each map

$$(R_{i_S}^{d-1-2i})_{i,0} \simeq \overline{\mathbb{F}}_\ell \longrightarrow (R_{i_J}^{d-1-2i})_{i,0} \simeq \overline{\mathbb{F}}_\ell \tag{4.1.6.2}$$

is an isomorphism. However, since we know that all the other maps $(R_{i_J}^{d-1-2i})_{i,0} \longrightarrow (R_{i_{J'}}^{d-1-2i})_{i,0}$ for $J' \subset J \subsetneq S$ are isomorphisms, it is sufficient to prove that (4.1.6.2) is an isomorphism for a single J. For this, we look at the special case $I = \emptyset$, so that the left corner is $(1-d, d-1-2i)$. If all the maps (4.1.6.2) were zero, we would have $E_2^{1-d,d-1-2i} \simeq \overline{\mathbb{F}}_\ell$, creating a nonzero $R_{\pi_\emptyset}^{-2i}$, which is absurd since $R_{\pi_\emptyset}^*$ has to vanish for $* < -i$, and i was supposed to be positive.

Therefore, we have identified the first page of our spectral sequence with that of [20, proof of Theorem 1] and [2, Chapter X, Proposition 4.7], up to shifts. Using their results, we get that E_2^{pq} is always zero except in the upper right corner of the triangle, where it is 1-dimensional. Therefore, we get

$$(R_{\pi_I}^*)_{i,0} \simeq \overline{\mathbb{F}}_\ell\big[-\big(2d-2-2i-2|J(i,I)|+|I|\big)\big].$$

We still need to compute the shift $a = -(2d-2-2i-2|J(i,I)|+|I|)$. Observe first that $2d-2-2|J(i,I)| = 2|\{1,\dots,i\} \setminus I|$. Then, using $i - |\{1,\dots,i\} \setminus I| = |\{1,\dots,i\} \cap I|$, we get $a = 2|\{1,\dots,i\} \cap I| - |I|$. Eventually, using the equality $i + |I| = |\{1,\dots,i\} \cap I| + |\{1,\dots,i\} \cup I|$, we get

$$a = |\{1,\dots,i\} \cap I| - |\{1,\dots,i\} \cup I| + i = \partial_I(i).$$

The proof of Theorem 4.1.3 is now complete. However, it will be important to keep some track of the isomorphism $(R_{\pi_I}^{-\partial_I(i)})_{i,0} \simeq \overline{\mathbb{F}}_\ell$ that we have just obtained when we study the Lefschetz operator in the next section. We may decompose this isomorphism in four steps:

(i) The spectral sequence provides the isomorphism $(R_{\pi_I}^{-\partial_I(i)})_{i,0} \simeq (R_{i_{J(i,I)}}^{-\partial_I(i)+|J(i,I)\setminus I|})_{i,0}$.

(ii) Corollary 2.3.3 and the following remark exhibit an isomorphism

$$(R_{i_{J(i,I)}}^{d-1-2i})_{i,0} \otimes \bigwedge^{\max} Y_{J(i,I)} \xrightarrow{\sim} (R_{i_{J(i,I)}}^{-\partial_I(i)+|J(i,I)\setminus I|})_{i,0}.$$

(iii) The inclusion $\overline{\mathbb{F}}_\ell = i_S \hookrightarrow i_{J(i,I)}$ induces the isomorphism $(R_{i_S}^{d-1-2i})_{i,0} \simeq (R_{i_{J(i,I)}}^{d-1-2i})_{i,0}$, as was shown in the above proof.

(iv) The geometric input from Section 4.1.4 provides the isomorphism $(R_{\overline{\mathbb{F}}_\ell}^{d-1-2i})_{i,0} \simeq \overline{\mathbb{F}}_\ell$.

4.2. The Lefschetz operator

We now study the Lefschetz operator recalled in the introduction. We refer the reader to [9, section 2.2.4] for the precise definition of this operator and will content ourselves with recalling the relevant details when necessary in the proof of Theorem 4.2.2 below.

4.2.1. Our aim is to describe the operator $L^*_\pi : R^*_\pi \longrightarrow R^*_\pi[2](1)$ for π a unipotent elliptic representation. Since this operator is $D^\times \times W_K$-equivariant, it decomposes as a sum $L^*_\pi = \sum_{i,j}(L^*_\pi)_{i,j}$ with

$$(L^*_\pi)_{i,j} : (R^*_\pi)_{i,j} \longrightarrow (R^{*+2}_\pi)_{i-1,j}.$$

It also satisfies the following compatibility with torsion:

$$(L^*_{\nu_G\pi})_{i,j} = (L^*_\pi)_{i-1,j-1}, \qquad (4.2.1.1)$$

where equality merely means that these morphisms are part of a commutative diagram involving isomorphisms (4.1.1.2). Thanks to (4.2.1.1), we may restrict our attention to the case $j = 0$.

Now, recall from Theorem 4.1.3 that each $(R^*_{\pi_I})_{i,0}$ is zero unless $I \subseteq S$. In the latter case, it is 1-dimensional and concentrated in degree $-\partial_I(i)$. Therefore, $(L^*_{\pi_I})_{i,0}$ is necessarily zero as soon as $\partial_I(i) \neq \partial_I(i-1)+2$, which by Fact 4.1.2 is equivalent to $i \notin I$. The following theorem asserts that $(L^*_{\pi_I})_{i,j}$ is nonzero in the remaining cases.

4.2.2.

THEOREM 4.2.2. *Let I be a subset of S, and let $i \in I$. Then the operator*

$$(L^*_{\pi_I})_{i,0} : (R^*_{\pi_I})_{i,0} \simeq \overline{\mathbb{F}}_\ell[\partial_I(i)] \longrightarrow (R^*_{\pi_I})_{i-1,0}[2] \simeq \overline{\mathbb{F}}_\ell[\partial_I(i-1)+2]$$

is nonzero and thus is an isomorphism.

Proof. As in the proof of Theorem 4.1.3, the crucial input comes from geometry, which rules out the case of the trivial representation $\overline{\mathbb{F}}_\ell = \pi_S$. Indeed, recall from Section 4.1.4 that the period map provides us with isomorphisms

$$R^*_{\pi_S}[1-d] \simeq \left(H^*(\mathbb{P}^{d-1,\mathrm{ca}}, \overline{\mathbb{F}}_\ell)\right)^\vee = \bigoplus_{i=0}^{d-1} \overline{\mathbb{F}}_\ell[2i](i).$$

But by its mere definition, the Lefschetz operator of [9, section 2.2.4] induces the tautological Lefschetz operator on $\mathbb{P}^{d-1}$, namely, that given by the Chern class of the tautological sheaf. It is well known to induce isomorphisms $H^i(\mathbb{P}^{d-1,\mathrm{ca}}, \overline{\mathbb{F}}_\ell) \xrightarrow{\sim} H^{i+2}(\mathbb{P}^{d-1,\mathrm{ca}}, \overline{\mathbb{F}}_\ell)(1)$ for $0 \leq i < 2d-2$, thereby proving the theorem for $I = S$.

We now consider a general $I \subset S$. We will use the four steps gathered in the end of Section 4.1.6, which summarize the origin of the isomorphism $(R^{-\partial_I(i)}_{\pi_I})_{i,0} \simeq \overline{\mathbb{F}}_\ell$. Motivated by step (iii) in that list, we consider for any $J \subset S$ the following commutative diagram, which is functorially induced by the inclusion map $i_S \hookrightarrow i_J$:

$$\begin{array}{ccc} (R^{d-1-2i}_{i_S})_{i,0} & \longrightarrow & (R^{d-1-2i}_{i_J})_{i,0} \\ {\scriptstyle L^{d-1-2i}_{i_S}}\Big\downarrow & & \Big\downarrow{\scriptstyle L^{d-1-2i}_{i_J}} \\ (R^{d-1-2i+2}_{i_S})_{i-1,0} & \longrightarrow & (R^{d-1-2i+2}_{i_J})_{i-1,0} \end{array}$$

The two horizontal maps were shown to be isomorphisms in Section 4.1.6, and the left vertical map has just been shown to be so. We conclude that $(L^{d-1-2i}_{i_J})_{i,0}$ is an isomorphism.

Further, let us consider the diagram for $k \in \mathbb{N}$

$$\begin{array}{ccc} (R^{d-1-2i}_{i_J})_{i,0} \otimes_{\overline{\mathbb{F}}_\ell} \bigwedge^k Y_J & \longrightarrow & (R^{d-1-2i+k}_{i_J})_{i,0} \\ {\scriptstyle L^{d-1-2i}_{i_J} \otimes \mathrm{Id}}\Big\downarrow & & \Big\downarrow{\scriptstyle L^{d-1-2i+k}_{i_J}} \\ (R^{d-1-2i+2}_{i_J})_{i-1,0} \otimes_{\overline{\mathbb{F}}_\ell} \bigwedge^k Y_J & \longrightarrow & (R^{d-1-2i+2+k}_{i_J})_{i-1,0} \end{array}$$

The horizontal maps are explained in the remark of Section 2.3.1, and the functoriality of these maps ensures that the diagram is commutative. It follows from the identification $(R^*_{i_J})_{i,0} \simeq \mathrm{Ext}^{*+i}_{G/\varpi^{\mathbb{Z}}}(v_{\{1,\ldots,i\}}, i_J)$ explained in the course of Section 4.1.6, together with the remark of Section 2.3.3, that these maps are isomorphisms. Since the left vertical map has just been shown to be an isomorphism, so is the right one, $(L^{d-1-2i+k}_{i_J})_{i,0}$.

Recall now the notation $J(i,I) = I \cup \{i+1, \ldots, d-1\}$ of Section 4.1.6, and observe that $J(i-1,I) = J(i,I)$ since we assume that $i \in I$. Recall also that $\partial_I(i) = \partial_I(i-1) + 2$ under this assumption, and consider the diagram

$$\begin{array}{ccc}
(R_{\pi_I}^{-\partial_I(i)})_{i,0} & \longrightarrow & (R_{i_{J(i,I)}}^{-\partial_I(i)+|J(i,I)\setminus I|})_{i,0} \\
{\scriptstyle L_{\pi_I}^{-\partial_I(i)}}\big\downarrow & & \big\downarrow{\scriptstyle L_{i_{J(i,I)}}^{-\partial_I(i)+|J(i,I)\setminus I|}} \\
(R_{\pi_I}^{-\partial_I(i-1)})_{i-1,0} & \longrightarrow & (R_{i_{J(i,I)}}^{-\partial_I(i-1)+|J(i,I)\setminus I|})_{i-1,0}
\end{array}$$

where the horizontal maps are provided by the spectral sequence considered in Section 4.1.6. (These are edge maps once we know enough on the support of the spectral sequence.) These maps were shown to be isomorphisms in Section 4.1.6, and we have just proved that the vertical right-hand map is also an isomorphism. We conclude that $L_{\pi_I}^{-\partial_I(i)}$ is an isomorphism, as desired. ∎

4.2.3. Recollection and proof of the main theorem. We now prove the theorem announced in the introduction. In particular, we forget all gradings. We first assume that π is a unipotent (or principal series) elliptic representation. Let I be the strict subset of $\widetilde{S}$ such that $\pi \simeq \pi_I$. By (4.1.1.1) and Section 4.1.3, we have

$$R_\pi^{*,\mathrm{ss}} \simeq \bigoplus_{i,j=0}^{d-1} (R_{\pi_I}^*)_{i,j} \simeq \bigoplus_{j \notin I} \nu_D^j \otimes (R_\pi^{*,\mathrm{ss}})_j \quad \text{with}$$

$$(R_\pi^{*,\mathrm{ss}})_j := \bigoplus_{i=0}^{d-1} (R_{\pi_I}^*)_{i,j} = \bigoplus_{i=0}^{d-1} \nu_W^i.$$

According to Theorem 4.2.2 and the explicit description of Proposition 2.2.3, we have

$$\big((R_\pi^{*,\mathrm{ss}})_0, L_\pi^*\big) \simeq \big(\sigma^{\mathrm{ss}}(\pi), L(\pi)\big).$$

Applying again Theorem 4.2.2 to $c^{-j}I$ and using compatibility with twisting (4.2.1.1), we get for any $j \notin I$:

$$\big((R_\pi^{*,\mathrm{ss}})_j, L_\pi^*\big) \simeq \nu_W^j \otimes \big(\sigma^{\mathrm{ss}}(\pi_{c^{-j}I}), L(\pi_{c^{-j}I})\big) \simeq \big(\sigma^{\mathrm{ss}}(\pi), L(\pi)\big).$$

Recalling now Proposition 2.2.1, we eventually get

$$(R_\pi^*, L_\pi^*)^{\mathrm{ss}} \simeq |\,\mathrm{LJ}(\pi)| \otimes \big(\sigma^{\mathrm{ss}}(\pi), L(\pi)\big),$$

as desired.

In order to finish the proof of the main theorem, we still have to deal with the case when π is not elliptic. In this case we must show that $R^*_\pi = 0$. Here we use the full force of the Vignéras-Zelevinski classification in [26, Theorem V.12]. Following this classification, there is a proper parabolically induced representation ι which contains π as a subquotient with multiplicity 1, and all other subquotients π' of which satisfy the condition $\lambda_{\pi'} < \lambda_\pi$. Here, λ_π is the partition associated to π via the successive highest derivatives. Hence, arguing by induction on λ_π, we see that it suffices to prove that $R^*_\iota = 0$. Write $\iota = i_P(\tau)$ for some proper standard parabolic subgroup $P = MU$ and some irreducible representation τ. Then $R^*_\iota = \bigoplus_{i=0}^{d-1} \mathrm{Ext}^*_M(r_P v_{\{1,\dots,i\}}, \tau)$. But since π is not elliptic, the cuspidal support of τ is disjoint from $W.\delta$. Therefore, Lemma 2.3.1(i) shows that each Ext group occurring in the above sum vanishes.

4.2.4. Remark on nonunipotent representations. The main theorem may remain true for any irreducible $\overline{\mathbb{F}}_\ell$-representation π of G, under the Coxeter congruence hypothesis. In fact, much is already known; Boyer [3] has described the cohomology of the whole tower and has announced that the integral cohomology is torsion-less. This allows the splitting of the full complex according to weights. Then our arguments, which are somehow inductive on the "Whittaker level," work fine for arbitrary elliptic representations, except that the induction has to be initialized at some point. For unipotent representations, the initialization was the computation of $(R^*_{\overline{\mathbb{F}}_\ell}, L^*_\pi)$ thanks to the period map.

All in all, our arguments show that the main theorem is true for any representation π, provided it holds true for any super-Speh representation, in the sense of [11, Définition 2.2.3].

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Université Pierre et Marie Curie
Institut de Mathématiques de Jussieu
4 place Jussieu
75005 Paris
France
dat@math.jussieu.fr