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THE FIRST LINE OF THE BOCKSTEIN SPECTRAL SEQUENCE ON A MONOCHROMATIC SPECTRUM AT AN ODD PRIME

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Abstract. The chromatic spectral sequence was introduced by Miller, Ravenel, and Wilson to compute the E_2-term of the Adams-Novikov spectral sequence for computing the stable homotopy groups of spheres. The E_1-term $E_1^{s,t}(k)$ of the spectral sequence is an Ext group of BP_*BP-comodules. There is a sequence of Ext groups $E_1^{s,t}(n-s)$ for nonnegative integers n with $E_1^{s,t}(0)=E_1^{s,t}$, and there are Bockstein spectral sequences computing a module $E_1^{s,*}(n-s)$ from $E_1^{s-1,*}(n-s+1)$. So far, a small number of the E_1-terms are determined. Here, we determine the $E_1^{1,1}(n-1)=\mathrm{Ext}^1\, M_{n-1}^1$ for $p>2$ and $n>3$ by computing the Bockstein spectral sequence with E_1-term $E_1^{0,s}(n)$ for $s=1,2$. As an application, we study the nontriviality of the action of α_1 and β_1 in the homotopy groups of the second Smith-Toda spectrum $V(2)$.

§1. Introduction

Let p be a prime number, let $\mathcal{S}_{(p)}$ be the stable homotopy category of p-local spectra, and let S be the sphere spectrum localized at p. Understanding homotopy groups $\pi_*(S)$ of S is one of the principal problems in stable homotopy theory. The main vehicle for computing $\pi_*(S)$ is the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum BP. Spectrum BP is the p-typical component of MU, the complex cobordism spectrum, and it has homotopy groups $BP_*=\pi_*(BP)=\mathbb{Z}_{(p)}[v_1,v_2,\ldots]$, where v_n is a canonical generator of degree $2p^n-2$. In order to study the E_2-term of the Adams-Novikov spectral sequence, Miller, Ravenel, and Wilson [7] introduced the chromatic spectral sequence. It was designed to compute the E_2-term but has the following deeper connotation. Let $L_n\colon \mathcal{S}_{(p)}\to\mathcal{S}_{(p)}$ denote the Bousfield-Ravenel localization functor with respect to $v_n^{-1}BP$ (see [11]). It gives rise to the chromatic filtration $\mathcal{S}_{(p)}\to\cdots\to L_n\mathcal{S}_{(p)}\to L_{n-1}\mathcal{S}_{(p)}\to$

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$\cdots \to L_0\mathcal{S}_{(p)}$ of the stable homotopy category of spectra, which is a powerful tool for understanding the category. The chromatic nth layer of the spectrum S can be determined from the homotopy groups of $L_{K(n)}S$, the Bousfield localization of S with respect to the nth Morava K-theory $K(n)$ that has homotopy groups $K(n)_* = v_n^{-1}\mathbb{Z}/p[v_n]$ for $n > 0$ and $K(0)_* = \mathbb{Q}$. By the chromatic convergence theorem of Hopkins and Ravenel [12], S is the inverse limit of the L_nS. Let $E(n)$ be the nth Johnson-Wilson spectrum $E(n)$ with $E(n)_* = v_n^{-1}\mathbb{Z}_{(p)}[v_1, \dots, v_n]$ for $n > 0$, and let $E(0) = K(0)$. It is Bousfield equivalent to $v_n^{-1}BP$ and also to $K(0) \vee \cdots \vee K(n)$; that is, $L_{E(n)} = L_n = L_{K(0)\vee\cdots\vee K(n)}$. We notice that $E(0) = H\mathbb{Q}$, the rational Eilenberg-MacLane spectrum, and that $E(1)$ is the p-local Adams summand of periodic complex K-theory. Furthermore, $E(2)$ is closely related to elliptic cohomology. So far, we have no geometric interpretation of homology theories $K(n)$ or $E(n)$ when $n > 2$.

From now on, we assume that the prime p is odd. We explain the E_1-term of the chromatic spectral sequence. The Brown-Peterson spectrum BP is a ring spectrum that induces the Hopf algebroid $(BP_*, BP_*(BP)) = (BP_*, BP_*[t_1, t_2, \dots])$ in the standard way (see [13]), and we have an induced Hopf algebroid

$$\big(E(n)_*, E(n)_*(E(n))\big) = \big(E(n)_*, E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*\big),$$

where $E(n)_*$ is considered to be a BP_*-module by sending v_k to zero for $k > n$. Then, the E_1-term is given by

$$E_1^{s,t}(n-s) = \mathrm{Ext}^t_{E(n)_*(E(n))}\big(E(n)_*, M^s_{n-s}\big).$$

Here, M^s_{n-s} denotes the $E(n)_*(E(n))$-comodule $E(n)_*/(I_{n-s} + (v_{n-s}^\infty, v_{n-s+1}^\infty, \dots, v_{n-1}^\infty))$, in which I_k denotes the ideal of $E(n)_*$ generated by v_i for $0 \le i < k$ ($v_0 = p$), and $M/(w^\infty)$ for $w \in E(n)_*$, and an $E(n)_*$-module M denotes the cokernel of the localization map $M \to w^{-1}M$. In order to study the stable homotopy groups $\pi_*(L_{K(n)}S)$, we study here the homotopy groups of the monochromatic component M_nS of S (see [11]). Then, the E_2-term $E_2^{s,t}(M_nS)$ of the Adams-Novikov spectral sequence for computing $\pi_*(M_nS)$ is the E_1-term $E_1^{n,s}(0)$ of the chromatic spectral sequence. In [7], the authors also introduced the v_{n-s}-Bockstein spectral sequence $E_1^{s-1,t+1}(n-s+1) \Rightarrow E_1^{s,t}(n-s)$ associated to a short exact sequence

$$0 \to M^{s-1}_{n-s+1} \xrightarrow{\varphi} M^s_{n-s} \xrightarrow{v_{n-s}} M^s_{n-s} \to 0$$

of $E(n)_*(E(n))$-comodules, where $\varphi(x) = x/v_{n-s}$. So far, the E_1-term $E_1^{s,t}(n-s)$ is determined in the following cases (see [13]):

$(s,t,n) = (0,t,n)$ for (a) $n \le 2$, (b) $n = 3$, $p > 3$, (c) $t \le 2$ by Ravenel [10] (Henn [2] for $n = 2$ and $p = 3$);
$\quad = (1,0,n)$ for $n \ge 0$ by Miller, Ravenel, and Wilson [7];
$\quad = (s,t,n)$ for $n \le 2$ by Shimomura ([14], [17], [18]) and his collaborators Arita [1], Tamura [19], Wang [20], and Yabe [21];
$\quad = (1,1,3)$ by Shimomura [15] and Hirata and Shimomura [3];
$\quad = (2,0,n)$ for $n > 3$ by Shimomura [16], for $n = 3$ by Nakai ([8], [9]).

In this paper, we determine the structure of $E_1^{1,1}(n-1)$ for $n > 3$. The case $n = 3$, which is special, is treated in [15] and [3]. The result is the first step to understanding $\pi_*(L_{K(n)}S)$ for $n > 3$ as explained above. We proceed to state the result.

In this paper, we consider only the cases $s = 0$ and $s = 1$, and hereafter, we put

$$v = v_n \qquad \text{and} \qquad u = v_{n-1}.$$

Furthermore, we put

$$F = \mathbb{Z}/p,$$

and we consider the coefficient ring $K(n)_* = F[v_n^{\pm 1}] = F[v^{\pm 1}] = E(n)_*/I_n$,

$$A = E(n)_*/I_{n-1} \qquad \text{and} \qquad B = M_{n-1}^1 = A/(u^\infty) = \mathrm{Coker}(A \to u^{-1}A).$$

Since the ideal I_{n-1} is invariant, $(A,\Gamma) = \big(A, E(n)_*(E(n))/I_{n-1}\big)$ is a Hopf algebroid, and we use the abbreviation

$$\mathrm{Ext}^s M = \mathrm{Ext}_\Gamma^s(A, M)$$

for a Γ-comodule M. Then, the chromatic E_1-terms are

$$E_1^{0,t}(n) = \mathrm{Ext}^t K(n)_* \qquad \text{and} \qquad E_1^{1,t}(n-1) = \mathrm{Ext}^t B.$$

We have the u-Bockstein spectral sequence

$$E_1 = \mathrm{Ext}^* K(n)_* \Longrightarrow \mathrm{Ext}^* B \tag{1.1}$$

associated to the short exact sequence

$$0 \to K(n)_* \xrightarrow{\varphi} B \xrightarrow{u} B \to 0, \tag{1.2}$$

where φ is a homomorphism defined by $\varphi(x) = x/u$.

Let R be a ring, and let $R\langle g\rangle$ denote the R-module generated by g. The E_1-term of the u-Bockstein spectral sequence was determined by Ravenel [10] as follows.

THEOREM 1.3. *We have* $\operatorname{Ext}^0 K(n)_* = K(n)_*$ *and*

$$\operatorname{Ext}^1 K(n)_* = K(n)_*\langle h_i, \zeta_n : 0 \le i < n\rangle,$$
$$\operatorname{Ext}^2 K(n)_* = K(n)_*\langle \zeta_n h_i, b_i, g_i, k_i, h_j h_k : 0 \le i < n, 0 \le j < k-1 < n-1\rangle.$$

In the theorem, the generators h_i and b_i are represented by $t_1^{p^i}$ and $\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} t_1^{kp^i} \otimes t_1^{(p-k)p^i}$ of the cobar complex $\Omega_\Gamma^* K(n)_*$, respectively, and g_i and k_i are given by the Massey products

$$g_i = \langle h_i, h_i, h_{i+1}\rangle \qquad \text{and} \qquad k_i = \langle h_i, h_{i+1}, h_{i+1}\rangle. \tag{1.4}$$

In order to determine the module $\operatorname{Ext}^0 B$, Miller, Ravenel, and Wilson [7] introduced elements x_i and integers a_i in [7, (5.11), (5.13)], where they denoted them by $x_{n,i}$ and $a_{n,i}$, such that $x_i \equiv v^{p^i} \bmod I_n$ with the action of the connecting homomorphism δ given in [7, (5.18)]:

$$\begin{aligned} \delta(v^s/u) &= s v^{s-1} h_{n-1} \qquad \text{and} \\ \delta(x_i^s/u^{a_i}) &= s v^{(sp-1)p^{i-1}} h_{[i-1]} \quad \text{for } i \ge 1. \end{aligned} \tag{1.5}$$

Hereafter, we let

$$[i] \in \{0, 1, \dots, n-2\}$$

be the principal representative of the integer i module $n-1$. The elements x_i and the integers a_i are defined inductively by $x_0 = v$ and $a_0 = 1$, and for $i > 0$,

$$\begin{aligned} x_i &= \begin{cases} x_{i-1}^p & \text{for } i = 1 \text{ or } [i] \ne 1, \\ x_{i-1}^p - u^{b_{n,i}} v^{p^i - p^{i-1}+1} & \text{for } i > 1 \text{ and } [i] = 1, \text{ and} \end{cases} \\ a_i &= \begin{cases} p a_{i-1} & \text{for } i = 1 \text{ or } [i] \ne 1, \\ p a_{i-1} + p - 1 & \text{for } i > 1 \text{ and } [i] = 1. \end{cases} \end{aligned} \tag{1.6}$$

Here, $b_{n,k(n-1)+1} = (p^n - 1)(p^{k(n-1)} - 1)/(p^{n-1} - 1)$. The result (1.5) determines the differentials of the Bockstein spectral sequence, which implies the following.

THEOREM 1.7 ([7, Theorem 5.10]). *As a k_*-module,*

$$\operatorname{Ext}^0 B = L_\infty \oplus \bigoplus_{p\nmid s, i\geq 0} L_{a_i}\langle x_i^s\rangle.$$

Here, $k_* = k(n-1)_* = F[u]$, $L_i = k_*/(u^i)$, and $L_\infty = k_*/(u^\infty) = \varinjlim_i L_i$.

This theorem together with (1.5) implies the following.

COROLLARY 1.8. *The cokernel of $\delta\colon \operatorname{Ext}^0 B \to \operatorname{Ext}^1 K(n)_*$ is the F-module generated by*

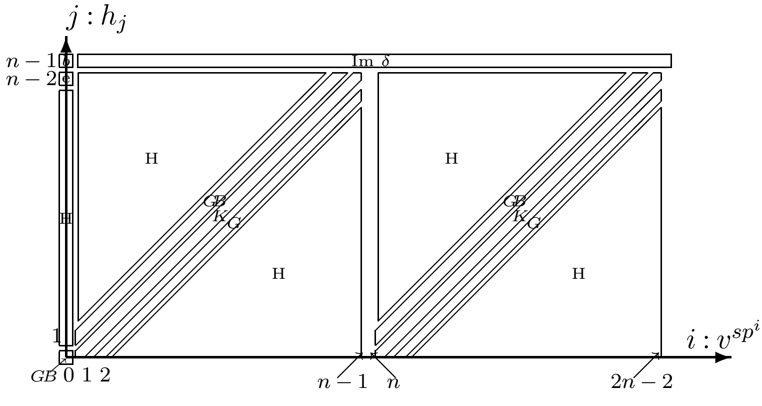
$$v^t\zeta_n, \qquad v^{tp-1}h_{n-1}, \qquad h_j \quad \text{for } 0 \leq j < n-1, \qquad \text{and}$$
$$v^{sp^k}h_j \quad \text{for } 0 \leq j < n-1, \text{ where } [k] \neq [j],\ s \not\equiv -1\ (p), \text{ or } s \equiv -1\ (p^2),$$

for integers s and t with $p \nmid s$.

By Theorem 1.3, the module $\operatorname{Ext}^1 K(n)_*$ is the direct sum of $\zeta_n \times \operatorname{Ext}^0 K(n)_* = \zeta_n K(n)_*$, $F\langle h_j\rangle$ for $j \in \mathbb{Z}/(n-1)$ and the modules

$$V_{(i,j,s)} = F\langle v^{sp^i}h_j\rangle$$

for $(i, j, s) \in \mathbb{N} \times \mathbb{Z}/n \times \overline{\mathbb{Z}}$. Here, $\mathbb{N}$ denotes the set of nonnegative integers, and $\overline{\mathbb{Z}} = \mathbb{Z} \setminus p\mathbb{Z}$. We partition $\mathbb{N} \times \mathbb{Z}/n$ as follows:



More precisely,

$$\begin{aligned}
H &= \{(0,j) : 1 \le j < n-2\} \\
&\quad \cup \{(i,j) : i > 0, [i] \ne n-3, n-2,\ 2+[i] \le j \le n-2\} \\
&\quad \cup \{(i,j) : i > 0,\ [i] \ne 0, 1, 0 \le j \le [i]-2\}, \\
GB &= \{(i,[i]) : i \ge 0\}, \\
K &= \{(i,[i]-1) : i > 0, [i] \ne 0\}, \qquad \text{and} \\
G &= \{(i,[i]-2) : i > 1, [i] \ne 0, 1\}.
\end{aligned}$$

We introduce notation

$$\begin{aligned}
V_{(0,n-2)} &= \bigoplus_{s\in\overline{\mathbb{Z}}'} V_{(0,n-2,s)}, \\
V_{(0,n-1)} &= \bigoplus_{t\in\mathbb{Z}} V_{(0,n-1,tp-1)} = F[v^{\pm p}]\langle v^{-1}h_{n-1}\rangle, \\
C_X &= \bigoplus_{(i,j)\in X, s\in\overline{\mathbb{Z}}} V_{(i,j,s)} \quad \text{for a subset } X \subset \mathbb{N}\times\mathbb{Z}/n, \\
\overline{C}_{GB} &= \bigoplus_{(i,j)\in GB} \left(\Big(\bigoplus_{s\in\overline{\overline{\mathbb{Z}}}} V_{(i,j,s)} \Big) \oplus \Big(\bigoplus_{t\in\mathbb{Z}} V_{(i,j,tp^2-1)} \Big) \right) \\
&= \bigoplus_{(i,[i],s)\in\widetilde{GB}} V_{(i,j,s)} \oplus \bigoplus_{i\ge 0} F[v^{\pm p^{i+2}}]\langle v^{-p^i} h_{[i]}\rangle, \qquad \text{and} \\
C_O &= F\langle \theta, h_j : j \in \mathbb{Z}/(n-1)\rangle.
\end{aligned}$$

Here, for $e(i) = (p^i-1)/(p-1)$, $\theta = v^{e(n-2)}h_{n-2}$,

$$\begin{aligned}
\overline{\mathbb{Z}}' &= \overline{\mathbb{Z}} \setminus \{e(n-2)\}, \qquad \overline{\overline{\mathbb{Z}}} = \{n \in \overline{\mathbb{Z}} : p \nmid (s+1)\}, \qquad \text{and} \\
\widetilde{GB} &= \{(i,[i],s) : s \in \overline{\overline{\mathbb{Z}}}\}.
\end{aligned}$$

We also consider the subset $\boldsymbol{T}$ of $\mathbb{N}\times\mathbb{Z}/n\times\overline{\mathbb{Z}}$ defined by

$$\begin{aligned}
\boldsymbol{T} = \{&(i,j,s) \in \mathbb{N}\times\mathbb{Z}/n\times\overline{\mathbb{Z}} : p \nmid (s+1) \text{ or } p^2 \mid (s+1) \text{ if } [i]=j, \\
&p \mid (s+1) \text{ if } (i,j)=(0,n-1), \text{ and } s \ne e(n-2) \text{ if } (i,j)=(0,n-2)\}.
\end{aligned}$$

In this notation, the cokernel of δ in Corollary 1.8 is given by

$$\begin{aligned}\operatorname{Coker}\delta &= \zeta_n K(n)_* \oplus C_O \oplus \bigoplus_{(i,j,s)\in \boldsymbol{T}} V_{(i,j,s)} \\ &= \zeta_n K(n)_* \oplus C_O \oplus V_{(0,n-2)} \oplus V_{(0,n-1)} \oplus C_H \oplus C_K \oplus C_G \oplus \overline{C}_{GB}.\end{aligned} \tag{1.9}$$

Finally, we consider the k_*-modules:

$$\begin{aligned}
W_{(i,j,s)} &= L_{a(i,j,s)}\langle x_i^s h_j\rangle, \\
W_{(0,n-2)} &= \bigoplus_{s\in\overline{\mathbb{Z}}'} W_{(0,n-2,s)}, \\
W_{(0,n-1)} &= \bigoplus_{t\in\mathbb{Z}} W_{(0,n-1,tp-1)}, \\
B_X &= \bigoplus_{(i,j)\in X,\ s\in\overline{\mathbb{Z}}} W_{(i,j,s)} \quad \text{for a subset } X\subset \mathbb{N}\times\mathbb{Z}/n, \\
\overline{B}_{GB} &= \bigoplus_{(i,j)\in GB}\Big(\Big(\bigoplus_{s\in\overline{\overline{\mathbb{Z}}}} W_{(i,j,s)}\Big)\oplus\Big(\bigoplus_{t\in\mathbb{Z}} W_{(i,j,tp^2-1)}\Big)\Big), \qquad \text{and} \\
C_\infty &= \big(K(n-1)_*/k_*\big)\langle \theta, h_j : j\in \mathbb{Z}/(n-1)\rangle.
\end{aligned}$$

Here, $a(i,j,s)$ denotes an integer defined as follows: for $(i,j)=(0,n-2)$, $a(0,n-2,s)=2$ if $p\nmid s(s-1)$, and

$$a(0,n-2,s)=\begin{cases} a_l, & p\nmid t, l>0, [l]\neq 0, n-2, \\ a_l+e(n-2)+p^{n-3}, & p\nmid t, l>0, [l]=n-2, \\ a_l+1, & p\nmid t, l>0, [l]=0,\end{cases}$$

if $s=tp^l+e(n-2)$; for $(i,j)\in\{(0,n-1)\}\cup H\cup K\cup G\cup GB$,

$$a(i,j,s)=\begin{cases} p-1, & (i,j)=(0,n-1), \\ a_i, & (i,j)\in H, \\ a_i+a_{i-1}, & (i,j)\in K\cup G, \\ 2a_i, & (i,j,s)\in \widetilde{GB}, \\ (p-1)a_{i+1}, & (i,j)\in GB,\ p^2\mid (s+1).\end{cases}$$

THEOREM 1.10. *The chromatic E_1-term* $\mathrm{Ext}^1 B = \mathrm{Ext}^1 M^1_{n-1}$ *is canonically isomorphic to the k_*-module*

$$\zeta_n \mathrm{Ext}^0 B \oplus C_\infty \oplus W_{(0,n-2)} \oplus W_{(0,n-1)} \oplus B_H \oplus B_K \oplus B_G \oplus \overline{B}_{GB}.$$

Let $V(n)$ be the nth Smith-Toda spectrum defined by $BP_*(V(n)) = BP_*/I_{n+1}$. As an application of the theorem, we study the action of α_1 and β_1 on the elements u^t $(t>0)$ in the Adams-Novikov E_2-term $E_2^*(V(n))$ in Section 6. In particular, it leads us a geometric result for $n=4$. Toda [22] constructed the self map γ on $V(2)$ to show the existence of $V(3)$ for the prime $p>5$. We notice that $\gamma^t i \in \pi_*(V(2))$ for the inclusion $i\colon S \to V(2)$ to the bottom cell is detected by $u^t = v_3^t \in BP_*(V(2))$ in the Adams-Novikov spectral sequence.

THEOREM 1.11. *Let $p>5$. Then $\gamma^t i\alpha_1$ and $\gamma^t i\beta_1$ are nontrivial in $\pi_*(V(2))$ for $t>0$.*

§2. Bockstein spectral sequence

We compute the Bockstein spectral sequence by use of the following lemma.

LEMMA 2.1. *Let $\delta\colon \mathrm{Ext}^s B \to \mathrm{Ext}^{s+1} K(n)_*$ be the connecting homomorphism associated to the short exact sequence (1.2). Suppose that* $\mathrm{Coker}\,\delta = \bigoplus_k V_k \subset \mathrm{Ext}^1 K(n)_*$ *and that $\bigoplus_k U_k \subset \mathrm{Ext}^2 K(n)_*$ for F-modules V_k and U_k, and suppose that there exist u-torsion k_*-modules W_k fitting in a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_k & \xrightarrow{\varphi'_*} & W_k & \xrightarrow{u} & W_k & \xrightarrow{\delta'} & U_k \\
 & & \downarrow & & \downarrow f_k & & \downarrow f_k & & \downarrow \\
0 & \longrightarrow & \mathrm{Coker}\,\delta & \xrightarrow{\varphi_*} & \mathrm{Ext}^1 B & \xrightarrow{u} & \mathrm{Ext}^1 B & \xrightarrow{\delta} & \mathrm{Ext}^2 K(n)_*
\end{array}$$

of exact sequences. Then, $\mathrm{Ext}^1 B = \bigoplus_k W_k$.

This follows immediately from [7, Remark 3.11].

Let $\widetilde{\theta}$ be an element of Corollary 5.8. Then, $\widetilde{\theta}/u^k$ and h_j/u^k for $j \in \mathbb{Z}/(n-1)$ belong to $\mathrm{Ext}^1 B$, and we define the map $f\colon C_\infty \to \mathrm{Ext}^1 B$ by $f((u^{-k})\theta) = \widetilde{\theta}/u^k$ and $f((u^{-k})h_j) = h_j/u^k$ for $(u^{-k}) \in K(n-1)_*/k_*$, so that the short exact sequence

$$0 \to C_O \xrightarrow{1/u} C_\infty \xrightarrow{u} C_\infty \to 0 \tag{2.2}$$

yields a summand of Lemma 2.1.

Note that if a cocycle z represents ζ_n, then so does z^p. Therefore, we have $\zeta_n/u^j \in \operatorname{Ext}^1 B$ represented by z^{p^j}/u^j. The exact sequence (1.2) induces the exact sequence $0 \to \operatorname{Ext}^0 K(n)_* \xrightarrow{\varphi_*} \operatorname{Ext}^0 B \xrightarrow{u} \operatorname{Ext}^0 B \xrightarrow{\delta} \operatorname{Ext}^1 K(n)_*$, and we have an exact sequence

$$0 \to \zeta_n \operatorname{Ext}^0 K(n)_* \xrightarrow{\varphi_*} \zeta_n \operatorname{Ext}^0 B \xrightarrow{u} \zeta_n \operatorname{Ext}^0 B \xrightarrow{\delta} \zeta_n \operatorname{Ext}^1 K(n)_*, \tag{2.3}$$

which is a summand of Lemma 2.1. Together with (2.2) and (2.3), Theorem 1.10 follows from Lemma 2.1 if the following sequence is exact for each $(i,j,s) \in \boldsymbol{T}$:

$$0 \to V_{(i,j,s)} \xrightarrow{\varphi'_*} W_{(i,j,s)} \xrightarrow{u} W_{(i,j,s)} \xrightarrow{\delta'} U_{(i,j,s)}, \tag{2.4}$$

where $U_{(i,j,s)}$ denotes an F-module generated by a single generator as follows: for $(i,j) = (0,n-2)$, $U_{(0,n-2,s)} = F\langle v^{s-2}k_{n-2}\rangle$ if $p \nmid s(s-1)$,

$$U_{(0,n-2,s)} = \begin{cases} F\langle v^{s-p^{l-1}} h_{[l-1]}h_{n-2}\rangle, & p \nmid t, l > 0, [l] \neq 0, n-2, \\ F\langle v^{s-p^{l-1}} b_{2n-5}\rangle, & p \nmid t, l > 0, [l] = n-2, \\ F\langle v^{s-p^{l-1}-1} g_{n-2}\rangle, & p \nmid t, l > 0, [l] = 0, \end{cases}$$

if $s = tp^l + e(n-2)$; for $(i,j) \in \{(0,n-1)\} \cup H \cup K \cup G \cup GB$,

$$U_{(i,j,s)} = \begin{cases} F\langle v^{s-p+1} b_{n-1}\rangle, & (i,j) = (0,n-1), \\ F\langle v^{(sp-1)p^{i-1}} h_{[i-1]}h_j\rangle, & (i,j) \in H, \\ F\langle v^{(s-2)p} k_{n-1}\rangle, & (i,j) = (1,0) \in K, \\ F\langle v^{(sp^2-p-1)p^{i-2}} k_{[i-2]}\rangle, & (i,j) \in K, i > 1, \\ F\langle v^{(sp^2-p-1)p^{i-2}} g_{[i-2]}\rangle, & (i,j) \in G, \\ F\langle v^{s-p-1} g_{n-1}\rangle, & (i,j,s) \in \widetilde{GB}, i = 0, \\ F\langle v^{(sp-2)p^{i-1}} g_{[i-1]}\rangle, & (i,j,s) \in \widetilde{GB}, i > 0, \\ F\langle v^{(s+1-p)p^i} b_j\rangle, & (i,j) \in GB, p^2 \mid (s+1). \end{cases}$$

Since the mapping $\boldsymbol{T} \to \{U_{(i,j,s)} : (i,j,s) \in \boldsymbol{T}\}$ assigning (i,j,s) to $U_{(i,j,s)}$ is an injection, we see the following.

LEMMA 2.5. *The direct sum of* $\zeta_n \operatorname{Ext}^1 K(n)_*$ *and* $U_{(i,j,s)}$ *for* $(i,j,s) \in \boldsymbol{T}$ *is a sub-*F*-module of* $\operatorname{Ext}^2 K(n)_*$.

The homomorphism f_k in Lemma 2.1 on $W_{(i,j,s)}$ for $(i,j,s) \in \boldsymbol{T}$ is explicitly given by

$$f_{(i,j,s)}(x) = x/u^{a(i,j,s)}.$$

It follows that the homomorphism δ' on it is given by the composite $\delta(1/u^{a(i,j,s)})$. Hereafter we denote it by $\delta'_{(i,j,s)}$, that is, $\delta'_{(i,j,s)} = \delta(1/u^{a(i,j,s)})$, and consider a condition:

$(2.6)_{(i,j,s)}$ $\qquad \delta'_{(i,j,s)}(x) = y$ for the generators $x \in W_{(i,j,s)}$ and $y \in U_{(i,j,s)}$.

Note that $\varphi'_*(\overline{x}) = u^{a(i,j,s)-1}x$ for the generators $\overline{x} \in V_{(i,j,s)}$ and $x \in W_{(i,j,s)}$, since $f_k\varphi'_*(\overline{x}) = \varphi_*(\overline{x}) = x/u$. Then, we have the following.

Lemma 2.7. *For each $(i,j,s) \in \boldsymbol{T}$, if the condition (2.6)$_{(i,j,s)}$ holds, then (2.4) for (i,j,s) is exact and yields a summand of Lemma 2.1.*

The relations in (1.5) show the following immediately.

(2.8) *The condition (2.6)$_{(i,j,s)}$ holds for $(i,j) \in H$.*

Proof of Theorem 1.10. The theorem follows from Lemmas 2.1, 2.5, and 2.7, together with (2.2), (2.3), (2.8), and Lemmas 3.7, 3.8, 4.1, and 5.9, which are proved below. Indeed, the direct sum of $\zeta_n \operatorname{Ext}^0 K(n)_*$, C_O, and $V_{(i,j,s)}$ for $(i,j,s) \in \boldsymbol{T}$ is the cokernel of δ by (1.9). $\square$

§3. The summands on $V_{(0,n-1)}$ and $\overline{C}_{GB}$

We begin by stating some formulas on the Hopf algebroid (A,Γ):

$$(3.1)\quad \begin{aligned} &0 = vt_k^{p^n} + ut_{k+1}^{p^{n-1}} - u^{p^{k+1}}t_{k+1} - t_k\eta_R(v^{p^k}) \in \Gamma \quad \text{for } k < n,\\ &\eta_R(u) = u, \quad \eta_R(v) = v + ut_1^{p^{n-1}} - u^p t_1,\\ &\Delta(t_k) = \sum_{i=0}^{k} t_i \otimes t_{k-i}^{p^i} \quad \text{for } k < n, \qquad \text{and}\\ &\Delta(t_n) = \sum_{i=0}^{n} t_i \otimes t_{n-i}^{p^i} - ub_{n-2}. \end{aligned}$$

Then the connecting homomorphism $\delta\colon \operatorname{Ext}^1 B \to \operatorname{Ext}^2 K(n)_*$ is computed by the differential $d\colon \Omega^1_\Gamma A \to \Omega^2_\Gamma A$ of the cobar complex modulo an ideal, which is defined by

$$(3.2)\qquad d(x) = 1 \otimes x - \Delta(x) + x \otimes 1.$$

We also use the differential $d\colon \Omega^0_\Gamma A \to \Omega^1_\Gamma A$ defined by $d(w) = \eta_R(w) - \eta_L(w)$. For $w, w' \in \Omega^0_\Gamma A$ and $x \in \Omega^1_\Gamma A$, these differentials satisfy

$$\begin{aligned} d(ww') &= d(w)\eta_R(w') + w\,d(w'), \\ d(wx) &= d(w) \otimes x + w\,d(x), \qquad \text{and} \\ d\big(x\eta_R(w)\big) &= d(x)\eta_R(w) - x \otimes d(w). \end{aligned} \tag{3.3}$$

We also use the Steenrod operations P^0 and βP^0 on $\operatorname{Ext}^* C(j)$ for $j \ge 1$ and $\operatorname{Ext}^* B$ (see [5], [13]). Here, $C(j)$ denotes the comodule $A/(u^j)$, and we notice that $C(1) = K(n)_*$. Let $\widetilde{\Omega}^s M = \Omega^s_{E(n)_*(E(n))} M$ for an $E(n)_*(E(n))$-comodule M. Given a cocycle $x(j)$ of $\widetilde{\Omega}^s C(j)$, $\widetilde{x}(j)$ denotes a cochain of $\widetilde{\Omega}^s E(n)_*$ such that $\pi_j(\widetilde{x}(j)) = x(j)$ for the projection $\pi_j\colon \widetilde{\Omega}^s E(n)_* \to \widetilde{\Omega}^s C(j)$. Since $x(j)$ is a cocycle, $d(\widetilde{x}(j)^p) = py_j + \sum_{i=1}^{n-2} v_i^p z_{j,i} + u^{jp} z_{j,n-1}$ for some elements y_j and $z_{j,i} \in \widetilde{\Omega}^{s+1} E(n)_*$. Under this situation, the Steenrod operations are defined by

$$\begin{aligned} P^0\big([x(j)]\big) &= [x(j)^p] \qquad \text{and} \\ \beta P^0\big([x(j)]\big) &= [y_j] \in \operatorname{Ext}^* C(jp), \qquad \text{and} \\ P^0\big([x(j)/u^j]\big) &= [x(j)^p/u^{jp}] \qquad \text{and} \\ \beta P^0\big([x(j)/u^j]\big) &= [y_j/u^{jp}] \in \operatorname{Ext}^* B. \end{aligned}$$

Here, $[x]$ denotes the homology class represented by a cocycle x. In particular, the operation acts on our elements as follows:

$$\beta P^0(x_i/u^{a_i}) = \begin{cases} v^{p-1} h_{n-1}/u^{p-1} & i = 0, \\ x_{i-1}^{p^2-1} h_{[i-1]}/u^{(p-1)a_i} & i > 0, \end{cases} \quad \text{in } \operatorname{Ext}^1 B; \tag{3.4}$$

$$\begin{aligned} P^0(x_i^s h_k/u^j) &= \begin{cases} x_{i+1}^s h_{k+1}/u^{jp} & k \ne n-2, \\ x_{i+1}^s h_0/u^{jp-p+1} & k = n-2, \end{cases} \quad \text{in } \operatorname{Ext}^1 B; \text{ and} \\ \beta P^0(x_i^s h_k) &= x_{i+1}^s b_k \quad \text{in } \operatorname{Ext}^2 K(n)_*. \end{aligned} \tag{3.5}$$

The following is a folklore (see [13, Corollary A1.5.5]):

$$P^0\delta = \delta P^0 \qquad \text{and} \qquad \beta P^0 \delta = -\delta \beta P^0 \quad \text{in } \operatorname{Ext}^* K(n)_*. \tag{3.6}$$

LEMMA 3.7. *The condition (2.6)$_{(i,j,s)}$ holds for each* $(i, j, s) \in \{(0, n-1, tp-1), (i, j, tp^2-1) : t \in \mathbb{Z}, (i, j) \in GB\}$.

Proof. For $k \geq -1$, consider a generator $x(k,t) = x_k^{tp^2-1} h_{[k]}$ for $k \geq 0$ and $x(-1) = x_0^{tp-1} h_{n-1}$, and $\overline{(k,t)}$ denotes a triple $(k,[k],tp^2-1)$ if $k \geq 0$ and $(0,n-1,tp-1)$ if $k=-1$. Then, $(1/u^{a\overline{(k,t)}})(x(k,t)) = x_{k+2}^{t-1}\beta P^0(x_{k+1}/u^{a_{k+1}})$ for $k \geq -1$ by (3.4). Now, $\delta'_{\overline{(k,t)}}(x(k,t))$ equals

$$x_{k+2}^{t-1}\delta\big(\beta P^0(x_{k+1}/u^{a_{k+1}})\big) = -x_{k+2}^{t-1}\big(\beta P^0(x_k^{p-1} h_{\overline{[k]}})\big) = -x_{k+1}^{\nu(t)} b_{\overline{[k]}}$$

by (3.6), (1.5), and (3.5). Here, $(\nu(t), \overline{[k]}) = (tp-1,[k])$ if $k \geq 0$, and it equals $((t-1)p, n-1)$ if $k=-1$. ∎

LEMMA 3.8. *The condition (2.6)$_{(i,[i],s)}$ holds for $(i,[i],s) \in \widetilde{GB}$.*

Proof. We prove this by induction on i. By (3.1) and (3.2), we compute mod (u^3)

$$\begin{aligned} d(v^{s+1-p}t_1^{p^n}) &\equiv (s+1)uv^{s-p}t_1^{p^{n-1}} \otimes t_1^{p^n} \\ &\quad + \binom{s+1}{2} u^2 v^{s-p-1} t_1^{2p^{n-1}} \otimes t_1^{p^n}, \\ d\big((s+1)uv^{s-p}t_2^{p^{n-1}}\big) &\equiv s(s+1)u^2 v^{s-p-1} t_1^{p^{n-1}} \otimes t_2^{p^{n-1}} \\ &\quad - (s+1)uv^{s-p} t_1^{p^{n-1}} \otimes t_1^{p^n} \end{aligned}$$

to obtain $\delta(v^s h_0/u^2) = s(s+1)v^{s-p-1}g_{n-1}$, and so

$$\delta'_{(0,0,s)}(v^s h_0) = s(s+1)v^{s-p-1}g_{n-1}.$$

Apply P^0 to it, and we obtain

$$\begin{aligned} \delta'_{(1,1,s)}(v^{sp}h_1) &= \delta\big(P^0(v^s h_0/u^2)\big) = P^0\delta(v^s h_0/u^2) = s(s+1)P^0(v^{s-p-1}g_{n-1}) \\ &= s(s+1)v^{sp-p^2-p}g_n = s(s+1)v^{sp-2}g_0. \end{aligned}$$

Here, we notice that $g_n = v^{p^2+p-2}g_0$ in $\mathrm{Ext}^2 K(n)_*$ by (3.1). Suppose inductively that $\delta'_{(i,1,s)}(x_i^s h_1) = s(s+1)v^{(sp-2)p^{i-1}}g_0$ for $[i]=1$, which is (2.6)$_{(i,1,s)}$. Note that $a_{i+j} = pa_{i+j-1}$ if $0 < j < n-2$, and we see that $P^0\delta'_{(i,j,s)} = \delta'_{(i+1,j+1,s)}P^0$ by (3.6). Therefore, $(P^0)^j$ for $j < n-2$ yields the equation for $\delta'_{a(i+j,j+1,s)}(x_{i+j}^s h_{j+1})$. At $i' = i+n-2$, for $t = (i',0,s)$, $\delta'_t(x_{i'}^s h_0) = \delta P^0(x_{i'-1}^s h_{n-2}/u^{a(i'-1,n-2,s)})$ (by (3.5)) $= s(s+1)v^{(sp-2)p^{i+n-3}}g_{n-2}$ by (3.6) and inductive hypothesis.

Note that $a_{i+n-1} = p^{n-1}a_i + p - 1$. Consider the connecting homomorphism $\delta_j\colon \operatorname{Ext}^1 M^1_{n-1} \to \operatorname{Ext}^2 C(j)$ associated to the short exact sequence $0 \to C(j) \xrightarrow{1/u^j} M^1_{n-1} \xrightarrow{u^j} M^1_{n-1} \to 0$. Then, $u^{j-1}\delta = \delta_j u^{j-1}$. Besides, $\delta_j(P^0)^k = (P^0)^k\delta$ if $p^k \geq j$. Now in $\operatorname{Ext}^2 C(p^2+p-1)$, $u^{p^2+p-2} \times \delta'_{(i+n-1,1,s)}(x^s_{i+n-1}h_1)$ equals

$$\begin{aligned} u^{p^2+p-2}\delta(x^s_{i+n-1}h_1/u^{p^{n-1}a+2(p-1)}) &= \delta_{p^2+p-1}(P^0)^{n-1}(x^s_i h_1/u^a) \\ &= (P^0)^{n-1}\big(s(s+1)v^{(sp-2)p^{i-1}}g_0\big) \\ &= s(s+1)v^{(sp-2)p^{i+n-2}}g_{n-1} \end{aligned}$$

for $a = a(i,[i],s)$, which equals $s(s+1)u^{p^2+p-2}v^{(sp-2)p^{i+n-2}}g_0$ by the relation $u^{p+2}g_{n-1} = u^{p^2+2p}g_0$. This relation follows from (1.4), and $uh_{n-1} = u^p h_0$ given by $d(v)$. ∎

§4. The summands C_G and C_K

We study the action of the connecting homomorphism δ by use of the Massey product. We notice that this is also shown by use of the P^0-operation considered in Section 3, but we use the Massey product for the sake of simplicity.

LEMMA 4.1. *The condition (2.6)$_{(i,j,s)}$ holds for* $(i,j) \in G \cup K$.

Proof. We consider the element $(1/u^{a(i,j,s)})(x^s_i h_j)$ the Massey product $\langle s x^{sp-1}_{i-1}/u^{a_{i-1}}, h_{[i-1]}, h_j\rangle$. Then, $\delta'_{(i,j,s)}(x^s_i h_j) = \delta\langle s x^{sp-1}_{i-1}/u^{a_{i-1}}, h_{[i-1]}, h_j\rangle = \langle s\delta(x^{sp-1}_{i-1}/u^{a_{i-1}}), h_{[i-1]}, h_j\rangle$, which equals $-\langle s v^{sp-2}h_{n-1}, h_0, h_0\rangle = -sv^{(s-2)p}k_{n-1}$ if $i=1$, and

$$-\langle s v^{(sp^2-p-1)p^{i-2}}h_{[i-2]}, h_{[i-1]}, h_j\rangle = \begin{cases} -sv^{(sp^2-p-1)p^{i-2}}k_{j-1} & j=[i-1], \\ -2sv^{(sp^2-p-1)p^{i-2}}g_j & j=[i-2] \end{cases}$$

otherwise. Here, we note that $\langle h_i, h_{i+1}, h_i\rangle = 2g_i$. ∎

§5. The summand $V_{(0,n-2)}$

Consider the elements $c_i = u^{p^i}h_{n-1+i}$ and $c'_i = u^{p^{i+1}}h_i$ of $\operatorname{Ext}^1 A$. The elements have internal degrees $|c_i| = |c'_i| = p^i e(n)q$ for $q = 2p-2$, and they satisfy

$$c_i = c'_i, \qquad c_ic_{i+1} = 0, \qquad h_{n+i}c_i = 0, \qquad \text{and} \qquad h_{i+1}c_i = h_{i+1}c'_i = 0.$$

We consider the cochains $\overline{w}_k = u^{e(k-1)}ct_k^{p^{n-1}}$ of the cobar complex $\Omega^1_\Gamma A$. Then,

$$\overline{w}_k = -\overline{w}_{k-1}^p \eta_R(v) + u^{pe(k-2)}v^{p^{k-1}}ct_{k-1} + u^{p^k+pe(k-2)}ct_k \tag{5.1}$$

for $k>1$ by (3.1). Let w_k be a cochain of the cobar complex $\Omega^1_\Gamma A$ defined inductively by

$$\begin{aligned} w_1 &= t_1^{p^{n-1}} - u^{p-1}t_1 = -\overline{w}_1 + u^{p-1}ct_1 \qquad \text{and} \\ w_k &= w_{k-1}^p\eta_R(v) + (-1)^k u^{pe(k-2)}v^{p^{k-1}}ct_{k-1}, \end{aligned} \tag{5.2}$$

and we put

$$\begin{aligned} m'_k &= -\sum_{i=1}^{k-1}(-1)^i u^{p^{i-1}} w_{k-i}^{p^i} \otimes \overline{w}_i \qquad \text{and} \\ m_k &= u^{p^{k-1}}w_k + \sum_{i=1}^{k-1}(-1)^i u^{p^{i-1}} v^{p^i e(k-i)}\overline{w}_i. \end{aligned} \tag{5.3}$$

Lemma 5.4. *We have $d(v^{e(k)}) = m_k$. Besides, $d(w_k) = m'_k$ if $k \le n$.*

Proof. We prove the lemma inductively. Since $d(v) = uw_1 = m_1$, we see the case for $k=1$. Indeed, $m'_1 = 0$.

Suppose that the equalities hold for $k-1$. Then, we compute by (3.3), (5.1), and (5.2),

$$\begin{aligned} d(v^{e(k)}) &= d(v^{pe(k-1)})\eta_R(v) + v^{pe(k-1)}d(v) \\ &= \Big(u^{p^{k-1}}w_{k-1}^p + \sum_{i=1}^{k-2}(-1)^i u^{p^i} v^{p^{i+1}e(k-1-i)}\overline{w}_i^p\Big)\eta_R(v) \\ &\quad - uv^{pe(k-1)}(\overline{w}_1 - u^{p-1}ct_1) \\ &= u^{p^{k-1}}\big(w_k - (-1)^k u^{pe(k-2)}v^{p^{k-1}}ct_{k-1}\big) - uv^{pe(k-1)}(\overline{w}_1 - u^{p-1}ct_1) \\ &\quad + \sum_{i=1}^{k-2}(-1)^i u^{p^i} v^{p^{i+1}e(k-1-i)}\big(-\overline{w}_{i+1} + (u^{pe(i-1)}v^{p^i}ct_i \\ &\quad + u^{p^{i+1}+pe(i-1)}ct_{i+1})\big), \end{aligned}$$

which equals m_k, and similarly,

$$\begin{aligned} d(w_k) &= -\sum_{i=1}^{k-2}(-1)^i u^{p^i} w_{k-1-i}^{p^{i+1}} \otimes \overline{w}_i^p \eta_R(v) + uw_{k-1}^p \otimes (\overline{w}_1 - u^{p-1}ct_1) \\ &\quad + (-1)^k u^{pe(k-2)}\big(u^{p^{k-1}} w_1^{p^{k-1}} \otimes ct_{k-1} + v^{p^{k-1}} d(ct_{k-1})\big) \\ &= -\sum_{i=1}^{k-2}(-1)^i u^{p^i} w_{k-1-i}^{p^{i+1}} \otimes (-\overline{w}_{i+1} + \underline{u^{pe(i-1)}v^{p^i}ct_i + u^{p^{i+1}+pe(i-1)}ct_{i+1}}) \\ &\quad + uw_{k-1}^p \otimes (\overline{w}_1 - \underline{u^{p-1}ct_1}) \\ &\quad + \underline{(-1)^k u^{e(k-2)}\big(u^{p^{k-1}} w_1^{p^{k-1}} \otimes ct_{k-1} + v^{p^{k-1}} d(ct_{k-1})\big)} = m'_k. \end{aligned}$$

Here, the underlined terms cancel each other if $k \leq n$ by (5.2) and (3.1), with the relation $\Delta(cx) = T(c \otimes c)\Delta(x)$ for the switching map $T\colon \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma$. ∎

We also introduce an element

$$\overline{c}_k = h_{n+k-1} - u^{(p-1)p^k} h_k \in \operatorname{Ext}^1 A.$$

COROLLARY 5.5. *For each $0 < k < n$, the Massey products $\mu_k = \langle u^{p^k}, \overline{c}_k, c_{k-1}, c_{k-2}, \dots, c_1, c_0\rangle$ and $\mu'_k = \langle \overline{c}_k, c_{k-1}, c_{k-2}, \dots, c_1, c_0\rangle$ are defined. In fact, the cocycles m_{k+1} and m'_{k+1} represent elements of the Massey products μ_k and μ'_k, respectively.*

In particular, we have the following.

COROLLARY 5.6. *The Massey product $\langle u^{p^{n-3}}, \overline{c}_{n-3}, c_{n-4}, \dots, c_0\rangle \subset \operatorname{Ext}^1 A$ is defined and contains zero.*

LEMMA 5.7. *The Massey product $\langle \overline{c}_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle \subset \operatorname{Ext}^2 A$ contains zero.*

Proof. The Massey product $\langle \overline{c}_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle$ contains

$$\langle h_{2n-4}, c_{n-4}, \dots, c_0, h_{n-2}\rangle - \langle u^{p^{n-2}-p^{n-3}} h_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle.$$

It suffices to show that the second term contains zero. Indeed, the first term does since a defining system cobounds $u^{e(n-3)}ct_{n-1}^{p^{n-2}}$. Since every Massey product $\langle h_j, h_{j-1}, \dots, h_{i+1}, h_i\rangle$ for $j - i \leq n - 2$ contains zero, all lower products contain zero, and we see that $\xi = \langle h_{n-3}, c_{n-4}, \dots, c_1, c_0, h_{n-2}\rangle$ is defined.

The statement of [4, Theorem 10] itself is applied to our case and says that there are elements $x_k \in \langle c_k, c_{k-1}, \dots, c_0, h_{n-2}, h_{n-3}, c_{n-4}, \dots, c_{k+1}\rangle$ for $0 \le k \le n-4$, $x_{n-3} \in \langle h_{n-3}, c_{n-4}, \dots, c_1, c_0, h_{n-2}\rangle$, and $x_{n-2} \in \langle h_{n-2}, h_{n-3}, c_{n-4}, \dots, c_1, c_0\rangle$ such that $\sum_{k=0}^{n-2} \pm x_k = 0$. Its proof tells us that we may take the elements x_k arbitrarily, and we take x_k so that $x_k = 0$ for $0 \le k \le n-4$ and $x_{n-2} = 0$, whose relations follow from $d(ct_{n-1})$. Therefore, $x_{n-3} = 0$ and the lemma follows. ∎

Corollary 5.8. *The Massey product* $\mu = \langle u^{p^{n-3}}, \overline{c}_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle$ *is defined and contains an element whose leading term is* $v^{e(n-2)} h_{n-2}$.

Lemma 5.9. *The condition (2.6)*$_{(i,j,s)}$ *holds for* $(i,j) = (0, n-2)$.

Proof. If $p \nmid s(s-1)$, it follows from the computation that

$$d(v^s t_1^{p^{n-2}}) \equiv suv^{s-1} t_1^{p^{n-1}} \otimes t_1^{p^{n-2}} + \binom{s}{2} u^2 t_1^{2p^{n-1}} \otimes t_1^{p^{n-2}} \mod (u^3),$$

$$\begin{aligned} d(suv^{s-1} ct_2^{p^{n-2}}) &\equiv s(s-1)u^2 t_1^{p^{n-1}} \otimes ct_2^{p^{n-2}} \\ &\quad - suv^{s-1} t_1^{p^{n-1}} \otimes t_1^{p^{n-2}} \mod (u^3). \end{aligned}$$

Suppose that $s = tp^l + e(n-2)$ with $p \nmid t$ and $l > 0$. Let $\widetilde{\theta}$ denote an element of Corollary 5.8. We take a generator corresponding to $v^s h_{n-2}$ to be $v^{s-e(n-2)}\widetilde{\theta}$. We denote a representative of $\widetilde{\theta}$ by m, which is congruent to $v^{e(n-2)} t_1^{p^{n-2}} + uv^{pe(n-3)} ct_2^{p^{n-2}}$ modulo (u^2). Then, $d(v^{s-e(n-2)}m) = tu^{a_l} v^{s-e(n-2)-p^{l-1}} t_1^{p^{[l-1]}} \otimes m \equiv tu^{a_l} v^{s-p^{l-1}} t_1^{p^{[l-1]}} \otimes t_1^{p^{n-2}}$. This shows the case for $[l] \neq 0, n-2$.

For $[l] = 0$, a similar computation shows that $d(v^{s-e(n-2)}m) \equiv tu^{a_l} \times v^{s-p^{l-1}}(t_1^{p^{n-2}} \otimes t_1^{p^{n-2}} + uv^{-1} t_1^{p^{n-1}+p^{n-2}} \otimes t_1^{p^{n-2}} + uv^{-1} t_1^{p^{n-2}} \otimes ct_2^{p^{n-2}})$, which yields $v^{s-1-p^{l-1}} g_{n-2}$. For $[l] = n-2$, $\widetilde{\theta} h_{n-3} \in u^{e(n-2)}\langle h_{2n-4}, h_{2n-5}, \dots, h_{n-2}, h_{n-3}\rangle = \{u^{e(n-2)+p^{n-3}} b_{2n-5}\}$ in $C(p^{n-2})$. Indeed, $u^{e(n-3)} t_n^{p^{n-3}}$ yields the equality by (3.1). ∎

§6. On the action of α_1 and β_1 on Greek letter elements

In this section, let H^*M for a $BP_*(BP)$-comodule M denote an Ext group $\mathrm{Ext}^*_{BP_*(BP)}(BP_*, M)$. Consider the comodule $N_{k-1}(j) = BP_*/(I_{k-1} + (v_{k-1}^j))$ $(v_0 = p)$, and the connecting homomorphism $\partial_{k,j}$ associated to the short exact sequence $0 \to BP_*/I_{k-1} \xrightarrow{v_{k-1}^j} BP_*/I_{k-1} \to N_{k-1}(j) \to 0$. We abbreviate $\partial_{k,1}$ to ∂_k. Here we consider the Greek letter elements of H^*BP_*/I_{n-1} defined by

$$\overline{\alpha}_t^{(n-1)} = u^t \in H^0BP_*/I_{n-1} \qquad \text{and}$$
$$\alpha_{(t/j)}^{(n)} = \partial_{n,j}(v^t) \in H^1BP_*/I_{n-1} \quad \text{for } v^t \in H^0N_{n-1}(j)$$

for $t > 0$, and

$$\alpha_1 = \partial_1(v_1) = h_0 \in H^1BP_* \qquad \text{and} \qquad \beta_1 = \partial_1\partial_2(v_2) = b_0 \in H^2BP_*.$$

PROPOSITION 6.1. *The elements α_1 and β_1 act on the Greek letter elements as follows:*

$$\alpha_1\overline{\alpha}_t^{(n-1)} \neq 0 \in H^1BP_*/I_{n-1}, \qquad \beta_1\overline{\alpha}_t^{(n-1)} \neq 0 \in H^2BP_*/I_{n-1};$$

and if the Greek letter elements $\alpha_{(sp^i/j)}^{(n)}$ have an internal degree greater than $2(p^n-1)(e(n-1)-1)$, then

$$\alpha_1\alpha_{(sp^i/j)}^{(n)} \neq 0 \in H^2BP_*/I_{n-1} \quad \text{if } [i] \neq 0,\ p \nmid (s+1) \text{ or } p^2 \mid (s+1); \qquad \text{and}$$
$$\beta_1\alpha_{(sp^i/j)}^{(n)} \neq 0 \in H^3BP_*/I_{n-1} \quad \text{if } n \neq 5,\ [i] \neq 1 \text{ or } p \nmid (s+1).$$

In order to prove this, we make a chromatic argument. Let N_k^0 denote the BP_*BP-comodule BP_*/I_k, and put $M_k^0 = v_k^{-1}N_k^0$. We denote the cokernel of the inclusion $N_k^0 \to M_k^0$ by N_k^1, so that $0 \to N_k^0 \to M_k^0 \xrightarrow{\psi} N_k^1 \to 0$ is an exact sequence. Let $\widetilde{\partial}_{k+1}\colon H^sN_k^1 \to H^{s+1}N_k^0$ be the connecting homomorphism associated to the short exact sequence. We notice that $N_k^1 = \operatorname{colim}_j N_k(j)$ with inclusion $\varphi_j\colon N_k(j) \to N_k^1$ given by $\varphi_j(x) = x/u^j$, and that the connecting homomorphism $\partial_{n,j}\colon H^sN_{n-1}(j) \to H^{s+1}N_{n-1}^0$ factorizes to $\widetilde{\partial}_n\varphi_j$.

LEMMA 6.2. *For an element $x_i^s/u^j \in H^0N_{n-1}^1$ for $0 < j \leq a_i$ ($j \leq p^i$ if $s = 1$), α_1 and β_1 act on it as follows:*

$$x_i^s\alpha_1/u^j \neq 0 \in H^1N_{n-1}^1 \quad \text{if } [i] \neq 0,\ p \nmid (s+1) \text{ or } p^2 \mid (s+1); \qquad \text{and}$$
$$x_i^s\beta_1/u^j \neq 0 \in H^2N_{n-1}^1 \quad \text{if } n \neq 5,\ [i] \neq 1 \text{ or } p \nmid (s+1).$$

Proof. A change of rings theorem of Miller and Ravenel [6] shows that the module $H^sM^1_{n-1}$ is isomorphic to $\mathrm{Ext}^s B$. By (1.5), we see that $x_i^s h_0/u \neq 0 \in \mathrm{Ext}^1 B$ unless $[i]=0$, $p \mid (s+1)$, and $p^2 \nmid (s+1)$. This shows the first nontriviality. Similarly, since we have shown that (2.4) is exact, we see that $x_i^s\beta_1/u \neq 0 \in \mathrm{Ext}^2 B$ unless $n=5$, $[i]=1$, and $p \mid (s+1)$. ∎

Lemma 6.3. *Let ξ_1 denote α_1 or β_1, let $x \in H^0N^1_{n-1}$, and suppose that $x\xi_1$ has an internal degree greater than $2(p^{n-1}-1)(e(n-1)-1)$. If $x\xi_1 \in H^sN^1_{n-1} \neq 0$, then $\widetilde{\partial}_n(x)\xi_1 \neq 0 \in H^{s+1}N^0_{n-1}$.*

Proof. It suffices to show that $x\xi_1$ is not in the image of $\psi_*\colon H^sM^0_{n-1} \to H^sN^1_{n-1}$. Again, the change of rings theorem shows that the module $H^sM^0_{n-1}$ is isomorphic to the module of Theorem 1.3 with substituting $n-1$ for n. Note that every generator of it except for ζ_{n-1} belongs to $H^sN^0_{n-1}$, and also is $u^{e(n-1)}\zeta_{n-1}$ (see [13]). It follows that every element of the image of ψ_* has an internal degree no greater than $2(e(n-1)-1)(p^{n-1}-1)$. Thus, the lemma follows. ∎

Proof of Proposition 6.1. The module $H^sM^0_{n-1}$ contains a submodule $k_*\langle h_0\rangle$ if $s=1$ and $k_*\langle b_0\rangle$ if $s=2$. Therefore, the first two relations hold. The other relations follow from Lemmas 6.2 and 6.3. ∎

Proof of Theorem 1.11. Note that $\overline{\alpha}^{(3)}_t = \overline{\gamma}_t = v_3^t$, and we obtain the theorem from Proposition 6.1 at $n=4$. ∎

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