# Vanishing theorems for vector bundles generated by sections

F. Laytimi and D. S. Nagaraj

**Abstract** In this article we give a vanishing result for the cohomology groups $H^{p,q}(X, S^{\nu}E \otimes L)$, where $E$ is a vector bundle generated by sections and $L$ is an ample line bundle on a smooth projective variety $X$. We also give an application related to a result of Barth-Lefschetz type. A general nonvanishing result under the same hypothesis is given to prove the optimality of the vanishing result for some parameter values.

## 1. Introduction

Throughout this article we denote by $X$ a smooth projective variety of dimension $n$ over the field of complex numbers, by $E$ a vector bundle of rank $e$, and by $L$ a line bundle on $X$.

For any integer $\nu \geq 0$, consider the Dolbeault cohomology group of type $H^{p,q}(X, S^{\nu}E \otimes L)$, where $S^{\nu}E$ denotes the $\nu$th symmetric power of $E$.

The classical theorem of Nakano, Akizuki, and Kodaira (see [1, Theorem 1]) gives conditions on parameters $n, e, p, q$ such that the cohomology group vanishes when $\nu = 0$ and $L$ is ample.

When $E$ is generated by sections and $L$ is ample, Le Potier [7], in the $\nu = 1$ case, gave generalized conditions on those parameters for the vanishing of the group.

Peternell, Le Potier, and Schneider [9] in the case $\nu \geq 2$, provided $n - q \leq 1$, gave a condition that assures a vanishing of the cohomology.

In this article, using a vanishing theorem of Laytimi and Nahm [6, Theorem 2.2], we give a condition for the vanishing of the cohomology group when $n - q \geq 2$.

In the particular case $p = n$, we improve the condition of the vanishing of the cohomology groups in question and give an application related to a result of the Barth-Lefschetz type.

In the second part, a general nonvanishing result under the same hypothesis on the vector bundles is given in order to prove the optimality of the obtained vanishing result for some parameter values.

All the results of this article are a generalization of the ones in [9].

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## 2. Vanishing theorem for vector bundles generated by sections

We need to recall some results. Let $\mathbb{N}$ denote the set of all natural numbers.

DEFINITION 2.1

Define the function $\delta : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$ by the following:

$$\binom{\delta(x)}{2} \leq x < \binom{\delta(x)+1}{2};$$

that is, $\delta(0) = 1$, $\delta(1) = \delta(2) = 2$, $\delta(3) = \delta(4) = \delta(5) = 3, \ldots$.

NOTATION 2.2

Let $\alpha \geq 1$, and let $e, n, p, q$ be nonnegative integers, where $p$ (and $q$) $\leq n$. Define the function $Q$ by

$$Q(\alpha, p, q) = r_0(e - 2 + \alpha) + (\alpha - 1)(e - 2)$$

$$\text{with } r_0 = \min\{\delta(n-q), \delta(n-p)\}.$$

THEOREM 2.3 ([6, THEOREM 2.2])

*Let $E$ be a vector bundle of rank $e$, and let $L$ be a line bundle on a smooth projective variety $X$ of dimension $n$. If $\nu \geq 2$ and $S^\nu E \otimes L$ is ample, then we have*

$$H^{p,q}(X, S^\nu E \otimes L) = 0 \quad \textit{for } q + p - n > Q(\nu, p, q),$$

*where $Q$ is defined in Notation 2.2.*

*In particular,*

$$H^{n,q}(X, S^\nu E \otimes L) = H^{q,n}(X, S^\nu E \otimes L) = 0 \quad \textit{for } q > \nu(e-1).$$

We need to use the following.

PROPOSITION 2.4 ([9, PROPOSITION 2.1])

*Let $X$ be a smooth projective variety of dimension $n$, let $E$ be a vector bundle generated by sections, let $F$ be a vector bundle on $X$. Let $k \leq n-1$ be a positive integer. If*

$$H^q(X, S^\nu E \otimes F) = 0 \quad \textit{for } n - q \leq k \textit{ and } 0 \leq \nu \leq k,$$

*then*

$$H^q(X, S^\nu E \otimes F) = 0 \quad \textit{for } n - q \leq k \textit{ and all } \nu \geq 0.$$

Our main result is the following.

THEOREM 2.5

*Let $X$ be a smooth projective variety of dimension $n$, let $E$ be a vector bundle of* *rank $e$ generated by sections, and let $L$ be an ample line bundle on $X$. Then*

$$H^{p,q}(X, S^{\nu}E \otimes L) = 0$$

(i) *if $n - q \leq 1$ for $p + q - n > e - 1$ and all $\nu \geq 1$;*

(ii) *if $n - q \geq 2$ for $p + q - n > Q(\beta, p, q)$, where $\beta = \min\{\nu, n - q\}$ (for the function $Q$; see Notation 2.2).*

*In particular,*

$$H^{n,q}(X, S^{\nu}E \otimes L) = 0 \quad \textit{for } q > \beta(e-1).$$

*Proof*

Part (i) is [9, Theorem 2.2]. For (ii), we apply Proposition 2.4 with $F = L \otimes \Omega_X^p$. Since the function $Q(k, p, q)$, for fixed $p$ and $q$ is increasing in $k$, it suffices according to Proposition 2.4 to prove

$$H^{p,q}(X, S^{\nu}E \otimes L) = 0 \quad \text{for } n - q = k,\ \nu = k.$$

This is true due to Theorem 2.3. □

When $p = n$, one can get a better result than Theorem 2.5.

THEOREM 2.6

*Let $X$ be a smooth projective variety of dimension $n$, let $E$ be a vector bundle of rank $e$ generated by sections, and let $L$ be an ample line bundle on $X$. Then*

$$H^{n,q}(X, S^{\nu}E \otimes L) = 0$$

(i) *if $n - q \leq 1$ for $q \geq e$ and all $\nu \geq 0$,*

(ii) *if $n - q \geq 2$ and $\nu \geq 2$ for*

$$q \geq \min\Big\{\beta(e-1) + 1, \frac{(e-1)n + 1}{e}\Big\},$$

*where $\beta = \min\{\nu, n - q\}$.*

*Proof*

The first part is a particular case of [9, Theorem 2.2].

For the second part, we apply Proposition 2.4 with $F = L \otimes K_X$ and $k = [(n-1)/e]$, where [ ] denotes the integral part. We have, therefore, to verify that

$$H^{n,q}(X, S^{\nu}E \otimes L) = 0 \quad \text{for } q \geq n - k \text{ and } \nu \leq k. \tag{1}$$

But $q \geq n - k$ implies $q \geq (e-1)k + 1 \geq (e-1)\beta + 1$. Hence (1) is a consequence of Theorem 2.3.

To finish the proof, notice that

$$q \geq n - k \Longleftrightarrow q \geq \frac{(e-1)n + 1}{e}.$$

□

Theorem 2.6 in the case $\mathrm{rk}(E) = 2$ was settled in [9, Theorem 2.3].

As an application of Theorem 2.6, we have the following.

THEOREM 2.7

*Let $X$ be a smooth subvariety of dimension $n$ in $\mathbb{P}^N$. Denote by $N_X$ the normal bundle of $X$ in $\mathbb{P}^N$. Then if $k \in \mathbb{Z}$ and $\nu \geq \max\{2, k+1\}$,*

$$H^q\big(X, S^\nu N_X^*(k)\big) = 0$$

(i) *if $q \leq 1$ for $q \leq 2n - N$,*

(ii) *if $q \geq 2$ for $q \leq \max\{(n-1)/(N-n), n-1-\gamma(N-n-1)\}$, where $\gamma = \min\{\nu, q\}$.*

*Proof*

The proof follows from Theorem 2.6 if one uses Serre duality and the fact that

$$S^\nu N_X(-k) = S^\nu\big(N_X(-1)\big) \otimes \mathcal{O}_X(\nu - k).$$

□

REMARK 2.8

For $k \leq 1$, using the amplitude of $N_X$, one has stronger results than Theorem 2.7. Indeed, with the notation

$$f(x) = n - 1 - x(N - n - 1),$$

it is shown in [10] that for $k \leq 0$,

$$H^q\big(X, S^\nu N_X^*(k)\big) = 0 \quad \text{for } q \leq f(1),$$

and in [3], for $k = 1$ the vanishing is obtained for $q \leq f(2)$.

However, in the case $k \geq 2$, Theorem 2.7 gives a new condition that assures a vanishing of the cohomology.

## 3. Nonvanishing theorem for vector bundles generated by sections

The aim of this section is to generalize the nonvanishing results in [9].

We need the following.

LEMMA 3.1 ([9, LEMMA 2.1])

*Let*

$$0 \to F \to F_k \to \cdots \to F_1 \to F_0 \to E \to 0$$

*be an exact sequence of sheaves on a smooth projective variety $X$.*
*Assume that*

$$H^q(X, F_i) = 0 \quad \textit{for } 0 \leq i \leq k,\ q \geq q_0.$$

*Then*

$$H^{q+k+1}(X, F) \simeq H^q(X, E) \quad \textit{for } q \geq q_0.$$

*Proof*

The lemma is proved by induction on $k$ and by cutting the exact sequence into two pieces. $\square$

PROPOSITION 3.2

*Let $X$ be a smooth projective variety of dimension $n$, let $E$ be a vector bundle of rank $e \geq 2$ generated by a vector space $V \subset H^0(X,E)$ of dimension $f+e$ for some positive integer $f$, and let $L$ be an ample line bundle. Consider the exact sequence of vector bundles*

$$(*) \qquad 0 \to F \to V \otimes \mathcal{O}_X \to E \to 0.$$

*Then*

(a) *if $f = 2$, and $\nu \geq 2$,*

$$S^{\nu-2}V \otimes H^{p,q+2}\big(X, (\det E)^* \otimes L\big) \simeq H^{p,q}(X, S^\nu E \otimes L)$$

(b) *if $f > 2$, $2 \leq \nu \leq e$, and either $p \geq q$ or $\nu \leq (\delta(n-q)/2)+1$,*

$$(**) \qquad H^{p,q+\nu}(X, \wedge^\nu F \otimes L) = 0 \quad \text{for } p+q-n > Q(\nu,p,q),$$

*where $Q$ is defined in Notation 2.2.*

*In particular, for $p = n$ we have if $f > 2$, and $2 \leq \nu \leq e$,*

$$H^{n,q+\nu}(X, \wedge^\nu F \otimes L) = 0 \quad \text{for } q > \nu(e-1),$$

(c) *if $f > 2$, $2 \leq \nu \leq e$, and either $p \geq q$ or $\nu \leq (\delta(n-q)/2)+1$,*

$$S^{\nu-f}V \otimes H^{p,q+f}\big(X, (\det E)^* \otimes L\big) \simeq H^{p,q}(X, S^\nu E \otimes L)$$

$$\text{for } p+q-n > Q(f-1,p,q).$$

*Proof*

Result (a) is [9, Proposition 3.1].

Note that, using the exact sequence $(*)$ tensored with $L \otimes \Omega_X^p$ and using the vanishing theorems of Nakano, Akizuki, Kodaira, and Le Potier, we get

$$H^{p,q+1}(X, F \otimes L) = 0 \quad \text{for } p+q-n \geq e.$$

Note also that if $\nu > f$, then the vanishing of $H^{p,q+\nu}(X, \wedge^\nu F \otimes L)$ holds because of $\wedge^\nu F = 0$.

For $\nu \leq f$, we use induction on $\nu$.

Tensor the exact sequence

$$0 \to \wedge^\nu F \to \wedge^{\nu-1} F \otimes V \otimes \mathcal{O}_X \to \cdots \to S^\nu V \otimes \mathcal{O}_X \to S^\nu E \to 0$$

with $L \otimes \Omega_X^p$, and denote

$$\mathcal{F}^{\nu-i} = \wedge^{\nu-i} F \otimes S^i V \otimes L \otimes \Omega_X^p.$$

This gives

$$0 \to \mathcal{F}^{\nu} \to \mathcal{F}^{\nu-1} \to \cdots \to \mathcal{F}^0 \to S^{\nu}E \otimes L \otimes \Omega_X^p \to 0.$$

Let us first treat the case $\nu = 2$.

We have $H^q(\mathcal{F}^0) = 0$ for $p+q-n > 0$ and $H^q(\mathcal{F}^1) = 0$ for $p+q-n > e$; it follows from Lemma 3.1 that for $p+q-n > e$,

$$H^{q+2}(\wedge^2 F \otimes L \otimes \Omega_X^p) \simeq H^q(S^2 E \otimes L \otimes \Omega_X^p),$$

but $H^q(S^2 E \otimes L \otimes \Omega_X^p) = 0$ for $p+q-n > Q(2,p,q)$ by Theorem 2.5, and $Q(2,p,q) \geq e$ since $e \geq 2$.

Now by the induction hypothesis, we have for $i = 1, 2, \ldots, \nu$,

$$H^q(\mathcal{F}^{\nu-i}) = 0 \quad \text{for } p+q-n > H(i),$$

where

$$H(i) = Q(\nu - i, p, q - \nu + i) + \nu - i,$$

$$H(i) = r_{\nu-i}(e - 2 + \nu - i) + (\nu - i - 1)(e - 2) + \nu - i,$$

and

$$r_{\nu-i} = \min\bigl\{\delta(n-p), \delta(n-q+\nu-i)\bigr\}.$$

Since

$$H(i) - H(i+1) = (r_{v-i} - r_{v-i-1})(e + \nu - i - 2) + r_{v-i-1} + e - 1 > 0,$$

we have for $i = 1, 2, \ldots, \nu$,

$$H^q(\mathcal{F}^{\nu-i}) = 0 \quad \text{for } p+q-n > H(1).$$

Now using Lemma 3.1, we get, under the condition $p+q-n > H(1)$,

$$H^{q+\nu}(\wedge^{\nu} F \otimes L \otimes \Omega_X^p) \simeq H^q(S^{\nu} E \otimes L).$$

But $H^q(S^{\nu} E \otimes L) = 0$ for $p+q-n > Q(\nu,p,q)$ by Theorem 2.5.

(1) If $\delta(n-p) \leq \delta(n-q)$ and thus $p \geq q$, then $r_{\nu-1} = r_0 = \delta(n-p)$ and

$$Q(\nu,p,q) - H(1) = (r_0 - 1) + (e - \nu),$$

which is nonnegative if $\nu \leq e$.

(2) If $\delta(n-p) > \delta(n-q)$, then $r_0 = \delta(n-q)$. By assumption, we have

$$\delta(n-q+\nu-1) \leq \delta\Bigl(n - q + \frac{\delta(n-q)}{2}\Bigr) \leq \delta\bigl(n - q + \delta(n-q)\bigr),$$

but $\delta(n-q+\delta(n-q)) = \delta(n-q)+1$ since for any integer $x$,

$$\delta\bigl(x + \delta(x)\bigr) = \delta(x) + 1.$$

This gives $r_{\nu-1} = \delta(n-q)$ or is equal to $\delta(n-q)+1$. If $r_{\nu-1} = \delta(n-q)$ as in (1),

$$Q(\nu,p,q) - H(1) = (r_0 - 1) + (e - \nu).$$

If $r_{\nu-1} = \delta(n-q)+1$,

$$H(1) = (r_0 + 1)(e - 3 + \nu) + (\nu - 2)(e - 2) + \nu - 1.$$

Hence $Q(\nu,p,q)-H(1)=r_0-2(\nu-1)$, but by hypothesis, $r_0-2(\nu-1)$ is non-negative.

Note that if $p=n$, then $r_{\nu-1}=r_0=1$, and

$$Q(\nu,n,q)-H(1)=e-\nu.$$

This finishes the proof of (b).

To prove (c), tensor the exact sequence

$$0\to S^{\nu-f}V\otimes\det F\to\cdots\to S^{\nu-1}V\otimes F\to S^{\nu}V\otimes\mathcal{O}_X\to S^{\nu}E\to 0$$

with $L\otimes\Omega^p_X$ $(\nu\geq f)$, and use the vanishing result $(**)$ in Proposition 3.2 and Lemma 3.1 to get

$$H^{p,q+f}(X,S^{\nu-f}V\otimes\det F\otimes L)\simeq H^{p,q}(X,S^{\nu}E\otimes L)$$

$$\text{for } p+q-n>Q(f-1,p,q).$$

To finish the proof we only have to note that $\det F=(\det E)^*$. □

For $p=n$ one can get this slightly better result.

PROPOSITION 3.3

*Let $X$ be a smooth projective variety of dimension $n$, let $E$ be a vector bundle of rank $e$ generated by a vector space $V\subset H^0(X,E)$ of dimension $f+e$ for some positive integer $f$, and let $L$ be an ample line bundle. Consider the exact sequence of vector bundles*

$$0\to F\to V\otimes\mathcal{O}_X\to E\to 0.$$

*Then*

(a) *if* $2\leq\nu\leq e$ *and* $(n(e-1)+1)/e\geq e+1$,

$$H^{n,q+\nu}(X,\wedge^{\nu}F\otimes L)=0$$

$$\textit{for } q\geq\min\left\{\nu(e-1)+1,\frac{(n-1)(e-1)}{e}+\frac{\nu(\nu-1)}{2}\right\};$$

(b) *if* $\nu\geq f$, $2\leq f-1\leq e$, *and* $(n(e-1)+1)/e\geq e+1$,

$$S^{\nu-f}V\otimes H^{n,q+f}\big(X,(\det E)^*\otimes L\big)\simeq H^{n,q}(X,S^{\nu}E\otimes L)$$

$$\textit{for } q\geq\min\left\{(f-1)e+1,\frac{(n-1)(e-1)}{e}+\frac{f(f-1)}{2}\right\}.$$

*Proof*

For (a) the vanishing of $H^{n,q+\nu}(X,\wedge^{\nu}F\otimes L)$ under the hypothesis $2\leq\nu\leq e$ is done in the particular case of Proposition 3.2(b). For the above vanishing under the hypothesis $(n(e-1)+1)/e\geq e+1$, we use induction on $\nu$.

With the notation

$$\mathcal{G}^{\nu-i}=\wedge^{\nu-i}F\otimes S^iV\otimes L\otimes K_X,$$

we get

$$0 \to \mathcal{G}^{\nu} \to \mathcal{G}^{\nu-1} \to \cdots \to \mathcal{G}^0 \to S^{\nu}E \otimes L \otimes K_X \to 0.$$

For $\nu = 2$, we have $H^q(\mathcal{G}^0) = 0$ for $q > 0$ and $H^q(\mathcal{G}^1) = 0$ for $q > e$; it follows from Lemma 3.1 that for $q \geq e+1$,

$$H^{n,q+2}(\wedge^2 F \otimes L) \simeq H^{n,q}(S^2 E \otimes L).$$

But $H^{n,q}(S^2E \otimes L) = 0$ for $q > (n(e-1)+1)/e$ by Theorem 2.6, and $(n(e-1)+1)/e \geq e+1$ by hypothesis.

Now by the induction hypothesis, with the notation $N_1 = (n(e-1)+1)/e$ we have for $i = 1, 2, \ldots, \nu$,

$$H^q(\mathcal{G}^{\nu-i}) = 0 \quad \text{for } q \geq N_1 - 1 + \frac{(\nu-i)(\nu-i+1)}{2}.$$

We have then, for $i = 1, 2, \ldots, \nu$,

$$H^q(\mathcal{G}^{\nu-i}) = 0 \quad \text{for } q \geq N_1 - 1 + \frac{(\nu-1)\nu}{2}.$$

Now using Lemma 3.1, we get, under the condition $q \geq N_1 - 1 + ((\nu-1)\nu)/2$,

$$H^{q+\nu}\Big(\bigwedge^{\nu} F \otimes L \otimes K_X\Big) \simeq H^q(S^{\nu}E \otimes L \otimes K_X).$$

But $H^q(S^{\nu}E \otimes L \otimes K_X) = 0$ for $q \geq N_1$ by Theorem 2.6. This finishes the proof of (a).

With the notation

$$\mathcal{G}^{f-i} = \bigwedge^{f-i} F \otimes S^{\nu-f-i}V \otimes L \otimes K_X,$$

we get

$$0 \to \mathcal{G}^{f} \to \mathcal{G}^{f-1} \to \cdots \to \mathcal{G}^0 \to S^{\nu}E \otimes L \otimes K_X \to 0.$$

We use the vanishing result of part (a) and Lemma 3.1 to get the wanted isomorphism in (b). □

COROLLARY 3.4

*Let $X$ be a smooth projective variety of dimension $n$, and let $E$ be a vector bundle of rank $e$ generated by a vector space $V \subset H^0(X,E)$ of dimension $f+e$. Assume that* $\det E$ *is ample. Then*

(a) *if $f = 2$ and $\nu \geq 2$,*

$$H^q(X, S^{\nu}E \otimes \det E \otimes \Omega_X^p) \simeq S^{\nu-2}V \otimes H^{q+2}(X, \Omega_X^p)$$

*for $p+q-n \geq e$;*

(b) *if $f > 2$, $2 \leq \nu \leq e$, and either $p \geq q$ or $\nu \leq (\delta(n-q))/2+1$,*

$$H^q(X, S^{\nu}E \otimes \det E \otimes \Omega_X^p) \simeq S^{\nu-f}V \otimes H^{q+f}(X, \Omega_X^p)$$

*for $p+q-n > Q(f-1,p,q)$ (see Notation 2.2);*

(c) *if* $\nu \geq f$, *and* $2 \leq f-1 \leq e$,

$$H^{n-f}(X, S^\nu E \otimes \det E \otimes K_X) \simeq S^{\nu-f}V$$

$$\textit{for } n \geq \min\Big\{e(f-1)+1+f, \frac{(n-1)(e-1)}{e} + \frac{f(f+1)}{2}\Big\}.$$

*Proof*
Part (a) is [9, Corollary 3.2]. Part (b) follows from Proposition 3.2(c) with $L = \det E$. To prove (c) we use Proposition 3.3(b) and the fact that $H^n(\Omega_X^n) \simeq \mathbb{C}$. □

For the sequel we need to recall the following result.

THEOREM 3.5 ([4, THEOREM 2.5])
*Let $X$ be a projective variety. Then for fixed $r \geq 1$ and $k \geq 0$, there is a natural bijection between the following two sets:*

(1) *the set of morphisms $f: X \longrightarrow \mathrm{Gr}(r, \mathbb{C}^{r+k})$ with $f$ finite (onto its image);*
(2) *the set of equivalence classes of vector bundle surjections*

$$\mathbb{C}^{r+k} \otimes \mathcal{O}_X \longrightarrow E$$

*with* $\mathrm{rank}(E) = r$ *such that* $\det(E)$ *is ample.*

EXAMPLE 3.6
Let $e, f$ be integers such that $3 \leq f \leq e-1$, and let

$$\mathrm{Gr}(e, \mathbb{C}^{f+e})$$

be the Grassmann manifold of $e$-dimensional quotient spaces of $\mathbb{C}^{f+e}$. Let $Q$ be the universal quotient bundle of rank $e$ on $\mathrm{Gr}(e, \mathbb{C}^{f+e})$. Let $Y$ be a nonsingular subvariety of $\mathrm{Gr}(e, \mathbb{C}^{f+e})$ of dimension greater than or equal to $e(f-1)+f+1$. Let $g: X \to Y$ be a finite surjective morphism, and assume that $X$ is nonsingular. Then the bundle $E = g^*(Q)$ satisfies the assumption of Corollary 3.4. Hence for any $\nu \geq f$, the cohomology group

$$H^{n-f}(X, S^\nu E \otimes \det E \otimes K_X) \neq 0.$$

Conversely, let $X$ be a smooth variety, and let $E$ be vector bundle on $X$ satisfying the hypothesis of Corollary 3.4 (i.e., the conditions that the vector bundle be generated by an $(e+f)$-dimensional subspace of $H^0(X, E)$, where $e = \mathrm{rank}(E)$, and that $\det(E)$ be ample). Then by Theorem 3.5 we get a finite (onto its image) morphism

$$g: X \to \mathrm{Gr}(e, \mathbb{C}^{f+e})$$

such that $g^*(Q) = E$. Note that $\dim(X) \leq ef$.

REMARK 3.7
In the above example, if we take $X$ to be $\mathrm{Gr}(e, \mathbb{C}^{f+e})$ and $L = \det(E)$, then we see that Theorem 2.5(ii) is optimal when $\beta$ in this theorem is equal to $n - q = f$.

We next show that for $X = \mathrm{Gr}(e, \mathbb{C}^{f+e})$ in the above example, the nonvanishing cohomology group can be independently computed using Bott's formula.

PROPOSITION 3.8

*Fix positive integers $f, r$. Let $\mathrm{Gr}(r,d)$ be the Grassmannian of all the codimensional $r$-subspaces of a vector space $V$ of dimension $d = f + r$.*

*Let $Q$ be the universal quotient bundle of rank $r$ on $X = \mathrm{Gr}(r,d)$.*

*Then for $\nu \geq f$,*

$$H^{n-f}(X, S^\nu Q \otimes \det Q \otimes K_X) = (\det V)^{\otimes f} \otimes S^{\nu - f} V.$$

*Proof*

For the universal subbundle on $\mathrm{Gr}(r,d)$, denoted by $S'$, we have $K_{\mathrm{Gr}(r,d)} = ((\det Q)^*)^{\otimes d} = (\det S')^{\otimes d}$. Thus

$$H^q\big(\mathrm{Gr}(r,d), S^\nu Q \otimes \det Q \otimes K_{\mathrm{Gr}(r,d)}\big) = H^q\big(\mathrm{Gr}(r,d), S^\nu Q \otimes (\det S')^{\otimes d-1}\big).$$

Let $\mathcal{S}_a$ be the Schur functor of the partition $a$ (for a definition, see [2, §6.1]). By a corollary of Bott's formula [8, Corollary 1], we have

$$H^q\big(\mathrm{Gr}(r,d), S^\nu Q \otimes (\det S')^{\otimes d-1}\big) = \delta_{q, i((a,b)-c(d))} \mathcal{S}_{\psi(a,b)} V,$$

where

$$a = (\nu, \underbrace{0, 0, \ldots, 0}_{r-1 \text{ times}}), \qquad b = (\underbrace{d-1, d-1, \ldots, d-1}_{d-r \text{ times}});$$

for a sequence $v = (v_1, v_2, \ldots)$,

$$i(v) = \mathrm{card}\big\{(i,j) \;\big|\; i < j, v_i < v_j\big\}.$$

For any sequence $u$, $\psi(u) = (u - c(d))^{\geq} + c(d)$, where

$$c(d) = (1, 2, \ldots, d),$$

and $(w)^{\geq}$ is the sequence obtained by rearranging the terms of the sequence $w$ in weakly decreasing order,

$$(a,b) = (\nu, \underbrace{0, 0, \ldots, 0}_{r-1 \text{ times}}, \underbrace{d-1, d-1, \ldots, d-1}_{d-r \text{ times}}).$$

$$\big((a,b) - c(d)\big) = (\nu - 1, -2, -3, \ldots, -r, f-2, f-3, \ldots, 0, -1),$$

we get $i((a,b) - c(d)) = f(r-1) = n - f$, and $\psi(a,b) = (\nu, \underbrace{f, f, \ldots, f}_{d-1 \text{ times}})$.

Thus $\mathcal{S}_{\psi((a,b)} V = (\det V)^{\otimes f} \otimes S^{\nu - f} V$. □

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## References

[1] Y. Akizuki and S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. Japan Acad. **30** (1954), 266–272.

[2] W. Fulton and J. Harris, *Representation Theory: A First Course*, Grad. Texts in Math. **129**, Springer, New York, 1991.

[3] F. Laytimi and D. S. Nagaraj, *Barth type vanishing theorem*, Geom. Dedicata **141** (2009), 87–92.

[4] ———, "Vector bundles generated by sections and morphisms to Grassmanian" in *Quadratic Forms, Linear Groups, and Cohomology*, Dev. Math. **18**, Springer, New York, 2010.

[5] F. Laytimi and W. Nahm, *A generalization of Le Potier's vanishing theorem*, Manuscripta Math. **113** (2004), 165–189.

[6] ———, *Vanishing theorems for product of exterior and symmetric powers*, preprint, arXiv:math.AG/9809064v2 [math.AG].

[7] J. Le Potier, *Annulation de la cohomologie à valeurs dans un fibré vectoriel holomorphe positif de rang quelconque*, Math. Ann. **218** (1975), 35–53.

[8] L. Manivel, *Un théorème d'annulation pour les puissances extérieures d'un fibré ample*, J. Reine Angew. Math. **422** (1991), 91–116.

[9] T. Peternell, J. Le Potier, and M. Schneider, *Vanishing theorems, linear and quadratic normality*, Invent. Math. **87** (1987), 573–586.

[10] M. Schneider and J. Zintl, *The theorem of Barth-Lefschetz as a consequence of Le Potier's vanishing theorem*, Manuscripta Math. **80** (1993), 259–263.

*Laytimi*: Mathématiques, Université Lille 1, F-59655 Villeneuve d'Ascq Cedex, France; fatima.laytimi@math.univ-lille1.fr

*Nagaraj*: Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India; dsn@imsc.res.in