The complex cobordism of BSO_n

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Abstract In this article, we compute $MU^*(BSO(2m))$ and show that it is generated as an MU^*-algebra by Conner-Floyd Chern classes c_i and one $2m$-dimensional element y_m. The case $BO(n)$ was studied by W. S. Wilson, and the case $BSO(2m+1)$ is derived directly from the result. We obtain the result for $BSO(2m)$ by using (equivariant) stratification methods introduced to compute Chow rings by Guillot, Molina, Vezzosi, and Vistoli.

1. Introduction

The complex cobordism of the classifying space of the nth orthogonal group was computed by W. S. Wilson [Wi], which is the simplest possible result that we can expect:

$$MU^*(BO_n) \cong MU^*[[c_1, \ldots, c_n]]/(c_1 - c_1^*, \ldots, c_n - c_n^*),$$

where c_k is the Conner-Floyd Chern class of complexification map $O(n) \to U(n)$ and c_k^* is the Chern class of the conjugate of the map.

The next problem is the case BSO_n. When n is odd, there is an isomorphism $O_n \cong SO_n \times \mathbb{Z}/2$, and we get $MU^*(BSO_{\mathrm{odd}})$ directly from Wilson's result,

$$MU^*(BSO_{2m+1}) \cong MU^*(BO_{2m+1})/(F_1),$$

where F_1 is the image of c_1 under $B\det^* : MU^*(B\mathbb{Z}/2) \to MU^*(BO_{2m+1})$.

Kono and Yagita [KY] and Inoue [In] computed $MU^*(BSO_{2n})$ for $n \le 3$ by using the Atiyah-Hirzebruch spectral sequence. The results are simple, but the Atiyah-Hirzebruch spectral sequence is very complicated even when $n = 3$ (see [In]).

On the other hand, Totaro [To] showed that for algebraic groups G, the classifying spaces BG are approximated by algebraic varieties. Molina and Vistoli [MVi] computed Chow rings $CH^*(BG)$ for classical groups G (e.g., GL_n, O_n, SO_n, $\ldots$) by using the stratification method introduced by Vezzosi [Ve]. Applying this method to $MU^*(-)$ theory (while we do not use results of algebraic geometry), we get the following theorems.

THEOREM 1.1
There is an element $y_m \in MU^{2m}(BSO_{2m})$ *with* $y_m^2 = (-1)^m 2^{2m-2} c_{2m} \bmod (v_1, \ldots)$

Kyoto Journal of Mathematics, Vol. 50, No. 2 (2010), 307–324
DOI 10.1215/0023608X-2009-015,
Received October 26, 2007. Revised July 6, 2009. Accepted October 29, 2009.
Mathematics Subject Classification: Primary 55N22, 57R75; Secondary 55N91, 57R85.

such that there is an MU^*-algebra isomorphism

$$MU^*(BSO_{2m}) \cong MU^*[[c_2, c_4, \ldots, c_{2m}]]\{y_m\} \oplus MU^*(BO_{2m})/(F_1)$$

with $c_{2i-1}y_m = 0 \bmod (v_1, \ldots)$ *for* $1 \le i \le m$.

We also prove

$$K(s)^*(BSO_{2m}) \cong K(s)^* \otimes_{MU^*} MU^*(BSO_{2m})$$

for the Morava K-theory $K(s)^*(X)$ for each $s \ge 0$. Hence from the main result of [RWY], we have the following.

THEOREM 1.2
The Küneth formula holds for all $n_i \ge 1$ *and* $1 \le i \le s$

$$MU^*(BSO_{n_1} \times \cdots \times BSO_{n_s}) \cong MU^*(BSO_{n_1}) \hat{\otimes}_{MU^*} \cdots \hat{\otimes}_{MU^*} MU^*(BSO_{n_s}).$$

Let $\Omega^*(X)$ be the algebraic cobordism defined by Levine and Morel [LM1,2] and $MGL^{2*,*}(X)$ be the $(2*,*)$-dimensional parts of $MGL^{*,*}(X)$ ([MoVo], [Vo]) the motivic cobordism defined by Voevodsky.

THEOREM 1.3
For all $n \ge 1$, *there are isomorphisms*

$$\Omega^*(BSO_n) \cong MGL^{2*,*}(BSO_n) \cong MU^{2*}(BSO_n).$$

In particular, we see that Totaro's conjecture [To, Introduction]

$$MU^{2*}(BG) \otimes_{MU^*} \mathbb{Z} \cong CH^*(BG)$$

holds for $G = SO_n$, while $CH^*(BSO_n)$ itself is computed by R. Field ([Fi], [Pa]) and recomputed by Molina and Vistoli by using the stratification methods.

In this article we use BP-theory assuming $p = 2$ instead of MU-theory. Indeed, there is the isomorphism $MU^*(X)_{(p)} \cong MU^*_{(p)} \otimes_{BP^*} BP^*(X)$.

Section 2 is a brief introduction of the stratification method (for Chow rings) by Molina and Vistoli. Section 3 is the application of this method for BP-theory when $X = BO_n$. The BP-theories for cases BSO_n are studied in §4 and §5 when $n =$ odd and $n =$ even, respectively. Morava K-theory of BSO_n is studied in §6. In §7, we note some results of BP-orientability as applications of the preceding sections.

2. Stratification method

We recall in this section the arguments by Molina and Vistoli [MVi] (see also [Gu], [Vi]). For a smooth algebraic set X over a field k of $\mathrm{ch}(k) = 0$, let $A^*(X) = CH^*(X)$ be the Chow ring generated by algebraic cycles modulo rational equivalence. Let G be an algebraic group over k. Suppose that G acts on X. Let $A_G^*(X)$ be the equivariant Chow ring (the Borel cohomology) defined by

Edidin and Graham [EG] (and by Totaro [To]) as follows. For each $i \geq 0$, choose a representation V of G with an open algebraic set U on which G acts freely, and $\operatorname{codim}_V(V-U) > i$. Then the quotient $(U \times X)/G$ exists as a smooth algebraic space, and we can define

$$A^i_G(X) = A^i\big((U \times X)/G\big).$$

This definition is independent of the choice of such V and U.

Of course, we identify $A^*_G = A^*_G(\mathrm{pt}) = A^*(BG)$. For a subgroup H of G, by the definition we see

$$A^*_G\big((X \times G)/H\big) \cong A^*_H(X).$$

One of the most important properties for $A^*_G(-)$-theory is the localization exact sequence; if Y is a closed G-equivariant algebraic subset of X of codimension s and $i: Y \subset X$ and $j: X - Y \subset X$ are the inclusions, then the following sequence is exact:

$$A^{*-s}_G(Y) \xrightarrow{i_*} A^*_G(X) \xrightarrow{j^*} A^*_G(X-Y) \to 0.$$

R. Field [Fi] computed the Chow ring of BSO_{2m}.

THEOREM 2.1

(Field) The Chow ring $A^*_{SO_{2m}} = CH^*(BSO_{2m})$ is isomorphic to

$$\mathbb{Z}[c_2, c_3, \dots, c_{2m}, y_m]/\big(y_m^2 - (-1)^m 2^{2m-2} c_{2m}, 2c_{\mathrm{odd}}, y_m c_{\mathrm{odd}}\big).$$

By using a Vezzosi stratification method (see [Ve]) Molina and Vistoli [MVi] give a very clear explanation of A^*_G for classical groups G; the outline of their arguments for $G = SO_{2m}$ is as follows.

Let $G = SO_n$, $n = 2m$. Recall that the (split) special orthogonal group SO_n is defined as the subgroup of SL_n generated by elements that preserve the quadratic form

$$q(x_1e_1 + \cdots + x_ne_n) = x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{2m}^2$$

for the basis $e_1, \dots, e_n$ of $V = \mathbb{A}^n$. Hence the sets

$$B = \big\{x \in \mathbb{A}^n \bigm| q(x) \neq 0\big\}, \qquad C = \big\{x \in \mathbb{A}^n - \{0\} \bigm| q(x) = 0\big\},$$

and $\mathbb{A}^n - \{0\}$ are all SO_n-invariant (and O_n-invariant) sets.

Thus we have the localization exact sequences

$$(1) \quad A^{*-n}_G(\{0\}) \xrightarrow{i_{1*}} A^*_G(\mathbb{A}^n) \longrightarrow A^*_G(\mathbb{A}^n - \{0\}) \to 0,$$

$$(2) \quad A^{*-1}_G(C) \xrightarrow{i_{2*}} A^*_G(\mathbb{A}^n - \{0\}) \longrightarrow A^*_G(B) \to 0.$$

Let $\mathbb{G}_m = \mathbb{A}^* = \mathbb{A} - \{0\}$ be the multiplicative group. The group $\mathbb{G}_m \times_{\mathbb{Z}/2} SO_n$ acts on B via $x \mapsto ks(x)$ for $(k,s) \in \mathbb{G}_m \times_{\mathbb{Z}/2} SO_n$ identifying $(k,s) = (-k,-s)$. The stabilizer of e_1 in B for this action is isomorphic to the group SO_{n-1}. Hence it is proven (the detailed proof is given for O_n in [MVi]) that

$$B \cong (\mathbb{G}_m \times_{\mathbb{Z}/2} SO_n)/(SO_{n-1}) \cong (\mathbb{G}_m \times SO_n)/(\mathbb{Z}/2 \times SO_{n-1}).$$

(We also note that $B \cong (\mathbb{G}_m \times_{\mathbb{Z}/2} O_n)/O_{n-1}$.) Hence we have the isomorphism

$$A^*_{SO_n}(B) \cong A^*_{SO_n}\big((\mathbb{G}_m \times SO_n)/(\mathbb{Z}/2 \times SO_{n-1})\big) \cong A^*_{\mathbb{Z}/2\times SO_{n-1}}(\mathbb{G}_m).$$

By using the facts that $\mathbb{G}_m \cong \mathbb{A} - \{0\}$ and $A^*_{\mathbb{Z}/2} \cong \mathbb{Z}[y]/(2y)$ and using the localization sequence again, we can prove (see [MVi])

$$A^*_{SO_n}(B) \cong A^*_{\mathbb{Z}/2\times SO_{n-1}}(\mathbb{G}_m) \cong A^*_{SO_{n-1}}.$$

Next, consider $A^*_{SO_n}(C)$. The stabilizer of the pair (e_1, e_{m+1}) is isomorphic to SO_{n-2}, and the action is transitive. Consider another basis $e'_i = 1/2(e_i + e_{m+i})$, $e'_{m+i} = 1/2(e_i - e_{m+i})$ for $1 \le i \le m$, so that $e'_i, e'_{m+i} \in C$ and

$$q(x_1e'_1 + \cdots + x_ne'_n) = x_1x_{m+1} + x_2x_{m+2} + \cdots + x_mx_{2m}.$$

The stabilizer of the one point e'_1 contains elements in SO_n which are represented by transformations

$$e'_{m+1} \mapsto e''_{m+1} = e'_{m+1} - \Big(\sum_{2\le i\le m} a_ia_{m+i}\Big)e'_1 + \sum_{j\ne 1,m+1} a_je'_j,$$

$$e'_1 \mapsto e'_1, \quad e'_i \mapsto -a_{i\pm m}e'_1 + e'_i \ (i \ne 1, m+1),$$

on C; indeed, $q(e''_{m+1}) = 0$ and $e''_{m+1} \in C$. Thus it is proven that (see [MVi, §4])

$$C \cong SO_n/(\mathbb{A}^{n-2} \rtimes SO_{n-2}),$$

where $\rtimes$ means the semidirect product. Since $A^*_{\mathbb{A}^{n-2}\rtimes G} \cong A^*_G$, we have the isomorphisms

$$A^*_{SO_n}(C) \cong A^*_{\mathbb{A}^{n-2}\rtimes SO_{n-2}} \cong A^*_{SO_{n-2}}.$$

Moreover, we know that $y_m = -i_{2*}(y_{m-1})$ by [MVi, Lemma 5.5] and $i_{1*}(1) = c_n$. By induction, we see that A^*_G is multiplicatively generated by $c_2, \dots, c_n, y_m$. Then Field's theorem is proved by considering restriction to $A^*_{T_G}$ for the maximal torus T_G of G.

These arguments work for $\Omega^*(X)$, the algebraic cobordism defined by Levine and Morel (see [LM1], [LM2]), or $MGL^{2*,*}(X)$, the $(2*,*)$-dimensional parts of $MGL^{*,*}(X)$ (see [MoVo], [Vo]), the motivic cobordism defined by Voevodsky. It is still known (see [LM2]) that

$$\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \cong CH^*(X),$$

and we may not have new information directly from the above arguments. However, if we can show the main theorem, Theorem 1.1, we then get Theorem 1.3 immediately.

Next, consider the case $BP^*(-)$, the Brown-Peterson cohomology. In general, $BP^{\mathrm{odd}}(X) \ne 0$, and there does not exist the localization exact sequence. (In general, j^* is not epic.) Moreover, $BP^{\mathrm{odd}}_{\mathbb{Z}/2\times SO_{n-1}}(\mathbb{G}_m) \ne 0$. However, we prove the main theorem by using the assumption that $BP^{\mathrm{odd}}_{SO_{n'}} = 0$ for $n' < n$ and $BP^*_{SO_{n'}}$ is 2-torsion free, in the next sections.

3. BP-theories of BO_n

In this section, we apply the stratification methods to BP^*-theory for $G = O_n$ by using the result of Wilson [Wi]. Of course, we consider the case $k = \mathbb{A} = \mathbb{C}$, the complex number field for $BP^*(BG)$. Moreover, there is Totaro's cycle map $\tilde{cl}$ (see [To]) such that the composition

$$CH^*(X)_{(p)} \xrightarrow{\tilde{cl}} BP^{2*}(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \xrightarrow{\rho} H^{2*}(X)_{(p)},$$

with the Thom map ρ, is the usual cycle map. (We will see that $\tilde{cl}$ are isomorphic for cases $X = BO_n, BSO_n$.)

REMARK

Totaro began to study $CH^*(BG)$ to show that the cycle map is not injective in general. He first showed that for $X = BSO(4)$ the mod 2 cycle map is not injective by using the result for $BP^*(SO(4))$ in [KY]; indeed, $\rho_{\mathbb{Z}/2}(y_2) = 0$ in $H^*(BSO(4);\mathbb{Z}/2)$.

For a compact Lie group G, we mainly consider its complexification $G_{\mathbb{C}}$ but not G itself. In fact, G is a maximal compact subgroup of $G_{\mathbb{C}}$, and we have the homotopy equivalence $G \cong G_{\mathbb{C}}$ and $BG \cong BG_{\mathbb{C}}$. Hence, hereafter in this article, the group G always means its complexification $G_{\mathbb{C}}$ but not the original (real) Lie group.

For example, SO_n is identified as the subgroup of $SL_n(\mathbb{C})$ generated by matrices A with $A^tA = I_n$, where A^t is the transposed matrix. Namely, A are matrices that preserve the quadratic form

$$q'(x_1e_1 + \cdots + x_ne_n) = x_1^2 + \cdots + x_n^2$$

for the basis $e_1, \dots, e_n$ of $\mathbb{C}^n$, as described in §2. Of course, these forms q' and q given in §2 are isomorphic over $\mathbb{C}$ but not over $\mathbb{R}$. In topology, SO_n usually means $SO_n(q')_{\mathbb{R}}$, the orthogonal group defined by q' over $\mathbb{R}$. We still know the homotopy equivalences

$$BSO_n(q')_{\mathbb{R}} \cong BSO_n(q')_{\mathbb{C}} \cong BSO_n(q)_{\mathbb{C}}.$$

The group $SO_n(q)_{\mathbb{C}}$ is written simply by SO_n in this article. However, note that it is unknown whether $CH^*(BSO_n(q'))$ for $k = \mathbb{R}$ is isomorphic or not to $CH^*(BSO_n(q))$ given in Theorem 2.1 for $n = 2 \bmod (4)$ (see [MVi, Remark 5.4]).

The topological counter part of the localization exact sequence given in §2 is the following long exact sequence. Let Y be a closed G-complex submanifold of G-complex manifold X of codimension s. It is well known that each complex bundle is $MU^*(-)$ orientable (see [Sw, page 400]), and so it is $BP^*(-)$-orientable. Hence we have the Thom isomorphism

$$BP^{*-2s}(Y) \cong BP^*\big(\mathrm{Th}_Y(X)\big) \cong BP^*\big(X/(X-Y)\big),$$

where $\mathrm{Th}_Y(X)$ is the Thom space for the normal bundle induced from $Y \subset X$. By the definition $BP_G^*(X) = BP^*((E_G \times X)/G)$, its G-equivariant version follows from the nonequivariant version. Thus we have the long exact sequence

$$\to BP_G^{*-2s}(Y) \xrightarrow{i_*} BP_G^*(X) \xrightarrow{j^*} BP_G^*(X-Y) \to BP_G^{*-2s+1}(Y) \to \cdots .$$

By Wilson's result, we know that $BP_{O_n}^{\mathrm{odd}} = 0$. The BP-version of the exact sequence (1) in §2 is given by

$$(1)' \quad 0 \to BP_{O_n}^{2*-1}(\mathbb{C}^n - \{0\}) \to BP_{O_n}^{2*-2n}(\{0\})$$

$$\xrightarrow{c_n} BP_{O_n}^{2*}(\mathbb{C}^n) \to BP_{O_n}^{2*}(\mathbb{C}^n - \{0\}) \to 0.$$

Next, we study the BP-version of the exact sequence (2) in §2. As for $C = \{x \in \mathbb{C}^n - \{0\} \mid q(x) = 0\}$, we have the similar results

$$(*) \quad BP_{O_n}^*(C) \cong BP_{\mathbb{C}^{n-2} \rtimes O_{n-2}}^* \cong BP_{O_{n-2}}^*.$$

As for $B = \{x \in \mathbb{C}^n - \{0\} \mid q(x) \neq 0\}$, we have the isomorphism

$$BP_{O_n}^*(B) \cong BP_{\mathbb{Z}/2 \times O_{n-1}}^*(\mathbb{C}^*)$$

from the the isomorphism for $A_{SO_n}^*(B)$, similarly. This isomorphism induces the long exact sequence

$$\to BP_{O_n}^{*-1}(B) \to BP_{\mathbb{Z}/2 \times O_{n-1}}^{*-2}(\{0\}) \xrightarrow{i_*} BP_{\mathbb{Z}/2 \times O_{n-1}}^*(\mathbb{C}) \xrightarrow{j^*} BP_{O_n}^*(B) \to \cdots .$$

Here we recall that

$$BP^*(B\mathbb{Z}/2) \cong BP^*[[y]]/\big([2](y)\big) \quad \text{with } |y| = 2,$$

where $y = c_1$; the first Chern class of the induced bundle from the natural inclusion $\mathbb{Z}/2 \subset \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$, and

$$[2](y) = 2y + v_1 y^2 + \cdots + \in BP^*[[y]]$$

is the sum of the formal group law for BP^*-theory. Since this BP^*-module satisfies the condition of the Landweber exact functor theorem (see [KY]) we know that

$$BP_{\mathbb{Z}/2 \times O_{n-1}}^* \cong BP_{O_{n-1}}^*[[y]]/\big([2](y)\big).$$

We also see that $i_*(x) = y \cdot x$ in the above exact sequence. Hence we have the isomorphisms

$$(**) \quad BP_{O_n}^*(B) \cong \begin{cases} BP_{O_{n-1}}^*[[y]]/([2](y), y) \cong BP_{O_{n-1}}^* & \text{for } * = \text{even}, \\ BP_{O_{n-1}}^{*-1}\{[2](y)/y\} \cong BP_{O_{n-1}}^{*-1} & \text{for } * = \text{odd}. \end{cases}$$

The BP-version of the exact sequence (2) is written as

$$\to BP_{O_n}^{2*-1}(\mathbb{C}^n - \{0\}) \to BP_{O_n}^{2*-1}(B)$$

$$\to BP_{O_n}^{2*-2}(C) \xrightarrow{i_{2*}} BP_{O_n}^{2*}(\mathbb{C}^n - \{0\}) \to BP_{O_n}^{2*}(B) \to \cdots .$$

From the isomorphisms $(*)$, $(**)$, we have

$$(2)' \quad 0 \to BP_{O_n}^{2*-1}(\mathbb{C}^n - \{0\}) \to BP_{O_{n-1}}^{2*-2}$$

$$\to BP_{O_{n-2}}^{2*-2} \xrightarrow{i_{2*}} BP_{O_n}^{2*}(\mathbb{C}^n - \{0\}) \to BP_{O_{n-1}}^{2*} \to 0.$$

LEMMA 3.1

We have $BP^{2*}_{O_n}(\mathbb{C}^n - \{0\}) \cong BP^{2*}_{O_{n-1}}$, *and* $i_{2*} = 0$ *in (2)'.*

Proof

From $(1)'$ and $(2)'$, we see the existence of epimorphisms

$$BP^*_{O_n}/(c_n) \to BP^*_{O_n}(\mathbb{C}^n - \{0\}) \to BP^*_{O_{n-1}}.$$

By Wilson, we still know that $BP^*_{O_n}/(c_n) \cong BP^*_{O_{n-1}}$. Hence we have the first isomorphism. Hence $i_{2*} = 0$ in $(2)'$. □

For ease of notation, let us write

$$\mathrm{Ker}(c_n)|BP^{2*}_{O_n} = \mathrm{Ker}(\times c_n : BP^{2*}_{O_n} \to BP^{2*+2n}_{O_n}),$$

$$BP^*_{O_n}(c_n) = \mathrm{Ideal}(c_n) \subset BP^*_{O_n}.$$

COROLLARY 3.2

We have $BP^*_{O_{n-1}}(c_{n-1}) \cong \mathrm{Ker}(c_n)|BP^*_{O_n}$.

Proof

Consider the maps in $(2)'$,

$$BP^{2*-2}_{O_{n-1}} \xrightarrow{j} BP^{2*-2}_{O_{n-2}} \xrightarrow{i_{2*}} BP^{2*}_{O_n}(\mathbb{C}^n - \{0\}).$$

Here $j(c_{n-1}) = c_{n-1}j(1) = 0$ since all maps in $(2)'$ are those of $BP^*_{O_n}$-algebras. Hence $BP^*_{O_{n-1}}(c_{n-1}) \subset \mathrm{Ker}(j)$. So j is decomposed as

$$BP^{2*}_{O_{n-1}} \to BP^{2*}_{O_{n-1}}/(c_{n-1}) \xrightarrow{j''} BP^{2*}_{O_{n-2}}.$$

Here note that $i_{2*} = 0$ and j is epic. So j'' is epic and hence is isomorphic also. Thus we see that $\mathrm{Ker}(j) = BP^*_{O_{n-1}}(c_{n-1})$.

From $(1)'$ and $(2)'$ again, we have

$$BP^*_{O_{n-1}}(c_{n-1}) \cong BP^{2*-1}_{O_n}(\mathbb{C}^n - \{0\}) \cong \mathrm{Ker}(c_n)|BP^*_{O_n}.$$

□

Here we recall the arguments and results of Kriz [Kr].

LEMMA 3.3 ([KR, THEOREM 6.2, LEMMA 6.3])

There is an element $q_n \in BP^*_{O_n}$ *such that*

$$c_n - c_n^* = c_n q_n \qquad \textit{and} \qquad q_{n-1} = 2 - q_n \mod(c_n).$$

Moreover, we have the isomorphism

$$BP^*_{O_n}(c_n) \cong BP^*[[c_1, \ldots, c_n]]/(c_1 - c_1^*, \ldots, c_{n-1} - c_{n-1}^*, q_n)\{c_n\}.$$

Proof of the second equation
Let $i=[-1]$ be the inverse map of the formal group laws over BP^*. Then

$$c_n^* = i(x_1)\cdots i(x_n),$$

identifying $c_n = x_1\cdots x_n$ over n variables x_i. Of course,

$$i(x) = -x - v_1x^2 + \cdots + \in BP^*[[x]].$$

So c_n divides c_n^*, and we put $t_n = c_n^*/c_n$. Consider the map fixing $x_1,\ldots,x_{n-1}$ and sending x_n to zero. Then $i(x_j)/x_j$ remains fixed for $j<n$ but $i(x_n)/x_n$ is sent to -1. Thus we get

$$t_{n-1} = -t_n \quad \mathrm{mod}(x_n).$$

Since $c_n - c_n^* = (1-t_n)c_n$, we get the desired equation identifying $q_n = (1-t_n)$. □

REMARK
S. Wilson also computed q_n in [Wi]:

$$q_n = \sum v_i s_{2^i-1} \quad \mathrm{mod}(2, v_1, \ldots)^2,$$

where $s_{2^i-1} = \sum x_i^{2^i-1}$ identifying $c_j = \sum x_{i_1}\cdots x_{i_j}$.

As a $BP^*_{O_{n-1}}$-module, $\mathrm{Ker}(c_n) \mid BP^*_{O_n}$ is generated by only one element q_n. Hence the isomorphism $BP^*_{O_{n-1}}(c_{n-1}) \cong \mathrm{Ker}(c_n) \mid BP^*_{O_n}$ in Corollary 3.2 is explicitly written by $xc_{n-1} \mapsto \lambda x q_n$ for $x \in BP^*_{O_{n-1}}$, where $0 \neq \lambda \in BP^*_{O_n}$ is a unit.

LEMMA 3.4
For $x \in BP^*_{O_{n-1}}$, the map

$$BP^*_{O_{n-1}}(c_{n-1}) \xrightarrow{\cong} \mathrm{Ker}(c_n) \mid BP^*_{O_n} \to BP^*_{O_{n-1}}$$

given by $xc_{n-1} \mapsto xq_n \mapsto xq_n \,\mathrm{mod}(c_n)$ is injective.

Proof
Let $0 \neq xc_{n-1} \in BP^*_{O_{n-1}}(c_{n-1})$. Consider the element

$$c_{n-1}xq_n = c_{n-1}x(2-q_{n-1}) \quad \mathrm{mod}(c_n).$$

Since $c_{n-1}q_{n-1} = 0$ in $BP^*_{O_{n-1}}$, the above element is $2c_{n-1}x$ in $BP^*_{O_{n-1}}$. But $BP^*_{O_{n-1}}$ is 2-torsion free (see [KY], [Kr]), and so $2c_{n-1}x \neq 0 \in BP^*_{O_{n-1}}$. Hence $c_{n-1}xq_n \neq 0 \in BP^*_{O_{n-1}}$, and so $xq_n \neq 0 \in BP^*_{O_{n-1}}$. Thus we get the desired result. □

Since $BP^*_{O_n}/(c_n) \cong BP^*_{O_{n-1}}$, we have the following corollary.

COROLLARY 3.5
We have $\mathrm{Ker}(c_n) \mid BP^*_{O_n} \cap BP^*_{O_n}(c_n) = 0$.

4. BP-theories of BSO_{odd}

In this section we consider the BP-theory for BSO_{2m+1}. Let $n=2m+1$ throughout this section. First, recall $O_n \cong SO_n \times \mathbb{Z}/2$ and the induced isomorphism

$$BP^*_{O_n} \cong BP^*_{SO_n} \otimes_{BP^*} BP^*_{\mathbb{Z}/2} \cong BP^*_{SO_n}[[y]]/([2](y)),$$

$$BP^*_{SO_n} \cong BP^*_{O_n}/(F_1).$$

Here $F_1 = B\det^*(y)$ under the map $B\det^* : BP^*_{\mathbb{Z}/2} \to BP^*_{O_n}$. We note that $F_1 = \sum_{BP} x_i$ but $c_1 = \sum x_i$, where $\sum_{BP}$ is the sum of the formal group over BP^*.

We consider the SO_n-version of $(1)'$.

LEMMA 4.1

We have $\mathrm{Ker}(c_n) \mid BP^*_{SO_n} \cong (\mathrm{Ker}(c_n) \mid BP^*_{O_n})/(F_1)$, *and hence*

$$BP^{2*-1}_{SO_n}(\mathbb{C}^n - \{0\}) \cong BP^{2*-1}_{O_n}(\mathbb{C}^n - \{0\})/(F_1).$$

Proof

Since $BP^*_{\mathbb{Z}/2}$ is BP^*-exact, we know that

$$\mathrm{Ker}(c_n)|BP^*_{O_n} \cong \big(\mathrm{Ker}(c_n)|BP^*_{SO_n}\big) \otimes_{BP^*} BP^*_{\mathbb{Z}/2}.$$

Taking the quotient ring by the ideal (F_1), we get the result. $\square$

Next consider the SO_n-version of $(2)'$. We first note that

$$BP^*_{SO_n}(C) \cong BP^*_{SO_{n-2}} \cong BP^*_{O_{n-2}}/(F_1).$$

Recall that

$$B \cong \mathbb{C}^* \times_{O_{n-1}} SO_n \cong (\mathbb{C}^* \times SO_n)/O_{n-1}.$$

Hence we see that $BP^*_{SO_n}(B) \cong BP^*_{O_{n-1}}(\mathbb{C}^*)$.

REMARK

Since $(1, 1 \oplus g) = (-1, -1 \oplus -g) \in G_m \times_{\mathbb{Z}/2} O_{n-1}$, we can identify $O_{n-1} \subset SO_n$.

We consider the exact sequence

$$\to BP^*_{O_{n-1}}(\{0\}) \overset{\times F_1}{\to} BP^*_{O_{n-1}}(\mathbb{C}^1) \to BP^*_{O_{n-1}}(\mathbb{C}^*) \to \cdots.$$

So we have the isomorphism

$$BP^*_{SO_n}(B) \cong \begin{cases} BP^*_{O_{n-1}}/(F_1) & \text{for } * = \text{even}, \\ \mathrm{Ker}(F_1) \mid BP^{*-1}_{O_{n-1}} & \text{for } * = \text{odd}. \end{cases}$$

Since $[2](F_1) = 0$ in $BP^*_{O_{n-1}}$, we see that

$$BP^*_{O_{n-1}}\big([2](F_1)/F_1\big) \subset \mathrm{Ker}(F_1) \mid BP^*_{O_{n-1}} \subset BP^{*+1}_{SO_n}(B).$$

For ease of notation, let us write $2_F = [2](F_1)/F_1$ and $2_y = [2](y)/(y)$ in $BP^*_{SO_n}(B)$.

LEMMA 4.2
We have the isomorphisms

$$BP^{2*-1}_{SO_n}(B) \cong BP^{2*}_{O_{n-1}}(2_F) \cong BP^{2*}_{O_{n-1}}/(F_1)\{2_F\},$$

where $BP^{2*}_{O_{n-1}}/(F_1)\{2_F\}$ *means the free* $BP^{2*}_{O_{n-1}}/(F_1)$*-module generated by* 2_F.

Proof
Compare the sequences *(2)'*:

$$\begin{CD}
0 \to BP^{2*-1}_{O_n}(\mathbb{C}^n-\{0\}) @>{a}>> BP^{2*-1}_{O_n}(B) @>{j}>> BP^{2*-2}_{O_{n-2}} @>{i_{2*}}>> 0\\
@V{c}VV @V{d}VV @V{e}VV\\
0 \to BP^{2*-1}_{O_n}(\mathbb{C}^n-\{0\})/(F_1) @>{a'}>> BP^{2*-1}_{SO_n}(B) @>{j'}>> BP^{2*-2}_{O_{n-2}}/(F_1) @>{i_{2*}}>> 0
\end{CD}$$

Here, from $(**)$ in §3, we still know that

$$BP^{2*-1}_{O_n}(B) \cong BP^{2*-2}_{O_{n-1}}\{2_y\}\ (\cong BP^{2*-2}_{O_{n-1}}).$$

Moreover, we also know from the above argument that

$$BP^{2*-1}_{SO_n}(B) \cong \mathrm{Ker}(F_1) \mid BP^{2*-2}_{O_{n-1}} \supset BP^{2*-2}_{O_{n-1}}(2_F).$$

Since $d(2_y)=2_F$, we see that $\mathrm{Im}(d) \subset BP^{2*-2}_{O_{n-1}}(2_F)$.

Since e and j are epic, j' is epic. As the above sequences are short exact sequences, then d is epic since e and c are epic. Thus we have

$$\mathrm{Ker}(F_1) \mid BP^{2*-2}_{O_{n-1}} \cong BP^{2*-1}_{SO_n}(B) \cong BP^{2*-2}_{O_{n-1}}(2_F).$$

Let us consider the following commutative diagram of short exact sequences:

$$\begin{CD}
0 \to BP^{2*-1}_{O_n}(\mathbb{C}^n-\{0\}) @>>> BP^{2*-1}_{O_n}(B) @>>> BP^{2*-2}_{O_n}(C) \to 0\\
@V{\cong}VV @V{\cong}VV @V{\cong}VV\\
0 \to BP^{2*-2}_{O_{n-1}}(c_{n-1}) @>>> BP^{2*-2}_{O_{n-1}} @>>> BP^{2*-2}_{O_{n-2}} \to 0\\
@V{pr_1}VV @V{pr_2}VV @V{pr_3}VV\\
0 \to BP^{2*-2}_{O_{n-1}}/(F_1)(c_{n-1}) @>>> BP^{2*-2}_{O_{n-1}}/(F_1) @>>> BP^{2*-2}_{O_{n-2}}/(F_1) \to 0\\
@V{\cong}VV @V{d'}VV @V{\cong}VV\\
0 \to BP^{2*-1}_{SO_n}(\mathbb{C}^n-\{0\}) @>>> BP^{2*-1}_{SO_n}(B) @>>> BP^{2*-2}_{SO_n}(C) \to 0
\end{CD}$$

Here d' is isomorphic. Hence $BP^*_{O_{n-1}}(2_F) \cong BP^*_{O_{n-1}}/(F_1)\{2_F\} \cong BP^*_{O_{n-1}}/(F_1)\{2\}$. □

Thus we get the exact sequence

$$0 \to BP^{2*-1}_{O_n}(\mathbb{C}^n-\{0\})/(F_1) \to BP^{2*-2}_{O_{n-1}}/(F_1)$$

$$\to BP^{2*-2}_{O_{n-2}}/(F_1) \overset{i_{2*}=0}{\to} BP^{2*}_{SO_n}(\mathbb{C}^n-\{0\}) \to BP^{2*}_{O_{n-1}}/(F_1) \to 0.$$

LEMMA 4.3

We have $BP^{2*}_{SO_n}(\mathbb{C}^n - \{0\}) \cong BP^{2*}_{O_{n-1}}/(F_1)$, *and*

$$\mathrm{Ker}(c_n) \mid BP^{2*}_{SO_n} \cong BP^{2*}_{O_{n-1}}/(F_1)(c_{n-1}).$$

5. BP-theories of BSO_{2m}

Now we study $BP^*_{SO_n}$ for $n = 2m$. By induction on m, we assume

$$BP^*_{SO_{n-2}} \cong BP^*_{O_{n-2}}/(F_1) \oplus BP^*[[c_2, \ldots, c_{2m-2}]]\{y_{m-1}\}.$$

For ease of notation, let us write $BP^*[[c_{\mathrm{even}}]]\{y_k\} = BP^*[[c_2, c_4, \ldots, c_{2k}]]\{y_k\}$. By this assumption $BP^{\mathrm{odd}}_{SO_{n-2}} = 0$ and the arguments similar to case $(2)'$, we have the BP_{SO_n}-version of the exact sequence

$$\begin{aligned}(2)'' \quad 0 &\to BP^{2*-1}_{SO_n}(\mathbb{C}^n - \{0\}) \to BP^{2*-2}_{SO_{n-1}} \\ &\to BP^{2*-2}_{SO_{n-2}} \xrightarrow{i_{2*}} BP^{2*}_{SO_n}(\mathbb{C}^n - \{0\}) \to BP^{2*}_{SO_{n-1}} \to 0.\end{aligned}$$

We also write the long exact sequence

$$\begin{aligned}(1)'' \quad &\to BP^{*-1}_{SO_n}(\mathbb{C}^n - \{0\}) \to BP^{*-2n}_{SO_n}(\{0\}) \\ &\xrightarrow{c_n} BP^*_{SO_n}(\mathbb{C}^n) \to BP^*_{SO_n}(\mathbb{C}^n - \{0\}) \to \cdots .\end{aligned}$$

Here we note the following.

LEMMA 5.1

There is an element $y_m \in BP^*_{SO_n}$ *such that*

$$y_m^2 = (-1)^m 2^{2m-2} c_{2m} \mod (v_1, \ldots)$$

and $BP^*[[c_{\mathrm{even}}]]\{y_m\} \subset BP^*_{SO_n}$.

Proof

From $(1)''$, we know that $BP^*_{SO_n}/(c_n) \subset BP^*_{SO_n}(\mathbb{C}^n - \{0\})$. Let us define $i_{2*}(y_{m-1}) = y_m \in BP^*_{SO_n}(\mathbb{C}^n - \{0\})$. We still know that $y_m \in CH^*(BSO_n)$ from the argument in §2 (Field's theorem). By Totaro's cycle map, we can take $y_m \in BP^*_{SO_n}$ (but only decided with $\mathrm{mod}(c_n, v_1, \ldots)$).

Moreover, considering the restriction on the BP^*-free algebra

$$BP^*(BT_{SO_n}) \cong BP^* \otimes H^*(BT_{SO_n})$$

for the maximal torus T_{SO_n}, we see $BP^*[[c_{\mathrm{even}}]]\{y_m\} \subset BP^*_{SO_n}$ and the equality of the lemma (see also the arguments (or Lemma 5.7) in [MVi, §5]). ☐

REMARK

The element y_m is also defined in BP^*-theories (but not as an image of Totaro's cycle map).

LEMMA 5.2

We have

$$BP^*_{SO_n}(\mathbb{C}^n-\{0\}) \cong \begin{cases} BP^{*+1}_{O_{n-1}}/(F_1)(c_{n-1}) & \text{if } *=\text{odd},\\ BP^*_{O_{n-1}}/(F_1)\oplus BP^*[[c_{\text{even}}]]\{y_m\}/(c_n) & \text{if } *=\text{even}.\end{cases}$$

Proof

Consider the exact sequence $(2)''$. For the element $1\in BP^*_{SO_{n-2}}$, the image $i_{2*}(1)=0$ since it is so in $BP^*_{O_{n-2}}$. Recall that

$$\mathrm{Ker}(BP^*_{SO_{n-1}}\to BP^*_{SO_{n-2}})\cong BP^*_{SO_{n-1}}(c_{n-1})\subset BP^*_{SO_{n-1}}.$$

From $(2)''$ and $BP^*_{SO_{n-1}}\cong BP^*_{O_{n-1}}/(F_1)$, we have the isomorphism for $*=\text{odd}$.

When $*=\text{even}$, the right-hand formula in this lemma is contained in the left-hand formula by $(2)''$ and the inductive assumption introduced in the earlier parts of this section. Since $i_{2*}(y_{m-1})=y_m$ and $i_{2*}(1)=0$ in $(2)''$, we see the isomorphism for $*=\text{even}$. □

From Lemma 5.2, we show that the map

$$BP^{2*}_{SO_n}(\mathbb{C}^n)\to BP^{2*}_{SO_n}(\mathbb{C}^n-\{0\})$$

in $(1)''$ is an epimorphism since $y_m\in BP^*_{SO_n}(\mathbb{C}^n)$.

LEMMA 5.3

In $(1)''$, the map $BP^{2*-1}_{SO_n}(\mathbb{C}^n-\{0\})\to BP^{2*-2n}_{SO_n}(\{0\})$ is injective.

Proof

Consider the composition of maps

$$BP^{2*}_{O_{n-1}}(c_{n-1})/(F_1)\cong BP^{2*+1}_{SO_n}(\mathbb{C}^n-\{0\})\to BP^{2*-2n}_{SO_n}\to BP^{2*-2n}_{O_{n-1}}/(F_1),$$

which sends xc_{n-1} to $xq_n \bmod (c_n)$ as the map in Lemma 3.4. Since $BP^*_{O_{n-1}}/F_1\cong BP^*_{SO_{n-1}}$ is 2-torsion free, we get the injection of the composition map from the same argument as the proof of Lemma 3.4. □

From Lemma 5.3 and $(1)''$, we have the exact sequence $0\to BP^{\text{odd}}_{SO_n}\xrightarrow{c_n} BP^{\text{odd}}_{SO_n}\to 0$ and $BP^{\text{odd}}_{SO_n}=0$.

Proof of Theorem 1.1

By $(1)''$ and $BP^{\text{odd}}_{SO_n}=0$, we see that $BP^*_{SO_n}$ is multiplicatively generated by $c_1,\dots,c_n$ and y_m.

Given the filtration by the ideal $(c_n^i)\subset BP^*_{O_n}$, we consider the associated graded algebra

$$\mathrm{gr}BP^*_{O_n}=\bigoplus_i (c_n^i)/(c_n^{i+1}).$$

Since $(c_n^i)/(c_n^{i+1})$ is a $BP^*_{O_{n-1}}$-module generated by only one element c_n^i, we can write it as $BP^*_{O_{n-1}}/A_i\{c_n^i\}$ for some ideal $A_i\subset BP^*_{O_{n-1}}$. The fact that

$BP^*_{O_n}/(c_n) \cong BP_{O_{n-1}}$ implies $A_0 = \{0\}$. From Lemma 3.3, we see that $\mathrm{Ker}(c_n) \mid BP^*_{O_n}(c_n) = (q_n c_n)$ and $A_1 = (q_n)$. Moreover, for all $i \geq 1$, we see that $A_i = (q_n)$ since

$$\times c_n : (c_n^i)/(c_n^{i+1}) \to (c_n^{i+1})/(c_n^{i+2})$$

is injective (so isomorphic) because $\mathrm{Ker}(c_n) \cap (c_n) = \{0\} \subset BP^*_{O_n}$ from Corollary 3.5. Thus

$$(*) \quad \mathrm{gr}BP^*_{O_n} \cong BP^*_{O_{n-1}} \oplus \bigoplus_{i=1} BP^*_{O_{n-1}}/(q_n)\{c_n^i\}.$$
$$\cong BP^*_{O_{n-1}} \oplus BP^*_{O_{n-1}}/(q_n)[c_n]\{c_n\}.$$

Next, consider the similar graded ring $\mathrm{gr}BP^*_{SO_n}$ for SO_n. From the $* = \text{even}$ case in Lemma 5.2, we see

$$\mathrm{gr}^0 BP^*_{SO_n} = BP^*_{SO_n}/(c_n) \cong BP^*_{O_{n-1}}/(F_1) \oplus BP^*[[c_{\mathrm{even}}]]/(c_n)\{y_m\}.$$

We still know $\mathrm{Ker}(c_n) \mid BP^*_{SO_n} \subset BP^*_{O_{n-1}}/(F_1)$ from Lemma 5.3. This shows that $\mathrm{Ker}(c_n) \cap (c_n) = 0$ also in $BP^*_{SO_n}$. Hence for all $i \geq 1$, the map $\times c_n : (c_n^i)/(c_n^{i+1}) \to (c_n^{i+1})/(c_n^{i+2})$ is isomorphic. Thus we have the isomorphism

$$(**) \quad \mathrm{gr}BP^*_{SO_n} \cong BP^*_{O_{n-1}}/(F_1) \oplus BP^*_{O_{n-1}}/(F_1, q_n)[c_n]\{c_n\}$$
$$\oplus \big(BP^*[[c_{\mathrm{even}}]]/(c_n)\big)[c_n]\{y_m\}.$$

Here, of course, the last term is isomorphic to $BP^*[[c_{\mathrm{even}}]]\{y_m\}$.

In general, $\mathrm{gr}(BP^*_{O_n}/F_1)$ is a quotient of $\mathrm{gr}(BP^*_{O_n})/(F_1)$. In this case, there is a map $BP^*_{O_n}/(F_1) \to BP^*_{SO_n}$, and there is the isomorphism $\mathrm{gr}(BP^*_{O_n}/F_1) \cong \mathrm{gr}(BP^*_{O_n})/(F_1)$ from $(*)$ and $(**)$. From the isomorphism $(**)$, we have

$$\mathrm{gr}BP^*_{SO_n} \cong \mathrm{gr}(BP^*_{O_n}/F_1) \oplus BP^*[[c_{\mathrm{even}}]]\{y_m\}.$$

Of course, this implies the isomorphism (without gr) in the theorem. □

Proof of Theorem 1.3
Recall that

$$\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \cong MGL^{2*,*}(X) \otimes_{MU^*} \mathbb{Z} \cong CH^*(X).$$

Hence $\Omega^*_{SO_n}$ and $MGL^{2*,*}_{SO_n}$ are generated by $c_1, \ldots, c_n$ and y_m as MU^*-algebras. Of course, there are relations $c_i - c_i^*, F_1$ also in $\Omega^*_{SO_n}$ and $MGL^{2*,*}_{SO_n}$. Thus we get Theorem 1.3 in the introduction. □

6. Integral Morava K-theory

The arguments of this section are suggested from the article by Kriz [Kr]. Let $K(s)^*(X)$ (resp., $\tilde{K}(s)^*(X)$) be the Morava (the integral Morava) K-theory with the coefficient ring

$$K(s)^* = \mathbb{Z}/2[v_s, v_s] \quad (\text{resp.}, \tilde{K}(s)^* = \mathbb{Z}_{(2)}[v_s, v_s^{-1}]).$$

For ease of notation, fixing s, we simply write

$$E = \tilde{K}(s), \qquad E/2 = K(s).$$

From [KY] and [Kr], it is known that $E/2^*(BO_n)$ and $E/2^*(BSO_{\mathrm{odd}})$ are generated by even-dimensional elements. Hence we know that $E^*(BO_n)$ and $E^*(BSO_{\mathrm{odd}})$ are also generated by even-dimensional elements and are 2-torsion free. Hence all arguments in §5 work well, and we see that $E^*(BSO_{2m})$ is also even-dimensionally generated, and

$$E^*(BSO_{2m}) \cong E^* \otimes_{BP^*} BP^*(BSO_{2m}).$$

We prove the the following lemma.

LEMMA 6.1
$E^*(BSO_{2m})$ is 2-torsion free.

If Lemma 6.1 holds, then we get $E/2^*(BSO_{2m}) \cong E^*(BSO_{2m})/2$ and hence $E/2^{\mathrm{odd}}(BSO_{2m}) = 0$. Then from the theorem of Ravenel, Wilson, and Yagita [RWY], we see that $BP^*(BSO_{2m})$ is BP^*-flat. Thus we get Theorem 1.2 in the introduction.

Now we prove Lemma 6.1. Since we have

$$E^*(BSO_{2m}) \cong E^*[[c_{\mathrm{even}}]]\{y_m\} \oplus E^*_{O_{2m}}/(F_1),$$

we only need to show that $E^*_{O_{2m}}/(F_1)$ is 2-torsion free. We consider the short exact sequence

$$(3) \quad 0 \to E^*_{O_n}/(F_1)(c_n) \to E^*_{O_n}/(F_1) \to E^*_{O_{n-1}}/(F_1) \to 0.$$

Since $E^*_{O_{n-1}}/(F_1) \cong E^*_{SO_{n-1}}$ is 2-torsion free, we only need to prove that $BP^*_{O_n}/(F_1)(c_n)$ is 2-torsion free.

We can show that we have the grading

$$(4) \quad \mathrm{gr} E^*_{O_n}/(F_1)(c_n) \cong E^*_{SO_{n-1}}/(q_n)[c_n]\{c_n\}$$

by the same reason as in the proof of Theorem 1.1 in §5. Here note that $E^*_{SO_{n-1}}/(q_n) \cong E^*_{SO_{n-1}}/(2 - q_{n-1})$ and

$$(5) \quad \mathrm{gr} E^*_{SO_{n-1}}/(2 - q_{n-1}) \cong E^*_{O_{n-2}}/(q_{n-2}, F_1) \oplus E^*_{SO_{n-1}}/(2)[c_{n-1}]\{c_{n-1}\}$$

since $q_{n-1}c_{n-1} = 0 \in BP^*_{O_{n-1}}$.

By induction and (3), we can assume that

$$E^*_{O_{n-2}}/(q_{n-2}, F_1) \cong E^*_{O_{n-2}}(c_{n-2})/(F_1) \subset E^*_{SO_{n-2}}$$

has no 2-torsion. Hence Lemma 6.1 is proved if we see the following lemma.

LEMMA 6.2
Let $xc^i_{n-1}c^j_n \in E^*_{O_n}/(F_1, q_n)$, $i, j \geq 1$, be an element such that $xc^i_{n-1} \neq 0 \in E^*_{O_{n-1}}/(2, F_1)$. Then $2xc^i_{n-1}c^j_n \neq 0 \in E^*_{O_n}/(F_1, q_n)$.

Proof
We consider the map

$$[c_{n-1}^{-1}]E^*_{O_n} \to [c_{n-1}^{-1}]E^*_{O_{n-1}} \otimes_{E^*} E^*[[x_n]]/\big(x_n - i(x_n)\big)$$

given by $c_i \mapsto c_i$ for $i \le n-1$ and $c_n \mapsto c_{n-1}x_n$. The map sends

$$\begin{aligned} c_n - c_n^* &\mapsto x_n c_{n-1} - i(x_n)c_{n-1}^* \\ &= i(x_n)(c_{n-1} - c_{n-1}^*) + \big(x_n - i(x_n)\big)c_{n-1} \end{aligned}$$

and is well defined, and moreover, it is isomorphic; indeed, $x_n = c_n/c_{n-1}$.

Recall that

$$x_n - i(x_n) = 2x_n + v_s x_n^{2^s} + \cdots \quad \text{in } E^*[[x_n]] = \tilde{K}(s)[[x_n]].$$

Then in $[c_{n-1}^{-1}]E^*_{O_n}$, we have

$$2c_n = 2x_n c_{n-1} = -v_s x_n^{2^s} c_{n-1} \quad \mathrm{mod}(x_n^{2^s+1}).$$

From $x_n = c_n/c_{n-1}$, the above equation means

$$2c_{n-1}^{2^s} c_n = -v_s c_n^{2^s} c_{n-1} \quad \mathrm{mod}(c_n^{2^s+1})$$

in $E^*_{O_n}$. Hence, note that it is so in $E^*_{O_n}/(F_1, q_n)$.

Let x be an element that satisfies the assumption of this lemma. Then

$$2xc_{n-1}^{i+2^s} c_n^j = -v_s x c_{n-1}^{i+1} c_n^{2^s+j} \quad \mathrm{mod}(c_n^{2^s+j+1}),$$

which is also nonzero from (4), (5), and $v_s^{-1} \in E^* = \mathbb{Z}_{(2)}[v_s, v_s^{-1}]$. □

REMARK
By using arguments in §3 and §4, we can inductively prove $E^{\mathrm{odd}}(BSO_n) = 0$ and $E^*(BSO_n)$ has no 2-torsion, without using Wilson's results. Logically it may be more simple than the arguments here. However, the proof that it is 2-torsion free is somewhat technical, and we include them in this section.

7. BP^*-orientability

Recall that an n-dimensional vector bundle $p : E \to X$ is BP^*-orientable if there is an element (Thom class) $\mathrm{th} \in BP^n(\mathrm{Th}_X(E))$ such that for each inclusion $i : \mathrm{pt} \to X$ the restriction image

$$i^*(\mathrm{th}) \in BP^*\big(\mathrm{Th}_{\mathrm{pt}}(p^{-1}(\mathrm{pt}))\big) \cong BP^*(S^n)$$

is a BP^*-module generator. If $p : E \to X$ is BP^*-orientable, we have the Thom isomorphism $BP^*(X) \cong BP^{*+n}(\mathrm{Th}_X(E))$ by the standard arguments using the Mayer-Vietoris sequence.

It is well known that each complex bundle is BP^*-orientable as stated in §3. Of course, there are SO_n-bundles which are not BP^*-orientable. Note that

$$BO_n \cong U \times_{O_n} D^n, \qquad BO_{n-1} \cong U \times_{O_n} O_n/O_{n-1} \cong U \times_{O_n} S^{n-1},$$

where U is an O_n-free contractible space, and D^n is the n-dimensional disk. Hence we can identify

$$\mathrm{Th}_{U/O_n}(U \times_{O_n} D^n) \cong BO_n/BO_{n-1}.$$

A similar fact also applies to SO_n. Let us write the Thom space of BO_n (resp., BSO_n) for the universal bundle as $MO_n = BO_n/BO_{n-1}$ (resp., $MSO_n = BSO_n/BSO_{n-1}$).

The cofibering $BO_{n-1} \to BO_n \to MO_n$ induces the exact sequence

$$0 \leftarrow BP^*_{O_{n-1}} \leftarrow BP^*_{O_n} \leftarrow \tilde{BP}^*(MO_n) \leftarrow 0.$$

Hence we see (cf. [Wi]) that

$$\tilde{BP}^*(MO_n) \cong \mathrm{Ker}(BP^*_{O_n} \to BP^*_{O_{n-1}}) \cong BP^*_{O_n}(c_n).$$

THEOREM 7.1 ([KR, THEOREM 6.2])

We have the short exact sequence

$$0 \to \tilde{BP}^{*+2n-2}(MO_{n-1}) \xrightarrow{i} BP^*_{O_n} \xrightarrow{\mathrm{Th}} \tilde{BP}^{*+2n}(MO_n) \to 0.$$

Proof

Consider the short exact sequence

$$0 \to \mathrm{Ker}(c_n) \to BP^*_{O_n} \xrightarrow{c_n} BP^*_{O_n}(c_n) \to 0.$$

By Corollary 3.2, we still know that $\mathrm{Ker}(c_n \mid B^*_{O_n}) \cong \tilde{BP}^*(MO_{n-1})$. Moreover, we know $BP^*_{O_n}(c_n) \cong \tilde{BP}^*(MO_n)$. □

Next, consider the case SO_n. The cofiber sequence $BSO_{n-1} \to BSO_n \to MSO_n$ induces the long exact sequence

$$0 \leftarrow \tilde{BP}^{*+1}(MSO_n) \leftarrow BP^*_{SO_{n-1}} \leftarrow BP^*_{SO_n} \leftarrow \tilde{BP}^*(MSO_n) \leftarrow 0.$$

Hence we have the isomorphism

$$BP^*(MSO_n) \cong BP^*_{O_n}(c_n)/(F_1) \oplus BP^\sharp[[c_{\mathrm{even}}]]\{y_m\},$$

where $\sharp = *$ for $n = 2m$ but $\sharp = * - 1$ for $n = 2m + 1$. Thus we have the SO_n-version of Theorem 7.1.

THEOREM 7.2

Let $n = 2m$ or $2m + 1$. Then there is an exact sequence

$$0 \to \tilde{BP}^{*+2n-2}(MO_{n-1})/(F_1) \xrightarrow{i} BP^*_{SO_n} \xrightarrow{\mathrm{Th}}$$

$$\tilde{BP}^{*+2n}(MSO_n) \to BP^\sharp[[c_{\mathrm{even}}]]/(c_n)\{y_m\} \to 0,$$

where $\sharp = *$ for $n = 2m$ but $\sharp = * - 1$ for $n = 2m + 1$.

REMARK

When $n = 2m + 1$, note that $BP^\sharp[[c_{\mathrm{even}}]]/(c_n) = BP^{*-1}[[c_{\mathrm{even}}]]$.

The kernel in $BP^*_{SO_n}$ of the map Th is generated by only one element q_n, which generates $\mathrm{Ker}(c_n) \mid BP_{SO_n}$ (recall Lemma 3.3). This element gives an obstruction for BP^*-orientability.

PROPOSITION 7.3

Let $p: E \to X$ be an SO_n-bundle, and let $f: X \to BSO_n$ be its classifying map. If $f^*(q_n) \neq 0 \in BP^*(X)$ then this bundle is not BP^*-orientable.

Proof

In $BP^*(MSO_n)$, we see $\mathrm{Th} \cdot q_n = 0$. Hence in $BP^*(\mathrm{Th}_X(E))$, the induced element $f^*(\mathrm{Th} \cdot q_n) = 0$. Thus the Thom isomorphism does not hold. □

Acknowledgments. Pierre Guillot and Angelo Vistoli showed their preprints to the authors. This is the starting point of this article. They also corrected some errors in the early version of this article. The authors thank them very much.

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