On L-functions of twisted 4-dimensional quaternionic Shimura varieties

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Abstract   In this article we prove the meromorphic continuation to the entire complex plane and also the functional equation of the zeta functions of twisted 4-dimensional quaternionic Shimura varieties in quite general cases.

1. Introduction

Let F be a number field, and let X be a smooth projective variety defined over F. For a prime number l, we denote by $H^i_{\mathrm{et}}(X, \bar{\mathbb{Q}}_l)$ the l-adic cohomology of $\bar{X} = X \times_F \bar{\mathbb{Q}}$. Then the Galois group $\Gamma_F := \mathrm{Gal}(\bar{\mathbb{Q}}/F)$ acts on $H^i_{\mathrm{et}}(X, \bar{\mathbb{Q}}_l)$ by a representation $\rho_{i,l}$, and the L-function $L^i(s, X_{/F})$ attached to the representation $\rho_{i,l}$ converges for $\mathrm{Re}(s) > 1 + i/2$. Moreover, it is conjectured that the L-function $L^i(s, X_{/F})$ converges and does not vanish in the semiplane $\mathrm{Re}(s) > 1 + i/2$, has a meromorphic continuation to the entire complex plane, and satisfies a functional equation.

In this article we prove, in quite general cases, that the zeta function of "twisted" quaternionic Shimura varieties can be meromorphically continued to the entire complex plane, satisfy a functional equation, do not vanish, and converge in the semiplane $\mathrm{Re}(s) > 3$. We prove all these results also for the base change of these Shimura varieties to arbitrary solvable extensions of totally real number fields which contain their field of definition.

More precisely, in this article we consider a totally real number field F with $[F : \mathbb{Q}] = d \geq 4$ and a quaternion algebra D over F, which is unramified at exactly 4 infinite places of F. We denote by G the algebraic group over F defined by the multiplicative group $D^\times$ of D, and we define $\bar{G} =: \mathrm{Res}_{F/\mathbb{Q}}(G)$. Fix a prime ideal $\wp$ of the ring of integers $O := O_F$ of F, such that $G(F_\wp)$ is isomorphic to $\mathrm{GL}_2(F_\wp)$, and denote by $S_{\bar{G}, \mathbf{K}} = S_\mathbf{K}$ the quaternionic Shimura variety associated to an open compact subgroup $\mathbf{K} := K_\wp \times H$ of $\bar{G}(\mathbb{A}_f)$, where $K_\wp$ is the set of elements of $\mathrm{GL}_2(O_\wp)$ which are congruent to 1 modulo $\wp$, H is an open compact subgroup of the restricted product of $(D \otimes_F F_\mathfrak{p})^\times$ where $\mathfrak{p}$ runs over all the finite places of F, $\mathfrak{p} \neq \wp$, and $\mathbb{A}_f$ is the finite part of the ring of adeles $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$. Then

the quasi-projective variety $S_{\mathbf{K}}$ is defined over a totally real finite extension $E/\mathbb{Q}$ called the canonical field of definition.

The group $\mathrm{GL}_2(O/\wp)$ acts on the variety $S_{\mathbf{K}}$. For H sufficiently small this action is free. We fix such a small group H, we consider a continuous Galois representation $\varphi : \Gamma_E \to \mathrm{GL}_2(O/\wp)$, and we denote by $S'_{\mathbf{K}}$ the variety defined over E obtained from $S_{\mathbf{K}}$ via twisting by φ composed with the natural action of $\mathrm{GL}_2(O/\wp)$ on $S_{\mathbf{K}}$ (see §2 for details).

In this article we assume that $D \neq M_2(F)$ and that $L := \bar{\mathbb{Q}}^{\mathrm{Ker}(\varphi)}$ is a solvable extension of a totally real number field. Under these assumptions we prove that (for some quaternion algebras D we have to assume the existence of some Asai representations of degree 3 or 4 for GL(2); see Theorem 7.1 for details) if k is an arbitrary solvable extension of a totally real number field containing E, then the L-function $L^4(s, S'_{\mathbf{K}/k})$ is holomorphic, does not vanish in the semiplane $\mathrm{Re}(s) > 3$, has a meromorphic continuation to the entire complex plane, and satisfies a functional equation. We remark that one could obtain these results also for general (not twisted) quternionic Shimura varieties of arbitrary level. In order to obtain these results, we use in particular the automorphy of degree two Asai representations of automorphic representations of GL(2) which appear in [R2] (for details, see Proposition 6.2) and also the meromorphic continuation of some degree 16 L-functions that appear in [R3] (see Proposition 6.3 for details). We remark that similar results were obtained by the author in [V1] in the case of twisted quaternionic Shimura varieties of dimension 3, and in that case the meromorphic continuation of the triple L-functions that appear in [I] was essential.

2. Twisted quaternionic Shimura varieties

Let F be a totally real number field such that $[F : \mathbb{Q}] = d \geqslant 4$, and let $O := O_F$ be its ring of integers. Let D be a quaternion algebra over F which is unramified at exactly 4 infinite places of F. In this article we assume that $D \neq M_2(F)$. Let S_∞ be the set of the infinite places of F. Then S_∞ is identified as a $\Gamma_{\mathbb{Q}}$-set with $\Gamma_F \backslash \Gamma_{\mathbb{Q}}$. Let S'_∞ be the subset of S_∞ at which D is ramified. Thus the cardinality of $S_\infty - S'_\infty$ is equal to 4.

We denote by G the algebraic group over F defined by the multiplicative group $D^\times$, and we let $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}} G$. For $v \in S_\infty - S'_\infty$, we fix an isomorphism of $G(F_v)$ with $\mathrm{GL}_2(\mathbb{R})$. We have $\bar{G}(\mathbb{R}) = \prod_{v \in S_\infty} G(F_v)$. Let $J = (J_v) \in \bar{G}(\mathbb{R})$, where

$$J_v = \begin{cases} 1 & \text{for } v \in S'_\infty, \\ 1/\sqrt{2} \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix} \right) & \text{for } v \in S_\infty - S'_\infty. \end{cases}$$

Let K_∞ be the centralizer of J in $\bar{G}(\mathbb{R})$. Set

$$X = \bar{G}(\mathbb{R})/K_\infty.$$

Then X is complex analytically isomorphic to $(\mathcal{H}_\pm)^4$, where $\mathcal{H}_\pm = \mathbb{C} - \mathbb{R}$. For each open compact subgroup $K \subset \bar{G}(\mathbb{A}_f)$, set

$$S_K(\mathbb{C}) = \bar{G}(\mathbb{Q}) \backslash X \times \bar{G}(\mathbb{A}_f)/K.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold that is the set of the complex points of a quasi-projective variety S_K of dimension 4 defined over a totally real number field E. More exactly, we have that Γ_E is the stabilizer of $S'_\infty \subseteq \Gamma_F \setminus \Gamma_\mathbb{Q}$.

We fix $\wp$ a prime ideal of O_F such that $G(F_\wp)$ is isomorphic to $\mathrm{GL}_2(F_\wp)$. Consider $\mathbf{K} := K_\wp \times H$, where $K_\wp$ is the set of elements of $\mathrm{GL}_2(O_\wp)$ which are congruent to 1 modulo $\wp$ and H is some open compact subgroup of the restricted product of $(D \otimes_F F_\mathfrak{p})^\times$, where $\mathfrak{p}$ runs over all the finite places of F, with $\mathfrak{p} \neq \wp$. Then it is well known (see, e.g., [C, Corollary 1.4.1.3]) that for H sufficiently small, the group $\mathrm{GL}_2(O/\wp)$ acts freely (see [V1]) on

$$S_\mathbf{K}(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/\mathbf{K}.$$

We fix such a small H.

Consider a continuous representation

$$\varphi : \Gamma_E \to \mathrm{GL}_2(O/\wp),$$

and denote by $S'_\mathbf{K}/\mathrm{Spec}(E)$ the twisted Shimura variety obtained from $S_\mathbf{K}$ via twisting by φ composed with the natural action of $\mathrm{GL}_2(O/\wp)$ on $S_\mathbf{K}$ (see [V1, §2] for details).

3. Zeta functions of twisted quaternionic Shimura fourfolds

From now on, if π is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, obtained from π by Jacquet-Langlands correspondence (usually denoted $\mathrm{JL}(\pi)$) by the same symbol π.

If π is a cuspidal automorphic representation of weight 2 of $\mathrm{GL}(2)/F$ which is a discrete series at infinity, then there exists (see [T], [C], [BR1]) a λ-adic representation

$$\rho_{\pi,\lambda} : \Gamma_F \to \mathrm{GL}_2(\mathcal{O}_\lambda) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

which satisfies $L(s - 1/2, \pi) = L(s, \rho_{\pi,\lambda})$ and is unramified outside the primes dividing $\mathbf{n}l$. Here $\mathbf{n}$ is the level of π, $\mathcal{O}$ is the integer ring of the coefficient field of π, and λ is a prime ideal of $\mathcal{O}$ above some prime number l. In order to simplify the notation, we denote by ρ_π the representation $\rho_{\pi,\lambda}$.

Let K be an open compact subgroup of $\bar{G}(\mathbb{A}_f)$, and let $\mathbb{H}_K$ be the Hecke algebra of convolutions of bi-K-invariant, $\bar{\mathbb{Q}}_l$-valued compactly supported functions on $\bar{G}(\mathbb{A}_f)$. If $\pi = \pi_\infty \otimes \pi_f$ is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote by π_f^K the space of K-invariants in π_f. The Hecke algebra $\mathbb{H}_K$ acts on π_f^K.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group Γ_E on the étale cohomology $H^4_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l)$ and these two actions commute. Then we know (see, e.g., [RT, Proposition 1.8]).

PROPOSITION 3.1

The representation of $\Gamma_E \times \mathbb{H}_K$ on the étale cohomology $H^4_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l)(2)$ is isomorphic to

$$\bigoplus_\pi \rho(\pi) \otimes \pi_f^K,$$

where $\rho(\pi)$ is a representation of the Galois group Γ_E. The above sum is over weight 2 irreducible cohomological automorphic representations π of $\bar{G}(\mathbb{A}_\mathbb{Q})$, and the $\mathbb{H}_K$-representations π_f^K are irreducible and mutually inequivalent.

The automorphic representations that appear in Proposition 3.1 are one-dimensional or cuspidal and infinite-dimensional. If π is one-dimensional, then $\rho(\pi)$ has dimension 6 and if π is infinite-dimensional, then $\rho(\pi)$ has dimension 16.

We fix an isomorphism $j : \bar{\mathbb{Q}}_l \to \mathbb{C}$ and define the L-function

$$L^4(s, S_K) := \prod_\pi \prod_q \det\big(1 - Nq^{-s+2}j(\rho(\pi)(\mathrm{Frob}_q))|H^4_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l)(2)^{I_q}\big)^{-1},$$

where Frob_q is a geometric Frobenius element at a finite place q of E and I_q is an inertia group at q. (In order to define the local factors at the places of E dividing l, one actually has to use the l'-adic cohomology for some $l' \neq l$ and [B, Theorem 3] which gives us the expression of the local factors of the zeta functions of quaternionic Shimura varieties.)

We have the following (for details, see the proof of [V1, Theorem 3.2]).

THEOREM 3.2

The L-function $L^4(s, S'_\mathbf{K})$ is given by the formula

$$L^4(s, S'_\mathbf{K}) = \prod_\pi L\big(s - 2, \rho(\pi) \otimes (\pi_f^\mathbf{K} \circ \varphi)\big),$$

where the product is taken over cohomological automorphic representations π of $\bar{G}(\mathbb{A}_\mathbb{Q})$ of weight 2, such that $\pi_f^\mathbf{K} \neq 0$.

4. Base change

We know the following result (see [V2, Theorem 1.1]).

THEOREM 4.1

If F is a totally real number field, π is a cuspidal automorphic representation of weight 2 of $\mathrm{GL}(2)/F$, and F' is a solvable extension of a totally real number field containing F, then there exists a Galois extension F'' of $\mathbb{Q}$ containing F', such that F'' is a solvable extension of a totally real field, and there exists a prime λ of the field coefficients of π, such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}}$ is modular; that is, there exists an automorphic representation π_1 of $\mathrm{GL}(2)/F''$ and a prime β of the field of coefficients of π_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}} \cong \rho_{\pi_1,\beta}$.

In this section we fix an automorphic representation π as in Theorem 4.1 and we denote $\omega := \pi_f^{\mathbf{K}} \circ \varphi$. In this article we assume that the field $L := \bar{\mathbb{Q}}^{\mathrm{Ker}(\varphi)}$ is a solvable extension of a totally real number field. Thus the field $K := \bar{\mathbb{Q}}^{\mathrm{Ker}(\omega)}$ is a solvable extension of a totally real number field.

Let k be a solvable extension of a totally real number field which contains E. From Theorem 4.1 we deduce that there exist a Galois extension F'' of $\mathbb{Q}$ containing FKk, such that F'' is a solvable extension of a totally real field, and a prime λ of the field coefficients of π, an automorphic representation π_1 of $\mathrm{GL}(2)/F''$, and a prime β of the field of coefficients of π_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}} \cong \rho_{\pi_1,\beta}$.

By Brauer's theorem (see [SE, Theorems 16, 19]), we can find some subfields $F_i \subset F''$ such that $\mathrm{Gal}(F''/F_i)$ are solvable, some characters $\chi_i : \mathrm{Gal}(F''/F_i) \to \bar{\mathbb{Q}}^{\times}$, and some integers m_i, such that the representation

$$\omega|_{\Gamma_k} : \mathrm{Gal}(F''/k) \to \mathrm{Gal}(Kk/k) \to \mathrm{GL}_N(\bar{\mathbb{Q}}_l)$$

can be written as $\omega|_{\Gamma_k} = \sum_{i=1}^{i=k} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i$ (a virtual sum). Then

$$
\begin{aligned}
L\big(s, (\rho(\pi) \otimes \omega)|_{\Gamma_k}\big) &= \prod_{i=1}^{i=k} L\big(s, \rho(\pi)|_{\Gamma_k} \otimes \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i\big)^{m_i} \\
&= \prod_{i=1}^{i=k} L\big(s, \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} (\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i)\big)^{m_i} \\
&= \prod_{i=1}^{i=k} L\big(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i\big)^{m_i}.
\end{aligned}
$$

If $F \subset F_i$, since $\rho_{\pi,\lambda}|_{\Gamma_{F''}}$ is modular and $\mathrm{Gal}(F''/F_i)$ is solvable, from Langlands base change for solvable extensions one can deduce easily that $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$ is modular, and in this case we denote by π_i the automorphic representation of $\mathrm{GL}(2)/F_i$ such that $\rho_{\pi,\lambda}|_{\Gamma_{F_i}} \cong \rho_{\pi_i}$.

5. Known results

The following is known (see [R1, Theorem M]).

PROPOSITION 5.1
If π_1 and π_2 are two cuspidal automorphic representations of $\mathrm{GL}(2)/L$, where L is a number field, then $\pi_1 \boxtimes \pi_2$ is an automorphic isobaric representation of $\mathrm{GL}(4)/L$.

We know the following (see [JS]).

PROPOSITION 5.2
If π_1 and π_2 are two cuspidal unitary automorphic representations of $\mathrm{GL}(n)/L$ and $\mathrm{GL}(m)/L$, where L is a number field, then the function $L(s, \pi_1 \times \pi_2)$ has a meromorphic continuation to the entire complex plane with possible simple poles

only at $\operatorname{Re}(s) = 0$ *and* 1, *and the completed L-function satisfies a functional equation* $s \leftrightarrow 1 - s$. *Also, the function* $L(s, \pi_1 \times \pi_2)$ *does not vanish in the semiplane* $\operatorname{Re}(s) > 1$.

6. Asai L-functions

Let K/F be an extension of number fields, and let $d := [K : F]$. We consider the algebraic group $G := \operatorname{Res}_{K/F} \operatorname{GL}_n / K$. Thus $G(F) = \operatorname{GL}_n(K)$, and the L-group of G is equal to the semidirect product

$$^L G = \operatorname{GL}_n(\mathbb{C})^d \rtimes \Gamma_F,$$

where $\operatorname{GL}_n(\mathbb{C})^d$ is indexed by the elements $\sigma \in \Gamma_K \setminus \Gamma_F$ and Γ_F acts on $\operatorname{GL}_n(\mathbb{C})^d$ by permuting the factors in the natural way. We consider the representation

$$r_{K/F} : {}^L G \to \operatorname{GL}\big((\mathbb{C}^n)^{\otimes d}\big) \cong \operatorname{GL}_{n^d}(\mathbb{C})$$

given by

$$r_{K/F}\big((g_\sigma); 1\big)\Big(\bigotimes_\sigma v_\sigma\Big) = \bigotimes_\sigma g_\sigma v_\sigma$$

and

$$r_{K/F}\big((I_\sigma); \tau\big)\Big(\bigotimes_\sigma v_\sigma\Big) = \bigotimes_\sigma v_{\tau\sigma},$$

where $(g_\sigma) \in \operatorname{GL}_n(\mathbb{C})^d$ and $\tau \in \Gamma_F$ and I_σ denotes the identity $(n \times n)$-matrix in the σth place.

For any automorphic representation $\pi = \bigotimes'_v \pi_v$ of $\operatorname{GL}(n)/K$ and any algebraic Hecke character χ of F, which by class field theory may be viewed as a character of Γ_F and thus as a character of ${}^L G$, one can define the *Asai L-function* $L(s, \pi, r_{K/F} \otimes \chi) = \prod_v L(s, \pi_v, r_{K/L} \otimes \chi_v)$, where the product is over all the finite places v of F, and if v is a finite place of F such that π_w is unramified at any place w of K above v and χ_v is unramified, then there is a semisimple conjugacy class $A(\pi_v)$ in ${}^L G$ such that

$$L(s, \pi_v, r_{K/L} \otimes \chi_v) = \det\big(I - \chi_v(\operatorname{Frob}_v) r_{K/F}(A(\pi_v)) N v^{-s}\big)^{-1},$$

where Frob_v is a geometric Frobenius at v.

We know the following (see [R3, Theorem 6.11]).

PROPOSITION 6.1

Let K/F be a quadratic extension of number fields, let n be a positive integer, and let π be a cuspidal automorphic representation of $\operatorname{GL}(n)/F$. Then $L(s, \pi, r_{K/F})$ admits a meromorphic continuation to the entire complex plane with the only possible poles at $s = 0$ and 1 and satisfies a functional equation of the form

$$L_\infty(1 - s, \pi^\vee, r_{K/F}) L(1 - s, \pi^\vee, r_{K/F})$$

$$= \epsilon(s, \pi, r_{K/F}) L_\infty(s, \pi, r_{K/F}) L(s, \pi, r_{K/F}),$$

where $\pi^\vee$ is the contragredient of π, $\epsilon(s,\pi,r_{K/F})$ is an invertible holomorphic function, and $L_\infty(1-s,\pi^\vee,r_{K/F})$ and $L_\infty(s,\pi,r_{K/F})$ are the infinity parts of the L-functions and are products of Γ-factors. Also, the function $L(s,\pi,r_{K/F})$ does not vanish in the semiplane $\mathrm{Re}(s) > 1$.

We know the following (see [R2, Theorem D]).

PROPOSITION 6.2
Let K/E be a quadratic extension of number fields, and let π be a cuspidal automorphic representation of $\mathrm{GL}(2)/K$. Then there is an isobaric automorphic representation $\mathrm{As}_{K/E}(\pi)$ of $\mathrm{GL}(4)/E$ such that

$$L\big(s,\mathrm{As}_{K/E}(\pi)\big) = L(s,\pi,r_{K/E}).$$

This identity is true for the completed L-functions, that is, for the infinity parts of these L-functions as well.

With the same notation as in Proposition 6.2, for π a cuspidal cohomological automorphic representation as in Proposition 3.1, we denote by

$$\rho_{\mathrm{As}_{K/E}(\pi)} : \Gamma_F \to \mathrm{GL}_4(\bar{\mathbb{Q}}_l)$$

the representation associated to $\mathrm{As}_{K/E}(\pi)$. Thus we have $L(s,\rho_{\mathrm{As}_{K/E}(\pi)}) = L(s,\mathrm{As}_{K/E}(\pi))$, and $\rho_{\mathrm{As}_{K/E}(\pi)}$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_K}^{\Gamma_E}(\rho_\pi \otimes \rho_\pi^\theta),$$

which satisfies

$$\rho_{\mathrm{As}_{K/E}(\pi)}|_{\Gamma_K} = \rho_\pi \otimes \rho_\pi^\theta,$$

where θ is the nontrivial automorphism of K over E.
   We know the following (see [R3, Proposition 7.3]).

PROPOSITION 6.3
Let $K/E/F$ be extensions of number fields with $[K:E]=2$ and $[E:F]=2$, and let π be a cuspidal automorphic representation of $\mathrm{GL}(2)/K$. Then

$$L(s,\pi,r_{K/F}) = L\big(s,\mathrm{As}_{K/E}(\pi),r_{E/F}\big).$$

Moreover, this identity is true also for the completed L-functions.

Actually, the following is conjectured.

CONJECTURE 6.4
Let K/E be an extension of number fields, with $[K:E]=d$ and π be a cuspidal automorphic representation of $\mathrm{GL}(n)/K$. Then there is an isobaric automorphic representation $\mathrm{As}_{K/E}(\pi)$ of $\mathrm{GL}(n^d)/E$ such that

$$L\big(s,\mathrm{As}_{K/E}(\pi)\big) = L(s,\pi,r_{K/E}).$$

Moreover if $K/E/F$ are extensions of number fields and π is a cuspidal automorphic representation of $\mathrm{GL}(n)/K$, then

$$L\big(s,\pi,r_{K/F}\big) = L\big(s,\mathrm{As}_{K/E}(\pi),r_{E/F}\big).$$

These identities are true for the completed L-functions as well.

7. L-functions of twisted quaternionic Shimura fourfolds

Assume that k is a solvable extension of a totally real number field which contains E and that π is an infinite-dimensional cuspidal automorphic representation of $\mathrm{GL}(2)/F$ which appears in Theorem 3.2.

We recall that in §4 we denoted $\omega =: \pi_f^{\mathbf{K}} \circ \varphi$, and we have assumed that the field $L := \bar{\mathbb{Q}}^{\mathrm{Ker}\,\varphi}$ is a solvable extension of a totally real number field.

We denote by $\rho(\pi)^{\mathrm{ss}}$ the semisimplification of $\rho(\pi)$. Since at all but a finite number of finite places of E the representations $\rho(\pi)|_{\Gamma_{F_i}}$ and $\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}}$ yield the same local L-factors, and because from the weight-monodromy conjecture which is true for $\rho(\pi)$ and $\rho(\pi)^{\mathrm{ss}}$ (see [B, Theorem 2]), we know that the poles of the local L-factors corresponding to $\rho(\pi)$ and $\rho(\pi)^{\mathrm{ss}}$ are on the line $\mathrm{Re}(s) = -1$ (see [B, Theorem 2]), we deduce that the order of the pole at some s with $\mathrm{Re}(s) > 1$ of $L\big(s,(\rho(\pi) \otimes \omega)|_{\Gamma_K}\big) = \prod_{i=1}^{i=k} L(s,\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}$ is equal to the order of the pole at s of $L\big(s,(\rho(\pi)^{\mathrm{ss}} \otimes \omega)|_{\Gamma_k}\big) = \prod_{i=1}^{i=k} L(s,\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}$.

In this article we show the following result.

THEOREM 7.1

If k is a solvable extension of a totally real number field containing E, and $K := \bar{\mathbb{Q}}^{\mathrm{Ker}(\omega)}$ is a solvable extension of a totally real number field, then the function $L\big(s,(\rho(\pi)^{\mathrm{ss}} \otimes \omega)|_{\Gamma_k}\big)$ has a meromorphic continuation and satisfies a functional equation $s \leftrightarrow 1 - s$, and it has no zeros and is holomorphic in the semiplane $\mathrm{Re}(s) > 1$. (In some subcases of cases (iv) and (v) below, we have to assume that the existence of Asai representations $\mathrm{As}_{F_2/F_1}\pi'$ defined in Conjecture 6.4 if $[F_2 : F_1] = 3$ or 4 and π' is an automorphic representation of $\mathrm{GL}(2)/F_2$).

Since $L\big(s,(\rho(\pi)^{\mathrm{ss}} \otimes \omega)|_{\Gamma_k}\big) = \prod_{i=1}^{i=k} L(s,\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}$, in order to prove Theorem 7.1 it is sufficient to show the following.

PROPOSITION 7.2

Under the same assumptions as in Theorem 7.1, we have that $L(s,\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i)$ is holomorphic away from the lines $\mathrm{Re}(s) = 0$ or 1, satisfies a functional equation $s \leftrightarrow 1 - s$, and does not vanish in the semiplane $\mathrm{Re}(s) > 1$.

We describe now the representation $\rho^{\mathrm{ss}}(\pi)$ with π infinite dimensional (see [BR2, §7.2]). Let G be a group, and let K and H be two subgroups of G. We consider a representation

$$\tau : H \to \mathrm{GL}(W)$$

and a double coset $H\sigma K$ such that $d(\sigma) = |H \setminus H\sigma K| < \infty$. We define a representation $\tau_{H\sigma K}$ of K on the $d(\sigma)$-fold tensor product $W^{\otimes d(\sigma)}$. Consider the representatives $\{\sigma_1, \ldots, \sigma_{d(\sigma)}\}$ such that $H\sigma K = \bigcup H\sigma_j$. If $\gamma \in K$, then there exists $\xi_j \in H$ and an index $\gamma(j)$ such that

$$\sigma_j \gamma = \xi_j \sigma_{\gamma(j)}.$$

We define the representation

$$\tau_{H\sigma K}(\gamma)(\omega_1 \otimes \cdots \otimes \omega_{d(\sigma)}) = \tau(\xi_1)\omega_{\gamma^{-1}(1)} \otimes \cdots \otimes \tau(\xi_{d(\sigma)})\omega_{\gamma^{-1}(d(\sigma))}.$$

One can prove easily that the equivalence class of $\tau_{H\sigma K}$ is independent of the choice of the representatives $\sigma_1, \ldots, \sigma_{d(\sigma)}$.

Let $S_\infty - S_\infty' = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, and let $S := \bigcup \Gamma_F \sigma_i$. We write S as a disjoint union of double cosets

$$S = \bigcup_{j=1}^{k} \Gamma_F \sigma_j \Gamma_E,$$

and we denote by ρ_j the representation of Γ_E defined by $\rho_{\pi,\lambda}$ and the double coset $\Gamma_F \sigma_j \Gamma_E$. Then our representation $\rho^{\mathrm{ss}}(\pi)$ is isomorphic to $\rho_1 \otimes \cdots \otimes \rho_k$.

Define $\Gamma_{E_1} := \sigma_1^{-1}\Gamma_F\sigma_1 \cap \sigma_2^{-1}\Gamma_F\sigma_2 \cap \sigma_3^{-1}\Gamma_F\sigma_3 \cap \sigma_4^{-1}\Gamma_F\sigma_4$.

We distinguish several cases (up to some permutations).

(i) We have $k = 4$ and $\Gamma_F\sigma_j\Gamma_E = \Gamma_F\sigma_j$ for $j = 1, \ldots, 4$. Then, for a fixed j, from $\Gamma_F\sigma_j\Gamma_E = \Gamma_F\sigma_j$ we get $\Gamma_E \subseteq \sigma_j^{-1}\Gamma_F\sigma_j$, and thus we obtain $E_1 = E$ and

$$\rho(\pi)^{\mathrm{ss}} \cong \rho_\pi|_{\Gamma_E}^{\sigma_1} \otimes \rho_\pi|_{\Gamma_E}^{\sigma_2} \otimes \rho_\pi|_{\Gamma_E}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_E}^{\sigma_4},$$

where

$$\rho_\pi|_{\Gamma_E}^{\sigma_j}(\gamma) = \rho_\pi(\sigma_j\gamma\sigma_j^{-1}).$$

Hence

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}},$$

where the representation $\pi_i^{\sigma_j}$ satisfies $\rho_{\pi_i^{\sigma_j}} \cong \rho_\pi|_{\Gamma_{F_i}}^{\sigma_j}$.

(ii) We have $k = 3$ and $\Gamma_F\sigma_1\Gamma_E = \Gamma_F\sigma_1$ and $\Gamma_F\sigma_2\Gamma_E = \Gamma_F\sigma_2$ and $\Gamma_F\sigma_3\Gamma_E = \Gamma_F\sigma_3 \cup \Gamma_F\sigma_4$. Then it is easy to see that $[E_1 : E] = 2$, E contains $\sigma_1(F)$ and $\sigma_2(F)$ but does not contain $\sigma_3(F)$ and $\sigma_4(F)$, that $\sigma_3 \cdot \sigma_4^{-1}$ is the nontrivial automorphism of E_1 over E, and that

$$\rho(\pi)^{\mathrm{ss}} \cong \rho_\pi|_{\Gamma_E}^{\sigma_1} \otimes \rho_\pi|_{\Gamma_E}^{\sigma_2} \otimes \rho(\pi)'',$$

where $\rho(\pi)''$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_{E_1}}^{\Gamma_E}\left(\rho_\pi|_{\Gamma_{E_1}}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_4}\right),$$

which satisfies

$$\rho(\pi)''|_{\Gamma_{E_1}} = \rho_\pi|_{\Gamma_{E_1}}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_4}.$$

We consider two subcases

(a) $E_1 \subseteq F_i$. Then

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}.$$

(b) $E_1 \nsubseteq F_i$. Then $[E_1 F_i : F_i] = 2$, and since $\rho_{\pi,\lambda}|_{\Gamma_{F''}}$ is modular, and F'' is a solvable extension of $E_1 F_i$, we obtain that $\rho_\pi|_{\Gamma_{E_1 F_i}}^{\sigma_3}$ is modular, and thus there exists an automorphic representation $\pi_i'^{\sigma_3}$ of $\mathrm{GL}(2)/FF_i$ such that $\rho_{\pi,\lambda}|_{\Gamma_{E_1 F_i}}^{\sigma_3} \cong \rho_{\pi_i'^{\sigma_3}}$. We get

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\mathrm{As}_{E_1 F_i / F_i}(\pi_i'^{\sigma_3})}.$$

(iii) We have $k = 2$ and $\Gamma_F \sigma_1 \Gamma_E = \Gamma_F \sigma_1 \cup \Gamma_F \sigma_2$ and $\Gamma_F \sigma_3 \Gamma_E = \Gamma_F \sigma_3 \cup \Gamma_F \sigma_4$. Then it is easy to see that

$$\rho(\pi)^{\mathrm{ss}} \cong \rho(\pi)' \otimes \rho(\pi)'',$$

where $\rho(\pi)'$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_{E_2}}^{\Gamma_E} (\rho_\pi|_{\Gamma_{E_2}}^{\sigma_1} \otimes \rho_\pi|_{\Gamma_{E_2}}^{\sigma_2}),$$

which satisfies

$$\rho(\pi)'|_{\Gamma_{E_2}} = \rho_\pi|_{\Gamma_{E_2}}^{\sigma_1} \otimes \rho_\pi|_{\Gamma_{E_2}}^{\sigma_2},$$

and $\rho(\pi)''$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_{E_3}}^{\Gamma_E} (\rho_\pi|_{\Gamma_{E_3}}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_{E_3}}^{\sigma_4}),$$

which satisfies

$$\rho(\pi)''|_{\Gamma_{E_3}} = \rho_\pi|_{\Gamma_{E_3}}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_{E_3}}^{\sigma_4},$$

where the fields E_2 and E_3 verify the following properties $[E_2 : E] = 2$; $[E_3 : E] = 2$; E does not contain $\sigma_1(F)$, $\sigma_2(F)$, $\sigma_3(F)$ and $\sigma_4(F)$; $E_1 = E_2 E_3$ and E_2 could be equal to E_3 and $\sigma_1 \cdot \sigma_2^{-1}$ is the nontrivial automorphism of E_2 over E and $\sigma_3 \cdot \sigma_4^{-1}$ is the nontrivial automorphism of E_3 over E.

Thus we have two cases.

(a) $E_1 = E_2 = E_3$. Then we consider two subcases.

(1) $E_1 \subseteq F_i$. Then

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}.$$

(2) $E_1 \nsubseteq F_i$. Then $[E_1 F_i : F_i] = 2$ and

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\mathrm{As}_{E_1 F_i / F_i}(\pi_i'^{\sigma_1})} \otimes \rho_{\mathrm{As}_{E_1 F_i / F_i}(\pi_i'^{\sigma_3})} \otimes \chi_i,$$

where $\pi_i'^{\sigma_1}$ and $\pi_i'^{\sigma_3}$ are automorphic representations of $\mathrm{GL}(2)/E_1 F_i$ defined as above.

(b) $E_2 \neq E_3$. Then we consider several subcases.

(1) $E_1 \subseteq F_i$. Then

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}.$$

(2) $E_3 \nsubseteq F_i$, but $E_2 \subseteq F_i$. Then $[E_3 F_i : F_i] = 2$, and

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\mathrm{As}_{E_3 F_i / F_i}(\pi_i'^{\sigma_3})},$$

where $\pi_i'^{\sigma_3}$ is an automorphic representation of $\mathrm{GL}(2)/E_3 F_i$ defined as above.

(3) $E_2 \nsubseteq F_i$, but $E_3 \subseteq F_i$. This case is similar to case (2).

(4) $F_i \cap E_1 = E$. Then $[E_2 F_i : F_i] = 2$ and $[E_3 F_i : F_i] = 2$ and

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\mathrm{As}_{E_2 F_i / F_i}(\pi_i'^{\sigma_1})} \otimes \rho_{\mathrm{As}_{E_3 F_i / F_i}(\pi_i'^{\sigma_3})} \otimes \chi_i,$$

where $\pi_i'^{\sigma_1}$ and $\pi_i'^{\sigma_3}$ are automorphic representations of $\mathrm{GL}(2)/E_2 F_i$ and of $\mathrm{GL}(2)/E_3 F_i$ defined as above.

(iv) We have $k = 2$ and $\Gamma_F \sigma_1 \Gamma_E = \Gamma_F \sigma_1$ and $\Gamma_F \sigma_2 \Gamma_E = \Gamma_F \sigma_2 \cup \Gamma_F \sigma_3 \cup \Gamma_F \sigma_4$. Then it is easy to see that $[E_1 : E] = 3$ or $[E_1 : E] = 6$ and E_1 is Galois over E, with $\mathrm{Gal}(E_1/E) \cong \mathbb{Z}_3$ or $\mathrm{Gal}(E_1/E) \cong \mathrm{S}_3$, and E contains $\sigma_1(F)$, but E does not contain $\sigma_2(F)$, $\sigma_3(F)$ and $\sigma_4(F)$. Thus we have two cases.

(a) $[E_1 : E] = 3$. Then $\sigma_3 \cdot \sigma_2^{-1}$ is a nontrivial automorphism of E_1 over E and $\sigma_3 \cdot \sigma_2^{-1} = \sigma_4 \cdot \sigma_3^{-1}$. We get

$$\rho(\pi)^{\mathrm{ss}} \cong \rho_\pi|_{\Gamma_E}^{\sigma_1} \otimes \rho(\pi)''',$$

where $\rho(\pi)'''$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_{E_1}}^{\Gamma_E} (\rho_\pi|_{\Gamma_{E_1}}^{\sigma_2} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_4})$$

which satisfies

$$\rho(\pi)'''|_{\Gamma_{E_1}} \cong \rho_\pi|_{\Gamma_{E_1}}^{\sigma_2} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_4}.$$

We consider two subcases.

(1) $E_1 \subseteq F_i$. Then

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}.$$

(2) $E_1 \nsubseteq F_i$. Then $[E_1 F_i : F_i] = 3$, and *we should have*

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\mathrm{As}_{E_1 F_i / F_i}(\pi_i'^{\sigma_2})},$$

where $\pi_i'^{\sigma_2}$ is an automorphic representation of $\mathrm{GL}(2)/E_1 F_i$ defined as above and $\mathrm{As}_{E_1 F_i / F_i}(\pi_i'^{\sigma_2})$ is the conjectured automorphic representation defined in Conjecture 6.4.

(b) $[E_1 : E] = 6$. Then it is easy to see that $\rho(\pi)^{\mathrm{ss}}$ satisfies the following properties:

(1) $E_1 \subseteq F_i$. Then

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}.$$

(2) $\sigma_2(F) \subseteq F_i$, but $E_1 \nsubseteq F_i$. Then $[E_1 F_i : F_i] = 2$ and $\sigma_3 \cdot \sigma_4^{-1}$ is the nontrivial automorphism of $E_1 F_i$ over F_i and

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\mathrm{As}_{E_1 F_i / F_i}(\pi_i'^{\sigma_3})}.$$

(3) $\sigma_3(F) \subseteq F_i$, but $E_1 \nsubseteq F_i$. This case is similar to case (2).

(4) $\sigma_4(F) \subseteq F_i$, but $E_1 \nsubseteq F_i$. This case is also similar to case (2).

(5) $\sigma_2(F) \nsubseteq F_i$ and $\sigma_3(F) \nsubseteq F_i$ and $\sigma_4(F) \nsubseteq F_i$. Then $[\sigma_2(F)F_i : F_i] = 3$ and *we should have*

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\mathrm{As}_{\sigma_2(F)F_i/F_i}(\pi_i'^{\sigma_2})},$$

where $\pi_i'^{\sigma_2}$ is an automorphic representation of $\mathrm{GL}(2)/\sigma_2(F)F_i$ defined as above and $\mathrm{As}_{\sigma_2(F)F_i/F_i}(\pi_i'^{\sigma_2})$ is the conjectured Asai automorphic representation defined in Conjecture 6.4.

(v) $k = 1$ and $\Gamma_F \sigma_1 \Gamma_E = \Gamma_F \sigma_1 \cup \Gamma_F \sigma_2 \cup \Gamma_F \sigma_3 \cup \Gamma_F \sigma_4$. Then it is easy to see that $[\sigma_1(F) : E] = 4$ and $\rho(\pi)^{\mathrm{ss}}$ satisfies, in particular,

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{E_1}} \cong \rho_\pi|_{\Gamma_{E_1}}^{\sigma_1} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_2} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_3} \otimes \rho_\pi|_{\Gamma_{E_1}}^{\sigma_4},$$

and hence if $E_1 \subseteq F_i$, then

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}.$$

The field E_1 is Galois over E, and $\mathrm{Gal}(E_1/E)$ is a subgroup of S_4 of order divisible by 4.

Anyway, $\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i$ has one of the forms described above at (i), (ii)(b), (iii)(b)(4), (iv)(b)(5), or

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\mathrm{As}_{K_i/F_i}(\pi_i'^{\sigma_j})} \otimes \chi_i,$$

where K_i is contained in $E_1 F_i$ and $K_i = \sigma_j(F)F_i$ for some j and $[K_i : F_i] = 4$ and $\pi_i'^{\sigma_j}$ is an automorphic representation of $\mathrm{GL}(2)/K_i$ defined as above and $\mathrm{As}_{K_i/F_i}(\pi_i'^{\sigma_j})$ is the conjectured Asai automorphic representation defined in Conjecture 6.4.

7.1. The proof of Proposition 7.2

In this section we prove Proposition 7.2.

In cases (i), (ii)(a), (iii)(b)(1), (iv)(a)(1), (iv)(b)(1), and so on, we know that

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}},$$

which, from Proposition 5.1, is a tensor product of two Galois representations $\rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}}$ and $\rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}$ arising from automorphic representations of $\mathrm{GL}(4)$. Thus from Proposition 5.2 we deduce Proposition 7.2. (We remark that when the representations $\rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}}$ and $\rho_{\pi_i^{\sigma_3}} \otimes \rho_{\pi_i^{\sigma_4}}$ are reducible, they are sums of cuspidal automorphic representations of $\mathrm{GL}(1)$, $\mathrm{GL}(2)$, or $\mathrm{GL}(3)$, and even in this case we can apply Proposition 5.2.)

In cases (ii)(b), (iii)(b)(2), (iv)(b)(2), and so on, we know that $[E_1 F_i : F_i] = 2$ and

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}} \otimes \rho_{\mathrm{As}_{E_1 F_i/F_i}(\pi_i'^{\sigma_3})},$$

where $\pi_i'^{\sigma_3}$ is an automorphic representation of $\mathrm{GL}(2)/E_1 F_i$. From Propositions 5.1 and 6.2, we deduce that $\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i$ is a tensor product of two Galois representations $\rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\pi_i^{\sigma_2}}$ and $\rho_{\mathrm{As}_{E_1 F_i/F_i}(\pi_i'^{\sigma_3})}$ arising from automorphic representations of $\mathrm{GL}(4)$. Thus from Proposition 5.2 we deduce Proposition 7.2.

In cases (iii)(b)(4), and so on, we know that $[E_2 F_i : F_i] = 2$ and $[E_3 F_i : F_i] = 2$ and

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\mathrm{As}_{E_2 F_i / F_i}}(\pi_i'^{\sigma_1}) \otimes \rho_{\mathrm{As}_{E_3 F_i / F_i}}(\pi_i'^{\sigma_3}) \otimes \chi_i,$$

where $\pi_i'^{\sigma_1}$ and $\pi_i'^{\sigma_3}$ are automorphic representations of $\mathrm{GL}(2)/E_2 F_i$ and of $\mathrm{GL}(2)/E_3 F_i$. Hence $\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i$ is a tensor product of two Galois representations $\rho_{\mathrm{As}_{E_2 F_i / F_i}}(\pi_i'^{\sigma_1})$ and $\rho_{\mathrm{As}_{E_3 F_i / F_i}}(\pi_i'^{\sigma_3}) \otimes \chi_i$ arising from automorphic representations of $\mathrm{GL}(4)$, and thus from Proposition 5.2 we deduce Proposition 7.2.

In the cases (iv)(a)(2), and so on, we know that $[E_1 F_i : F_i] = 3$, and if we assume the existence of the Asai automorphic representation $\mathrm{As}_{E_1 F_i / F_i}(\pi_i'^{\sigma_2})$ which appears in Conjecture 6.4, we have

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\pi_i^{\sigma_1} \otimes \chi_i} \otimes \rho_{\mathrm{As}_{E_1 F_i / F_i}}(\pi_i'^{\sigma_2}).$$

Hence under the above assumption, $\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i$ is a tensor product of two Galois representations $\rho_{\pi_i^{\sigma_1} \otimes \chi_i}$ and $\rho_{\mathrm{As}_{E_1 F_i / F_i}}(\pi_i'^{\sigma_2})$ arising from automorphic representations, and thus from Proposition 5.2 we deduce Proposition 7.2.

In some subcases of case (v), we have $[K_i : F_i] = 4$ and

$$\rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i \cong \rho_{\mathrm{As}_{K_i / F_i}}(\pi_i'^{\sigma_j}) \otimes \chi_i$$

if we assume the existence of the automorphic representation $\mathrm{As}_{K_i / F_i}(\pi_i'^{\sigma_j})$ which appears in Conjecture 6.4. Hence under this assumption we deduce Proposition 7.2.

Also, we observe that in case (v), when $[K_i : F_i] = 4$, if we assume that K_i contains a quadratic extension F_i' of F_i, then

$$L\big(s, \rho(\pi)^{\mathrm{ss}}|_{\Gamma_{F_i}} \otimes \chi_i\big) = L\big(s, \mathrm{As}_{K_i / F_i'}(\pi_i'^{\sigma_j} \otimes \chi_i|_{\Gamma_{K_i}}), r_{F_i' / F_i}\big),$$

and in this case, from Proposition 6.1, we deduce Proposition 7.2. $\qquad\square$

If π is one-dimensional, then $\rho(\pi)$ is a 6-dimensional representation, that is, a virtual sum of representations induced from characters, and Proposition 7.1 is trivial.

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