or, equivalently, $\gtrsim$ fails to be basic binary. (If we include Levi's lexicographic "security" considerations in determining admissible choices, then the choice function violates Sen's (1977, page 64) Property α, as Example 2 illustrates. With "security," the insurance is uniquely admissible in a choice between it and one of the two bets. But the insurance is inadmissible as a choice among the three options.)

In *Foundations* (Section 7.2) Savage uses a group decision rule that fails to satisfy his **P1**: the postulate that preference is a weak order, where an option is admissible if it maximizes utility for some p in **P**. (The set **P** corresponds to the convex combination of personal probabilities held by the individuals in the group.) Again, in Section 13.5, he defends group decision rules that violate **P1**. Is it not wise to propose the same norms in group and in individual choices? I think so. In that case, we can adopt Savage's own reasons to argue for the liberalization of strict Bayesian norms proposed by Smith, Levi, et al. But what is left of Savage's project to complete the reduction of quantitative personal probability to choice (without extraneous notions of probability)? Can the set **P** be recovered from choice behavior without the tacit assumption of a utility function as used in the "Dutch Book" arguments, i.e., without requiring that the conjunction of favorable gambles be favorable? I believe that remains an open question.

ADDITIONAL REFERENCES

GIRON, F. J. and RIOS, S. (1980). Quasi-Bayesian behavior: A more realistic approach to decision making? In *Bayesian Statistics II* (J. M. Bernardo, M. H. DeGroot, D. V. Lindley, and A. F. M. Smith, eds.) 17–38. University Press, Valencia.

LEVI, I. (1974). On indeterminate probabilities. *J. Phil.* **71** 391–418.

LEVI, I. (1980). *The Enterprise of Knowledge*. MIT Press, Cambridge.

SEN, A. K. (1977). Social choice theory: a re-examination. *Econometrica* **45** 53–89.

SHIMONY, A. (1955). Coherence and the axioms of confirmation. *J. Symbolic Logic* **20** 8–20.

Comment

Mervyn Stone

Dr. Fishburn's review is both authoritative and timely. It is good to see a paper that dares, in this new journal, to propagate by style and content the best *Annals* tradition—clear exposition and comparison of important mathematical structures, unclouded by the polemical discussion that inevitably arises when mathematical concepts are ultimately related to the problems of induction and decision.

It will be interesting to see if the present discussants let him get away with it. They may not—for the simple reason that there is a sizeable school of "infinitarians" who will be disposed to sift through Dr. Fishburn's fine deposits for items to advance their cause (see Scozzafava (1984) for examples of the art). For the purposes of this discussion, an "infinitarian" is one who will not countenance the restriction to countable additivity, and is prepared to defend any implications of this stand, including those that are regarded by some as manifest counterexamples to the view that finite additivity rules OK.

Dr. Fishburn raises a polemical little finger, as it were, when he states that the assumption of monotone continuity is "quite appealing." It is somewhat paradoxical that the effect of the monotone continuity axiom, whose statement involves countable infinities, is to allow one then to forget about "infinity" as a point in the sample space, and get on quietly with the job of using infinity, in the sample space as a whole, as a framework for useful approximation of necessarily finite, practical induction and decision. In contrast, the axioms for merely finite additivity do not explicitly involve infinity, but have unresolved problems of infinity that ought, I think, to disturb the practical inferencer or decision-maker who adopts a finitely additive P of the type in question.

One widely considered example has

$$S = \{(x, \theta): x = 1, 2, 3, \ldots, \qquad \theta = 0, 1\}$$

with

$$P(\Theta = 0) = P(\Theta = 1) = 1/2,$$
$$P(X = x \mid \Theta = \theta) = 2^{-(x+\theta)}.$$

Note the missing probability 1/4 in the countable union of $(x, 1)$, $x = 1, 2, 3, \ldots$. Formally, P is "nonconglomerable in the x margin." The setup implies

$$(1)\quad \left.\begin{array}{l} P(X > 12) > 1/4 \\ P(\Theta = 1 \mid X > 12)/P(\Theta = 0 \mid X > 12) > 1000 \end{array}\right\}.$$

Mervyn Stone is Professor of Statistics, Department of Statistical Science, University College London, Gower Street, London WC1E 6BT, United Kingdom.

Defining q as the set of conditional distributions

$$\{(P(\Theta = 0 \mid X = x) = 2/3,\ P(\Theta = 1 \mid X = x) = 1/3)): x = 1, 2, 3, \ldots\},$$

Kadane, Schervish, and Seidenfeld (1985) assert that it is "reasonable to claim that q is the posterior for θ given X once finite additivity is accepted," and that the example "makes clear the need for a less restrictive definition of posterior distribution that will allow inference even when a probability cannot be made conglomerable in a specific partition." For q,

$$(2) \qquad \frac{P(\Theta = 1 \mid X = x)}{P(\Theta = 0 \mid X = x)} = 1/2, \quad x = 1, 2, \ldots.$$

To ignore the conflict between (1) and (2), on the grounds that this is merely an expression of acceptable nonconglomerability, is to turn a blind eye to the problem that it raises in the use of P to approximate honest opinion about what odds to quote for Θ given $12 < X < 12^{144}$.

Of course, not all the applications of finitely non-countably additive probabilities are unattractive. As yet, there appear to be no axioms that will discriminate either the probabilities or the applications that are acceptable. It is not easy to see how the necessary weakening or replacement of monotone continuity might be engineered. There may be one or two clues in the work of Seidenfeld and Schervish (1983). Let us hope that Dr. Fishburn will return once again to the topic, in a survey that will remove the remaining obscurities.

ADDITIONAL REFERENCES

KADANE, J. B., SCHERVISH, M. AND SEIDENFELD, T. (1985). Statistical implications of finitely additive probability. In *Bayesian Inference and Decision Techniques with Applications: Essays in Honor of Bruno de Finetti* (P. K. Goel and A. Zellner, eds.). Elsevier Science Publishers B. V. (North-Holland), Amsterdam.

SCOZZAFAVA, R. (1984). A survey of some common misunderstandings concerning the role and meaning of finitely additive probabilities in statistical inference. *Statistica* **44** 21–45.

Comment

William D. Sudderth

Most of Professor Fishburn's interesting article treats axiom schemes for the relations *is more probable than* and *is at least as probable as*, and the question of when these schemes lead to a compatible probability measure. There are two other approaches to formulating axioms for probabilities interpreted as degrees of belief. The first is due to de Finetti (1937, 1949) and gives a direct economic interpretation to probability numbers. The second was developed by Cox (1961) and Jaynes to formulate axioms for rational beliefs and for how such beliefs should be modified. Perhaps some readers will be interested in a brief description of these two alternative routes.

One version of the de Finetti theory begins with a function P which assigns a real number $P(A)$ to certain events A. Think of $P(A)$ as your price in dollars for a ticket worth \$1 if A occurs and \$0 if not. You are required to be willing to buy or sell a finite number of tickets on any of the events in a given collection $\mathscr{A}$. (There is no need to assume $\mathscr{A}$ is an algebra.) Then de Finetti shows that you are *coherent* in the sense that you cannot be made a sure loser if and only if P is a finitely additive probability measure (or can be extended to be one if $\mathscr{A}$ is not an algebra). An advantage of this approach is that the conditional probability $P(A \mid B)$ can be defined directly as the price of a \$1 ticket on A with the provision that the transaction is called off if B does not occur. A requirement of coherence for these conditional transactions leads to the formula

$$P(AB) = P(B)P(A \mid B)$$

which in turn implies the finite form of Bayes' formula given in Section 7. All of this is explained in detail by de Finetti (1949). There are extensions of de Finetti's result which yield Bayes' formula for infinite partitions (cf. Heath and Sudderth (1978) and Lane and Sudderth (1984)). These extensions involve a strengthening of the coherence condition which is not acceptable to all of de Finetti's followers.

In the Cox-Jaynes theory it is assumed that the *plausibility* of A on the evidence B can be represented by a real number $(A \mid B)$. Qualitative arguments are given for a postulate stating that the plausibility number $(AB \mid C)$ should be some function F of $(B \mid C)$ and $(A \mid BC)$. Because AB is the same as BA, the function F is required to give the same answer if its arguments

William D. Sudderth is Professor of Theoretical Statistics, School of Statistics, 270 Vincent Hall, University of Minnesota, Minneapolis, Minnesota 55455.