and processing only via probabilistic means (e.g., Bayes theorem), while explicitly recognizing the inexactness of probability elicitation. This approach, long advocated by I. J. Good (cf. Good, 1983), is reviewed and discussed (as the "robust Bayesian" viewpoint) in Berger (1984, 1985). Of particular note, in terms of axiomatics, is that Smith (1961), Good (1962), Giron and Rios (1980), and others show that possible noncomparability, together with a reasonable set of other axioms, essentially yield the robust Bayesian approach.

As a second example of how "reality" might impact on axiomatics, consider the issue of finitely additive versus countably additive probabilities. Axiomatically, additional assumptions must be made to guarantee countably additive probabilities, assumptions which tend to be somewhat obscure and noncompelling. Attempts to work with finitely additive probabilities, however, encounter the difficulty that conditional distributions (or posterior probabilities) are often not well-defined, so that additional assumptions end up being needed anyway. And the nature of these assumptions is perhaps even more obscure than those leading to countable additivity; one might well conclude that the countably additive domain is the least objectionable arena in which to perform.

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Comment

Terrence L. Fine

My remarks focus on the themes of extension, tolerance for limited precision, the restricted applicability of the familiar concept of numerical probability, and the possibilities for other concepts of probability that are suggested by the axiomatic measurement-theoretic approach to comparative probability. Dr. Fishburn provides us with an authoritative survey of several axiom systems for binary relations of comparative (qualitative) probability that have been developed in the context of an interpretation of subjective probability based upon the degrees of belief of an individual. One might hope that a study of such axiom systems for comparative probability would lead us closer to the conceptual issues and roots of probabilistic reasoning and rational beliefs about uncertainty and thereby also enable us to develop such reasoning processes and model such beliefs through a probability-like mathematical structure. A process of axiomatization enables us to decompose a complex issue into a related set of simpler component issues that can then be examined closely on their merits. When properly engaged in, such a study does not prejudge its outcome. By observing the nature and strength of the axioms necessary to insure that the resulting model is a finitely or countably additive numerical probability measure, we can gain insight into the limitations of this familiar and often reliable model. By eliminating those axioms that appear to be objectionable in particular application domains, we can develop alternative concepts of probability useful for fairly representing probabilistic reasoning about either individual beliefs or objective nondeterministic phenomena, as appropriate for the domain. Clearly, the process of axiom selection must be guided by sound interpretations of the probability concept.

Regrettably, but understandably, few of these issues are addressed with sufficient emphasis either in this survey or in much of the related literature cited therein. While the opening quotations might lead us to anticipate an analysis of the link between belief or expectation (on the subjective interpretation) and the mathematical apparatus that is then deployed, this is

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largely not the case. In Section 3, we do encounter examples of uncertain situations wherein we may doubt the adequacy of the axioms previously introduced for comparability, transitivity, and additivity. It is here that we can see the merits of the conceptual decomposition encouraged by the axiomatic approach. Yet, notwithstanding the cogency of these examples, there has been little work on following out their implications for probability models that differ from the familiar numerical one and that are suitable, say, for modeling individual beliefs (but see Walley and Fine (1979)). Albeit, notice is taken of upper and lower probability in connection with interval orders.

The familiar numerical probability model can only represent a given comparative probability ordering if the ordering satisfies the infinite axiom schema detailed in Section 2 in the definition of strong additivity. Contemplation of the complicated necessary and sufficient condition of strong additivity should reduce our bias toward accepting the numerical probability structure as a natural model capable of representing all of the random (chance, uncertainty, indeterminate) phenomena about which we attempt to reason probabilistically.

In Sections 2 and 4 we find embedding, extension, scaling, Archimedean, and partition axioms whose purpose is to force a unique representing probability measure. Such axioms have little grounding in our intuitions about probabilistic reasoning. While the Caratheodory extension theorem allows us to embed a given probability algebra in the particular extension that is its σ-algebra, there is no reason to expect this to hold in all mathematical or real world settings; indeed we know that we cannot usually extend probability measures on a given algebra of subsets of a sample space to, say, the power set of the sample space. Again, the usual numerical probability theory does allow us to independently combine conventional random experiments through a product measure on a product space. However, there is little reason to expect this convenient property to hold over the full range of probabilistic phenomena. For example, what in our understanding of the relative likelihoods of various individuals being elected President in 1988 enables us to compare these likelihoods with, say, those we might attach to the outcomes of personal medical treatments and thereby form a joint model for the two random phenomena?

While embeddings, scalings, etc. appear as innocent and uncontroversial assumptions from the conventional viewpoint, the powerful conclusions deducible from them demonstrate that they imply substantial commitments about the nature of probabilistic phenomena. As was shown in Kaplan and Fine (1977) not all comparative probability relations satisfying the basic axioms of Section 1 can be either extended to larger order relations or combined with each other as a joint order. Qualitatively speaking, the reason for this is that the ability to form an extension or joint order presumes more precise knowledge of the original probabilistic relations between events than may either be available or, more importantly, mathematically consistent with the given ordering.

There is little reason to assume that all random phenomena or sets of beliefs admit of gradation into arbitrarily fine degrees. Naive introspection into our own beliefs certainly suggests that there are limits beyond which we do not feel we can refine what we know about particular events. In this regard, chance and uncertainty differ from such arbitrarily divisible notions as mass and length even though our common reliance on measure theory blurs this distinction. The usual concept of numerical probability prejudges this issue to assert that, at least in principle, degrees of belief may be measured arbitrarily precisely. It is an overlooked virtue of the comparative probability formulation that it lays bare the fact that this is a highly problematic postulate.

A theory of rational belief needs to respect the limited precision with which we can scale our confidence in our beliefs. The absence of unicity in the class of probability measures representing a comparative probability ordering should be welcomed as a sign of realism and only cautiously eliminated through the invocation of joint orders with side experiments, standard series, etc. Ignorance should be given its due. A serious interest in the implications of the axiomatic approach to probability through order relations should also lead to the serious consideration of such alternative probability concepts as those of comparative probability itself and interval valued or upper and lower probability. Particular cases of the latter have been studied under the names of belief functions (Shafer, 1976), lower envelopes (Levi, 1980; Kyburg, 1974), and undominated lower probability (Kumar and Fine, 1985; Papamarcou and Fine, 1986; Grize and Fine, 1986). Walley (1987) provides a masterful development of subjective probability in terms of lower envelopes that fully discusses many of the issues to which we have referred.

Section 7 briefly discusses some of the issues regarding conditional probability. This discussion is in keeping with the tenor of the preceding measurement-theoretic treatments. Conditioning is also a covert way of extending a given comparative probability ordering. While one can naturally extend to comparisons of the form $A \mid B$ versus $C \mid B$, it requires additional information to compare $A \mid B$ to $C \mid D$. Indeed, as was first noted by Kaplan (1971), it is not always possible to extend a comparative probability ordering satisfying the axioms of Section 1 to any conditional comparative ordering. We might also note recent activity concerning the definition of conditioning and its relation to the familiar Bayes rule. When one interprets conditional probabilities as updates of probabilities in the light of new evidence, then it is suggested that we may have more flexibility in the choice of updated conditional probabilities than is allowed in classical probability (Diaconis and Zabell, 1982; Shafer, 1982). Conditional probability in a lower envelope setting is thoroughly treated in Walley (1987, Chapter 7).

Finally, subjective probability is complementary to objective, frequentist-based probability, but the two approaches taken together neither exhaust the domain of random phenomena nor the possible interpretations for the axioms surveyed (Fine, 1983). Neither theory accounts for intrinsic limits to the precision with which we can model random phenomena when we need to account for hesitancy in the case of individual beliefs and unstable relative frequencies in the frequentist case. Nor do they exhaust the possibilities for interpreting probability and reasoning about random phenomena.

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Comment

Teddy Seidenfeld

This essay provides an informative overview of axiomatic theories whose common theme is a development of quantitative personal probability from the qualitative or comparative binary relation '$A > B$', understood as "A is subjectively more probable than B." The pathbreaking works of Ramsey, de Finetti, and Savage contribute to this project by giving the comparative probability relation an operational, decision theoretic basis. Roughly put, they propose that A is subjectively more probable than B provided the lottery L_A, having a desirable prize awarded if event A occurs and status quo otherwise, is preferred to ($\succ$) the lottery L_B which has the desirable prize awarded if B occurs.

Definition

$$(1) \qquad A > B \quad \text{iff} \quad L_A \succ L_B.$$

In Savage's hands, quantitative personal probability is reduced to the qualitative relation $\geq$ which, in turn, is reduced to (weak) preference among lotteries $\succsim$.

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For Ramsey and de Finetti, quantitative probability may be "elicited" directly from choices among gambles and agrees with the comparative relation $>$ (defined above). But the common thread is that rational belief is constrained by coherent preference, and binary choices reveal preferences.

In Section 3 of his paper, Professor Fishburn turns his attention to theories of personal probability different from the strict Bayesian position of Ramsey, de Finetti, and Savage. Specifically, he rightly considers a liberalization which relaxes the assumption that $>$ is a weak order. To understand why this is a reasonable change from the norms of strict Bayesianism recall, e.g., Smith's (1961) idea for "medial odds," to permit a spread in the odds as Levi (1980, Section 7.3) so aptly puts it.

Consider a wager on event A with a combined stake s at odds $p : 1 - p$ ($0 \leq p \leq 1$). You bet on A by putting up ps (which is lost in case A fails to occur) with the prospect of winning $(1 - p)s$ in case A occurs. (These wagers are a special case of Smith's bets "on A against B," obtained by letting B be the sure event.) Also, there is the associated wager against A, equivalent to a bet on $\neg A$ at odds of $(1 - p) : p$, where you place $(1 - p)s$ on $\neg A$, lost in case A occurs, with the prospect