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Optimal Designs for Comparing Test Treatments with Controls

A. S. Hedayat, Mike Jacroux and Dibyen Majumdar

Abstract. This article outlines existing knowledge on optimal designs for comparing test treatments with controls under 0-, 1- and 2-way elimination of heterogeneity models. The results are motivated through numerical examples.

Key words and phrases: A-optimal designs, MV-optimal designs, BTIB designs, BIB designs, controls, test treatments, model robust designs.

1. INTRODUCTION

In this article we consider the problem of comparing a set of test treatments with a control or standard treatment. Such a problem arises, for example, in screening experiments or in the beginning of a long term experimental investigation where it is initially desired to determine the relative performance of the new test treatments with respect to the control or standard treatment. For specificity, suppose four new methods for performing a certain task become available and we wish to conduct an experiment to compare the new methods to the standard procedure currently being used to perform the given task. Further, suppose there are 18 experimental units available for conducting the study. Thus we need to design an experiment for comparing the control, denoted by 0, with four test treatments, denoted by 1, 2, 3 and 4. Any particular allocation of treatments to experimental units is called a design and is denoted by d.

As a statistical problem, the question of how to compare the test treatments with the control cannot be answered unless it is asked in a more precise manner. To begin with we need to postulate a model for the response observed upon application of a treatment, test treatment or control, to an experimental unit. In this article we shall consider three possible models: 0-way elimination of heterogeneity model in which all experimental units are homogeneous before application of treatments:

$$y_{ij} = \mu + t_i + \varepsilon_{ij}; \qquad (1.1)$$

1-way elimination of heterogeneity model in which experimental units can be divided into several homogeneous blocks:

$$y_{ij} = \mu + t_i + \beta_j + \varepsilon_{ij}; \qquad (1.2)$$

2-way elimination of heterogeneity model in which the experimental units can be conceptually arranged according to rows and columns:

$$y_{ijl} = \mu + t_i + \beta_j + \rho_l + \varepsilon_{ijl}. \qquad (1.3)$$

In models (1.1), (1.2) and (1.3) the y's denote observations obtained after applying treatment i to an experimental unit occurring in block j or column j and row l, t_i represents the effect of treatment i, β_j the effect of block or column j, ρ_l the effect of row l, and the ε's are independent random error terms having expectation zero and constant variance σ^2.

Now we can be more precise about what we mean by comparing test treatments with a control. In particular, because our primary goal is to determine which among the test treatments might be better than the control, we would like to estimate the magnitude of each $t_i - t_0$ with as much precision as possible. More precise comparisons among test treatments found to perform better than the control at this initial stage is generally left to later experimentation. Under the assumptions made above, the method of least squares yields the best linear unbiased estimators $\hat{t}_{di} - \hat{t}_{d0}$ for the contrasts $t_i - t_0$ under a given design d. In assigning treatments to experimental units, we have to make sure that the contrasts $t_i - t_0$ are estimable. A design satisfying this latter condition is said to be treatment connected and we shall restrict our attention to such designs. Clearly there are a number of designs available for the situation being considered here and we

A. S. Hedayat is Professor and Dibyen Majumdar is Associate Professor, Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Box 4348, Chicago, Illinois 60680. Mike Jacroux is Professor, Department of Pure and Applied Mathematics, Washington State University, Pullman, Washington 99164.

want to choose one which is best in some sense. For example, we might choose a design that gives the minimal value among all available designs of

$$\sum_{i=1}^{4} \operatorname{var}(\hat{t}_{di} - \hat{t}_{d0}) \tag{1.4}$$

or

$$\max_{1 \le i \le 4} \operatorname{var}(\hat{t}_{di} - \hat{t}_{d0}) \tag{1.5}$$

where $\operatorname{var}(\hat{t}_{di} - \hat{t}_{d0})$ denotes the variance of $\hat{t}_{di} - \hat{t}_{d0}$. A design which gives the minimum in (1.4) is called an A-optimal design and one which gives the minimum in (1.5) is called an MV-optimal design.

Without further ado we give designs which are A- and MV-optimal under each of the three models:

A- and MV-optimal design under model (1.1):
Assign three experimental units to each of the four test treatments and six to the control.

A- and MV-optimal design under model (1.2), when there are six blocks of size 3 each:
Take each column of the following array as a block:

0	0	0	0	0	0
1	1	1	2	2	3
2	3	4	3	4	4

A- and MV-optimal design under model (1.3), where there are three rows and six columns:
Assign the treatments according to the following array:

1	0	3	4	2	0
0	3	4	2	0	1
4	2	0	0	1	3

We note that even in a small experiment such as the example given above, the determination of an optimal design is usually not easy. During the past several years there has been a concentrated effort to identify and construct optimal designs for the general problem of comparing v test treatments with a control. The A- and MV-optimality criteria defined in (1.4) and (1.5) have been the most widely studied criteria with regard to the construction of such designs. Finding an A-optimal design corresponds to minimizing mean square error in inference and finding an MV-optimal design is analogous to finding a minimax procedure. We point out that D-optimality is not a natural criterion for this type of problem; see the paragraph just below Definition 2.1. Although other optimality criteria have also been considered for comparing test treatments with a control, it is our view that the published literature on these other criteria has not reached a level of generality for summarization. In this article we shall attempt to summarize the known results on A- and MV-optimal designs, which we hope will be useful to both the theoretician and the practitioner.

In Sections 2 and 3 we give general results for A- and MV-optimal designs for comparing v test treatments with a control in each of the three models (1.1), (1.2) and (1.3). In Section 4 we give model robust A- and MV-optimal designs. In Section 5 we suggest various approaches for finding efficient designs in those cases where A- and MV-optimal designs are unknown. In Section 6 we give A- and MV-optimal designs for comparing test treatments with two or more controls. In Section 7 we outline Bayes A-optimal designs. In Section 8 we give an overview of the literature of optimal designs for comparing test treatments with controls.

2. A-OPTIMAL DESIGNS

We shall give A-optimal designs for comparing v test treatments with a control separately for the zero-way, one-way and two-way elimination of heterogeneity. Throughout this section the control will be denoted by the symbol 0 and the test treatments by $1, 2, \cdots, v$.

2.0 A-optimal Designs for the Zero-way Elimination of Heterogeneity

Our statistical set-up consists of n experimental units, and our model of response under a design d is

$$y_{dij} = \mu + t_i + \varepsilon_{ij}, \tag{2.1}$$

where $j = 1, \cdots, r_{di}$, $i = 0, 1, \cdots, v$. Here and throughout the sequel r_{di} is the number of experimental units receiving treatment i under a particular design d. We assume the model to be homoscedastic. The symbols in equation (2.1) have the same meaning as described in Section 1. The A-optimal design minimizes

$$\sum_{i=1}^{v} \left(\frac{1}{r_{d0}} + \frac{1}{r_{di}}\right) \tag{2.2}$$

subject to the restriction that $r_{d0} + r_{d1} + \cdots + r_{dv} = n$. It is easily seen that for a fixed value of r_{d0}, (2.2) is minimized when $r_{di} = p(r_{d0})$ or $p(r_{d0}) + 1$ for $i = 1, \cdots, v$ where $p(r_{d0}) = [(n - r_{d0})/v]$ and $[x]$ denotes the integral part of the decimal expansion for $x > 0$. Thus the problem of finding an A-optimal design in this case reduces to that of finding the value of r_{d0}

that minimizes

$$(2.3)\quad \frac{v}{r_{d0}}+\frac{v-n+r_{d0}+vp(r_{d0})}{p(r_{d0})}+\frac{n-r_{d0}-vp(r_{d0})}{p(r_{d0})+1}.$$

The minimization of (2.3) can easily be done using a calculator. In the case v is a square and $n = m(v + v^{1/2})$ for an integer m, the A-optimal design d^* is

$$r_{d^*1} = \cdots = r_{d^*v} = m,\ r_{d^*0} = mv^{1/2}.$$

2.1 A-optimal Designs for One-way Elimination of Heterogeneity

Our statistical setup consists of b blocks on size k each, $k \le v$. The model of response under a design d is

$$y_{dijp} = \mu + t_i + \beta_j + \varepsilon_{ijp},$$

where $i = 0, 1, \cdots, v; j = 1, \cdots, b$ and $p = 0, 1, \cdots, n_{dij}$. Here n_{dij} is the number of times treatment i is used in block j and the matrix $N_d = (n_{dij})$ is called the incidence matrix of the design. We note that

$$r_{di} = \sum_{j=1}^{b} n_{dij}.$$

We shall also let

$$C_d = \operatorname{diag}(r_{d0}, r_{d1}, \cdots, r_{dv}) - k^{-1}N_dN_d'$$

and

$$P = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix},$$

P being a $v \times (v+1)$ matrix. The matrix C_d is called the information matrix of d and PC_d^-P' is the covariance matrix of the vector of least squares estimators of the contrasts $t_1 - t_0, \cdots, t_v - t_0$; here C_d^- is a generalized inverse of C_d. We note that PC_d^-P' is proportional to the inverse of the matrix obtained by eliminating the first row and first column of C_d (see Bechhofer and Tamhane, 1981; Constantine, 1983). Then an A-optimal design minimizes

$$(2.4)\quad \operatorname{trace} PC_d^-P'$$

over all possible designs with parameters v, b and k.

Experience has shown that this minimization is usually not easy. As in other cases of exact design theory, it is highly unlikely that we can obtain one method which is capable of producing A-optimal designs for arbitrary values of v, b and k. Recently several families of A-optimal designs have been discovered.

At this point it is useful to recall a celebrated result. If there is no control and if we are interested in comparing v test treatments among themselves, then a balanced incomplete block (BIB) design (definition follows) in the v test treatments would be A-optimal (Kiefer, 1958; Kshirsagar, 1958; Roy, 1958; Kiefer, 1975). Here the A criterion is defined by expression (2.4) with P denoting any $(v-1) \times v$ matrix having normalized rows orthogonal to each other and to the vector $(1, \cdots, 1)'$, and C_d denoting the $v \times v$ information matrix of the test treatments. In fact, it has been proved by Kiefer (1975) that BIB designs are "universally optimal" in the sense that they are optimal under a large family of criteria, which includes the A- and MV-optimality criteria.

Definition 2.1. A BIB design with parameters v, b, r, k, λ is a block design with b blocks each containing $k < v$ distinct treatments such that each treatment is replicated r times and each pair of treatments appears in λ blocks.

Unlike the case of BIB designs, the structure of optimal designs for treatment-control comparisons seems to depend heavily on the criterion used. Although A- and MV-optimal designs are often the same, other criteria (see Majumdar and Notz, 1983) yield different designs, usually requiring either fewer or more replications of the control, but otherwise balanced with respect to test treatments. For example, the D-optimality criterion selects a block design from within a given class of designs which minimizes the determinant of the matrix PC_d^-P' defined in (2.4) and it can be shown that a BIB design is always D-optimal when such a design exists. But for the problem of comparing test treatments with a control, the D-optimality criterion does not seem to be either an intuitively or statistically suitable criterion because the designs it selects as being optimal generally do not provide any more information about treatment-control comparisons than they do about comparisons among the test treatments. On the other hand, the A- and MV-optimality criteria each have a natural and statistically meaningful interpretation as given above. Unfortunately, with the presence of a control and for the set of contrasts of interest a BIB design is almost never an A- or MV-optimal design. However, we can sometimes utilize BIB designs in the test treatments to construct an A- or MV-optimal design for our problem. We shall shortly give some sufficient conditions that can be used to establish the A- and MV-optimality of some families of such designs. For convenience, we introduce the notation ABIB $(v, b, k-t; t)$ to denote a BIB design in the v test treatments in b blocks of size $k-t$ each argumented by t replications of the control in each block. The following three families of ABIB designs are A-optimal for comparing a set of v test treatments to a control.

Family 1. An ABIB(v, b, $k - 1$; 1) is A-optimal whenever $(k-2)^2 + 1 \le v \le (k-1)^2$. An example of an A-optimal design when $v = 7$, $b = 7$, $k = 4$ is given below, where the columns are the blocks:

0	0	0	0	0	0	0
1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

For each (v, k) satisfying $(k-2)^2 + 1 \le v \le (k-1)^2$, there are an infinite number of values of b for which A-optimal ABIB(v, b, $k - 1$; 1) designs exist. This can be seen as follows. For each (v, k), form all $b = \binom{v}{k-1}$ subsets of size $k - 1$ out of the v test treatments. Augment each subset with a copy of the control. Then these b augmented subsets form an ABIB(v, b, $k - 1$; 1) design. Further, we note that the design which consists of the b_1 blocks of an ABIB(v, b_1, $k - 1$; 1) design and the b_2 blocks of an ABIB(v, b_2, $k - 1$; 1) design is an ABIB(v, b, $k - 1$; 1) design with $b = b_1 + b_2$. For more details concerning family 1 of designs, the reader is referred to Hedayat and Majumdar (1985).

Stufken (1987) has generalized the preceding idea to:

Family 2. An ABIB(v, b, $k - t$; t) is A-optimal whenever $(k-t-1)^2 + 1 \le t^2 v \le (k-t)^2$.

An example of an A-optimal design when $v = 8$, $b = 28$, $k = 8$ is given below:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	4	3	3	3	3	3	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
5	5	5	4	4	4	4	5	5	4	4	4	4	5	4	4	4	4	3	3	3	3	3	3	3	3	3	3
6	6	6	6	5	5	5	6	6	6	5	5	5	6	6	5	5	5	6	5	5	5	4	4	4	4	4	4
7	7	7	7	7	6	6	7	7	7	7	6	6	7	7	7	6	6	7	7	6	6	7	6	6	5	5	5
8	8	8	8	8	8	7	8	8	8	8	8	7	8	8	8	8	7	8	8	8	7	8	8	7	8	7	6

Sometimes we can use two BIB designs to construct an A-optimal design for our problem. We give below one such family, which is taken from Cheng, Majumdar, Stufken and Türe (1988):

Family 3. For $v = \alpha^2 - 1$, $b = \gamma(\alpha + 2)(\alpha^2 - 1)$ and $k = \alpha$, the union of an ABIB(v, $\gamma(\alpha + 1)(\alpha^2 - 2)$, $\alpha - 1$; 1) and a BIB design in all the $v + 1$ treatments, test treatments and control, in $\gamma\alpha(\alpha + 1)$ blocks of size k each is A-optimal whenever α is a prime power, and γ is any integer.

An example when $\alpha = 3$, $\gamma = 1$, $v = 8$, $b = 40$, $k = 3$ is:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	4	4
2	3	4	5	6	7	8	3	4	5	6	7	8	4	5	6	7	8	5	6

0	0	0	0	0	0	0	0	1	4	7	1	2	3	1	2	3	1	2	3
4	4	5	5	5	6	6	7	2	5	8	4	5	6	5	6	4	6	4	5
7	8	6	7	8	7	8	8	3	6	0	7	8	0	0	7	8	8	0	7

Stufken (1987) has some more families of A-optimal designs.

To establish the optimality of these families, the starting point is a result due to Majumdar and Notz (1983), which we shall shortly state. However, before stating the result we need some definitions.

Definition 2.2. d is a balanced treatment incomplete block (BTIB) design if

$$\lambda_{d01} = \cdots = \lambda_{d0v},$$

$$\lambda_{d12} = \cdots = \lambda_{d,v-1,v},$$

where $\lambda_{dij} = \sum_{p=1}^{b} n_{dip} n_{djp}$. This definition is due to Bechhofer and Tamhane (1981).

Definition 2.3. For integers $t \in \{0, 1, \cdots, k-1\}$ and $s \in \{0, 1, \cdots, b-1\}$, d is a BTIB((v, b, k; t, s) if it is a BTIB design with the additional property that

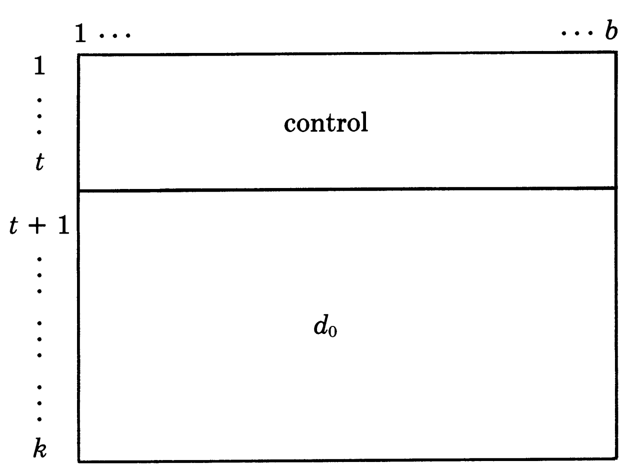
$$n_{dij} \in \{0, 1\}, \quad i = 1, \cdots, v; \qquad j = 1, \cdots, b,$$

$$n_{d01} = \cdots = n_{d0s} = t + 1,$$

$$n_{d0,s+1} = \cdots = n_{d0b} = t.$$

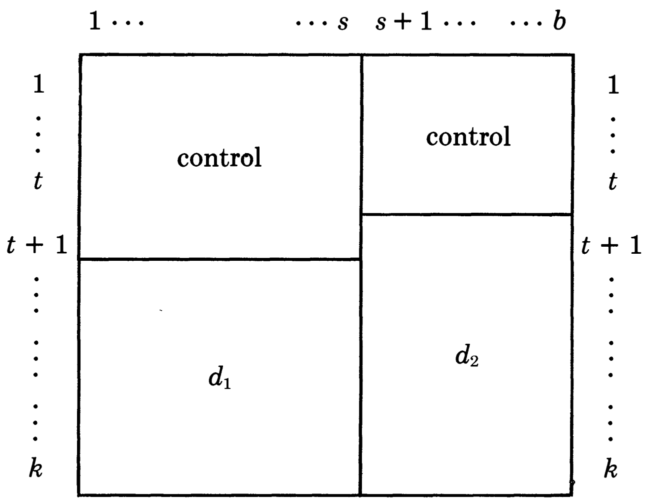
A BTIB(v, b, k; t, s) is called a rectangular (R-) type design when $s = 0$, and a step (S-) type design when $s > 0$. The layout of these designs can be pictured as follows, with columns as blocks, in each of the two cases R-type and S-type:

(i) R-type.



d_0 is a BIB design in the test treatments.

(ii) S-type.



d_1 and d_2 are components of the design which involve the test treatments only.

Now we are ready to state the result of Majumdar and Notz (1983).

THEOREM 2.1. Let v, b, k be integers with $k \le v$. A BTIB(v, b, k; t, s) is A-optimal in the class of all designs with the same values of v, b, and k if

$$g(t, s) = \min\{g(x, z): (x, z) \in \Lambda\},$$

where

$$\Lambda = \{(x, z): x = 0, \cdots, [k/2] - 1; \quad z = 0, \cdots, b \text{ with } z > 0 \text{ when } x = 0\}, \tag{2.5}$$

$$g(x, z) = a/A(x, z) + 1/B(x, z),$$

$$a = (v-1)^2, \quad c = bvk(k-1), \quad p = v(k-1) + k,$$

$$A(x, z) = \{c - p(bx + z) + bx^2 + 2xz + z\}/vk,$$

$$B(x, z) = \{k(bx + z) - (bx^2 + 2xz + z)\}/vk.$$

We note that there are many parameter combinations (v, b, k) which do not belong to any of the three families of A-optimal BTIB designs given previously for which the result of Majumdar and Notz (1983) can still be used to get an optimal design. Hedayat and Majumdar (1984) have devised an algorithm for obtaining A-optimal designs based on Theorem 2.1 and gave a list of all designs available by this result when $2 \le k \le 8$, $k \le v \le 30$, $v \le b \le 50$. Jacroux (1988b) has generalized this algorithm. His algorithm is often capable of producing A-optimal designs which are not necessarily BTIB in their structure. In particular, the algorithm given by Jacroux often produces A-optimal group divisible treatment designs (GDTD's).

Definition 2.4. d is a GDTD with parameters m, n, λ_0, λ_1 and λ_2 if the treatments $1, \cdots, v$ can be divided into m groups $V_1, \cdots, V_m$ of size n such that

(i) $\lambda_{d0i} = \lambda_0$ for $i = 1, \cdots, v$ and for some constant λ_0,
(ii) if $i, j \in V_p$, $i \ne j$, $\lambda_{dij} = \lambda_1$ for some constant λ_1,
(iii) if $i \in V_p$, $j \in V_q$, $p \ne q$, $\lambda_{dij} = \lambda_2$ for some constant λ_2.

Definition 2.5. For integers $t \in \{0, 1, \cdots, k-1\}$ and $s \in \{0, 1, \cdots, b-1\}$, d is a GDTD(v, b, k; t, s) if it is a GDTD with the additional property that

$$n_{dij} \in \{0, 1\}, \quad i = 1, \cdots, v, \quad j = 1, \cdots, b,$$

$$n_{d01} = \cdots = n_{d0s} = t + 1,$$

$$n_{d0,s+1} = \cdots = n_{d0b} = t.$$

A GDTD(v, b, k; t, s) is called an R-type design when $s = 0$ and an S-type design when $s > 0$.

Jacroux's generalization of Theorem 2.1 can be stated as follows.

THEOREM 2.2. Let v, b, k be integers with $k \le v$. A BTIB(v, b, k; $\bar{t}$, $\bar{s}$) or a GDTD(v, b, k; $\bar{t}$, $\bar{s}$) having $m = 2$, $n = v/2$ and $\lambda_2 = \lambda_1 + 1$ is A-optimal in the class of all designs if

$$n(\bar{t}, \bar{s}) = \min\{n(x, z): (x, z) \in \bar{\Lambda}\},$$

where a, c, p, $A(x, z)$, $B(x, z)$ and $g(x, z)$ are as defined in (2.5) and where

$$\bar{\Lambda} = \{(x, z): x = 0, \ldots, k-2; \quad z = 0, \ldots, b \text{ with } z > 0 \text{ when } x = 0\}$$

and

$$n(x, z) = \min\{h(x, z), m(x, z)\}, \tag{2.6}$$

with

$$h(x, z) = a/(A(x, z) - 2/k) + 1/B(x, z),$$

$$m(x, z) = \begin{cases} g(x, z), \\ \quad \text{if } B(x, z) > \{A(x, z) - ((v-1)/(v-2))^{1/2} P(x, z)\}/(v-1), \\ 1/B(x, z) \\ \quad + (v-2)(v-1)/\{A(x, z) - ((v-1)/(v-2))^{1/2} P(x, z)\} \\ \quad + (v-1)/\{A(x, z) + ((v-1)(v-2))^{1/2} P(x, z)\}, \text{ otherwise.} \end{cases}$$

The quantities a, $A(x, z)$ and $B(x, z)$ are as defined in

(2.5) and

$$P(x, z) = \{C(x, z) - B^2(x, z) - A^2(x, z)/(v-1)\}^{1/2},$$

$$\begin{aligned} C(x, z) = &\{bk - bx - z - vR(x, z)\} \\ &\cdot \{(R(x, z) + 1)(k-1)\}^2 \\ &+ \{v - bk + bx + z + vR(x, z)\} \\ &\cdot \{R(x, z)(k-1)\}^2/k^2 \\ &+ \{A(x, z) - v(v-1)\lambda(x, z)\} \\ &\cdot \{\lambda(x, z) + 1\}^2/k^2 \\ &+ \{v(v-1) - A(x, z) \\ &\quad + v(v-1)\lambda(x, z)\}\lambda^2(x, z)/k^2, \end{aligned}$$

$$R(x, z) = [(bk - bx - z)/v],$$

$$\lambda(x, z) = [\{(bk - bx - z)(k-1) - vkB(x, z)\}/v(v-1)].$$

We note that many BTIB(v, b, k; $\bar{t}, \bar{s}$) designs not satisfying the conditions of Theorem 2.1 can be shown to satisfy the conditions of Theorem 2.2. In addition, Theorem 2.2 can be used to establish the A-optimality of GDTD(v, b, k; $\bar{t}, \bar{s}$)'s having $m = 2$, $n = v/2$ and $\lambda_2 = \lambda_1 + 1$. Using some more elaborate computational techniques, Jacroux (1988b) has also developed some sufficient conditions for GDTD(v, b, k; $\bar{t}, \bar{s}$)'s having $\lambda_2 = \lambda_1 + 1$ or $m = v/2$, $n = 2$ and $\lambda_2 = \lambda_1 - 1$ to be A-optimal among all designs with parameters v, b and k. One example of an A-optimal GDTD is that GDTD(9, 9, 4; 1, 0) having $m = 3$, $n = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$ and treatment groups (1, 2, 3), (4, 5, 6) and (7, 8, 9) given below:

0	0	0	0	0	0	0	0	0
1	1	1	2	2	2	3	3	3
4	5	6	4	5	6	4	5	6
7	8	9	8	9	7	9	7	8

2.2 A-optimal Designs for Two-way Elimination of Heterogeneity

Our statistical set-up consists of bk experimental units arranged in a $k \times b$ array, and the model of response under design d is

(2.7) $$y_{dijl} = \mu + t_i + \beta_j + \rho_l + \varepsilon_{ijl},$$

$i = 0, 1, \cdots, v; j = 1, \cdots, b; l = 1, \cdots, k$, if treatment i is applied to the experimental unit in cell (l, j).

Let

n_{dij} = number of times treatment i occurs in column j,

m_{dil} = number of times treatment i occurs in row l,

$r_{di} = \sum_{j=1}^{b} n_{dij}$,

$N_d = (n_{dij})$, a $(v+1) \times b$ matrix,

$M_d = (m_{dij})$, a $(v+1) \times k$ matrix,

P is the $v \times (v+1)$ matrix defined in Subsection 2.1,

$r_d = (r_{d0}, r_{d1}, \cdots, r_{dv})'$,

$$C_{d(2)} = \text{diag}(r_{d0}, r_{d1}, \cdots, r_{dv}) - k^{-1}N_dN_d' - b^{-1}M_dM_d' + (bk)^{-1}r_dr_d'.$$

Then an A-optimal design minimizes trace $PC_{d(2)}^{-}P'$. We shall now highlight some of the results from recent literature.

Family 1. Let p be an integer and $v = p^2$, $b = k = p^2 + p$. A $b \times b$ array in which each test treatment appears once in each row and in each column and the control appears p times in each row and in each column is A-optimal.

One easy way to construct members of this family is to start with a Latin square of order $p^2 + p$ and change symbols $p^2 + 1, \cdots, p^2 + p$ to 0 (control). We illustrate this in the following example with $v = 4$, $b = k = 6$:

1	2	3	4	5	6		1	2	3	4	0	0
6	1	2	3	4	5		0	1	2	3	4	0
5	6	1	2	3	4		0	0	1	2	3	4
4	5	6	1	2	3	→	4	0	0	1	2	3
3	4	5	6	1	2		3	4	0	0	1	2
2	3	4	5	6	1		2	3	4	0	0	1

This and some more general results are available in Notz (1985).

Majumdar (1986) has generalized the preceding family of A-optimal designs to also include the following designs.

Family 2. Let p, α and γ be integers, $v = p^2$, $k = \alpha(p^2 + p)$ and $b = \gamma(p^2 + p)$. A $k \times b$ array in which each test treatment appears α times in each column and γ times in each row, and the control appears αp times in each column and γp times in each row is A-optimal.

One way to construct members of this family is to form the array

$$(L_{ij}), \quad i = 1, \cdots, \alpha; \quad j = 1, \cdots, \gamma$$

where each L_{ij} is a member of Family 1.

Family 3. A $k \times b$ array is A-optimal if

(2.8)
(i) it is an A-optimal block design for 1-way elimination of heterogeneity with columns as blocks, and
(ii) the total number of replications for each treatment, test treatment or control, is divided equally among the k rows.

This has been given by Jacroux (1986). The following is an example when $v = 9$, $b = 24$, $k = 3$:

0	4	1	0	2	5	7	8	3	0	9	6	0	0	8	0	7	9	1	6	2	3	4	5
1	0	5	8	0	2	0	0	5	3	0	4	4	6	6	9	0	7	2	1	3	8	7	9
3	1	0	1	4	0	2	2	0	7	3	0	9	5	0	6	8	0	9	7	6	4	5	8

We note that when considering this last array as a block design with columns acting as blocks, it is a BTIB (9, 24, 3; 0, 18) design which satisfies the conditions of Theorem 2.1 (hence it is A-optimal) and because the total number of replications for each treatment are divided equally among the rows, condition (ii) of (2.8) is satisfied.

In general, the method used to construct most of the families of row-column designs given in this section is to find an optimal block design having parameters v, b and k and having the numbers of replications assigned to the test treatments and control divisible by k, then arrange treatments within blocks so that condition (ii) of (2.8) is satisfied. The fact that an arrangement within blocks satisfying (2.8(ii)) can always be found when the numbers of replications assigned to the test treatments and control are divisible by k essentially follows from Hall's (1935) "marriage lemma."

3. MV-OPTIMAL DESIGNS

In this section we give a number of results concerning the MV-optimality of designs for comparing v test treatments with a control for the zero-way, one-way and two-way elimination of heterogeneity. The notation introduced in Section 2 is also used throughout this section.

3.0 MV-optimal Designs for Zero-way Elimination of Heterogeneity

Our statistical setup is the same as given in Subsection 2.0. An MV-optimal design minimizes

$$\max_{1\le i\le v} (1/r_{d0} + 1/r_{di}) \tag{3.1}$$

subject to the restriction $r_{d0} + r_{d1} + \cdots + r_{dv} = n$. It is easily seen that an MV-optimal design d^* has

$$r_{d^*i} = \tilde{r} \qquad \text{for } i = 1, \cdots, v$$

and

$$r_{d^*0} = n - v\tilde{r},$$

where

$$\tilde{r} = \begin{cases} \hat{r}, & \text{if } \hat{r} \text{ is an integer,} \\ [\hat{r}] + 1, & \text{otherwise,} \end{cases}$$

and

$$\hat{r} = \frac{2n + v - v^2 - (v^4 + v^2 - 2v^3 + 4n^2 v)^{1/2}}{2v(v-1)}.$$

We note that for a fixed value of n, the A- and MV-optimality criteria may select substantially different optimal designs from those available. For example, when $n = 30$ and $v = 15$, an A-optimal design $\bar{d}$ will have $r_{\bar{d}0} = 5$ and $r_{\bar{d}i} = 1$ or 2 for $i = 1, \cdots, 15$ whereas the MV-optimal design d^* will have $r_{d^*0} = 15$ and $r_{d^*i} = 1$ for $i = 1, \cdots, 15$.

3.1 MV-optimal Designs for the One-way Elimination of Heterogeneity

Our statistical setup is the same as given in Subsection 2.1. We note that any design which is A-optimal among all designs having parameters v, b and k and which estimates all contrasts of the form $t_i - t_0$ with the same variance will also be MV-optimal. Thus we see that all designs given in Subsection 2.1 as being A-optimal are also MV-optimal because all GDTD($v, b, k; \bar{t}, \bar{s}$)'s estimate contrasts of the form $t_i - t_0$ with the same variance. However, Jacroux (1987a) has developed some additional sufficient conditions which can be used to establish the MV-optimality of various GDTD($v, b, k; x, z$)'s which cannot be proven to be A-optimal using any known results. As an example of the types of results which can be proven for MV-optimality, we have the following.

THEOREM 3.1. Using the same notation as introduced in Theorem 2.2, let d^* be a BTIB($v, b, k; \tilde{t}, \tilde{s}$) where

$$n(\tilde{t}, \tilde{s}) = \min\{n(x, z): (x, z) \in \bar{\Lambda}, \\ (bk - bx - z)/v \text{ is an integer}\}$$

and for positive integers p and q, let $\bar{B}(p, q)$ denote the smallest value of y such that

$$(1, -1)\begin{pmatrix} p/k & -y/k \\ -y/k & q/k \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \le \operatorname{var}(\hat{t}_{d^*i} - \hat{t}_{d^*0}).$$

If for any $(x, z) \in \bar{\Lambda}$ such that $n(x, z) < n(\tilde{t}, \tilde{s})$, it holds that

$$\begin{aligned} vkB(x, z) < {} & (bk - bx - z - vR(x, z)) \\ & \cdot \bar{B}(vkB(x, z), (R(x, z) + 1)(k - 1)) \\ & + (v - bk + bx + z + vR(x, z)) \\ & \cdot \bar{B}(vkB(x, z), R(x, z)(k - 1)), \end{aligned}$$

then d^* is MV-optimal among all designs.

Using some more complex computational techniques, Jacroux (1987a) has obtained some further results similar to Theorem 3.1 which can be used to establish the MV-optimality of various GDTD($v, b, k; \tilde{t}, \tilde{s}$)'s having $0 \le \lambda_2 - \lambda_1 \le 1$ or $m = v/2$, $n = 2$ and $\lambda_2 = \lambda_1 - 1$ whose A-optimality remains unknown. For example, when $v = 6$, $b = 11$ and $k = 3$, as well as when $v = 6$, $b = 16$ and $k = 4$, an A-optimal design is unknown. However, we are able to give designs whose MV-optimality can be established using results such as Theorem 3.1. These are exhibited below.

Example 3.1. $v = 6$, $b = 11$ and $k = 3$.

0	0	0	0	0	0	0	0	0	1	4
1	1	1	2	2	2	3	3	3	2	5
4	5	6	4	5	6	4	5	6	3	6

This design is a BTIB (6, 11, 3; 0, 9) design.

Example 3.2. $v = 6$, $b = 16$ and $k = 4$.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	3	1	3	1	2	1	2	1	2	1	3	1	2
2	5	4	4	2	5	4	3	3	5	4	3	2	4	3	4
3	6	6	5	4	6	5	6	4	6	6	5	5	6	6	5

This design is a GDTD (6, 16, 4; 1, 0) having treatment groups (1, 4), (2, 5) and (3, 6).

It is interesting to note that the design in Example 3.1 is an S-type BTIB design, while the design given in Example 3.2 is a GDT.

3.2 MV-optimal Designs for the Two-way Elimination of Heterogeneity

Again our statistical set-up is the same as that given in Subsection 2.2. Using arguments similar to those used in Subsection 3.1, we see that all of the A-optimal row-column designs which estimate treatment contrasts $t_i - t_0$ with the same variance will also be MV-optimal. Thus all the A-optimal row-column designs listed in Subsection 2.2 are also MV-optimal. In addition, a $k \times b$ array is MV-optimal if

(i) it is an MV-optimal block design for 1-way elimination of heterogeneity with columns as blocks, and
(ii) the total number of replications for each treatment, test treatment or control, is divided equally among the k rows.

Example 3.3. For $v = 6$, $b = 16$ and $k = 4$, the row-column design given by

0	6	4	5	0	3	1	2	0	5	6	2	0	3	1	4
1	0	6	4	2	0	5	3	1	0	4	3	5	0	6	2
2	1	0	3	4	6	0	6	3	2	0	5	1	4	0	5
3	5	2	0	1	5	4	0	4	6	1	0	2	6	3	0

is MV-optimal as shown in Jacroux (1986).

We close this section by noting that all of the optimal designs given in this and the preceding section possess a high degree of balance in many respects. For example, in terms of the number of replications for test treatments, in terms of the number of joint appearances between the test treatments and the control and between the test treatments themselves in blocks or rows and columns, etc.

4. MODEL ROBUST OPTIMAL DESIGNS

There are circumstances in which the experimenter is not sure whether to fit a one-way or a two-way elimination of heterogeneity model to the data. For example, the performance of several technicians are being compared to the incumbent (control) and the days of the week as well as the hours within each day are the two possible sources of heterogeneity. In such a situation it would be highly desirable to obtain a design which is A- or MV-optimal under each of these models. Hedayat and Majumdar (1988) studied this aspect of the problem and gave some families of model robust designs. The families were constructed using the Euclidean plane, the projective plane and some other geometrical structures. The exact description of the families are somewhat involved; some typical examples are given below.

Example 4.1. Let $v = 4$, $k = 3$ and $b = 6$. The following design is A- and MV-optimal for both 1- and 2-way elimination of heterogeneity models:

1	0	3	4	2	0
0	3	4	2	0	1
4	2	0	0	1	3

In fact, this design is A- and MV-optimal for the 0-way elimination of heterogeneity model as well.

Example 4.2. Let $v = 7$, $k = 4$ and $b = 28$. The following design is A- and MV-optimal for both 1- and 2-way elimination of heterogeneity models:

0 0 0 0 0 0 0	1 2 3 4 5 6 7
1 2 3 4 5 6 7	0 0 0 0 0 0 0
2 3 4 5 6 7 1	2 3 4 5 6 7 1
4 5 6 7 1 2 3	4 5 6 7 1 2 3
1 2 3 4 5 6 7	1 2 3 4 5 6 7
2 3 4 5 6 7 1	2 3 4 5 6 7 1
0 0 0 0 0 0 0	4 5 6 7 1 2 3
4 5 6 7 1 2 3	0 0 0 0 0 0 0

Before closing this section we would like to mention that the designs in families 1 and 2 in Subsection 2.2 are A- and MV-optimal under zero-way, one-way and two-way elimination of heterogeneity models, whereas the designs in family 3 are A- and MV-optimal at least under one-way and two-way elimination of heterogeneity models. We would also like to mention that in our settings, designs that are known to us to be A- or MV-optimal in the n-way elimination of heterogeneity are also A- or MV-optimal in the $(n-1)$-way elimination of heterogeneity where $n = 2, 1$. However, a general result of this nature has not been proven.

5. OTHER EFFICIENT DESIGNS

Even though, for each set of v test treatments there is an A- or MV-optimal design for a zero-, one- or two-way elimination of heterogeneity model, the task of finding this design can be very difficult indeed. For situations where an A- or MV-optimal design is unknown, there are several alternative ways of planning an experiment. Here are some possibilities.

5.1 Limit the Class of Competing Designs to a "Reasonably Rich" Subclass, So That an A- or MV-optimal Design within This Subclass Can Be Constructed

For example, under a one-way elimination of heterogeneity model the BTIB designs form such a subclass. For $v = 3$, $b = 12$, $k = 2$ the A- or MV-optimal design is not available in the literature, but a design which is A- and MV-optimal among all BTIB designs is given by

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 3 & 3 \end{matrix}$$

This approach has been studied in Hedayat and Majumdar (1984) under the A-optimality criterion, and some series of such designs have been catalogued. Numerical evidence indicates that optimal designs obtained in this fashion are generally highly efficient in the class of all designs. There are, however, isolated instances where they perform poorly. A similar study for the MV-optimality criterion has been carried out by Jacroux (1987b).

5.2 Search for an Approximately A- or MV-optimal Design

This can be carried out in two steps. First compute

$$(5.1) \qquad \text{(i) } g(t, s) \quad \text{or} \quad \text{(ii) } g(t, s)/v$$

with

$$g(t, s) = \min\{g(x, z) : (x, z) \in \Lambda\},$$

where the function $g(x, z)$ has been defined in Subsection 2.1. These give lower bounds to the values of the A- and MV-criteria, respectively. The second step consists of guessing a good design $\bar{d}$ based on available theory. Compute the corresponding value of the expression (2.4) or $\max_{1\le i\le v} \operatorname{var}(\hat{t}_{\bar{d}i} - \hat{t}_{\bar{d}0})$ for this design and compare with the appropriate minimum value given in (5.1). If the comparison is poor in the opinion of the experimenter, then he should modify his guess and try again.

Let us demonstrate this approach by an example. Let $v = 21$, $b = 30$ and $k = 9$. Here the minimum given by (5.1(i)) is 2.589 and that given by (5.1(ii)) is .1233. Our experience shows that BIB designs in the test treatments augmented by one or more replications of the control in each block are often highly efficient, as seen in families 1 and 2 of Subsection 2.1. In our case we can try a design, $\bar{d}$, which is an ABIB(21, 30, 7; 2). For this design the value of (2.4) is 2.618 and $\max_{1\le i\le v} \operatorname{var}(\hat{t}_{\bar{d}i} - \hat{t}_{\bar{d}0}) = .1247$, giving an efficiency of at least 98.87% for both the A- and MV-optimality criteria. So this is indeed a highly efficient design. This approach of approximating an A- and MV-optimal design by an augmented BIB design has been studied by Stufken (1988).

Another method of tracking down a good approximation has been given in Cheng, Majumdar, Stufken and Türe (1988). It consists of first determining the point (t, s) which minimizes the function $g(x, z)$ given in (5.1). In case a BTIB$(v, b, k; t, s)$ exists, it is both A- and MV-optimal. If it does not, then at least one of the following two designs is expected to be a good approximation:

(i) A design with the same number $(bt + s)$ of replications of the control as a BTIB$(v, b, k; t, s)$ and which is "combinatorially close" to a BTIB design.
(ii) A BTIB design with the number of replications of the control "close" to $bt + s$.

We demonstrate the idea by an example when $v = 5$, $b = 7$ and $k = 4$. Here $(t, s) = (1, 0)$ and $g(t, s) = 2.04$. There is no BTIB(5, 7, 4; 1, 0). Consider the following two designs:

$$d_1: \begin{matrix} 0&0&0&0&0&0&0 \\ 1&1&1&1&1&2&2 \\ 2&2&3&3&4&3&3 \\ 4&5&4&5&5&4&5 \end{matrix} \qquad d_2: \begin{matrix} 0&0&0&0&0&0&1 \\ 0&0&1&1&2&2&2 \\ 1&3&3&4&3&4&3 \\ 2&4&5&5&5&5&4 \end{matrix}$$

Here $bt + s = 7$, d_1 is a non-BTIB design with seven replications of the control, whereas d_2 is a BTIB design with eight replications of the control. The value for expression (2.4) for d_1 is 2.058 giving it an efficiency of 99.2%. The value for expression (2.4) for d_2 is 2.143 giving it an efficiency of 95.2%. Thus both of these designs are highly efficient under the A-criterion, d_1 being the better of the two.

Finally, with the availability of today's high-speed computers and supercomputers, one can find an A- or MV-optimal design by a complete search among all possible designs if the parameters v, b and k are not too large.

6. A- AND MV-OPTIMAL DESIGNS FOR TWO OR MORE CONTROLS

So far we have been discussing optimal designs for comparing test treatments with one control. There are circumstances when the test treatments have to be compared with two or more controls. Suppose we denote by S the set of all controls and by T the set of all test treatments. Then, an A-optimal design is one which minimizes

$$\sum_{g\in S}\sum_{h\in T}\operatorname{var}(\hat{t}_{dg}-\hat{t}_{dh}),$$

and an MV-optimal design is one which minimizes

$$\max_{g\in S, h\in T}\operatorname{var}(\hat{t}_{dg}-\hat{t}_{dh}),$$

among all designs under a zero-way, one-way or two-way elimination of heterogeneity model. Majumdar (1986) has studied this problem, and has identified designs which are A- and MV-optimal in various settings. In this set-up the mathematical structure of optimal designs is more complex and further research needs to be done to better understand them. To give the reader an idea about the nature of these designs, we present an example of an A- and MV-optimal design for one-way elimination of heterogeneity and an example of a design which is optimal for each of zero-, one- and two-way elimination of heterogeneity.

Example 6.1. Suppose four test treatments are to be compared with three controls in 30 blocks of size 3 each under the one-way elimination of heterogeneity model. Denoting the test treatments by A, B, C, D and the controls by 1, 2, and 3 the A- and MV-optimal design is:

1	1	2	1	1	2	1	1	2	1	1	2	1	1	1	1	1	1
2	3	3	2	3	3	2	3	3	2	3	3	A	A	A	B	B	C
A	A	A	B	B	B	C	C	C	D	D	D	B	C	D	C	D	D

2	2	2	2	2	2	3	3	3	3	3	3
A	A	A	B	B	C	A	A	A	B	B	C
B	C	D	C	D	D	B	C	D	C	D	D

Example 6.2. Suppose eight test treatments are to be compared with two controls in a 12×12 array under a two-way elimination of heterogeneity model. Denoting the test treatments by A, B, C, D, E, F, G, H, and the controls by 1, 2, the A- and MV-optimal design is given by

A	B	C	D	E	F	G	H	1	1	2	2
B	C	D	E	F	G	H	1	1	2	2	A
C	D	E	F	G	H	1	1	2	2	A	B
D	E	F	G	H	1	1	2	2	A	B	C
E	F	G	H	1	1	2	2	A	B	C	D
F	G	H	1	1	2	2	A	B	C	D	E
G	H	1	1	2	2	A	B	C	D	E	F
H	1	1	2	2	A	B	C	D	E	F	G
1	1	2	2	A	B	C	D	E	F	G	H
1	2	2	A	B	C	D	E	F	G	H	1
2	2	A	B	C	D	E	F	G	H	1	1
2	A	B	C	D	E	F	G	H	1	1	2

This design is model robust in the sense of being A- and MV-optimal for zero- and one-way elimination of heterogeneity models as well.

7. BAYES OPTIMAL DESIGNS

In this section we look at the problem of finding optimal designs for comparing test treatments with a control, when prior information is available on the parameters of the model. For mathematical tractability, it is convenient to consider this approach in the framework of "continuous" designs. This means conceptually allowing the r_{di}'s and the n_{dij}'s to be real numbers and carrying out the optimization. In implementing such optimal designs for the zero-way elimination of heterogeneity model it is necessary to round the nonintegral r_{di}'s to the nearest integers, keeping the total number of observations in mind; for the one-way elimination of heterogeneity model we have to round the nonintegral n_{dij}'s to the nearest integers, keeping the total number of observations in mind.

7.0 Bayes A-optimal Designs for Zero-way Elimination of Heterogeneity

The model for the observations is assumed to be

$$y_{dij} = \mu + t_i + \varepsilon_{ij},$$

as in Subsection 2.0. We shall utilize prior information available on the parameters

$$\mu + t_0, \mu + t_1, \cdots, \mu + t_v,$$

which represent the expected responses under the treatments. We start by rewriting the above model in matrix notation:

$$Y_d = A_d t + \varepsilon,$$

where $t = (\mu + t_0, \cdots, \mu + t_v)'$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)'$ and A_d is an $n \times (v + 1)$ matrix of zeros and ones. Here n is the total number of observations. The (j, i) entry of A_d is 1 if the jth observation receives treatment i, $i = 0, 1, \cdots, v$, and 0 otherwise.

The error distribution given t in the representation of Y_d is assumed multivariate normal,

$$Y_d \mid t \sim N(A_d t, C_1); \tag{7.1}$$

and so is the prior distribution of t,

$$t \sim N(\mu_t, C_2). \tag{7.2}$$

Then the posterior distribution of t is (cf. (7) of Lindley and Smith, 1972)

$$t \sim N(B_d\lambda_d, B_d),$$

where

$$B_d^{-1} = A_d' C_1^{-1} A_d + C_2^{-1}$$

and

$$\lambda_d = A_d' C_1^{-1} Y_d + C_2^{-1} \mu_t.$$

Our objective is to estimate the vector of control-test treatment contrasts, which can be written as

$$\theta = Pt$$

where P is the $v \times (v+1)$ matrix defined in Subsection 2.1. To estimate θ in a decision theoretic framework, we assume the squared error loss,

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)'(\hat{\theta} - \theta)$$

where $\hat{\theta}$ is an estimator of θ. It is well known that for a given experiment the standard Bayes estimator of θ is its posterior expectation $PB_d\lambda_d$ and the expected loss with this estimator is $\mathrm{tr}PB_dP'$, the trace of the posterior dispersion matrix PB_dP'. Therefore, we shall choose a design d which minimizes

$$\mathrm{tr}PB_dP'.$$

This approach was taken, for example, by Owen (1970) in the context of one-way elimination of heterogeneity model. This criterion is called Bayes A-optimality.

Let us illustrate this method for the special case

$$C_1 = \sigma^2 I, \quad C_2 = \mathrm{diag}(c_0, c_1, \cdots, c_1).$$

Here the problem reduces to minimizing

$$v(p_{d0} + qm)^{-1} + \sum_{i=1}^{v} (p_{di} + m)^{-1}$$

where $p_{di} = r_{di}/n$, $i = 0, 1, \cdots, v$, $q = c_1/c_0$ and $m = \sigma^2/nc_1$. The quantities p_{di} are nonnegative real numbers satisfying

$$p_{d0} + p_{d1} + \cdots + p_{dv} = 1,$$

because we are allowing the r_{di}'s to be real numbers in this continuous design approach. It is not difficult to see that a Bayes A-optimal design is given by

$$r_{d_0 i} = np_{d_0 i}, \quad i = 0, 1, \cdots, v; \qquad p_{d_0 0} = 1 - p_{d_0 1},$$

$$p_{d_0 1} = \cdots = p_{d_0 v} = (q(m - \sqrt{v}) + 1)(v + \sqrt{v})^{-1}.$$

Example 7.1. Let $v = 4$ and $n = 36$; $m = 1/6$, $q = 3$. Then $p_{d_0 1} = p_{d_0 2} = p_{d_0 3} = p_{d_0 4} = 7/36$, $p_{d_0 0} = 8/36$. The Bayes A-optimal design is $r_{d_0 1} = r_{d_0 2} = r_{d_0 3} = r_{d_0 4} = 7$ and $r_{d_0 0} = 8$; the control is applied to 8 units and each test treatment to 7 units. In case no prior information is available, using the results stated in Subsection 2.0, the A-optimal design consists of applying the control to 12 units and each test treatment to 6 units.

Smith and Verdinelli (1980) took a somewhat different approach to Bayes A-optimal designs. Following Lindley and Smith (1972), they modeled μ_t in (7.2) as

$$\mu_t = A_2\Psi; \ A_2 \text{ is a known matrix and } \Psi \sim N(\eta, C_3). \tag{7.3}$$

This is known as the hierarchical linear model. It follows from (7.1), (7.2) and (7.3) that the posterior distribution of t is (see (12) and (13) of Lindley and Smith, 1972)

$$t \sim N(B_{1d}^* \lambda_{1d}^*, B_{1d}^*)$$

where

$$B_{1d}^{*-1} = A_d' C_1^{-1} A_d + (C_2 + A_2 C_3 A_2')^{-1}$$

and

$$\lambda_{1d}^* = A_d' C_1^{-1} Y + (C_2 + A_2 C_3 A_2')^{-1} A_2 \eta.$$

In case the prior information at the third stage is vague, i.e. $C_3^{-1} \to 0$, these expressions reduce to

$$B_{1d}^{-1} = A_d' C_1^{-1} A_d + C_2^{-1} - C_2^{-1} A_2 (A_2' C_2^{-1} A_2)^{-1} A_2' C_2^{-1}$$

and

$$\lambda_{1d} = A_d' C_1^{-1} Y.$$

The posterior distribution of t becomes

$$t \sim N(B_{1d}\lambda_{1d}, B_{1d}).$$

For the hierarchical linear model with vague prior at the third stage, a Bayes A-optimal design for estimating $\theta = Pt$ minimizes $\mathrm{tr}PB_{1d}P'$. For the special case

$$C_1 = \sigma^2 I, \quad C_2 = \mathrm{diag}(c_0, c_1, \cdots, c_1),$$

$$A_2' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{pmatrix},$$

Smith and Verdinelli (1980) shows that the problem can be reduced to minimizing

$$(1 - vp_d)^{-1} + (p_d + m)^{-1} + m[vp_d(p_d + m)]^{-1}$$

over p_d in the interval $(0, v^{-1})$, where $m = \sigma^2/nc_1$, as defined earlier. The value of p_d which minimizes the above expression is given as the unique root, p^*, in the interval $(0, v^{-1})$, of the following equation in p_d:

$$(p_d + m)^2(2vp_d - 1) - (v - 1)p_d^2(1 - vp_d)^2 = 0.$$

We point out that this quartic equation differs from the corresponding equation given on page 617 in Smith and Verdinelli (1980). A Bayes A-optimal design d^* is given by

$$r_{d^*1} = \cdots = r_{d^*v} = np^*, \ r_{d^*0} = n(1 - vp^*).$$

Example 7.2. Let $v = 4$, $n = 36$, $m = 1/6$; the same values as in Example 7.1. In this case $p^* = .141$ approximately. After rounding off to the nearest integer, the Bayes A-optimal design d^* is given by $r_{d^*1} = r_{d^*2} = r_{d^*3} = r_{d^*4} = 5$, $r_{d^*0} = 16$. This means d^* allocates 5 units to each test treatment and 16 units to the control.

7.1 Bayes A-optimal Designs for One-way Elimination of Heterogeneity

The model for the observations is assumed to be

$$y_{dijp} = \mu + t_i + \beta_j + \varepsilon_{ijp},$$

as in Subsection 2.1. We shall utilize prior information on $\gamma_j = \mu + t_0 + \beta_j$, $j = 1, \cdots, b$, and $\theta_i = t_i - t_0$, $i = 1, \cdots, v$. In many experimental situations, the control is a standard treatment which has been previously studied in the blocks under consideration. This naturally gives us a prior on $\gamma_1, \cdots, \gamma_b$. On the other hand, the prior on $\theta_1, \cdots, \theta_v$ consists of our belief, or the prior information available about the relative performance of the test treatments with respect to the control. The model for the observations can be rewritten as

$$y_{dijp} = \theta_i + \gamma_j + \varepsilon_{ijp}$$

with $\theta_0 = 0$, $i = 0, 1, \cdots, v$ and $j = 1, \cdots, b$.

In matrix notation, the model is

$$Y_d = F_d\theta + G\gamma + \varepsilon$$

where $\theta = (\theta_1, \cdots, \theta_v)'$, $\gamma = (\gamma_1, \cdots, \gamma_b)'$, F_d is an $n \times v$ matrix and G is an $n \times b$ matrix where n is the total number of observations. The (l, i) entry of F_d is one if the lth unit receives treatment i, $i = 1, \cdots, v$ and zero otherwise. The matrix G can be written as

$$G = \begin{pmatrix} 1_{k_1} & 0 & \cdots & 0 \\ 0 & 1_{k_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1_{k_b} \end{pmatrix},$$

by permuting the entries of Y_d, if necessary. The symbol 1_k stands for a $k \times 1$ vector of 1's. Here $k_1, k_2, \cdots, k_b$ are the block sizes. We are allowing for the possibility that all blocks are not of the same size.

The error distribution given θ and γ in the representation of Y_d is assumed to be multivariate normal,

$$Y_d \mid \theta, \gamma \sim N(F_d\theta + G\gamma, E);$$

so is the joint prior distribution of θ and γ,

$$\begin{pmatrix} \theta \\ \gamma \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_\theta \\ \mu_\gamma \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & B \end{pmatrix}\right),$$

where the vectors μ_θ, μ_γ and the positive definite matrices E, T and B are assumed known. The posterior distribution of θ is

$$\theta \sim N(D_dM_d, D_d)$$

where

$$D_d^{-1} = F_d'(E + GBG')^{-1}F_d + T^{-1}$$

and

$$M_d = F_d'(E + GBG')^{-1}(Y_d - G\mu_\gamma) + T^{-1}\mu_\theta.$$

It has been pointed out by Owen (1970, page 1921) and Giovagnoli and Verdinelli (1983, page 697) that this derivation of the posterior with D_d positive definite is possible even in limiting situations when $T^{-1} \to 0$ and/or $B^{-1} \to 0$ provided rank $(F_d: G) = v + b$, which means that all the parameters $\theta_1, \cdots, \theta_v, \gamma_1, \cdots, \gamma_b$ are estimable. The limiting cases correspond to assuming a vague prior on the parameters.

Our object is to estimate θ, the vector of control-test treatment contrasts. The Bayes estimator of θ is D_dM_d with expected loss trD_d under the squared error loss $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)'(\hat{\theta} - \theta)$. A Bayes A-optimal design is a design d which minimizes trD_d. A study of the problem of determining Bayes A-optimal designs has been made by Owen (1970). We shall now outline Owen's results.

We start by assuming a structure for the matrix E,

$$E = G\hat{E}G' + \Delta$$

where $\hat{E} = (e_{ij})$ is a symmetric matrix and

$$\Delta = \text{diag}(e_1I_{k_1}, \cdots, e_bI_{k_b}),$$

a diagonal matrix with diagonal blocks $e_1I_{k_1}, \cdots, e_bI_{k_b}$. This means that the errors within block j have variance $(e_j + e_{jj})$ and covariance e_{jj}, whereas the covariance between errors in blocks j and h is e_{jh}, $j = 1, \cdots, b$; $h = 1, \cdots, b$.

This structure of E implies that

$$D_d^{-1} = -F_d'(GHG' - \Delta^{-1})F_d + T^{-1}$$

where $H^{-1} = \text{diag}(k_1e_1, \cdots, k_be_b) + \text{diag}(e_1, \cdots, e_b)(B + \hat{E})^{-1}\text{diag}(e_1, \cdots, e_b)$. Clearly, $F_d'G$ is $\bar{N}_d$, the matrix obtained from the incidence matrix N_d by deleting the row corresponding to the control. Hence,

$$D_d^{-1} = -\bar{N}_dH\bar{N}_d' + \text{diag}(\alpha_1, \cdots, \alpha_v) + T^{-1}$$

where $\alpha = (\alpha_1, \cdots, \alpha_v)' = \bar{N}_dp$ with $p = (e_1^{-1}, \cdots, e_b^{-1})$. We note that in the special case where $T^{-1} \to 0$, $B^{-1} \to 0$, $k_1 = \cdots = k_b$, $e_1 = \cdots = e_b = \sigma^2$, we get $\sigma^2D_d^{-1} \to (PC_d^-P')^{-1}$, which appears in expression (2.4).

In addition to the specified structure on E, we assume that the θ_i's have equal prior variances and prior covariances. This means that for some δ^2 and ρ $(-(v-1)^{-1} < \rho < 1)$,

$$T = \delta^2((1 - \rho)I_v + \rho J_{vv}).$$

As mentioned in the beginning of the section, we will extend the definition of the incidence matrix N_d to include all $(v + 1) \times b$ matrices with non-negative real entries. These matrices have been called approximate or continuous block designs by Giovagnoli and Wynn (1981).

The following theorem is due to Owen (1970, Theorem 3).

THEOREM 7.1. If $T = \delta^2((1-\rho)I_v + \rho J_{vv})$ and if the numbers of observations to be allocated to the control in each block is fixed, i.e., $n_{d01}, \cdots, n_{d0b}$ are given, then the Bayes A-optimal design is given by

$$n_{d^*ij} = (k_j - n_{d0j})/v, \qquad i = 1, \cdots, v, \quad j = 1, \cdots, b.$$

In view of Theorem 7.1, the problem of finding a Bayes A-optimal design reduces to finding an optimal allocation of the control or any one test treatment to blocks. Let $x_j = n_{dij}$, $i = 1, \cdots, v$; $j = 1, \cdots, b$. Then $\mathrm{tr}D_d$ can be expressed solely as a function of $x = (x_1, \cdots, x_b)'$. Owen (1970) has given an algorithm for determining the optimal x. This algorithm needs at most b steps. Owen also gave explicit forms for x in special cases. Here we quote the example given in Owen (1970, page 1932).

Example 7.3. Let $b = 4$, $k_1 = 100$, $k_2 = 120$, $k_3 = 130$, $k_4 = 140$. There are $v = 9$ test treatments and one control, $e_1 = 10$, $e_2 = 20$, $e_3 = 30$, $e_4 = 40$ and

$$\hat{E} = \begin{pmatrix} 1 & -0.2 & 0.2 & -0.1 \\ -0.2 & 3 & 0 & 0.1 \\ 0.2 & 0 & 2 & 0.3 \\ -0.1 & 0.1 & 0.3 & 5 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.4 & -0.2 & 0.3 & 0.2 \\ -0.2 & 0.6 & 0.1 & 0 \\ 0.3 & 0.1 & 0.8 & -0.1 \\ 0.2 & 0 & -0.1 & 1.0 \end{pmatrix},$$

$\delta^2 = 0.25$, $\rho = 0.11$.

The optimal allocation is $x_1 = 10.98$, $x_2 = 13.00$, $x_3 = 13.93$, $x_4 = 14.91$. The nearest integer allocation gives a design with the following blocks:

Block 1 has 11 units for each test treatment and 1 for control,
Block 2 has 13 units for each test treatment and 3 for control,
Block 3 has 14 units for each test treatment and 4 for control,
Block 4 has 15 units for each test treatment and 5 for control.

If no prior information is available then the optimal allocation will be

$$n_{d^*0j} = \sqrt{v}\, n_{d^*ij} = 3n_{d^*ij} \qquad \text{for } i = 1, \cdots, v; j = 1, \cdots, b.$$

A careful study of the problem of finding optimal block designs for comparing test treatments with a control using the "continuous" design approach, in case no prior information is available has been made by Giovagnoli and Wynn (1985a).

Giovagnoli and Verdinelli (1985) investigated Bayes A-optimal designs for the one-way elimination of heterogeneity model in the hierarchical linear model of Lindley and Smith (1972). Toman and Notz (1987) have recently extended Bayes A-optimality results to two-way elimination of heterogeneity models.

8. HISTORICAL REVIEW

Cox (1958, page 238) advocated augmenting a BIB design in test treatments with one or more replications of controls in each block as a means of getting good designs. He neither formally mathematized the problem nor gave any justification for his suggestion. However, based on what has been developed during the past several years, we know that this is an excellent method of getting efficient designs in many cases. Fieller (1947) gave a solution for A-optimal designs for the zero-way elimination of heterogeneity model which is applicable when v is a square. Pearce (1960) proposed a class of designs for comparing test treatments with a control and gave their analysis for the one-way elimination of heterogeneity model. Freeman (1975) studied some designs for comparing two sets of treatments for the two-way elimination of heterogeneity model. Pesek (1974) compared a BIB design with an augmented BIB design, as suggested by Cox (1958), in estimating control-test treatment contrasts and noticed that the latter was more efficient. Das (1958) has also looked at augmented BIB designs.

Bechhofer and Tamhane (1981) were the first to study the problem of obtaining optimal block designs. However their optimality consideration was neither A- nor MV-optimality, but for the problem of obtaining optimal simultaneous confidence intervals under a one-way elimination of heterogeneity model. Their discoveries led to the concept of BTIB designs; Notz and Tamhane (1983) studied their construction.

Constantine (1983) showed that a BIB design in test treatments augmented by a replication of the control in each block is A-optimal in the class of designs with exactly one replication of the control in each block. Jacroux (1984) showed that Constantine's conclusion remains valid even when the BIB designs are replaced by some group divisible designs.

Majumdar and Notz (1983) gave a method of obtaining A- and MV-optimal designs among all designs for the one-way elimination of heterogeneity model. Hedayat and Majumdar (1984) gave an algorithm and a catalog of A- and MV-optimal designs and studied approximations. Türe (1982, 1985) also studied A-optimal designs and their approximations and construction. Hedayat and Majumdar (1985) gave families of A- and MV-optimal designs. Notz (1985) studied optimal designs for the two-way elimination of heterogeneity model. Majumdar (1986) considered the problem of finding optimal designs for comparing the test treatments with two or more controls.

Jacroux (1987a, b, 1988a) gave new methods for obtaining MV-optimal designs under one-way elimination of heterogeneity models, gave catalogs and studied approximations. Jacroux (1986) studied optimal designs for two-way elimination of heterogeneity models, utilizing techniques of Hall (1935) and Agarwal (1966). Hedayat and Majumdar (1988) studied designs simultaneously optimal under both one- and two-way elimination of heterogeneity models. Jacroux (1988b) generalized the Hedayat and Majumdar (1984) algorithm for finding A-optimal designs. Cheng, Majumdar, Stufken and Türe (1988) gave new families of A- and MV-optimal designs and some approximations for one-way elimination of heterogeneity models. Stufken (1986, 1987, 1988) studied A- and MV-optimal designs for one-way elimination of heterogeneity models, gave families and studied approximations.

There are many other design settings in which it would be useful to identify optimal designs for comparing test treatments with controls. One such setting is that of repeated measurements designs. Some aspects of optimality and construction of designs in this area have been investigated by Pigeon (1984), Pigeon and Raghavarao (1987) and Majumdar (1988).

Giovagnoli and Wynn (1985a) studied A-optimality of designs for one-way elimination of heterogeneity models set in the context of approximate theory, i.e., with an infinite number of observations. Christof (1987) made some further investigations along these lines. Spurrier and Edwards (1986) did a similar study for optimal designs for finding simultaneous confidence intervals.

Bayes optimal designs have been studied in the context of approximate theory. Owen (1970) studied Bayes A-optimal designs, Giovagnoli and Verdinelli (1983) studied Bayes ϕ_p criteria, including D- and E-optimality. Verdinelli (1983) gave methods for computing Bayes D- and A-optimal block designs, and Giovagnoli and Verdinelli (1985) investigated the Bayesian approach under a hierarchical linear model.

This area of research continues to grow in several directions. Among some recent technical reports are: Toman and Notz (1987) on Bayes A-optimal designs for two-way elimination of heterogeneity models; Ting and Notz (1987a) on optimal designs for two-way elimination of heterogeneity models; Ting and Notz (1987b, 1988) and Jacroux and Majumdar (1987) on optimal designs for one-way elimination of heterogeneity models with $k > v$.

It seems appropriate to make a comment on randomization. In running optimal designs we often have to follow a well structured pattern. This does not, however, mean that there will be no room for randomization. The labelling of the treatments, experimental units under a zero-way elimination of heterogeneity model, blocks under a one-way elimination of heterogeneity model and rows and columns under a two-way elimination of heterogeneity model can be randomized.

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