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The Life and Work of Gustav Elfving

Kenneth Nordström

Abstract. This article outlines the scientific work and life of the Finnish statistician, probabilist, and mathematician Gustav Elfving (1908–1984). Elfving's academic career, scientific contacts, and personal life are sketched, and his main research contributions to the fields of statistics, probability, and mathematics are reviewed. (Elfving's pioneering work in optimal design of experiments is not covered, as this topic will be treated elsewhere in this issue.) A chronological bibliography of Gustav Elfving is also given.

Key words and phrases: Bayesian statistics, complex analysis, decision theory, expansion of distributions, Markov chains, multivariate distribution theory, nonparametric statistics, optimal design of experiments, optimal stopping, order statistics, quality control, sufficiency and completeness.

1. GUSTAV ELFVING—HIS LIFE

1.1 Childhood and Formative Years

Erik Gustav Elfving was born on June 25, 1908, in Helsinki, Finland. The son of Fredrik Elfving and Thyra Ingman, Gustav was the youngest of four children; there were two daughters and two sons in the family.

His father was for many years a Professor of Botany at the University of Helsinki. He is considered the grand old man of botany in Finland, who brought new directions of research, including plant physiology, into botany as an academic subject in Finland. Fredrik Elfving had a widespread reputation for being a very colorful and strong personality as well as an excessively demanding teacher. Indeed, Gustav Elfving later met people who, upon hearing the name Elfving, immediately recalled the feared oral examination given by his father some decades ago, which they had flunked several times! At these oral examinations particular emphasis was placed not on book learning, but on displaying a general scientifically critical mind capable of creative solutions. Thus a student might be required to identify the age of a beet, lying sliced on a plate. If the poor soul started counting rings to determine the age, it normally meant a new attempt at the oral examination after a full year.

Fredrik Elfving was also a very dynamic person. Largely due to his determined efforts, a new building was created in 1903 for the Institute of Botanics in the Botanical Garden, in which he both worked and lived with his family. Gustav thus spent his childhood in the Professor's residence in the Botanical Garden—not a bad beginning for an academic.

Fredrik Elfving taught the children to express themselves concisely. Gustav later recalled that, as a little boy sitting at the dinner table, his father would point at, say, the salt, saying just "G" (as in Gustav). This had been agreed upon as standing for "Gustav, please pass me the salt." Other than this somewhat unorthodox behavior, there is no sign of Fredrik Elfving having been a severe father. On the contrary, as a boy Gustav was several times caught climbing on the rooftops of the greenhouses in the Botanical Garden!

In the spring of 1926 Elfving graduated with excellent marks from Svenska normallyceum, a renowned gymnasium in Helsinki for Swedish-speaking boys. The pursuit of an academic career must have been a natural choice for the gifted young man. In the fall of 1926 he enrolled at the University of Helsinki, planning to major in astronomy. While attending compulsory mathematics courses for astronomy students, he became interested in mathematics. Thus, he switched to mathematics, graduating in 1930 in mathematics, with astronomy and physics as minor subjects. Later on he attributed the change to mathematics largely to

Kenneth Nordström holds the Chair in Statistics, Department of Mathematical Sciences/Statistics, University of Oulu, FIN-90401 Oulu, Finland (e-mail: kenneth.nordstrom@oulu.fi).

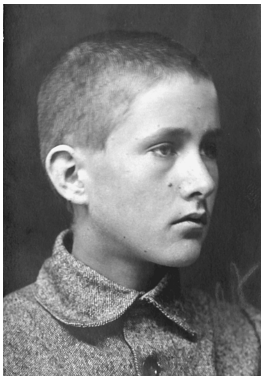

FIG. 1. *Young Gustav.*

the superb teaching of Ernst Lindelöf (1870–1946), a leading complex analyst and the founder of the Finnish school of function theory.

For outstanding master's and licentiate theses Gustav Elfving received the Donner and Lindelöf prizes in 1930 and 1934, respectively. The 1934 dissertation was written under the supervision of Rolf Nevanlinna (1895–1980), and treated Riemann surfaces and their uniformization, a topic within the subfield of complex analysis now known as the Nevanlinna value distribution theory (see Section 2.1). This had been preceded by a study trip to Göttingen in 1931.

Another influential teacher was Jarl Waldemar Lindeberg (1876–1932) of central limit theorem fame, whose course in probability theory Elfving attended in 1929. Elfving later noted that, as a teacher, "... Lindeberg was painstaking and helpful, but somewhat dry" (1985, page 6). He goes on to confess that he found Lindeberg's lectures "a little boring" and had been asked very kindly by Lindeberg at the end: "Hasn't the matter interested you!" (1985, page 6).

Nonetheless, after two additional studies in complex analysis (1935a, b), Elfving's interests turned to probability and, later, to statistics. Elfving himself attributed this complete change of field to a scientific expedition in which he took part as a result of a tragic event in his personal life. At the time he was engaged to a girl from Denmark, who had, however, died of tuberculosis. Probably feeling the need for some distance from this event, and with the help of the girl's parents, Elfving was put in contact with the Danish Geodetic Institute and qualified as the mathematician member on its cartographic expedition to Western Greenland in the summer of 1935. The fact that he had in 1927–1929 worked

FIG. 2. *Graduating from the gymnasium in Helsinki, 1926.*

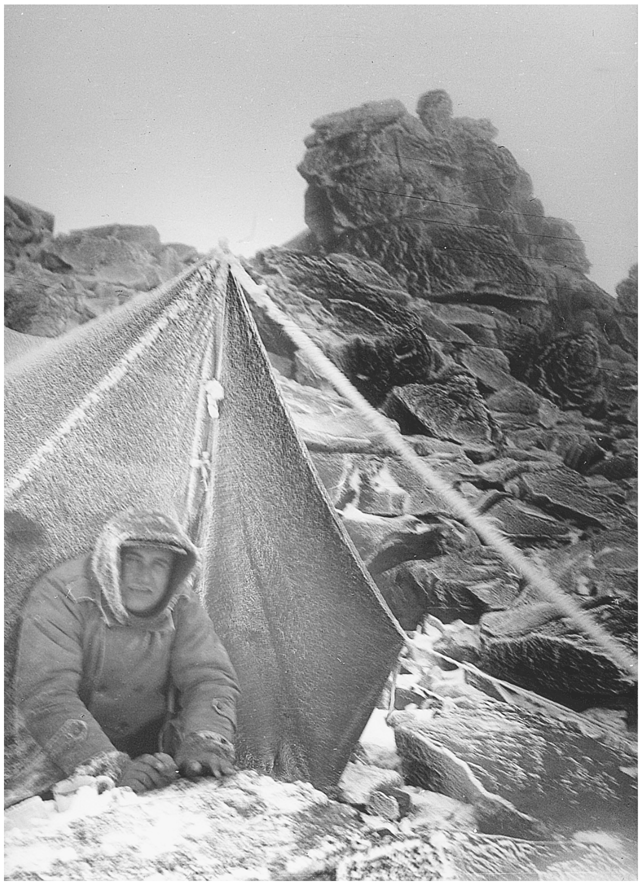

FIG. 3. *Least squares conditions, Western Greenland, 1935.*

FIG. 4. *Making theodolite measurements, Western Greenland 1935.*

as an assistant at the astronomical observatory of the University of Helsinki, responsible for astrophotographic calculations, was probably helpful in this respect. During a rainy spell in Greenland, the expedition members were forced to remain in their tents for three full days, and to pass time Elfving started thinking about least squares problems.

1.2 Academic Life in Helsinki and Stockholm

Having returned from Greenland, Elfving obtained in 1935 his first tenured academic position as a lecturer at Åbo Akademi, a small Swedish-language university in Åbo (Turku), Finland. In 1936 he married Irene (Ira), née Aminoff. They had three children, all boys (Johan, Jörn and Tord, born in 1938, 1941 and 1946, respectively). In 1938 he made a longer trip to Stockholm, resulting, among others, in lifelong warm friendships with Harald Cramér and Herman Wold. The same year he returned from Åbo to his hometown, Helsinki, where he had been appointed Lecturer at the Helsinki University of Technology. From this period one finds Elfving's first two substantial publications in probability theory (1937, 1938), in which he formulates (and solves in a special case) an important problem to be known later as the embedding problem for Markov chains, and which to date remains unsolved in full generality (see Section 2.2).

During the 1939–1940 and 1941–1944 wars with the Soviet Union, Elfving served in the Finnish coast artillery as a geodesist, and when possible also taught mathematics at the University of Technology. In a set of lecture notes in probability theory (1946c), there is evidence also of Elfving's ongoing wartime scientific activities at the University of Helsinki, where he had been appointed Docent in mathematics in 1937. In these lecture notes (1946c, page 39) there is a reference to dittoed seminar notes by Lars Ahlfors and others on the theory of functions of a real variable (spring term 1942) and by Elfving on the set-theoretic foundations of probability (fall term 1942). Although written at an intermediate level, the lecture notes place probability explicitly within the framework of general measure theory and Lebesgue integration on Euclidean spaces, with due reference to the monographs of Saks and de la Vallée Poussin. This period also marks the beginning of Elfving's lifelong interest in the writing of articles directed to a wide nonspecialist audience (1941, 1942a, 1945b), the publication in 1942 being Elfving's first one in statistics.

In 1945 one of the chairs in mathematics at the University of Helsinki became vacant when Elfving's good friend and fellow mathematics student in the 1920s, Fields medalist Lars Ahlfors (1907–1996), was offered an *Extraordinariat* in applied mathematics at the University of Zürich. [In 1946 Ahlfors received an invitation to go to Harvard, where he remained throughout the rest of his career; see Ahlfors (1982) for an interesting account.] When Ahlfors's chair became vacant, the applicants, including Elfving, were granted a one-year period in which to qualify themselves for the position. One sees evidence of this in the form of quite a few substantial publications from this period: (1946a, b, 1947a) on problems in stochastic processes, and (1947b, e) on distributions for order statistics (see Section 2.3) and multivariate sampling distributions (see Section 2.4), respectively.

A substantial part of this fruitful period Elfving spent at Stockholm University, where he was locum tenens professor during the academic year 1946–1947. That year he was substituting for Harald Cramér, who had been invited to lecture at Princeton University in the fall of 1946, when Princeton celebrated its Bicentennial Year, and at Yale University in the spring of 1947. During the stay in Sweden, Elfving and his family lived in Cramér's villa in Djursholm, outside of Stockholm. Once when trying to get some work done at home in the midst of three children, Gustav is said to have taken his papers up to the rooftop of Cramér's villa, but this ingenious hideaway was soon discovered by the resourceful children.

In the final evaluation of the professorial candidates for the chair in Helsinki, each of the referees Lars Ahlfors (Harvard), Arne Beurling (Uppsala) and Harald Bohr (Copenhagen) chose Elfving as their top candidate, and in 1948 Elfving was appointed Professor of Mathematics at the University of Helsinki.

This position, which Elfving held until he retired in 1975, was a Swedish-language chair in mathematics, the holder of which was expected to teach the whole breadth of pure mathematics courses to Swedish-speaking mathematics majors at the University of Helsinki. On the other hand, at the time of the appointment Elfving's interests had already shifted fully to probability and statistics. Indeed, except for Kari Karhunen, who after a short and promising academic career left academia for a career in an insurance company, Elfving remained throughout the 1950s and 1960s the single notable and internationally recognized scientist in Finland in the fields of probability theory and statistics. Thus he ended up with the twofold task of being responsible for pure mathematics courses, to be given to the Swedish-speaking mathematics majors, as well as being responsible for introducing and developing single-handedly courses in probability and mathematical statistics at the University of Helsinki.

Throughout the 1950s Elfving therefore delivered lecture series on a number of topics in mathematics, such as calculus of variations, differential equations, functions of a complex variable, non-Euclidean geometry and number theory and algebra, in addition to developing and teaching new courses in probability and statistics. From the perspective of today's all-pervading narrow specialization, it is remarkable that in the middle of a highly creative period during which Elfving produced some of his most fundamental research contributions to statistics (1952a, b, 1954a, c, 1955a), one finds him giving series of lectures on, among others, differential geometry (spring 1952) and the history of mathematics based on van der Waerden's well-known 1954 treatise *Science Awakening* (fall 1955).

Being appointed to a Swedish-language position, Elfving also delivered his lectures in probability and statistics in Swedish. However, for the benefit of the attending Finnish-speaking students, he early on developed the habit of giving intermittently concise and elegant summaries in Finnish of the main points of a lecture. Not surprisingly, these summaries soon served other purposes as well. The importance of Elfving's work in promoting probability and statistics in Finland was later recognized by the University of Helsinki through the founding of a professorial chair in this area in 1971.

1.3 Visits to the United States

On several occasions Elfving took leave from the University of Helsinki and went abroad as a visiting professor. The first and longest such visit was the two-year period 1949–1951, which he spent at Cornell University, accompanied by his family. The visit came about at the invitation of William Feller, whom Elfving had apparently met in Sweden in the 1930s. The invitation was part of an endeavor to establish a center for probability theory at Cornell. Indeed, there was a very strong group of probabilists in residence at Cornell, including Joe Doob, William Feller and Mark Kac as the senior people, and Kai Lai Chung, Monroe Donsker, Gilbert Hunt and Murray Rosenblatt as younger colleagues.

Elfving and his family clearly enjoyed themselves wholeheartedly in the scientifically and socially stimulating atmosphere in Ithaca. As agreed upon with Feller, Elfving assumed the main responsibility for the teaching of courses in mathematical statistics. Interestingly, in the lecture notes from one such course which he taught during the academic year 1949–1950, one finds an early use of the symbol $\perp\!\!\!\perp$ for stochastic independence. When asked in the 1960s about the origin of this symbol, Elfving referred to it as being "homemade."

Scientifically the stay at Cornell was certainly very fruitful. During this period, Elfving produced his arguably most influential paper (1952a), in which he laid the foundations for the whole subject of optimal design of experiments as we know it today. [Elfving's contributions to the design of ex-

FIG. 5. *The Elfving family about to sail off to the United States, 1949.*

periments will only be mentioned in passing here, as this topic will be treated elsewhere in this issue; see also Draper, Mäkeläinen, Nordström and Pukelsheim (1999) and the references therein.] Another notable paper from this period is (1950), which develops an often-cited nonparametric trend test (the Elfving–Whitlock test); see Section 2.5. Yet another important contribution from this period is the paper (1952b) on sufficiency and completeness in decision function theory (see Section 2.6).

At that time there were no books on probability theory or statistics available in Finnish, the only exception being an elementary textbook *Todennäköisyyslasku ja sen käytäntö tilastotieteessä*: *alkeellinen esitys* (*The Calculus of Probability and Its Application to Statistics*: *An Elementary Treatment*) by Lindeberg, which had appeared in 1927. Although well written, this book was understandably showing definite signs of age. During the summer of 1950, most of which was spent with the family in Florida, Elfving started writing a book on probability in Finnish. The result appeared some years later (1956a). It is a carefully written intermediate-level text in probability, a large portion of which is actually devoted to topics from the theory of statistics. In the preface Elfving mentions explicitly the influence of Harald Cramér's 1949 textbook *Sannolikhetskalkylen och några av dess användningar* (*The Calculus of Probability and Some of Its Applications*) and Cramér's well-known 1945 treatise *Mathematical Methods of Statistics*. For many years courses in probability at Finnish universities were based on (1956a), which appeared in both a second and a third edition in 1964 and 1966, respectively.

Among other activities while at Cornell, one finds visits of Elfving to Chapel Hill, at the invitation of Herbert Robbins, and to Princeton, at the invitation of William Feller (who left Cornell soon after Elfving arrived), both visits taking place in the spring of 1951.

Elfving made three further longer visits to the United States. At the joint invitation of T. W. Anderson, Herbert Robbins and Herbert Solomon, he spent the spring term 1955 as a visiting professor at Columbia University, teaching and carrying out research within a research project on discriminatory analysis [cf. the interview of Herbert Solomon in Switzer (1992, page 397) and Draper et al. (1999)]. Research around this project continued throughout the rest of the 1950s, with Elfving (back in Finland) as a part-time research associate. The research project culminated in a volume on item analysis and prediction, edited by Herbert Solomon. Elfving's contribution to this volume was quite substantial: he was the single author of four chapters, a fifth chapter being coauthored with Rosedith Sitgreaves and Herbert Solomon. However, three of these contributions had actually appeared elsewhere earlier (1956c, 1957a, 1959b), while (1961b) builds on a paper (1955a) on the expansion of distribution functions (see Section 2.7).

During the visit to Columbia, Elfving was an invited speaker at the Third Berkeley Symposium and also visited, among others, the University of Chicago, where he met William Kruskal and Jimmie Savage. Elfving continued to have regular contact with Savage throughout the 1950s and 1960s. In addition to warm friendship, Elfving and Savage also shared common scientific interests: allocation problems and the comparison of experiments in the 1950s and Bayesian statistics in the 1960s.

Both Savage and Kruskal later visited the Elfvings in Finland, along with many other prominent statisticians, probabilists and mathematicians, including T. W. Anderson, Herman Chernoff, Bradley Efron, Wolfgang Fuchs, Joe Gani, Dennis Lindley, Ingram Olkin, Murray Rosenblatt, Frank Spitzer and Helmut Wielandt (with apologies to anyone unintentionally left out). (T. W. Anderson had already visited the Elfvings at the end of the 1940s during his sojourn in Sweden.) Quite a few of the visitors were taken to Sundholm, the summer residence of the Elfvings, located near Nystad (Uusikaupunki) on the western coastline some 50 miles northwest of the city of Åbo. The old parts of Sundholm, which is a large mansion belonging to his wife's aristocratic family (the Aminoffs), date back to the year 1487, and it is said that Gustav Elfving occasionally joked with his American visitors remarking that Sundholm was built five years before Christopher Columbus supposedly discovered the New World!

Not all of these visits passed entirely uneventfully. Thus, when Jimmie Savage, who had poor eyesight, visited Sundholm, a kitten belonging to the mansion barely escaped being crushed when Savage was taking a seat. (This was presumably unintentional.) On another occasion, when Johan Fellman visited Sundholm for a couple of weeks during a summer in the 1960s, preparing his dissertation under Elfving's supervision, Elfving wanted to show his student the beautiful archipelago outside of Sundholm. As it happened, he had forgotten to check the gas supply in the small motorboat, and, of course, they ran out of gas in the outer part of the archipelago. The return trip was then travelled at a leisurely pace using a small bucket as a paddle.

The fall of 1960 and spring of 1966 Elfving spent at Stanford University, at the invitation of Herbert

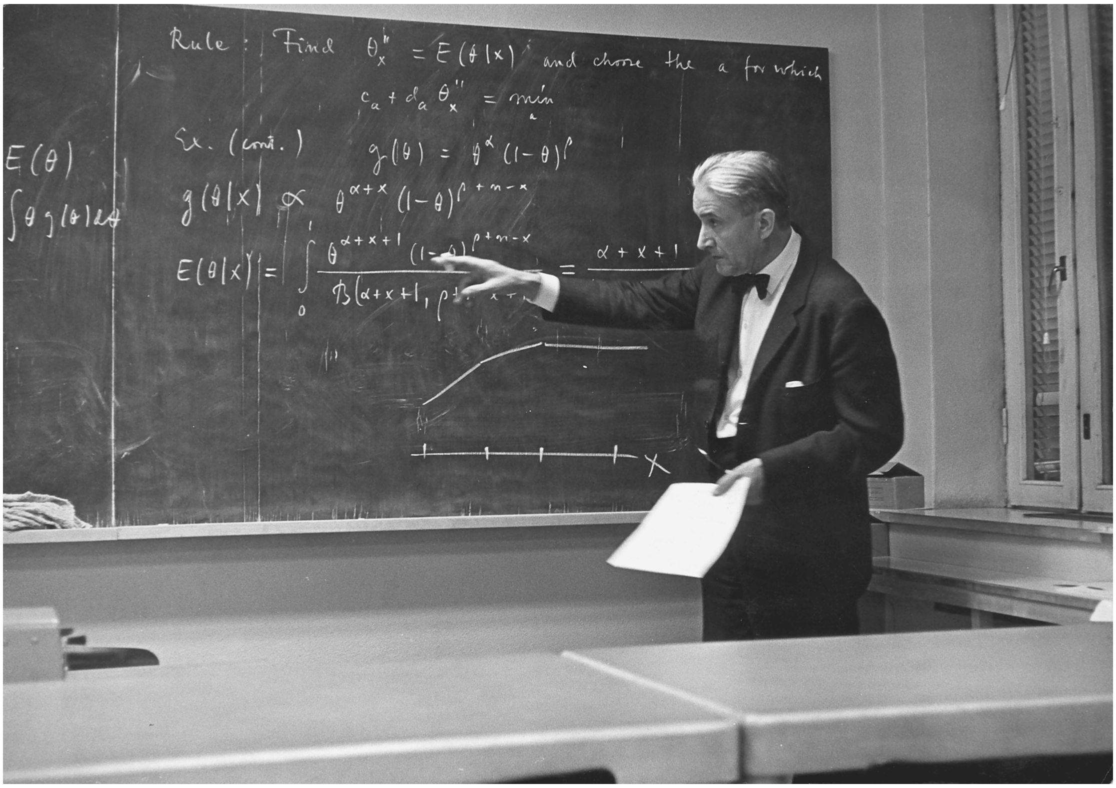

FIG. 6. *Elfving giving a talk in his seminar at the University of Helsinki in 1966, after his Stanford visit.*

Solomon, who had moved from Columbia to become chairman of the Department of Statistics at Stanford in 1959. Another colleague at Stanford with whom Elfving had had common scientific interests (optimal design of experiments) was Herman Chernoff. During the visit in 1960, Elfving gave lecture series on elementary probability theory and on information theory, and participated in a Stanford research project on quality control and acceptance sampling. A notable publication from this period is the paper (1962a) and the unpublished technical report (1962b), both on continuous sampling plans (see Section 2.8). Another interesting piece of work is the report (1962c) on two-person Markov games, the work on which was started at Stanford in 1960 and was completed when Elfving visited the Institute of Mathematics, University of Århus, Denmark, in the spring of 1962. In the report (1962c), Elfving makes an attempt at overcoming some of the well-known inherent difficulties associated with non-zero-sum games, such as the nonexistence of a unique "solution" of the game, by extending the notion of a game in a number of interesting ways.

During his 1966 visit to Stanford, Elfving taught an elementary statistics course and also gave a more advanced lecture series on Bayesian statistics. Indeed, around this time he was very interested in Bayesian statistics and wrote an early (unpublished) Bayesian robustness study, which appeared as a Stanford technical report (1966a); see Section 2.9. Another substantial publication from this period is (1967a), in which Elfving formulates and solves a new stopping problem involving a point process (see Section 2.10).

Throughout his career from the 1950s onwards, Elfving received several offers for permanent positions from universities in the United States. Although he must have been tempted to move from the relative scientific isolation in Finland to the stimulating environment at major departments in the United States, his concerns about his family and his characteristic sense of duty and loyalty toward his country and his friends made him decline these offers. In one of the letters regarding an offer he had received in the 1950s one reads:

> . . . there are so many things that keep me here [in Finland], in the long run: ties of friendship and kinship, loyalty towards a small and poor country, the psychological problems that would arise in transplanting the children, and ourselves, in a new ground.

1.4 From the 1960s into Retirement

At the time of the first visit to Stanford in 1960, Elfving was looking for new fields of research, having essentially summarized his work in design of experiments in the Cramér Festschrift publication (1959a). Whereas quality control and optimal stopping problems were clearly topics of a more incidental character that happened to occupy Elfving during his Stanford visits (see the preceding section and Sections 2.8 and 2.10), the areas in which he was particularly interested throughout the 1960s and 1970s were Bayesian statistics and foundational questions, as well as decision theory, game theory and information theory.

Thus, for example, during the 1962 visit to Århus mentioned above, Elfving gave a series of lectures on non-zero-sum games, based in part on the material in the report (1962c), and on the foundations of statistics, based in part on L. J. Savage's book *The Foundations of Statistics*. Also, at the Nordic Conference on Mathematical Statistics held in Århus in 1965, Elfving gave an invited lecture series on Bayesian inference and subjective probability, which later appeared as (1968b); see Section 2.9.

On decision theory, game theory and information theory (as well as on other topics), Elfving wrote several illuminating articles directed to a wide nonspecialist audience (1953b, 1955b, 1959c, 1962d, 1963b, c, 1965a, 1968a, 1973a). The writing of nontechnical articles, as well as the delivery of such lectures, was an integral part of Elfving's scientific activities throughout his career and was, as pointed out by Mäkeläinen (1984, page 202) (translated from Finnish), "... partly an acquired habit of working, which involved a certain effort to put new lines of thought into a general perspective." A typical feature of these articles is broad coverage combined with transparent exposition. Through a sequence of well-chosen examples, often from everyday life and the surrounding society, the reader is guided almost effortlessly, and with a minimal use of mathematical machinery, to the key concepts and ideas of a mathematically abstracted theory. Indeed, Elfving was always particularly fascinated by the *ideas* in science and by their relationship to, and possible impact on, the surrounding society. Consequently, he was interested not only in the most recent technical results as such, but always sought to put new developments and results in a broader perspective. This is especially visible in his nontechnical writings.

The article (1965a), which Elfving prepared for a Festschrift for Rolf Nevanlinna, led Elfving to some interesting early ideas on randomness, regularity and symmetry in observed patterns. Some of these rather original ideas were expanded further in (1968a). Elfving was specifically interested in using the theory of stochastic processes to study the connections between information theory and regularity of patterns, and in combining ideas from information theory and the theory of stochastic processes to provide models for a quantitative study of esthetic evaluation. Together with a graduate student, Pentti Suomela, he investigated the possibilities of using computer-generated two-dimensional Markov random fields to produce random patterns, exhibiting a certain symmetry without being deterministic. Indeed, in the spring of 1968, Elfving gave a talk on Markov properties of two-dimensional random fields at University College London, as part of the London Joint Statistics Seminar.

The period from the 1960s until retirement was a period filled with many kinds of academic activities. Besides the usual everyday activities of teaching, seminars and guiding of students as well as research, Elfving's expertise was called upon in other academic matters at an increasing rate, abroad as well as in Finland. He was an opponent at numerous doctoral defences (the equivalent of external examiner at the thesis defence procedure employed in many European universities), and on several occasions evaluated candidates for professorial chairs, in particular in the Nordic countries. He also sat on the editorial board of *The Annals of Mathematical Statistics* (1964–1967) and *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* (1962–1975), and was the regional editor of *Mathematica Scandinavica* (1953–1972).

Elfving also served the Institute of Mathematical Statistics (IMS) and the International Statistical Institute (ISI) in various other capacities in the 1960s and 1970s. He was a member of the IMS Committee on Nomination of Fellows as well as the Committee on Special Invited Papers, and also served the IMS as a member of its Committee on the European Region. Interestingly, when voices were raised in Europe against American dominance and plans were put forth for a new society, Elfving was rather moderate in his views. In particular, he expressed in correspondence the hope for "... a universal IMS with various regional chapters over the world," rather than the formation of yet another society. He served the ISI in 1965–1969 and 1971–1975 as a council member of the International Association for Statistics in Physical Sciences (IASPS), which later became the Bernoulli Society for Mathematical Statistics and Probability.

Another kind of academic activity, which Elfving took particular pleasure in, was his involvement with the Åbo Nation student body at the University of Helsinki, of which he was the "inspector" during the years 1964–1975. The collection (1975) includes the annual addresses given by Elfving to the student body. These entertaining and humorous addresses were cherished by the students and reveal Elfving's broad cultural interests.

Elfving retired in 1975. As with many active academics, retirement simply meant being able to carry out research more intensively, free of the earlier routine duties. Elfving thus turned with vigor to full-time research in the history of mathematics and statistics. The topic was by no means new to him. The early paper (1948) on the historical developments in probability and (1967b) as well as the book reviews (1966b) and (1969) are all clear indications of his long-time interest also in historical developments.

Elfving's major research project in retirement was the preparation of a history of mathematics in Finland from 1828 until 1918, the year of Finland's independence, a task that had been given to him by the Finnish Society of Sciences and Letters. The choice of the Society could hardly have been better. The result appeared as (1981a), a volume displaying Elfving's true craftmanship, beautifully written as well as meticulously researched. Not only does Elfving treat the historical developments of mathematics in Finland and abroad during that period, but, in keeping with his own broad cultural interests, manages to embed these developments seamlessly into the history of learning and culture in Finland. This volume is warmly recommended to anyone interested in Elfving's writings as well as the subject itself. The publications (1980c, 1981b, c, 1982) are offsprings of this treatise, describing briefly some of the notable mathematicians among the total of 52 mathematicians considered in detail in (1981a).

FIG. 7. *Gustav Elfving on his seventieth birthday, 1978.*

When the work on (1981a) was finished, Elfving was asked by the Finnish Society of Sciences and Letters to prepare an historical account of the third half-century 1938–1987 of the Society. Having served the Society in various capacities for decades, Elfving was again singularly qualified for the task. Interestingly, the previous account, covering the first century of the Society, had been written by his father. Except for the final chapter, concerned with the developments from 1980 onward, Elfving had completed the work at the time of his death. The writing of the final chapter was undertaken by Gösta Mickwitz, and the result appeared posthumously as (1988).

Gustav Elfving died at his home in Helsinki on March 25, 1984, outliving his beloved wife, Ira, by five years.

1.5 Gustav Elfving—the Man

In a short note prepared for a Festschrift in honor of Herman Wold (1970a), Elfving gives an indication of how he felt about academia, research and publications, as well as his own scientific contributions. Regarding his motive for doing research, he writes (1970a, page 41)

> I think my *motive* for sitting down to read and write instead of doing something else is in the first place a sense of duty towards the institution at which I happen to be working. This sense of duty has, of course, crystallized into a habit.

Interestingly, the importance of doing one's *duty* is something which colleagues and students have stressed in their writings about his father Fredrik Elfving. As to his own scientific contributions he notes (1970a, page 41)

> I have very little belief in the importance of my research work; however, I have

> some belief in the perpetuation of a certain tradition of clarity of thought and expression.

As pointed out by Mäkeläinen (1984, page 205), this kind of self-effacing criticalness and scepticism was characteristic of Elfving. Indeed, during the 1966 Stanford visit Elfving is remembered as having said something like "I don't know why anyone besides Doob bothers to do probability, or why anyone besides Charles Stein bothers to try statistics," a humorous and self-deprecating statement typical of Elfving.

In the note (1970a) Elfving expresses also his concern about the "rising flood of scientific publications"—this was a theme to which he returned repeatedly in private correspondence—and suggests reconsidering the purpose and the form of scientific publications. He notes that many papers are "... really exercises, written for the legitimate but ephemeral purpose of proving ability and adding to the writer's list of qualifications." Elfving firmly believed that one should have substantial results of hopefully lasting value before publishing and would certainly have agreed with Raj Bahadur's well-known maxim "publish and perish." Indeed, in (1970a, page 42) he writes admiringly of the

> ... mature treatises, simple and final, of which great scientists may bring forth maybe a dozen, while ordinary vineyard men might produce perhaps one in their life time, if any.

Although Gustav Elfving was a man who, in discussion, would never have put forward his own achievements, his contributions to the fields of statistics and probability and his other services to the profession did not go without recognition. In addition to being elected to various Finnish scientific societies, he was in 1955 elected Fellow of the Institute of Mathematical Statistics and became an elected member of the International Statistical Institute in 1963. Also, in 1974 he was elected an Honorary Fellow of the Royal Statistical Society and a Foreign Member of the Royal Academy of Sciences (Sweden).

The following story gives an indication of Elfving's general moral standards. During the Christmas holidays 1967–1968, Elfving spent some time in Rome at the Finnish Rome Institute, Villa Lante, accompanied by his wife, Ira. The purpose of the trip was to carry out research and to meet with Bruno de Finetti and some of de Finetti's visitors (including Allan Birnbaum) while in Rome. When Elfving returned to Finland, he felt that he had not accomplished very much concrete in terms of research during his stay in Rome. Consequently, he contacted the foundation from which he had obtained a grant for the stay, and volunteered to return the grant money.

Also, when Elfving stepped down from the editorial board of one of the journals with which he had been associated for more than a decade, his letter of resignation contains a sentence where he explicitly declines being sent further free issues of the journal.

Gustav Elfving was a dignified gentleman of the old school and an exceptionally civilized and humane individual. In the company of casual acquaintances, he was courteous and soft-spoken but somewhat reserved. (However, see the anecdote at the end of this section.) Colleagues and friends came to see the more personal side of Gustav Elfving: a considerate and warm human being as well as an open-minded man who enjoyed conversation, contributing to it with charm and wit. He was a connoisseur of culture in its broadest sense, with a keen interest in philosophy, history, literature, languages and society at large, as well as a *bon vivant*, who enjoyed a glass of wine in good company. Occasionally he combined a glass of wine with being a gentleman, as in the case when he and Ingram Olkin had taken the wrong train from Zürich to Basel on their way to the 1964 meeting of the IMS in Bern. Upon asking a French-speaking lady in Basel for the directions to the train to Bern, Elfving most eloquently invited the lady to join Ingram Olkin and him for a glass of wine while they were waiting for the train. On another occasion, after a dinner party at Stanford, Elfving turned to Kai Lai Chung saying: "A meal without wine is like a day without sunshine."

Elfving wrote poems and enjoyed philosophical discussions, for example, with his close friend Georg Henrik von Wright, the well-known Finnish philosopher who in 1948 was appointed as successor to Ludwig Wittgenstein at Cambridge University. Nearly every summer from the beginning of the 1950s onward he would go sailing for a week or two, writing poems and enjoying the unique Finnish archipelago in the company of von Wright and Eric Bargum, a lifelong friend, mathematician and the captain of the boat. Besides Swedish (his native language) and Finnish, Gustav Elfving had an excellent command of German, Latin, French and English.

The following anecdote from an Oberwolfach conference, held in the spring of 1960, shows that, if the occasion so demanded, Gustav Elfving could be anything but soft-spoken and reserved. It was communicated to the author by Arthur Albert of Boston University. As is still the case today, participation in the conference was by invitation only, and some two dozen distinguished statisticians and probabilists, including Gustav Elfving, had been invited. At the time, Albert was an NSF Postdoctoral Fellow working with Ulf Grenander in Stockholm. At the suggestion of Grenander, who was unable to attend, Albert went to the conference instead. Here is Albert's recollection (freely adapted from an e-message) of the farewell banquet of the Oberwolfach conference in March 1960.

> On the last night, there was a farewell banquet with lots of good food. Much wine was consumed as well, because it was known in advance that each participant in the conference had to stand up and give a performance of some sort. There were no rules concerning the nature of the performance, but everyone was morally bound to do something.
>
> It was a polyglot performance. Alfred Rényi and a Hungarian colleague stood up in front of everyone. Rényi told a joke . . . in Hungarian. The colleague convulsed. This completely cracked up the rest of us even though we were clueless about the joke itself. Someone recited a stirring Teutonic poem. I sang a few limericks of various shades of gray.
>
> Then it was Elfving's turn. I must preface what happened next by stating for those who never met him that Elfving was a dignified gentleman of the old school. He was very soft-spoken and self-effacing in one-on-one encounters. In preparation for the evening's performance, he had fortified himself with ample quantities of wine. He announced in several languages that he was about to sing a hunting song from Lapland, explaining that the song recounted the tracking and killing of a large warm-blooded mammal of some sort (a bear perhaps). Without further ado, Gustav climbed up on a table top and cut loose. There was chanting, arm waving, foot stomping, shouting, guttural animal sounds, guttural Lappish (or Finnish) sounds. The tracking of the animal was acted out at length, and when it came time to kill the beast, he alternated playing the part of the hunter and the hunted. Although the narrative was being chanted in a completely incomprehensible language, it was clear that the bear wasn't anxious to become freezer meat, and therefore put up quite a fight. Naturally, so did the hunter, who apparently was only armed with a spear. For about 10 minutes, Elfving commanded everyone's rapt attention, amazed as we were by the incredible transformation that we were witnessing. At the end, the hunter prevailed, Gustav stepped down off the table, straightened his tie and put on his suit jacket. It took a couple of seconds for us to transit back to the present. The ovation was tumultuous. Elfving looked slightly embarrassed. Superman became Clark Kent once again.

2. GUSTAV ELFVING'S SCIENTIFIC CONTRIBUTIONS

In this section Gustav Elfving's main research contributions to statistics (excluding design of experiments), probability and mathematics are reviewed. Elfving's original results are outlined (in our notation) and appear grouped into research areas that are presented in chronological order. Short surveys of related literature are given mainly in those areas that have had either substantial later developments, building directly on Elfving's results and ideas, or that contain interesting applications of Elfving's results.

2.1 Complex Analysis

Elfving's dissertation (1934) was prepared under the supervision of Rolf Nevanlinna and is on a topic within function theory now known as the inverse problem of the Nevanlinna theory. Broadly speaking, this problem arises when one studies the interplay between the distribution of values of a meromorphic function and the structure of the Riemann surface associated with the corresponding inverse function.

In the inverse problem one attempts, starting from a Riemann surface, to construct meromorphic functions with prescribed (Nevanlinna) deficiency and ramification indices, subject to certain necessary conditions (inequalities) that follow from the two fundamental theorems of Nevanlinna. In a celebrated 1932 paper, Nevanlinna solved a restricted inverse problem for the class of Riemann surfaces with finitely many logarithmic branch points. In his dissertation, Elfving extends these results to Riemann surfaces with finitely many branch points that are logarithmic or algebraic. He also simplifies Nevanlinna's construction of Riemann surfaces by using certain line complexes (topological trees). For an appreciation of Elfving's thesis, see Bakken (1977) and Drasin (1977).

Of the two other papers (1935a, b) that Elfving published in this area, the first one is based on a presentation that Elfving gave at the Eighth Scandinavian Congress of Mathematicians held in Stockholm in August 1934. In the conference publication (1935a), Elfving studies the connection between the convergence of Riemann surfaces and the locally uniform convergence of meromorphic functions. The second paper (1935b) illustrates the connection between the distribution of values of a meromorphic function and the structure of the corresponding Riemann surface by way of a simple example.

Although Elfving turned to probability and statistics after this, one sees clear traces of his strong research background in pure mathematics in some of his later work, where he is not shy of using his substantial analytic skills, combined with, for example, results from function theory, when needed.

2.2 Markov Chains

Having moved away from research in pure mathematics, Elfving's first two subsequent publications are in probability, specifically on problems in the theory of Markov chains. In the first one (1937), Elfving formulates what has become known as the embedding problem for Markov chains (stochastic matrices). Although extensively studied since, this important problem remains unsolved in full generality. The essence of Elfving's embedding problem and the results in (1937) are as follows.

Consider a discrete-time Markov chain with, say, n states, and corresponding transition matrix $P = (p_{ij})$. Does there exist a continuous-time Markov chain such that P occurs as its transition matrix? More formally, let $\mathscr{P} = \{P\}$ denote the set of $n \times n$ stochastic matrices, and consider the set of stochastic matrices $\{P(s,t)\colon 0 \le s \le t < t_0 \le \infty\}$, each element of which satisfies the continuity condition

$$\lim_{t\downarrow s} P(s,t) = \lim_{s\uparrow t} P(s,t) = I,$$

and the Chapman–Kolmogorov equations

$$P(s,t) = P(s,u)P(u,t), \quad 0 \le s \le u \le t < t_0.$$

Given $P_0 \in \mathscr{P}$, does there exist a $P(s,t)$ satisfying the above conditions and such that $P(0,1) = P_0$? If this is the case, then P_0, and the corresponding discrete-time Markov chain, is said to be embeddable. In Elfving's original terminology the chain P_0 is "interpolated" by the class $\{P(s,t)\}$.

In (1937) Elfving considers the situation where the continuous-time Markov chains are time-homogeneous, that is, the transition matrix $P(s,t)$ depends on the time difference $t - s$ only. For this case (and assuming throughout that P_0 has distinct eigenvalues), he shows that there is no matrix, a unique matrix or finitely many matrices interpolating P_0. He also shows that, for P_0 to be embeddable, it is necessary that P_0 have no eigenvalue other than 1 of unit modulus, and that every negative eigenvalue be of even algebraic multiplicity.

Assuming a state distribution $Q(t_0)$, fixed at some time point t_0, and allowing both discrete- and continuous-time chains, Elfving also considers in (1937) the problem of whether the distributions of all the previous time points are completely determined by $Q(t_0)$, that is, whether there is a definite "starting point" of the process in the past or whether it extends infinitely into the past. Essentially he shows that the former is generally the case, unless $Q(t_0)$ is the stationary distribution of the chain.

Elfving's paper (1937) (which is written in German) was later translated into English and appeared in the Mimeo Series of the Department of Experimental Statistics, North Carolina State College, as Report 103 (May 1954).

In (1938) Elfving extends the above results to the case of nonhomogeneous continuous-time chains and gives some specific results for the two-state ($n = 2$) case and the case of cyclical chains.

After Elfving's pioneering work in this area, the embedding problem appears to have remained dormant until the beginning of the 1960s. The problem was revived by Chung (1960, page 203), who formulated the embedding problem for countable chains and emphasized the importance of the problem. A characterization of embeddability when $n = 2$ was obtained by D. G. Kendall [see Kingman (1962)], and a number of partial results for the case $n = 3$ have been given since, much less being known about the structure of the set of embeddable stochastic matrices when $n > 3$. For a list of key references, see Goodman (1983).

It is of interest to note that Elfving's embedding problem has not only been looked upon from the point of view of theoretical probabilists. When modelling, for example, social processes by continuous-time Markov structures, one is faced with the question of whether an empirical process could have arisen as a result of an evolving continuous-time Markov process. Thus one is naturally led to Elfving's embedding problem; see, for example, Singer and Spilerman (1976) and Geweke, Marshall and Zarkin (1986).

2.3 Order Statistics

The 1947 *Biometrika* paper (1947b) is Elfving's first one in a mainstream statistical journal. In this paper he establishes a distributional result that has become part of the standard body of results on order statistics; see, for example, Stuart and Ord (1987, Sections 14.27–14.29) or David (1970, Section 9.4).

Elfving is interested in the distribution of the sample range when the sample has been drawn from a standard normal distribution. Referring to earlier work where the exact calculation of the finite-sample distribution had been found intractable, he turns instead to the problem of determining the asymptotic distribution of the sample range. More precisely, letting $X_{(1)}$ and $X_{(n)}$ denote the smallest and largest observations in the sample $X_1, \ldots, X_n$, and letting $W_n = X_{(n)} - X_{(1)}$ denote the corresponding sample range statistic, Elfving determines the asymptotic law of the probability integral transformation of W_n, when $X_1, \ldots, X_n$ forms an iid sample from $N(0, 1)$. The essence of his results are as follows.

Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cdf and density of $N(0, 1)$. The joint density of the extremes $X_{(1)}$ and $X_{(n)}$ then takes the well-known form

$$\begin{aligned} &f_{X_{(1)},X_{(n)}}(x_{(1)}, x_{(n)}) \\ &\quad = n(n-1)\phi(x_{(1)})\phi(x_{(n)})[\Phi(x_{(n)}) - \Phi(x_{(1)})]^{n-2}. \end{aligned}$$

Letting $S_n = (X_{(1)} + X_{(n)})/2$ denote the midrange, Elfving introduces the transformation

$$\begin{aligned} U_n &= 2n[\Phi(X_{(1)})\Phi(-X_{(n)})]^{1/2} \\ &= 2n[\Phi(S_n - W_n/2)\Phi(-S_n - W_n/2)]^{1/2} \end{aligned}$$

and

$$V_n = \frac{1}{2}\log\frac{\Phi(X_{(1)})}{\Phi(-X_{(n)})} = \frac{1}{2}\log\frac{\Phi(S_n - W_n/2)}{\Phi(-S_n - W_n/2)},$$

arriving at the joint density

$$f_{U_n,V_n}(u, v) = \frac{n-1}{2n}u\left(1 - \frac{u\cosh(v)}{n}\right)^{n-2}, \quad u \geq 0,\ u\cosh(v) \leq n. \tag{1}$$

Elfving notes the stochastic convergence

$$S_n \to_{\mathbb{P}} 0 \tag{2}$$

and points out that U_n tends to coincide with $U_n^* = 2n\Phi(-W_n/2)$ for large n, by virtue of (2). He substantiates this by showing effectively the asymptotical equivalence of (U_n^*) and (U_n), so that the sequences (U_n^*) and (U_n) share the same limiting distribution. He goes on to determine the limiting form of the joint density (1), from which the marginal limiting density of (U_n) obtains as

$$f_U(u) = u\int_1^\infty \frac{e^{-ut}}{\sqrt{t^2-1}}\,dt.$$

From this he derives the corresponding mean, variance and cdf, respectively, as

$$\mathbb{E}(U) = \pi/2, \quad \mathrm{Var}(U) = 4 - \pi^2/4$$

[cf. Stuart and Ord (1987, Exercise 14.20)] and

$$\begin{aligned} \mathbb{F}_U(u) &= 1 - \int_1^\infty \frac{1+ut}{t^2\sqrt{t^2-1}}\,e^{-ut}\,dt \\ &= 1 + \frac{\pi u}{2}H_1^{(1)}(iu), \end{aligned} \tag{3}$$

where $H_1^{(1)}(z)$ is the first-order Bessel function, which tends to zero as $-(\pi z/2i)^{-1/2}e^{iz}$ for $z \to i\infty$ (i denoting the imaginary unit). He also derives upper bounds for the remainder $|\mathbb{F}_{U_n}(u) - \mathbb{F}_U(u)|$, valid for $n \geq 12$, and provides a short table of f_U and $\mathbb{F}_U$. As the sequences (U_n^*) and (U_n) share the same limiting distribution (in this setup), (3) is also the cdf of the asymptotic distribution of (U_n^*), thus yielding effectively the asymptotic law of the probability integral transformation of the range.

Elfving indicates how a part of the results could be modified to cover distributions other than the standard normal distribution, but notes that the crucial convergence property

$$U_n^*/U_n \to_{\mathbb{P}} 1,$$

which is proved using (2), restricts the class of possible distributions; for a discussion and examples, see Stuart and Ord (1987, Section 14.28).

This suggests that a more general solution would be of interest. Indeed, independently of Elfving, Gumbel (1947) addressed this problem in a more general setting, deriving the asymptotic distribution of a linear transformation of the range (the "reduced range") for a general symmetric exponential-type underlying distribution. [For a comparison of the two approaches, see also Gumbel (1949, 1958).] This result was also obtained independently by Cox (1948) and leads to a Bessel function approximation of the finite-sample distribution of the range W_n. Cox (1948) further suggests a more accurate steepest descent approximation and carries out a numerical comparison of these approximations with the approximation put forth by Elfving (1947b), when the underlying distribution is standard normal. He finds Elfving's approximation to be the most accurate one under normality, but notes the theoretical disadvantage of Elfving's method as involving a nonlinear transformation of the range.

2.4 Multivariate Distribution Theory

The paper (1947e) in *Skandinavisk Aktuarietidskrift* was written while Elfving was in Stockholm and is an elegant early study of various distributions which arise when a sample is drawn from a multivariate normal distribution. Elfving derives the sampling distributions in an elementary and unified manner, using the close relationship between "multivariate correlation theory" and the more elementary "ordinary regression theory," where the explanatory variables are considered nonrandom.

Essentially Elfving treats the ordinary regression setup as a special case of the general situation, corresponding to fixing the values of some of the random variables in the joint distribution, and considers the conditional distribution defined in this way. He notes that if, within this conditional framework, the distribution of a statistic happens to be functionally independent of the values of the conditioning variables, then this is also the distribution of the statistic within the corresponding unconditional framework. A noteworthy feature is that, in contrast with the presentation in (1947b) which focused on deriving explicit densities and cdf's, Elfving adopts here the more modern approach of providing representations in terms of random variables, governed by prescribed probability laws.

In this manner, formulating multivariate regression problems as corresponding "linear model" ones, Elfving derives a number of known results, such as the distribution of the regression coefficients and their independence of the residual sum of squares, when regressing one or a subset of normal variables on the rest. In a similar way, he derives the distribution of the sample multiple correlation coefficient (when the true correlation is zero) and the partial correlation coefficient.

Letting A and Σ denote, respectively, the (undivided) sample covariance matrix and the covariance matrix of the underlying normal distribution, Elfving further gives a representation of the sample generalized variance $|A|$ as the product of $|\Sigma|$ and independent χ^2-distributed random variables [cf., e.g., Anderson (1984, Theorem 7.5.3)]. He notes that some earlier results of Wilks (1932) on the moments of $|A|$, and the characteristic function of $\log|A|$, follow at once from this representation. En route, he also obtains a representation of the transformation $(1+T^2/(n-1))^{-1}$ of Hotelling's T^2-statistic as a product of independent beta-distributed random variables, from which the null beta distribution of $(1+T^2/(n-1))^{-1} = \lambda^{2/n}$ (where λ is the likelihood ratio) is immediate [cf. Anderson (1984, Section 5.2)].

The two most novel contributions in (1947e) are in the final part. There, Elfving uses again the same elementary regression approach to construct, in a very elegant way, a matrix transformation which yields directly the Bartlett decomposition theorem (for the details of this transformation, refer to the source). Whereas Bartlett (1933) derives the decomposition starting from the Wishart distribution of the sample moments, Elfving proceeds in the opposite direction: from the regression framework he derives via a matrix transformation the Bartlett decomposition, from which the Wishart distribution is obtained. Elfving's derivation of the Wishart distribution is often referenced in multivariate texts, along with several other derivations; see, for example, Anderson (1984, page 251), Kshirsagar (1972, page 58) or Srivastava and Khatri (1979, page 73).

As an interesting offspring of the Bartlett decomposition, Elfving derives finally a "simple representation" of the ordinary sample correlation coefficient r, corresponding to a sample from a bivariate normal distribution. Denoting by ρ the population correlation coefficient, Elfving shows that the quantity $r(1-r^2)^{-1/2}$ admits the representation

$$\frac{r}{\sqrt{1-r^2}} = \frac{\rho(1-\rho^2)^{-1/2}X+Z}{Y}, \tag{4}$$

where X, Y and Z are mutually independent random variables distributed as $\mathscr{L}(X)=\chi_{n-1}$, $\mathscr{L}(Y)=\chi_{n-2}$ and $\mathscr{L}(Z)=N(0,1)$.

Representation (4) was later rediscovered by Fraser and Sprott [see Fraser (1963, Section 4)] and by Ruben (1966), and, although apparently due to Elfving, is often attributed to these authors. In Rao (1973, page 207), the representation (4) is referred to as a "...fundamental equation due to Fraser and Sprott," whereas Johnson, Kotz and Balakrishnan (1995, page 573) refer to it as a "...representation...constructed by Ruben (1963, 1966)" [cf. also Stuart and Ord (1987, page 539)]. For some later extensions of (4), see, for example, Lee (1971) and Gurland and Asiribo (1991).

2.5 Nonparametrics

The *Biometrics* paper (1950) is the outcome of collaboration with J. H. Whitlock during Elfving's two-year visit to Cornell University in 1949–1951. Associated with the New York State Veterinary College at Cornell, Whitlock was involved in research on the determination of possible changes in shape and size of erythrocytes to diagnose dietary deficiencies in sheep. Specifically, it was desirable to detect a possible change in erythrocyte cell volume in a series of measurements before the consequences became lethal to an anemic sheep.

Noting that regression methods would be "...too laborious to be worthwhile," Elfving and Whitlock adopt a rank method. (In a footnote it is acknowledged that this approach had been suggested to Whitlock by William Feller and that John Tukey suggested improvements in the computing scheme.) The essence of their method is the ranking of the observations (measured cell volumes) in consecutive measurements on an individual (sheep), followed by the counting of the number of inverted pairs in the rank sequence. (For technical reasons this quantity is multiplied by the factor 2.) As pointed out by the authors, this statistic is equivalent to the usual Kendall rank correlation coefficient (tau). However, as there is information on several (say r) individuals, Elfving and Whitlock suggest pooling the information over the individuals (blocks), leading to a simple statistic of the form $K = K_1 + \cdots + K_r$, where K_i is twice the number of inverted pairs in the rank sequence corresponding to individual i. The authors note that the distribution of K is complicated even under the null distribution of no trend, but observe that, for moderate r, it is well approximated by a normal distribution. To facilitate the computation of the mean and the standard deviation of the normal distribution, a table is given.

In the final part, Elfving and Whitlock study the efficiency of the proposed test for trend. They consider the regression model

$$x_i(t) = \alpha_i + \beta t + \xi_{it}, \quad i = 1, \ldots, r,$$

where the ξ_{it}'s are assumed iid $N(0, \sigma^2)$. As a measure of "essential trend," they consider the trend compared with the standard deviation of the ξ_{it}'s, and define $\theta = \beta/\sigma$, the null hypothesis thus being $\theta = 0$. Expanding the expectation $\mathbb{E}(K)$ in a Taylor series and neglecting terms of second and higher order, they construct an unbiased estimator of θ, based on K and valid for small values of θ. The variance of this estimator is then compared with the theoretical lower bound.

This simple nonparametric trend test, sometimes referred to as the Elfving–Whitlock test, is often used in applied work; see, for example, Scarborough and McLaurin (1964), Szabó (1967), Ury (1968), Wilcoxon (1973), Whitlock and Georgi (1976) and Newell, Blick and Hjort (1993). For some recent work on the "blocked tau," see Korn (1984) and Taylor (1987).

2.6 Sufficiency and Completeness

The paper (1952b), which appeared in a series published by the Finnish Academy of Sciences, is an elegant study of the relationship between a sufficient statistic and the completeness of a class of decision functions. Elfving notes that the concept of a sufficient statistic suggests that the statistic be "...for any purpose at least as good as any other [statistic]" and proceeds to formalize this in terms of decision functions.

In line with his general preference for the main ideas of a theory rather than its more technical aspects, Elfving restricts himself to the case where the space of possible distributions, the sample space and the decision space are all finite, and the procedures are nonsequential. Nevertheless, he introduces a new concept of completeness that yields an interesting characterization of sufficiency in this all-finite nonsequential case, and for which the paper (1952b) is (rightly) cited in the literature; see, for example, Blackwell and Girshick (1954), Savage (1972) and Lehmann (1986).

To state Elfving's result formally (in our terminology and notation), let $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$ denote a sample space on which a family of probability measures $\mathscr{P} = \{\mathbb{P}_\theta\colon \theta \in \Theta\}$ is defined, let $\mathscr{D}$ denote a class of (possibly randomized) decision functions $\delta\colon (\mathscr{X}, \mathscr{B}_{\mathscr{X}}) \to (D, \mathscr{B}_D)$ and let $L(\theta, \delta)$ and $R(\theta, \delta)$ denote a loss function and the associated risk function, respectively.

Given a statistic T defined on $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$, let $\mathscr{D}_T$ denote the class of all those $\delta \in \mathscr{D}$ which depend on $x \in \mathscr{X}$ only through $T(x)$. Recall that the class $\mathscr{D}_T$ is said to be essentially complete if, for any $\delta \in \mathscr{D}$, there exists a $\delta^* \in \mathscr{D}_T$ such that $R(\theta, \delta^*) \le R(\theta, \delta)$ for all $\theta \in \Theta$. Elfving defines the class $\mathscr{D}_T$ to be *uniformly essentially complete* if $\mathscr{D}_T$ is essentially complete for all loss functions $L(\cdot, \cdot)$, and proves the following result:

THEOREM (Elfving, 1952b). *Suppose the sample space $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$, the family $\mathscr{P} = \{\mathbb{P}_\theta\colon \theta \in \Theta\}$ of probability measures on $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$ and the set $(D, \mathscr{B}_D)$ of possible decisions are all finite. Then the class $\mathscr{D}_T$ is uniformly essentially complete if and only if the statistic T is sufficient for $\mathscr{P}$.*

An extension of Elfving's result was later established by Raj Bahadur, who showed that the result remains valid under the conditions that the family $\mathscr{P}$ be dominated and contain at least two distinct measures, and that the decision space $(D, \mathscr{B}_D)$ be of "real-line type" $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and contain at least two points; see Bahadur (1955) for the details.

2.7 Expansion of Distributions

In the paper (1955a), which appeared in the same series as (1952b), Elfving introduces an expansion principle for distribution functions. In the special case of the t-distribution, Elfving derives a two-term expansion of its cdf in terms of the normal cdf and density, which gives an accuracy of order n^{-2}. The paper (1955a) was written while Elfving was visiting Columbia University and is also reproduced as the first two sections of (1961b), where a similar approximation is given of the cdf of the Wald–Anderson classification statistic. The essence of Elfving's approach is as follows.

Using the ordinary t-statistic and the Wald–Anderson classification statistic as motivating examples, Elfving considers the following general situation. Let X be a random variable with continuous cdf $\mathbb{F}_X(\cdot)$, and let (Y_n) be a sequence of random vectors, independent of X and converging stochastically to some constant vector y_0. Also, let t be a measurable function such that $t(x, y) \longrightarrow x$ as $y \longrightarrow y_0$ for all x, let $\mathbb{F}_n(\cdot)$ denote the cdf of Y_n and write the cdf of $T_n = t(X, Y_n)$ as

$$\mathbb{F}_{T_n}(t) = \int \mathbb{P}\{t(X, y) \le t\}\, d\mathbb{F}_n(y).$$

Now, expanding $\mathbb{P}\{t(X, y) \le t\}$ suitably in the neighborhood of y_0 as

$$\mathbb{P}\{t(X, y) \le t\} = \sum_j h_j(t, y) \tag{5}$$

and integrating term-by-term with respect to the cdf of Y_n, yields an asymptotic series representation in the form

$$\mathbb{F}_{T_n}(t) = \sum_j \int h_j(t, y)\, d\mathbb{F}_n(y). \tag{6}$$

Provided the expansion in (5) has good asymptotic properties, and provided the integrals in (6) can be explicitly evaluated, representation (6) can thus be expected to give a good working representation of $\mathbb{F}_{T_n}(t)$.

Using the above method, Elfving goes on to derive an approximation of the cdf of the ordinary t-statistic $T_n = X/S_n$, resulting eventually in the two-term approximation

$$\begin{aligned} \mathbb{F}_{T_n}(t) &= \Phi(\alpha t) + \frac{5}{96}\alpha t^5 \phi\left(\frac{\alpha t}{\sqrt{2}}\right) \\ &\quad \times \left(1 + \frac{t^2}{2n}\right)^{-n/2} \left(n + \frac{t^2}{2}\right)^{-2}, \end{aligned} \tag{7}$$

where

$$\alpha = \sqrt{\frac{n - 1/2}{n + (1/2)t^2}}.$$

When the second term (and the higher-order terms not included) can be neglected, Elfving notes that (7) is equivalent to saying that the random variable

$$T_n \sqrt{\frac{n - 1/2}{n + (1/2)T_n^2}} \tag{8}$$

is normally distributed and points out the resemblance to the Fisher z-transformation.

In the literature, this approximation of the t-distribution is sometimes used in a form that, as noted also by Elfving, follows directly from (8):

$$T_p = Z_p \sqrt{\frac{2n}{(2n-1) - Z_p^2}},$$

where T_p and Z_p are the percentage points of the t-distribution and the standard normal distribution, respectively; see, for example, Paulson (1969, page 511). Approximation (7) is reproduced in Johnson, Kotz and Balakrishnan (1995, page 377) (where there appears to be a misprint, however).

2.8 Quality Control

When Elfving visited Stanford in the fall 1960, he had essentially finished his work on optimal design of experiments and was looking around for new areas of research. The outcome of this was two seminal contributions to quality control that have become standard references in the field: the *Zeitschrift für Wahrscheinlichkeitstheorie* paper (1962a) and the unpublished Stanford technical report (1962b). [Another piece of work, which was started while at Stanford, is the study (1962c) on two-person Markov games; see Section 1.3 and also Section 7 of the annotated bibliography in Draper et al. (1999).]

In (1962a) Elfving considers so-called multilevel plans (MLPs) for continuous sampling in quality control, introduced in the mid-1950s. For various variants of MLPs, he derives the (unrestricted) average outgoing quality limit (AOQL) when the process is not assumed to be "in control," and a general probability sampling scheme is employed. These results provide a substantial extension of earlier results on MLPs. The essence of the problem and Elfving's approach are as follows.

Consider a production process where consecutive produced items are classified as either good or defective. The general problem is one of keeping the proportion of defectives among outgoing items as low as possible (in some sense to be specified), while inspecting as few items as possible. In the MLPs considered by Elfving, the Statistician's strategy (sampling procedure), say Σ, is to sample at n_0+1 different levels, described by decreasing sampling rates $f_0 = 1 > f_1 > \cdots > f_{n_0}$ and relaxation numbers $l_0, l_1, \dots, l_{n_0-1}$. When sampling at rate f_n, each item is inspected with probability f_n, and once l_n consecutive good items have been inspected, sampling is "relaxed" from level n to level $n+1$. When a defective item is found, sampling is switched to level $n-1$ (under plan MLP–1), to the starting level 0 (under MLP–T), or to level $\max(0, n-r)$ (under MLP–r), and the defective item is replaced by a good one.

The process may be formalized by considering a sequence of random variables (X_t), $t = 1, 2, \dots$, the value of the X_t's being 1 or 0 as the tth item is sampled or not. The output of items from the production process may similarly be described by a sequence (Y_t), $t = 1, 2, \dots$, where the Y_t's are 1 or 0 as the tth item is defective or not. A general strategy, say S, of Nature is a rule prescribing the probability that the $(t+1)$st item is defective, as a function of the whole past of the process. The pair (Σ, S) thus defines a stochastic process (X_t, Y_t).

As the variable $Y_t(1-X_t)$ is 1 when the tth item is defective but is passed without inspection, one may, for a particular realization of the process, define outgoing quality as the proportion of defectives not sampled up to item T, that is,

$$Z_T = T^{-1} \sum_{t=1}^{T} Y_t(1-X_t).$$

The average outgoing quality (AOQ) is defined as the least upper bound on the long-run proportion of defectives remaining in the output, that is, as the smallest number z such that

$$\mathbb{P}\Big\{\limsup_{T\to\infty} Z_T \le z\Big\} = 1.$$

The AOQL is then the supremum of AOQ, taken over all possible sequences of defective and good items that Nature can come up with, that is, a worst-case scenario. This is the "unrestricted" AOQL, as Nature is not restricted to put out defectives at a constant rate, that is, the process is not "in control."

Elfving divides the process into states (sampling levels) and stages (sampling within levels) and assumes that Nature's strategies are restricted to "stage strategies," in which the stage decision rules depend only on the stage reached. Under this restriction on the class of strategies $\{S\}$ of Nature (which Elfving motivates in general terms), one is led to study certain Markov chains and associated transition matrices, describing the transitions between states as well as between stages. By an elegant use of properties of Markov chains, Elfving simplifies the complex setup in a number of intricate steps (for the details, see the source), ending up with the (unrestricted) AOQLs for the above variants of MLPs.

At the end, Elfving notes that "Intuitively, the Statistician should rather not temptate Nature to postpone her output of defects to a very low sampling level (high k_n $[= 1/f_n]$), lest there be a catastrophy" and that "... a reasonable principle for the Statistician seems to be to make Nature's choice indifferent... ." For the case of MLP–1 and MLP–T, Elfving notes that "... [his result] seems to provide a motivation for the commonly used 'geometric' sequence of sampling rates... ."

Containing a new type of model and perhaps more novelty in terms of ideas, the technical report (1962b) appears to be even more cited and is summarized at length in several publications; see, for example, Lieberman (1965, page 290), Phillips (1969, pages 210–211) and Chiu and Wetherill (1973, page 364). [In the latter, (1962b) is listed among the main papers in the area.]

Indeed, in this unpublished report Elfving introduces a model for the continuous control of the manufacturing process of some expensive item when the purpose of inspection is to check the state of the process rather than replacement of defective items. Elfving distinguishes between two fundamentally different situations, depending on whether the true state of the process is observable or not. In both cases, a finite-state Markov model is adopted, the states indicating the true quality levels of the production process.

For the case of observable states (1962b, Part I), Elfving gives a detailed analysis, incorporating various costs such as the loss suffered from turning out an item of quality level i, $i = 0, 1, \dots, k$ (state 0 calling for immediate revision of the process), the inspection cost per item, the cost of revising the production process and the time delay caused by revision. The decision rule is to carry out the next inspection after d_i time units (= produced items) when the process is observed in state i, and the criterion for selection of the vector $d = (d_1, \dots, d_k)$ is the minimization of expected cost per unit of time. Elfving discusses also the necessary estimation of the transition probabilities from one state to another. For the case of two states, Elfving derives an explicit optimum inspection rule, but the general case is too complicated to admit anything but rough guidelines.

In Elfving's model for unobservable states (Part II), the Markovian character of the process is maintained, and the inspection procedure consists of a decision rule as outlined above. However, under this setup the argument of the decision function is no longer the true state (which is not observable), but rather the accumulated information in the earlier observations.

Specifically, let X_t denote the true state of the process at time t, and let Y_t denote the observation of the process at time t, the variable Y_t being discrete or continuous and one-dimensional or possibly multidimensional. Under the condition $X_t = i$, Elfving assumes that Y_t has a known distribution, depending only on i and independent of any past values of X or Y. We thus have an interesting early example of a hidden Markov model.

Further, letting $t_1 < \cdots < t_m$ denote the m first time points of inspection (in a cycle between two subsequent revisions of the process), all the information available at time t_m is contained in the observation vector $\mathbf{Y}_{t_m} = (Y_{t_1}, \ldots, Y_{t_m})$, combined with the vector of time points of inspection $T_{t_m} = (t_1, \ldots, t_m)$. However, fixing the inspection rule up to the mth inspection, T_{t_m} is completely determined by $\mathbf{Y}_{t_m}$, so that the decision rule of the statistician at time t_m is a function $d(\mathbf{Y}_{t_m})$. Elfving notes that the vector $\mathbf{Y}_{t_m}$ (whose dimension increases) is a rather awkward argument for a decision function, and goes on to show that the information contained in $\mathbf{Y}_{t_m}$ can indeed be condensed. Specifically, letting $g_{t_m}(\cdot, \cdot)$ denote the joint probability function of X_{t_m} and $\mathbf{Y}_{t_m}$, and defining $q_i(t_m) = g_{t_m}(i, \mathbf{Y}_{t_m})$, Elfving shows that the vector $q_{t_m} = (q_1(t_m), \ldots, q_k(t_m))$ is a sufficient statistic for making a decision after the mth inspection. A recursion formula is given for q_{t_m}, but Elfving concludes that the final minimization problem (which would yield the optimum inspection rule) "...remains a forbiddingly complex problem. It seems that, in practice, hardly more than a comparison of a few tentative rules can be attempted."

Apparently Elfving was not entirely satisfied with the answers he obtained, as he decided not to publish this report. On the other hand, employing such a general model, it is not surprising that not much definitive in terms of concrete results emerged, the value of this part being clearly the novelty of the model and the general discussion pertaining to it. Elfving acknowledges the influence of a fundamental paper by Girshick and Rubin (1952) on the second part of his report. In fact, his model can be seen as a twofold extension of the model put forth in Girshick and Rubin (1952), in that a finite number of states are allowed and the quality is allowed to improve after deterioration.

2.9 Bayesian Statistics

As indicated in Sections 1.3 and 1.4, throughout the 1960s and the 1970s Elfving was very much interested in the foundations of statistics and the Bayesian approach to statistics. Unfortunately, he did not publish much in this area, the exceptions being the *Skandinavisk Aktuarietidskrift* publication (1968b) and a contribution (1978b) to a discussion paper. In addition, there is an interesting early Bayesian robustness study (1966a), which appeared as a Stanford technical report only, and which appears to be the only research work in this area that Elfving left for posterity.

The paper (1968b) is based on an invited series of lectures that Elfving delivered at the Nordic Conference in Mathematical Statistics, held in Århus in June 1965. The preparation of the published version of these lectures was undertaken during Elfving's visit to Stanford in the spring of 1966.

As pointed out by Elfving in the introductory section, the aim of the paper is to give "...a short survey—for non-specialists—of problems, methods, and achievements, in Bayesian decision and inference theory, as developed mainly during the last fifteen years." Elfving presents the fundamental ideas of Bayesian decision theory and its axiomatic justification, with particular emphasis on the utility function and prior probabilities. A section (Section 2) is devoted to the more technical side of this theory, along the lines of Raiffa and Schlaifer (1961). The second part of the paper contains a presentation of the principles of Bayes inference, and the paper concludes with a section devoted to alternative approaches and critical discussion.

As one can expect, the last section reveals more of Elfving's personal views on the Bayesian approach. He notes that "One of the most important tasks for the workers on Bayes theory will probably be to investigate the *robustness* of the bayesian methods with respect to variations in the prior distribution and in the utility function...", with a reference to his 1966 Stanford report, and ends the paper by noting that "...the main merit of the Bayes approach lies in the conceptual unification of a hitherto rather scattered methodology."

Although lots of ground has been covered in the Bayesian camp since the mid-1960s, Elfving's survey is still eminently readable, not the least because it is so well written (as are all publications of Elfving).

When Dennis Lindley gave a similar series of lectures 12 years later at the Nordic Conference in Ystad, Sweden, Elfving was one of the invited discussants, and his comments are published in (1978b).

The starting point of Elfving's Stanford report (1966a) is the well-known fact that, under suitable regularity conditions, the asymptotic posterior distribution is normal, with parameters independent of the prior. Elfving notes that, as a consequence of this, the outcome (in terms of utility or loss) of any Bayes decision procedure can be expected to become insensitive to the choice of prior distribution as the sample size grows. The report is "... an attempt at a quantitative evaluation of this approximation."

Elfving considers the asymptotic behavior of the difference in loss under two priors, the difference being evaluated for a fixed true state of nature θ_0, consistent with the two priors. Under appropriate regularity and identifiability conditions, he shows in essence that, asymptotically, the difference consists of an error term with expectation 0 and of order $n^{-3/2}$, and a systematic term of order n^{-2}.

More formally, Elfving establishes the following result. Consider a Bayes decision problem. Let θ be the "state of nature" with prior density $g(\theta)$, let $X = (X_1, \dots, X_n)$ be a sample of independent observations from a distribution with density $f(\cdot|\theta)$, let a be the action taken by the statistician and let $L(a, \theta)$ be the loss function. Any particular choice of $g(\theta)$ thus yields a Bayes decision function $a(X)$ and an actual loss $L(a(X), \theta_0)$. Denote, for brevity, $\Delta L = L(a_{g_1}(X), \theta_0) - L(a_{g_2}(X), \theta_0)$, that is, the difference in actual losses when X is observed and Bayes decision procedures are based on the priors g_1 and g_2. For one-dimensional θ, Elfving then shows that, under appropriate regularity conditions, ΔL is asymptotically of the form

$$A\xi n^{-3/2} + Bn^{-2},$$

where ξ is asymptotically $N(0, 1)$, and the nonrandom coefficients A and B depend on the local behavior of $f(x|\theta)$, $L(a, \theta)$, $g_1(\theta)$ and $g_2(\theta)$ in θ_0. The expansion of the loss function from which this follows (1966a, Theorem 5.1) occurs also in a similar form in a paper by Bickel and Yahav (1969).

2.10 Optimal Stopping

In the paper (1967a), also written while at Stanford University in 1966, Elfving presents the following interesting model for a sales problem. A man owns some commodity which is for sale. Offers arrive every now and then, and the longer he waits, the more he loses because of "... deterioration, interest losses, or the like." At each offer, he must decide whether to accept it or wait for a better one, with past offers no longer available. [Elfving notes that "(A more picturesque example would be that of a girl scrutinizing successive suitors.)."] Elfving formalizes this by representing the incoming offers as a suitable stochastic process, the loss due to postponement being accounted for by a discount function. The resulting stopping problem and its solution, to be briefly outlined below, are discussed at length by Chow, Robbins and Siegmund (1971), who devote a section (Section 5.4) to "A Problem of G. Elfving."

Let (N_t), $t \geq 0$, be a Poisson process, with intensity function $p(t)$. Hence, events (offers) in disjoint time intervals are assumed independent and the probability of an event in the interval $]t, t+\delta[$ is $\delta p(t) + o(\delta)$. Let $\tau_1, \tau_2, \dots$ denote the time points of successive events (offers), that is,

$$\tau_n = \inf\{t\colon t \geq \tau_{n-1},\ N_t \neq N_{\tau_{n-1}}\} \quad (\tau_0 = 0),$$

and associate with each τ_n a nonnegative random variable Y_n. The random variables (Y_n) (the offered amounts) are assumed to be iid with finite mean. Furthermore, there is given a nonnegative, nonincreasing and right-continuous discount function $r(\cdot)$, defined on $[0, \infty[$, with $r(0) = 1$.

Elfving considers decision rules of the following form. Let $y(\cdot)$ be a nonnegative, at least piecewise continuous function defined on $[0, \infty[$. Also, let

$$M = \inf\{n\colon Y_n \geq y(\tau_n)\},$$

that is, Y_M is the first accepted offer, after which the process terminates, and the value is recorded as $r(\tau_M)Y_M$. If no offer is ever accepted, the value is considered to be zero. Elfving's problem is one of choosing the critical curve $y(\cdot)$ so as to maximize the expected gain $\mathbb{E}\, r(\tau_M)Y_M$.

Elfving derives an integral equation for the optimal critical curve, from which a differential equation is obtained that has infinitely many solutions. Under the convergence condition

$$\int_0^\infty p(t)r(t)\,dt < \infty, \tag{9}$$

and using elaborate, entirely analytic techniques, he is able to establish the existence and uniqueness of a solution $y(\cdot)$, and hence the existence and uniqueness of an optimal decision rule. The result is illustrated with several examples, where exact solutions are computed. The possibilities which arise when the integral in (9) diverges are also considered. Furthermore, Elfving shows that, by truncating the process at some time $t = T$, the analysis goes through without the convergence assumption (9), but then a maximizer $y(\cdot)$ need not exist. However, when a maximizer does exist, it may be approximated using solutions from the truncated problem. Some of Elfving's assumptions were later removed by Siegmund (1967).

An extension to the case of $k > 1$ commodities has been given by Stadje (1987). For some further methodological developments, see also Enns and Ferenstein (1990), Collins and McNamara (1993) and Stadje (1996). Elfving's formulation of this problem has been found fruitful in a number of applications; see, for example, Hayes (1969), Ioannides (1975), Weibull (1978) and David and Yechiali (1985).

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The titles of Elfving's publications in Finnish and Swedish have been translated into English, the original language of publication appearing in parentheses. For the compilation of this bibliography, an earlier chronological list of Elfving's publications by Mäkeläinen (1984) has been very helpful; the present one updates and completes the earlier one, and should now be fairly complete.

1930

Zur Reduktion von Echolotungen. *Havsforskningsinstitutets Skrift* **69** 11 pages.

1934

Über eine Klasse von Riemannschen Flächen und ihre Uniformisierung. *Acta Societatis Scientiarum Fennicæ Nova Series A* **2**(3) 60 pages.

1935

[a] Über Riemannsche Flächen und Annäherung von meromorphen Funktionen. In *Åttonde Skandinaviska Matematikerkongressen i Stockholm 1934* 96–105. Håkan Ohlssons Boktryckeri, Lund.

[b] Zur Flächenstruktur und Wertverteilung. Ein Beispiel. *Acta Academiae Aboensis Mathematica et Physica* **8**(10) 13 pages.

1937

Zur Theorie der Markoffschen Ketten. *Acta Societatis Scientiarum Fennicæ Nova Series A* **2**(8) 17 pages.

1938

Über die Interpolation von Markoffschen Ketten. *Societas Scientiarum Fennica Commentationes Physico-Mathematicae* **10**(3) 8 pages.

1941

Foundations of probability from a modern viewpoint. *Matemaattisten Aineiden Aikakauskirja* **1** 8–21. (In Swedish.)

1942

[a] On statistical analysis. In *Tekniska Läroverket i Helsingfors 25-års Jubileumsskrift* 121–129. Liiketieto, Helsinki. (In Swedish.)

[b] The relationship between high schools and universities in the present exceptional circumstances. *Skola och Hem* **5**(1) 34–39. (In Swedish.)

1944

[a] Die Radialzuwachsvariationen der Waldgrenzkiefer. *Societas Scientiarum Fennica Commentationes Biologicae* **9**(8) 18 pages (with I. Hustich).

[b] A collection of mathematical problems for the courses Mathematics I–II at the Helsinki University of Technology. Helsinki Univ. Technology, 56 pages. (In Finnish.)

1945

[a] School mathematics for polytechnic students. *Matemaattisten Aineiden Aikakauskirja* **9** 41–49. (In Swedish.)
[b] Foundations of and developments in probability. *Matemaattisten Aineiden Aikakauskirja* **9** 141–150. (In Swedish.)

1946

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[c] The calculus of probability. Univ. Helsinki, 118 pages. (Lecture notes in Finnish).

1947

[a] On a class of elementary Markoff processes. In *Den 10. Skandinaviske Matematiker Kongres i København 1946* 149–159. Jul. Gjellerups Forlag, Copenhagen.
[b] The asymptotical distribution of range in samples from a normal population. *Biometrika* **34** 111–119.
[c] The mathematical concept of probability. *Elementa* **30** 19–33. (In Swedish.)
[d] Mathematical statistics—problems, methods and applications. *Elementa* **30** 209–225. (In Swedish.)
[e] A simple method of deducing certain distributions connected with multivariate sampling. *Skandinavisk Aktuarietidskrift* **30** 56–74.

1948

Some aspects of the history of probability. *Finsk Tidskrift* **4** 186–198. (In Swedish.)

1950

A simple trend test with application to erythrocyte size data. *Biometrics* **6** 282–288 (with J. H. Whitlock).

1951

A collection of mathematical problems for the courses Mathematics I–II at the Helsinki University of Technology. Helsinki Univ. Technology, 147 pages (with O. Lokki). (In Finnish.)

1952

[a] Optimum allocation in linear regression theory. *Ann. Math. Statist.* **23** 255–262.
[b] Sufficiency and completeness in decision function theory. *Ann. Acad. Sci. Fenn. Ser. A I Math.–Phys.* **135** 9 pages.

1953

[a] Life at American universities. *Arkhimedes* **5**(1) 28–31. (In Swedish.)
[b] Von Neumann's game theory. *Arkhimedes* **5**(2) 17–21. (In Swedish.)
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