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ON THE INEQUALITY FOR BIBDs WITH SPECIAL PARAMETERS

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For a μ-resolvable Balanced Incomplete Block Design (BIBD) with parameters v, $b = mt$, $r = \mu t$, k and λ, Kageyama (1973) obtained an inequality $b \geqq v + t - 1$. The main purpose of this note is to improve $b \geqq v + t - 1$ to $b \geqq \max\{v + t - 1, (m^2\lambda + m)/\mu^2\}$. This inequality is also improved further for a μ-resolvable BIBD which is not affine μ-resolvable.

1. Introduction and summary. For a BIBD with parameters v, b, r, k and λ, if the blocks can be separated into t sets of m blocks each such that each set contains every treatment exactly μ times, then the design is called μ-resolvable. Moreover, if any pair of blocks belonging to the same set contain q_1 treatments in common, whereas any pair of blocks belonging to different sets contain q_2 treatments in common, then the design is called affine μ-resolvable. Then we have the following relations (cf. [3], [5]):

$$\text{(1.1)} \qquad vr = bk\,, \quad \lambda(v-1) = r(k-1)\,, \quad b \geqq v\,, \quad b = mt\,, \quad r = \mu t\,,$$
$$q_1 = (\mu - 1)k/(m-1) = k + \lambda - r\,, \qquad q_2 = \mu k/m = k^2/v\,.$$

Shrikhande and Raghavarao [5] proved that the necessary and sufficient condition for a μ-resolvable BIBD to be affine μ-resolvable is $b - v = t - 1$. Kageyama [3] and Raghavarao [4] showed that if there exists a μ-resolvable BIBD with parameters v, $b = mt$, $r = \mu t$, k and λ, then $b \geqq v + t - 1$. Further, when $v \leqq r$, this inequality was improved to $b \geqq 2(v-1)/\mu + r$ without the assumption of μ-resolvability, but with the assumption of $b = mt$ and $r = \mu t$ [3]. In this note these inequalities are improved further.

2. Statement. A BIBD with parameters v, $b = mt$, $r = \mu t$, k and λ is considered throughout this section. From (1.1), since $(\mu r - m\lambda)k = \mu(r - \lambda) > 0$, we obtain

$$\text{(2.1)} \qquad \mu r - m\lambda \geqq 1\,.$$

Further, we have

$$\text{(2.2)} \qquad b = vr/k = \{v(\mu r - m\lambda) + m\lambda\}/\mu$$
$$= (\mu r - m\lambda)(v-1)/\mu + r\,.$$

Multiplying (2.1) by m, we obtain $b \geqq (m^2\lambda + m)/\mu^2$. Moreover, (2.1) and (2.2) imply $b \geqq (v-1)/\mu + r$. It follows from (2.2) that $b \geqq (m^2\lambda + m)/\mu^2$ is equivalent to $b \geqq (v-1)/\mu + r$ and both the equality signs hold at the same time. As a comparison of these two inequalities, we have

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LEMMA 2.1. *In a* BIBD *with parameters* v, $b = mt$, $r = \mu t$, k *and* λ, *the following relation holds*:

$$b \geqq (m^2\lambda + m)/\mu^2 \geqq (v - 1)/\mu + r .$$

PROOF. Let $F = (m^2\lambda + m)/\mu^2 - (v - 1)/\mu - r$. Then $\mu^2 F = m^2\lambda + m - \mu(v - 1) - \mu^2 r$. From (1.1) and (2.1) we have $\mu\lambda(v - 1) \geqq (m\lambda + 1)(k - 1)$. Hence $m\lambda \geqq \mu\lambda + k - 1$, i.e., if $\mu r - m\lambda = 1$, then $m\lambda = \mu\lambda + k - 1$. Thus $\mu^2 F = \mu(m\lambda + 1 - \mu r) = 0$ provided $\mu r - m\lambda = 1$. Therefore $(m^2\lambda + m)/\mu^2 = (v - 1)/\mu + r$ provided $\mu r - m\lambda = 1$. In the case of $\mu r - m\lambda > p$ for a positive integer p, it is sufficient to consider $\mu r - m\lambda \geqq p + 1$. From (1.1), $\mu r - m\lambda \geqq p + 1$ leads to $m\lambda \geqq \mu\lambda + (p + 1)k - (p + 1)$, i.e., if $\mu r - m\lambda = p + 1$, then $m\lambda = \mu\lambda + (p + 1)k - (p + 1)$. Thus $\mu^2 F = mp\{k - (1 + \mu/m)\} > 0$ by $k \geqq 2$ and $m > \mu$ provided $\mu r - m\lambda = p + 1$. Therefore $(m^2\lambda + m)/\mu^2 > (v - 1)/\mu + r$ provided $\mu r - m\lambda = p + 1$ for a positive integer p. Repeated applications of this procedure completes the theorem.

Further, from the result that for a μ-resolvable BIBD with parameters v, $b = mt$, $r = \mu t$, k and λ, an inequality $b \geqq v + t - 1$ holds, we have

THEOREM 2.1. *For a μ-resolvable* BIBD *with parameters* v, $b = mt$, $r = \mu t$, k *and* λ, *an inequality*

$$b \geqq \max\{v + t - 1, (m^2\lambda + m)/\mu^2\}$$

holds.

Note that $(v - 1)/\mu + r \geqq v + t - 1$ provided $v \leqq r$. Then from Lemma 2.1 we have $b \geqq (m^2\lambda + m)/\mu^2$ for $v \leqq r$ in Theorem 2.1. In particular, when $\mu = 1$, for a resolvable BIBD with parameters v, $b = mr$, r, k and λ an inequality $b \geqq m^2\lambda + m$, which is more stringent than Bose's inequality $b \geqq v + r - 1$ [1], always holds from Lemma 2.1.

EXAMPLE 1. Consider an affine 4-resolvable BIBD with parameters $v = 169$, $b = 182$, $r = 56$, $k = 52$ and $\lambda = 17$ where $t = 14$ and $m = 13$ [3]. Then $b \geqq v + t - 1$ and $b \geqq (m^2\lambda + m)/\mu^2$ imply $182 \geqq 182$ and $182 \geqq 181$, respectively.

EXAMPLE 2. Consider a 2-resolvable BIBD with parameters $v = 6$, $b = 15$, $r = 10$, $k = 4$ and $\lambda = 6$ where $t = 5$ and $m = 3$ [3]. Then $b \geqq v + t - 1$ and $b \geqq (m^2\lambda + m)/\mu^2$ imply $15 \geqq 10$ and $15 \geqq 15$, respectively.

EXAMPLE 3. Consider an affine 2-resolvable BIBD with parameters $v = 9$, $b = 12$, $r = 8$, $k = 6$ and $\lambda = 5$ where $t = 4$ and $m = 3$ [3]. Then $b \geqq v + t - 1$ and $b \geqq (m^2\lambda + m)/\mu^2$ imply the same $12 \geqq 12$.

EXAMPLE 4. Consider a resolvable BIBD with parameters $v = 12$, $b = 44$, $r = 11$, $k = 3$ and $\lambda = 2$ where $m = 4$. Then $b \geqq v + r - 1$ and $b \geqq m^2\lambda + m$ imply $44 \geqq 22$ and $44 \geqq 36$, respectively.

As a generalization of Lemma 2.1, we obtain

THEOREM 2.2. *For a* **BIBD** *with parameters* v, $b = mt$, $r = \mu t$, k *and* λ, *if* $\mu r - m\lambda > p$ *for a nonnegative integer* p, *then*

$$b \geqq \{m^2\lambda + (p+1)m\}/\mu^2 \geqq (p+1)(v-1)/\mu + r .$$

The proof of this theorem is similar to that of Lemma 2.1 and hence it is omitted. When $p = 0$, from (2.1) Theorem 2.2 implies Lemma 2.1.

Note that from (2.2), $b \geqq \{m^2\lambda + (p+1)m\}/\mu^2$ is equivalent to $b \geqq (p+1)(v-1)/\mu + r$, in particular, the equality sign $b = \{m^2\lambda + (p+1)m\}/\mu^2$ holds, if and only if the equality sign $b = (p+1)(v-1)/\mu + r$ holds. Theorem 2.2 also shows that if $b > (m^2\lambda + pm)/\mu^2$, then $b \geqq \{m^2\lambda + (p+1)m\}/\mu^2$. That is to say, we can improve the bound of b in turn.

COROLLARY 2.1. *In a* **BIBD** *with parameters* v, $b = mt$, $r = \mu t$, k *and* λ, *if* $b > v + t - 1$, *then* $b \geqq (m^2\lambda + 2m)/\mu^2 \geqq 2(v-1)/\mu + r$.

PROOF. Case I, i.e., $v \leqq r$. Then from Theorem 4.2 of Kageyama [3], $b \geqq 2(v-1)/\mu + r$ holds without the assumption $b > v + t - 1$. From (2.2), $b \geqq 2(v-1)/\mu + r$ implies $\mu r - m\lambda \geqq 2$. Hence from Theorem 2.2, we obtain $b \geqq (m^2\lambda + 2m)/\mu^2 \geqq 2(v-1)/\mu + r$. Case II, i.e., $v > r$. From (2.1), assume on the contrary that $\mu r - m\lambda = 1$. Then (2.2) implies $b = (v-1)/\mu + r$ which is less than or equal to $v + t - 1$ provided $v > r$. This is a contradiction since $b > v + t - 1$. Hence we have $\mu r - m\lambda \geqq 2$, i.e., from Theorem 2.2, we have $b \geqq (m^2\lambda + 2m)/\mu^2 \geqq 2(v-1)/\mu + r$.

As an implication of Corollary 2.1, we have the following corollary from a necessary and sufficient condition for a μ-resolvable **BIBD** to be affine μ-resolvable:

COROLLARY 2.2. *For a* μ-*resolvable* **BIBD** *with parameters* v, $b = mt$, $r = \mu t$, k *and* λ *which is not affine* μ-*resolvable, a relation*

$$b \geqq (m^2\lambda + 2m)/\mu^2 \geqq 2(v-1)/\mu + r$$

holds.

EXAMPLE 5. Consider a 4-resolvable **BIBD** with parameters $v = 9$, $b = 12$, $r = 8$, $k = 6$ and $\lambda = 5$ which is not affine 4-resolvable for $t = 2$ and $m = 6$. Then $b \geqq (m^2\lambda + 2m)/\mu^2$ implies $12 \geqq 12$.

When $\mu = 1$, Corollary 2.2 shows that $b \geqq m^2\lambda + 2m$ is more stringent than $b \geqq 2v + r - 2$ [2] for a resolvable **BIBD** with parameters v, $b = mr$, r, k and λ which is not affine resolvable. As an example of this result, one should be referred to Example (iii) of [2]. Finally, it is interesting to note that when $\mu = 1$, since from (1.1) and (2.1) we have $m\lambda \geqq \lambda + k - 1$, from $\lambda \geqq 1$ we have $m\lambda \geqq k$, i.e., $\lambda \geqq k/m$, which implies that for an affine resolvable **BIBD** with parameters v, $b = mr$, r, k and λ, the number of treatments common to any two blocks belonging to different sets is not greater than λ.

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