The Annals of Statistics
1973, Vol. 1, No. 1, 54–66

ON THE CENTERING OF A SIMPLE LINEAR RANK STATISTIC[1]

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Hájek (1968) proved that under weak conditions the distribution of a simple linear rank statistic S is asymptotically normal, centered at the mean ES. He left open the question whether under the same conditions the centering constant ES may be replaced by a simpler constant μ, as was found to be true in the two-sample case and under different conditions by Chernoff and Savage (1958) and Govindarajulu, LeCam and Rhagavachari (1966). In this paper it is shown that the replacement of ES by μ is permissible if one of Hájek's conditions is slightly strengthened.

1. Introduction and statement of results. Hájek (1968) studied the asymptotic distribution of the sum

$$S = \sum_{i=1}^{N} c_i a_N(R_i)\,, \tag{1.1}$$

called a simple linear rank statistic, where $c_1, \cdots, c_N$ are real numbers, $R_1, \cdots, R_N$ are the respective ranks of N independent random variables $X_1, \cdots, X_N$ whose distribution functions $F_1, \cdots, F_N$ are continuous, and the so-called scores $a_N(i)$ are generated by a function $\phi(t)$, $0 < t < 1$, in either of the following two ways:

$$a_N(i) = \phi(i/(N+1))\,, \qquad i = 1, \cdots, N\,, \tag{1.2}$$

$$a_N(i) = E\phi(U_N^{(i)})\,, \qquad i = 1, \cdots, N\,. \tag{1.3}$$

Here $U_N^{(i)}$ denotes the ith order statistic in a random sample of size N from the uniform distribution on (0, 1). Hájek proved four theorems asserting the asymptotic normality of S under different conditions, of which we quote

HÁJEK'S THEOREM 2.3. *Let $\phi(t) = \phi_1(t) - \phi_2(t)$, $0 < t < 1$, where $\phi_1(t)$ and $\phi_2(t)$ are both non-decreasing, square integrable, and absolutely continuous inside* (0, 1). *Then for every $\varepsilon > 0$ and $\eta > 0$ there exists $N(\varepsilon, \eta)$ such that*

$$N > N(\varepsilon, \eta)\,, \qquad \operatorname{Var} S > \eta N \max_{1\le i\le N} (c_i - \bar{c})^2 \tag{1.4}$$

entails

$$\sup_x |P\{S - ES < x(\operatorname{Var} S)^{\frac{1}{2}}\} - \Phi(x)| < \varepsilon\,. \tag{1.5}$$

The assertion remains true if we replace Var S *in* (1.4) *and* (1.5) *by*

$$\sigma^2 = \sum_{i=1}^{N} \operatorname{Var} l_i(X_i)\,, \tag{1.6}$$
$$l_i(x) = N^{-1} \sum_{j=1}^{N} (c_j - c_i) \int \{u(y - x) - F_i(y)\}\phi'(H(y))\, dF_j(y)\,.$$

Received July 28, 1971; revised May 12, 1972.

[1] Research partially sponsored by the Mathematics Division of the Air Force Office of Scientific Research Grant No. AFOSR-68-1415.

AMS 1970 *subject classifications.* Primary 62G10, 60F05.

Key words and phrases. Simple linear rank statistic, asymptotically normal, centering constant.

Here $\bar{c} = N^{-1} \sum_{i=1}^{N} c_i$, $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-y^2/2)\, dy$, $u(x) = 1$ or 0 according as $x \geqq 0$ or $x < 0$, ϕ' denotes the derivative of ϕ, and

$$(1.7) \qquad H(x) = N^{-1} \sum_{i=1}^{N} F_i(x) .$$

Hájek's Theorem 2.4 states that under the conditions of Theorem 2.3, for every $\varepsilon > 0$ and $\eta > 0$ there exist $N(\varepsilon, \eta)$ and $\delta(\varepsilon, \eta)$ such that the conclusion of Theorem 2.3 holds with Var S replaced by

$$d^2 = \sum_{i=1}^{N} (c_i - \bar{c})^2 \int_0^1 \{\phi(t) - \int_0^1 \phi(s) ds\}^2\, dt ,$$

provided that the condition $\max_{i,j,x} |F_i(x) - F_j(x)| < \delta(\varepsilon, \eta)$ is added to (1.4).

Hájek's theorems are extensions of the earlier results of Chernoff and Savage (1958) and of Govindarajulu, LeCam and Raghavachari (1966), which are concerned with the special case $c_1 = \cdots = c_m$, $c_{m+1} = \cdots = c_N$, $F_1 = \cdots = F_m$, $F_{m+1} = \cdots = F_N$ (two-sample case). Apart from different sets of assumptions (which, in essential parts, are more restrictive than Hájek's), the theorems in those earlier papers differ from Hájek's theorems in that the centering constant ES is replaced by

$$(1.8) \qquad \mu = \sum_{i=1}^{N} c_i \int_{-\infty}^{\infty} \phi(H(x))\, dF_i(x) .$$

The problem of whether ES may be replaced by μ is of interest since μ has a simpler structure and is easier to evaluate. Hájek observed ((1968) page 330) that he did not succeed in showing that this replacement is possible under the conditions of Theorems 2.3 and 2.4.

In this paper it is shown that if the condition of square integrability of ϕ_1 and ϕ_2 is slightly strengthened, then the conclusions of Theorems 2.3 and 2.4 remain true with ES replaced by μ or by

$$(1.9) \qquad \mu' = \mu + \bar{c}\{\sum_{i=1}^{N} \phi(i/(N+1)) - N \int_0^1 \phi(t)\, dt\} .$$

Explicitly, the following result is proved.

THEOREM 1. *Let $\phi(t) = \phi_1(t) - \phi_2(t)$ satisfy the conditions of Hájek's Theorem* 2.3 *with the square integrability condition on ϕ_1 and ϕ_2 replaced by*

$$(1.10) \qquad \int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\, d\phi_k(t) < \infty , \qquad k = 1, 2 .$$

Then the conclusions of Hájek's Theorems 2.3 *and* 2.4 *remain true with ES replaced by μ' in the case of the scores* (1.2) *and by μ in the case of the scores* (1.3). *If $|\bar{c}|/\max_i |c_i - \bar{c}|$ is bounded, ES may be replaced by μ also in the case* (1.2).

Concerning condition (1.10) we observe the following. For ϕ non-decreasing let

$$(1.11) \qquad J(\phi) = \int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\, d\phi(t) .$$

Integrating by parts, we obtain

$$(1.12) \qquad J(\phi) = \int_0^1 \phi(t)(t - \tfrac{1}{2})t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\, dt .$$

Hence condition (1.10) is equivalent to

$$\int_0^1 |\phi_k(t)| t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt < \infty\,, \qquad k = 1, 2\,. \tag{1.10'}$$

In the Appendix it is shown that if ϕ is non-decreasing, then the condition $J(\phi) < \infty$ implies square integrability of ϕ and is implied by $\int_0^1 \phi^2(t)\{\log(1 + |\phi(t)|)\}^{1+\delta}\,dt < \infty$ for some $\delta > 0$. In this sense condition (1.10) is not much stronger than square integrability.

The results of Hájek (1968) have been extended by Dupač and Hájek (1969) to cases where the function ϕ is not required to be continuous. The permissibility of replacing ES by μ or μ' in the appropriate theorems of that paper when the condition (1.10) is added has been proved by Dupač (1970), referring to an earlier version of the present paper.

Theorem 1 is proved in Section 4. The proof depends on the following two propositions which are demonstrated in Sections 2 and 3.

PROPOSITION 1. *There is a numerical constant C_1 such that if ϕ is non-decreasing, then*

$$\sum_{i=1}^N |E\phi(U_N^{(i)}) - \phi(i/(N+1))| \leqq C_1 N^{\frac{1}{2}} J(\phi)\,. \tag{1.13}$$

PROPOSITION 2. *There is a numerical constant C_2 such that if ϕ is non-decreasing and $F_1, \cdots, F_N$ are any continuous distribution functions, then*

$$\sum_{i=1}^N |E\phi(R_i/(N+1)) - \int_{-\infty}^{\infty} \phi(H(x))\,dF_i(x)| \leqq C_2 N^{\frac{1}{2}} J(\phi)\,. \tag{1.14}$$

The author does not know whether the conclusion of Theorem 1 is true under the conditions of Hájek's Theorem 2.3. The role of condition (1.10) in the proof of the theorem is discussed in Section 5.

2. Proof of Proposition 1. It is sufficient to show that if ϕ is non-decreasing, then

$$\sum_{i=1}^N \left| E\phi(U_N^{(i)}) - \phi\left(\frac{i}{N+1}+\right)\right| \leqq C_3 N^{\frac{1}{2}} J(\phi) \tag{2.1}$$

and

$$\sum_{i=1}^N \left| \phi\left(\frac{i}{N+1}+\right) - \phi\left(\frac{i}{N+1}\right)\right| \leqq C_4 N^{\frac{1}{2}} J(\phi)\,, \tag{2.2}$$

where C_3 and C_4 are numerical constants.

Let $G_{N,i}(\cdot)$ denote the distribution function of $U_N^{(i)}$. Then for $i = 1, \cdots, N$,

$$\begin{aligned}
E\phi(U_N^{(i)}) - \phi\left(\frac{i}{N+1}+\right) &= \int_0^1 \left\{\phi(u) - \phi\left(\frac{i}{N+1}+\right)\right\} dG_{N,i}(u) \\
&= -\int_0^{i/(N+1)} \int_u^{i/(N+1)+} d\phi(t)\,dG_{N,i}(u) \\
&\quad + \int_{i/(N+1)}^1 \int_{i/(N+1)+}^u d\phi(t)\,dG_{N,i}(u) \\
&= -\int_0^{i/(N+1)+} G_{N,i}(t)\,d\phi(t) \\
&\quad + \int_{i/(N+1)+}^1 \{1 - G_{N,i}(t)\}\,d\phi(t)\,.
\end{aligned}$$

Hence if we define

$$(2.3)\qquad H_{N,i}(t) = G_{N,i}(t) \quad \text{if} \quad 0 \leqq t \leqq \frac{i}{N+1}\,,$$

$$H_{N,i}(t) = 1 - G_{N,i}(t) \quad \text{if} \quad \frac{i}{N+1} < t \leqq 1\,,$$

then

$$(2.4)\qquad \sum_{i=1}^{N} \left| E\phi(U_N^{(i)}) - \phi\left(\frac{i}{N+1}+\right)\right| \leqq \int_0^1 \sum_{i=1}^{N} H_{N,i}(t)\, d\phi(t)\,.$$

Inequality (2.1) will be proved if we show that

$$(2.5)\qquad \sum_{i=1}^{N} H_{N,i}(t) \leqq C_3 N^{\frac{1}{2}} t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,, \qquad 0 \leqq t \leqq 1\,.$$

From definition (2.3) we have

$$\sum_{i=1}^{N} H_{N,i}(t) = \sum_{i=1}^{j} \{1 - G_{N,i}(t)\} + \sum_{i=j+1}^{N} G_{N,i}(t) \quad \text{if} \quad \frac{j}{N+1} < t \leqq \frac{j+1}{N+1}\,,$$

for $j = 0, 1, \cdots, N$. Now $G_{N,i}(t) = P\{W_N(t) \geqq i\}$, where $W_N(t)$ has the binomial (N, t) distribution. Hence we obtain

$$(2.6)\qquad \sum_{i=1}^{N} H_{N,i}(t) = E|W_N(t) - j| \quad \text{if} \quad j < (N+1)t \leqq j+1\,, \qquad j = 0, \cdots, N\,.$$

But $E|W_N(t) - j| \leqq \{E|W_N(t) - j|^2\}^{\frac{1}{2}} = \{Nt(1-t) + (Nt-j)^2\}^{\frac{1}{2}}$, and $(Nt-j)^2 \leqq Nt(1-t)$ in the range specified in (2.6). Thus (2.5) holds true with $C_3 = 2^{\frac{1}{2}}$, and inequality (2.1) is proved.

Finally, inequality (2.2) holds since

$$\sum_{i=1}^{N} \left|\phi\left(\frac{i}{N+1}+\right) - \phi\left(\frac{i}{N+1}\right)\right|$$
$$\leqq \sum_{i=1}^{N} \int_{t=i/(N+1)} d\phi(t)$$
$$\leqq \max_{1\leqq i\leqq N}\left(\frac{i}{N+1}\right)^{-\frac{1}{2}}\left(1-\frac{i}{N+1}\right)^{-\frac{1}{2}} \sum_{i=1}^{N} \int_{t=i/(N+1)} t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\, d\phi(t)$$
$$\leqq 2N^{\frac{1}{2}} J(\phi)\,.$$

3. Proof of Proposition 2. We can write $R_i = \sum_{k=1}^{N} u(X_i - X_k)$, where $u(x) = 1$ or 0 according as $x \geqq 0$ or $x < 0$. Thus if we let

$$(3.1)\qquad V(x) = \sum_{k=1}^{N} u(x - X_k)\,,$$
$$V_i(x) = 1 + \sum_{k=1, k\neq i}^{N} u(x - X_k) = 1 - u(x - X_i) + V(x)\,,$$

then

$$(3.2)\qquad E\left\{\phi\left(\frac{R_i}{N+1}\right)\middle| X_i = x\right\} = E\phi\left(\frac{V_i(x)}{N+1}\right).$$

Hence, for $i = 1, \cdots, N$,

$$(3.3)\qquad \left|E\phi\left(\frac{R_i}{N+1}\right) - \int \phi(H(x))\, dF_i(x)\right| \leqq \int \left|E\phi\left(\frac{V_i(x)}{N+1}\right) - \phi(H(x))\right| dF_i(x)$$

$$(3.4)\qquad \leqq \int E\left|\phi\left(\frac{V_i(x)}{N+1}\right) - \phi(H(x))\right| dF_i(x)\,.$$

Since $V_i(x) \leqq V(x) + 1$ and $V_i(x) \leqq N$, we have

$$\phi\left(\frac{V_i(x)}{N+1}\right) \leqq g(V(x)) , \tag{3.5}$$

where

$$g(v) = \min\left\{\phi\left(\frac{v+1}{N+1}\right), \phi\left(\frac{N}{N+1}\right)\right\} , \qquad 0 \leqq v \leqq N . \tag{3.6}$$

We now show that

$$\begin{aligned} \textstyle\sum_{i=1}^N \int E\left|\phi\left(\frac{V_i(x)}{N+1}\right) - \phi(H(x))\right| dF_i(x) \\ \leqq N \textstyle\int E|g(V(x)) - \phi(H(x))|\, dH(x) + \phi(N/(N+1)) \\ - \phi(1/(N+1)) . \end{aligned} \tag{3.7}$$

Recall that $\sum_{i=1}^N F_i(x) = NH(x)$, so that the first term on the right of (3.7) is equal to the sum on the left with $\phi(V_i(x)/(N+1))$ replaced by $g(V(x))$ for all i. Hence, and due to (3.5), the absolute value of the difference between the first term on the right and the sum on the left does not exceed

$$\textstyle\sum_{i=1}^N \int E\{g(V(x)) - \phi(V_i(x)/(N+1))\}\, dF_i(x) .$$

But $g(V(x)) \leqq g(V_i(x))$, so that the latter sum is not larger than

$$\begin{aligned} \textstyle\sum_{i=1}^N E\{g(R_i) - \phi(R_i/(N+1))\} &= \textstyle\sum_{i=1}^N \{g(i) - \phi(i/(N+1))\} \\ &= \phi(N/(N+1)) - \phi(1/(N+1)) . \end{aligned}$$

This proves (3.7).

It is easy to verify that

$$\begin{aligned} \textstyle\int E|g(V(x)) - \phi(H(x))|\, dH(x) \leqq \int E|g(V(x)) - g([NH(x)])|\, dH(x) \\ + \textstyle\int_0^1 |g([Nt]) - \phi(t)|\, dt , \end{aligned} \tag{3.8}$$

where $[u]$ denotes the largest integer $\leqq u$, and

$$\begin{aligned} E|g(V(x)) - g([NH(x)])| \\ = \textstyle\sum_{i=1}^{[NH(x)]} \{g(i) - g(i-1)\} P\{V(x) \leqq i-1\} \\ + \textstyle\sum_{i=[NH(x)]+1}^{N} \{g(i) - g(i-1)\} P\{V(x) \geqq i\} . \end{aligned} \tag{3.9}$$

Let $W_N(p)$ denote a binomial (N, p) random variable. By Theorem 5 of [7],

$$\begin{aligned} P\{V(x) \leqq j\} &\leqq P\{W_N(H(x)) \leqq j\} \quad \text{if } j \leqq NH(x) - 1 , \\ P\{V(x) \geqq j\} &\leqq P\{W_N(H(x)) \geqq j\} \quad \text{if } j \geqq NH(x) + 1 . \end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$E|g(V(x)) - g([NH(x)])| \leqq E|g(W_N(H(x)) - g([NH(x)])| . \tag{3.11}$$

Note that the right-hand side depends on x only through $H(x)$. On integrating both sides of (3.11) with respect to $dH(x)$, we have

$$\int E|g(V(x)) - g([NH(x)])|\, dH(x) \leqq \int_0^1 E|g(W_N(t)) - g([Nt])|\, dt\,. \tag{3.12}$$

From (3.9) with $V(x)$ replaced by $W_N(H(x))$ and $H(x)$ by t we obtain after simplification

$$\begin{aligned} &\int_0^1 E|g(W_N(t)) - g([Nt])|\, dt \\ &\quad = \sum_{i=1}^{N-1} \{g(i) - g(i-1)\}\Big(\int_{i/N}^1 P\{W_N(t) \leqq i-1\}\, dt \\ &\qquad + \int_0^{i/N} P\{W_N(t) \geqq i\}\, dt\Big)\,. \end{aligned} \tag{3.13}$$

In the notation used in Scetion 2, $P\{W_N(t) \geqq i\} = P\{U_N^{(i)} \leqq t\} = G_{N,i}(t)$. Thus

$$\begin{aligned} &\int_{i/N}^1 P\{W_N(t) \leqq i-1\}\, dt + \int_0^{i/N} P\{W_N(t) \geqq i\}\, dt \\ &\quad = \int_{i/N}^1 \{1 - G_{N,i}(t)\}\, dt + \int_0^{i/N} G_{N,i}(t)\, dt \\ &\quad = \int_0^1 \left|u - \frac{i}{N}\right| dG_{N,i}(u) \\ &\quad \leqq \left\{\int_0^1 \left(u - \frac{i}{N}\right)^2 dG_{N,i}(u)\right\}^{\frac{1}{2}} \\ &\quad = \left\{\frac{1}{N+2}\frac{i}{N+1}\left(1 - \frac{i}{N+1}\right) + \left(\frac{i}{N+1} - \frac{i}{N}\right)^2\right\}^{\frac{1}{2}} \\ &\quad \leqq \left(\frac{2}{N}\right)^{\frac{1}{2}}\left(\frac{i}{N+1}\right)^{\frac{1}{2}}\left(1 - \frac{i}{N+1}\right)^{\frac{1}{2}} \\ &\quad \leqq 2N^{-\frac{1}{2}}\left(\frac{i}{N}\right)^{\frac{1}{2}}\left(1 - \frac{i}{N}\right)^{\frac{1}{2}} \end{aligned} \tag{3.14}$$

for $1 \leqq i \leqq N-1$. Since $g(i) = \phi((i+1)/(N+1))$ for $0 \leqq i \leqq N-1$, we have from (3.13) and (3.14)

$$\begin{aligned} &\int_0^1 E|g(W_N(t)) - g([Nt])|\, dt \\ &\quad \leqq 2N^{-\frac{1}{2}} \sum_{i=1}^{N-1} \left\{\phi\left(\frac{i+1}{N+1}\right) - \phi\left(\frac{i}{N+1}\right)\right\}\left(\frac{i}{N}\right)^{\frac{1}{2}}\left(1 - \frac{i}{N}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

The sum on the right is equal to $J(\phi_N^*)$, where $\phi_N^*(t) = \phi(([Nt]+1)/(N+1))$. Since ϕ is non-decreasing, it is easy to show that

$$\sum_{i=1}^{N-1} \left\{\phi\left(\frac{i+1}{N+1}\right) - \phi\left(\frac{i}{N+1}\right)\right\}\left(\frac{i}{N}\right)^{\frac{1}{2}}\left(1 - \frac{i}{N}\right)^{\frac{1}{2}} = J(\phi_N^*) \leqq 2J(\phi)\,. \tag{3.16}$$

By (3.7), (3.8), (3.12), (3.15) and (3.16),

$$\begin{aligned} &\sum_{i=1}^{N} \int E\left|\phi\left(\frac{V_i(x)}{N+1}\right) - \phi(H(x))\right| dF_i(x) \\ &\quad \leqq 4N^{\frac{1}{2}}J(\phi) + N\int_0^1 |g([Nt]) - \phi(t)|\, dt \\ &\qquad + \left\{\phi\left(\frac{N}{N+1}\right) - \phi\left(\frac{1}{N+1}\right)\right\}. \end{aligned} \tag{3.17}$$

Finally,

$$\begin{aligned} N \int_0^1 |g([Nt]) - \phi(t)|\, dt \\ (3.18) \qquad &= N \int_0^1 |\phi(([Nt]+1)/(N+1)) - \phi(t)|\, dt \\ &\leqq N \int_0^1 \left\{\phi\left(\frac{Nt+1}{N+1}\right) - \phi\left(\frac{Nt}{N+1}\right)\right\} dt \\ &= (N+1)\{\textstyle\int_{N/(N+1)}^1 \phi(t)\, dt - \int_0^{1/(N+1)} \phi(t)\, dt\}\,, \end{aligned}$$

$$\begin{aligned} (3.19) \qquad \phi(N/(N+1)) - \phi(1/(N+1)) \\ &\leqq (N+1)\{\textstyle\int_{N/(N+1)}^1 \phi(t)\, dt - \int_0^{1/(N+1)} \phi(t)\, dt\}\,, \end{aligned}$$

and

$$\begin{aligned} (3.20) \qquad \int_{N/(N+1)}^1 \phi(t)\, dt - \int_0^{1/(N+1)} \phi(t)\, dt &= \int_0^1 \min(u, (N+1)^{-1}, 1-u)\, d\phi(u) \\ &\leqq N^{-\frac{1}{2}} \int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\, d\phi(u) = N^{-\frac{1}{2}} J(\phi)\,. \end{aligned}$$

Proposition 2 now follows from (3.3), (3.4) and (3.17) through (3.20).

4. Proof of Theorem 1. The following lemma will be used.

LEMMA 1. *If ϕ satisfies the conditions of Theorem* 1, *then for every $\alpha > 0$ there exists a decomposition*

$$(4.1) \qquad \phi(t) = \psi(t) + \phi^{(1)}(t) - \phi^{(2)}(t)\,, \qquad 0 < t < 1\,,$$

such that ψ is a polynomial, $\phi^{(1)}$ and $\phi^{(2)}$ are non-decreasing, and

$$(4.2) \qquad J(\phi^{(1)}) + J(\phi^{(2)}) < \alpha\,.$$

Lemma 1 is an analog of Lemma 5.1 of Hájek (1968), which differs from Lemma 1 in that ϕ is assumed to satisfy the conditions of Theorem 2.3, and (4.2) is replaced by $\sum_{k=1}^2 \int_0^1 \phi^{(k)}(t)^2\, dt < \alpha$. Hájek's proof of Lemma 5.1 serves without change to prove Lemma 1.

It will be sufficient to prove the assertion of Theorem 1 concerning Theorem 2.3 since for Theorem 2.4 the proof is analogous. First let S be defined with $a_N(i) = \phi(i/(N+1))$. To prove the statement of the theorem with centering constant μ', it is enough to show that for every $\beta > 0$ and $\eta > 0$ there exists a number $N' = N'(\beta, \eta)$ such that

$$(4.3) \qquad N > N'\,, \qquad \operatorname{Var} S > \eta\, N \max_{1 \leqq i \leqq N}(c_i - \bar{c})^2$$

implies

$$(4.4) \qquad |ES - \mu'|/(\operatorname{Var} S)^{\frac{1}{2}} < \beta\,.$$

Indeed, given $\varepsilon > 0$ and $\eta > 0$, choose $\beta = \beta(\varepsilon)$ so that $\max_x |\Phi(x \pm \beta) - \Phi(x)| < \varepsilon/2$. Let $N''(\varepsilon, \eta) = \max\{N'(\beta(\varepsilon), \eta), N(\varepsilon/2, \eta)\}$, with $N(\cdot, \cdot)$ defined in Hájek's Theorem 2.3. Then (1.4) with $N(\varepsilon, \eta)$ replaced by $N''(\varepsilon, \eta)$ implies (1.5) with ES replaced by μ'.

We write $S(\phi)$, $\mu'(\phi)$ for S, μ' to exhibit the dependence on ϕ. Since $\sum_{i=1}^N \phi(R_i/(N+1)) = \sum_{i=1}^N \phi(i/(N+1))$ and $\sum_{i=1}^N \int \phi(H(x))dF_i(x) = N \int_0^1 \phi(t)\, dt$, we have from (1.9)

$$S(\phi) - \mu'(\phi) = \sum_{i=1}^N (c_i - \bar{c})\{\phi(R_i/(N+1)) - \int \phi(H(x)\, dF_i(x)\}\,.$$

Hence

$$|ES(\phi) - \mu'(\phi)| \leqq \max_{1\leqq i\leqq N}|c_i - \bar{c}| \sum_{i=1}^N |E\phi(R_i/(N+1)) - \int \phi(H(x))\, dF_i(x)| \,. \tag{4.5}$$

We apply Lemma 1 with α to be specified later. Clearly

$$|ES(\phi) - \mu'(\phi)| \leqq |ES(\psi) - \mu'(\psi)| + \sum_{k=1}^2 |ES(\phi^{(k)}) - \mu'(\phi^{(k)})| \,. \tag{4.6}$$

Since ψ has a bounded second derivative, it follows by a Taylor expansion (see Hájek (1968) page 340) that there is a constant $K(\psi)$ such that

$$|E\psi(R_i/(N+1)) - \int \psi(H(x)\, dF_i(x))| < K(\psi)N^{-1}\,, \qquad i = 1, \cdots, N\,. \tag{4.7}$$

Hence, from (4.5) with $\phi = \psi$,

$$|ES(\psi) - \mu'(\psi)| \leqq K(\psi) \max_{1\leqq i\leqq N}|c_i - \bar{c}| \,. \tag{4.8}$$

From (4.5) with $\phi = \phi^{(k)}$, Proposition 2, and (4.2),

$$\sum_{k=1}^2 |ES(\phi^{(k)}) - \mu'(\phi^{(k)})| \leqq C_2 N^{\frac{1}{2}}\alpha \max_{1\leqq i\leqq N}|c_i - \bar{c}| \,. \tag{4.9}$$

If $\operatorname{Var} S > \eta N \max_i (c_i - \bar{c})^2$, it follows from (4.6), (4.8) and (4.9) that

$$|ES(\phi) - \mu'(\phi)|/(\operatorname{Var} S)^{\frac{1}{2}} \leqq \eta^{-\frac{1}{2}}K(\psi)N^{-\frac{1}{2}} + C_2\eta^{-\frac{1}{2}}\alpha \,. \tag{4.10}$$

Now, given $\beta > 0$ and $\eta > 0$, choose α in Lemma 1 so that $C_2\eta^{-\frac{1}{2}}\alpha = \beta/2$. This choice fixes $K(\psi) = K_1(\beta, \eta)$. Define $N' = N'(\beta, \eta)$ by $\eta^{-\frac{1}{2}}K(\psi)(N')^{-\frac{1}{2}} = \beta/2$. Then (4.3) implies (4.4), as was to be proved.

To prove the last part of Theorem 1 concerning the case (1.2), note that, by (1.9),

$$\begin{aligned} |\mu' - \mu| &\leqq |\bar{c}|\,|\textstyle\sum_{i=1}^N \phi(i/(N+1)) - N\int_0^1 \phi(t)\, dt| \\ &= |\bar{c}|N\left|\int_0^1 \left\{\phi\left(\frac{[Nt]+1}{N+1}\right) - \phi(t)\right\} dt\right| \,. \end{aligned}$$

Assume for the moment that ϕ is non-decreasing. Then (compare (3.18))

$$\begin{aligned} N\int_0^1 \left|\phi\left(\frac{[Nt]+1}{N+1}\right) - \phi(t)\right| dt &\leqq (N+1)\{\textstyle\int_{N/(N+1)}^1 \phi(t)\, dt - \int_0^{1/(N+1)} \phi(t)\, dt\} \\ &\leqq (N+1)^{\frac{1}{2}}\{\textstyle\int_{N/(N+1)}^1 |\phi(t)|t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\, dt \\ &\qquad + \textstyle\int_0^{1/(N+1)} |\phi(t)|t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\, dt\} \,. \end{aligned}$$

Since $\phi = \phi_1 - \phi_2$ is the difference of two non-decreasing functions which satisfy condition (1.10′), it follows that

$$|\mu' - \mu| \leqq |\bar{c}|N^{\frac{1}{2}}K_N \,,$$

where $K_N = K_N(\phi) \to 0$ as $N \to \infty$. Hence if $\operatorname{Var} S > \eta N \max(c_i - \bar{c})^2$, then

$$|\mu' - \mu|/(\operatorname{Var} S)^{\frac{1}{2}} < \eta^{-\frac{1}{2}}K_N|\bar{c}|/\max|c_i - \bar{c}| \,,$$

which is arbitrarily small for N large enough if $|\bar{c}|/\max|c_i - \bar{c}|$ is bounded. This implies the last part of the theorem.

Finally consider S with $a_N(i) = E\phi(U_N^{(i)})$. In this case $\sum_{i=1}^N a_N(i) = N\int_0^1 \phi(t)\, dt$, hence

$$(4.11)\qquad \begin{aligned} S(\phi) - \mu(\phi) &= \textstyle\sum_{i=1}^N (c_i - \bar{c})\{a_N(R_i) - \int \phi(H(x))\, dF_i(x)\}\,, \\ |ES(\phi) - \mu(\phi)| &\leqq \max_i |c_i - \bar{c}| \textstyle\sum_{i=1}^N |Ea_N(R_i) - \int \phi(H(x))\, dF_i(x)|\,. \end{aligned}$$

Now it is easily seen that

$$(4.12)\qquad \begin{aligned} \textstyle\sum_{i=1}^N |Ea_N(R_i) - \int \phi(H(x))\, dF_i(x)| \\ \leqq \textstyle\sum_{i=1}^N |E\phi(R_i/(N+1)) - \int \phi(H(x))\, dF_i(x)| \\ + \textstyle\sum_{i=1}^N |E\phi(U_N^{(i)}) - \phi(i/(N+1))|\,. \end{aligned}$$

For $\phi = \psi$ we apply Taylor's formula to the last term. Since $EU_N^{(i)} = i/(N+1)$ and $\operatorname{Var} U_N^{(i)} < N^{-1}$ for all i, we find that there is a constant $K'(\psi)$ such that $|E\psi(U_N^{(i)}) - \psi(i/(N+1))| < K'(\psi)N^{-1}$, $i = 1, \cdots, N$. Together with (4.7) this implies an inequality analogous to (4.8). Applying Propositions 1 and 2 to (4.12) with $\phi = \phi^{(k)}$, $k = 1, 2$, and using Lemma 1, we obtain an inequality analogous to (4.9). Now the conclusion follows as in the first part of the proof.

5. Remarks on the condition $J(\phi) < \infty$. Theorem 1 shows that if the condition $J(\phi_k) < \infty$, $k = 1, 2$, is added to the assumptions of Hájek's Theorem 2.3, then the conclusion of that theorem holds with ES replaced by μ or μ'. The analogous result has been proved by Dupač (1970) concerning Theorem 2 of Dupač and Hájek (1969), where ϕ is not required to be continuous, but restrictions beyond continuity are imposed on the distribution functions $F_1, \cdots, F_N$. The remarks of this section are intended to throw some light on the role of the added condition, although the question whether the condition is needed remains open.

In the case of the scores $\phi(i/(N+1))$ the proof of Theorem 1 depends on Proposition 2. Let

$$(5.1)\qquad D_i = D_i(\phi, F_1, \cdots, F_N) = E\phi(R_i/(N+1)) - \int \phi(H(x))\, dF_i(x)\,.$$

Proposition 2 implies that if ϕ is non-decreasing and $J(\phi)$ is finite, then $N^{-\frac{1}{2}} \sum_{i=1}^N |D_i|$ is bounded, and is small if $J(\phi)$ is small. Recall that Proposition 2 is applied only with $\phi = \phi^{(1)}$ or $\phi^{(2)}$, the irregular components in the decomposition (4.1). An inspection of the proof shows that for the function ϕ of Theorem 1, the $D_i = D_i(\phi)$ satisfy

$$(5.2)\qquad \textstyle\sum_{i=1}^N |D_i| = o(N^{\frac{1}{2}}) \qquad \text{as } N \to \infty\,.$$

The following suggests that condition (5.2) is essential for the conclusion of Theorem 1. If $(S - ES)(\operatorname{Var} S)^{-\frac{1}{2}}$ has a limit distribution, then $(S - \mu')(\operatorname{Var} S)^{-\frac{1}{2}}$ has the same limit distribution if and only if

$$(5.3)\qquad (ES - \mu')(\operatorname{Var} S)^{-\frac{1}{2}} = o(1)\,.$$

Now

$$(5.4)\qquad ES - \mu' = \textstyle\sum_{i=1}^N (c_i - \bar{c})D_i\,,$$

whence $|ES - \mu'| \leqq \max_i |c_i - \bar{c}| \sum_{i=1}^N |D_i|$. If assumption (1.4) of Hájek's Theorem 2.3 is satisfied, that is, if

$$\max_i |c_i - \bar{c}| (\operatorname{Var} S)^{-\frac{1}{2}} = O(N^{-\frac{1}{2}}) ,$$

then (5.2) implies (5.3).

On the other hand, if (A) ϕ is non-decreasing and square integrable, and (B) the $F_i = F_{i,N}$ are such that condition (5.2) is not satisfied, then there exist constants $c_i = c_{i,N}$ such that (5.3) is not satisfied. To see this, note first that $\sum_{i=1}^N D_i$ does not depend on the F_i and may be written as

$$\sum_{i=1}^N D_i = N \int_0^1 \left\{\phi\left(\frac{[Nt]+1}{N+1}\right) - \phi(t)\right\} dt . \tag{5.5}$$

Assumption (A) implies that the right-hand side of (5.5) is $o(N^{\frac{1}{2}})$; this follows from (3.18) and Schwarz's inequality. Hence if we define the c_i by

$$c_i = \operatorname{sgn} D_i , \tag{5.6}$$

then, from (5.4),

$$ES - \mu' = \sum_{i=1}^N |D_i| + o(N^{\frac{1}{2}}) . \tag{5.7}$$

Also, Since ϕ is non-decreasing, we have, by Theorem 3.1 of Hájek (1968), $\operatorname{Var} S \leqq 21 \max_i (c_i - \bar{c})^2 \sum_{i=1}^N (a_i - \bar{a})^2$, where $a_i = \phi(i/(N+1))$ and $\bar{a} = N^{-1} \sum_{i=1}^N a_i$. Since ϕ is also square integrable, $\sum (a_i - \bar{a})^2 = O(N)$. Combined with (5.6) this implies $\operatorname{Var} S = O(N)$. It now follows from (5.7) that if (5.2) is violated, so is (5.3).

We now turn to a review of the proof of Proposition 2. The proof starts with the inequalities (3.3) and (3.4), which imply

$$\begin{aligned} \sum_{i=1}^N |D_i| &\leqq \sum_{i=1}^N \int \left| E\phi\left(\frac{V_i(x)}{N+1}\right) - \phi(H(x))\right| dF_i(x) \\ &\leqq \sum_{i=1}^N \int E\left|\phi\left(\frac{V_i(x)}{N+1}\right) - \phi(H(x))\right| dF_i(x) . \end{aligned} \tag{5.8}$$

The proof is completed with inequalities (3.17) through (3.20) which show that

$$\sum_{i=1}^N \int E\left|\phi\left(\frac{V_i(x)}{N+1}\right) - \phi(H(x))\right| dF_i(x) \leqq C_2 N^{\frac{1}{2}} J(\phi) . \tag{5.9}$$

This last inequality is best possible in the sense that in the special case $F_1 = \cdots = F_N$ the left-hand side of (5.9) is asymptotically proportional to the right-hand side (asymptotically equal to $(2/\pi)^{\frac{1}{2}} N^{\frac{1}{2}} J(\phi)$) if $J(\phi)$ is finite. Indeed, if $F_1 = \cdots = F_N$, then the sum on the left of (5.9) is nearly equal to N times $\int_0^1 E|g(W_N(t)) - g([Nt])|\, dt$, and an elaboration on the calculations following (3.13) yields the stated result.

Thus if the condition $J(\phi) < \infty$ can be avoided, we must go back to one of the first two sums in (5.8). In the i.i.d. case, $F_1 = \cdots = F_N$, the second of those sums is equal to

$$N \int_0^1 \left| E\phi\left(\frac{W_{N-1}(t)+1}{N+1}\right) - \phi(t)\right| dt . \tag{5.10}$$

Note that

$$E\phi\left(\frac{W_{N-1}(t)+1}{N+1}\right) = \sum_{k=0}^{N-1} \phi\left(\frac{k+1}{N+1}\right)\binom{N-1}{k} t^k(1-t)^{N-1-k} \tag{5.11}$$

is a slightly modified version of the Bernstein polynomial of ordered $N-1$ which approximates the function $\phi(t)$, and (5.10) is N times the L_1 norm of the error of approximation. Theorem 2 of [8], which is concerned with polynomials closely related to the polynomials (5.11), implies the following. If ϕ is a non-decreasing square integrable step function having finitely many steps in every closed sub-interval of (0, 1), then

$$\lim_{N\to\infty} N^{\frac{1}{2}} \int_0^1 \left| E\phi\left(\frac{W_{N-1}(t)+1}{N+1}\right) - \phi(t)\right| dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} J(\phi) , \tag{5.12}$$

irrespective of whether $J(\phi)$ is finite or infinite.

In Theorem 2 of Dupač and Hájek (1969), ϕ is allowed to be a step function of the type here assumed, and the stated result shows that the behavior even of the first upper bound in (5.8) does not permit us to decide whether ES may be replaced by μ' under the conditions of that theorem.

It should be noted that in the case $F_1 = \cdots = F_N$ the sum $\sum |D_i|$ itself does not behave in a way similar to (5.12). In fact, $\sum |D_i|$ is *minimized* with respect to $F_1, \cdots, F_N$ when the F_i are all equal, since equality in $\sum |D_i| \geqq |\sum D_i|$ holds in the latter case. And, as noted after (5.5), $|\sum D_i| = o(N^{\frac{1}{2}})$ if ϕ is non-decreasing and square integrable.

It can be shown (proof omitted) that the following analog of (5.12) with $J(\phi) = \infty$ holds for $\sum |D_i|$. Let ϕ be a non-decreasing square integrable step function having finitely many steps in every closed sub-interval of (0, 1), and let $J(\phi)$ be infinite. Then there exist continuous distribution functions $F_i = F_{i,N}$ such that

$$N^{-\frac{1}{2}} \sum_{i=1}^N |D_i| \to \infty \qquad \text{as } N \to \infty .$$

However, the distribution functions F_i in the author's proof do not satisfy the conditions of Theorem 2 of Dupač and Hájek.

APPENDIX

Here the following is proved.

(A1) If ϕ is non-decreasing and $J(\phi) < \infty$ then $\int_0^1 \phi^2(t)\, dt < \infty$.

(A2) If ϕ is non-decreasing and $\int_0^1 \phi^2(t)\{\log(1+|\phi(t)|)\}^{1+\delta}\, dt < \infty$ for some $\delta > 0$ then $J(\phi) < \infty$.

(A3) There exists a non-decreasing function ϕ such that the integral in (A2) with $\delta = 0$ is finite but $J(\phi) = \infty$.

PROOF OF (A1). If ϕ is non-decreasing and $J(\phi) < \infty$ then ϕ is integrable on (0, 1), as can be seen from (1.12). Hence it is sufficient to prove:

(A1′) If ϕ is non-decreasing and integrable on (0, 1) then

$$\int_0^1 \{\phi(t) - \int_0^1 \phi(s)\, ds\}^2\, dt \leqq \{\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\, d\phi(u)\}^2 = J^2(\phi)\,.$$

To see this, we note that

$$\int_0^1 \{\phi(t) - \int_0^1 \phi(s)\, ds\}^2\, dt = \int\int_{0<s<t<1} \{\phi(t) - \phi(s)\}^2\, ds\, dt\,.$$

If ϕ is non-decreasing then $\phi(t) - \phi(s) = \int_{s<u<t} d\phi(u)$ almost everywhere for $0 < s < t < 1$, and interchanging the order of integration we obtain

$$\int\int_{0<s<t<1} \{\phi(t) - \phi(s)\}^2\, ds\, dt = \int_0^1 \int_0^1 \{\min(u, v) - uv\}\, d\phi(u)\, d\phi(v)\,.$$

But $\min(u, v) - uv \leqq u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}v^{\frac{1}{2}}(1-v)^{\frac{1}{2}}$ for u and v in [0, 1], and the inequality in (A1′) follows.

PROOF OF (A2). The assumption that ϕ is non-decreasing is here made only because this was assumed in the definition of $J(\phi)$. It is sufficient to prove:

(A2′) If ϕ is Lebesgue measurable and $\int_0^1 \phi^2(t)\{\log(1 + |\phi(t)|)\}^{1+\delta}\, dt < \infty$ for some $\delta > 0$ then $\int_0^1 |\phi(t)| t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\, dt < \infty$.

Let $\delta > 0$,

$$f(u) = (1+u)\{\log(1+u)\}^{1+\delta}\,, \qquad F(v) = \int_0^v f(u)\, du\,, \qquad G(v) = \int_0^v f^{-1}(u)\, du\,,$$

where f^{-1} denotes the inverse of f. As $v \to \infty$, the function $F(v)$ is asymptotically proportional to $v^2\{\log(1+v)\}^{1+\delta}$. Hence the hypothesis of (A2′) implies $\int_0^1 F(|\phi(t)|)\, dt < \infty$. From Young's inequality (see Theorem 237 in [6]) we have

$$\int_0^1 |\phi(t)| t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\, dt \leqq \int_0^1 F(|\phi(t)|)\, dt + \int_0^1 G(t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}})\, dt\,.$$

Thus it is sufficient to show that the second integral on the right is finite. The latter is equivalent to the finiteness of each of the following integrals:

$$\int_0^{\frac{1}{2}} G(t^{-\frac{1}{2}})\, dt\,, \qquad \int_1^\infty G(x)x^{-3}\, dx\,, \qquad \int_1^\infty f^{-1}(y)y^{-2}\, dy\,, \qquad \int_1^\infty uf(u)^{-2}f'(u)\, du\,.$$

The last integral is finite since $uf(u)^{-1}f'(u)$ is bounded and $f(u)^{-1}$ is integrable on $(1, \infty)$.

PROOF OF (A3). The function

$$\phi(t) = (1-t)^{-\frac{1}{2}}\{4 - \log(1-t)\}^{-1}\{\log(4 - \log(1-t))\}^{-1}$$

may be shown to have the properties stated in (A3).

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