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TESTING THAT A GAUSSIAN PROCESS IS STATIONARY

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A class of procedures is proposed for testing the stationarity of a Gaussian process or the homogeneity of independent processes. Requiring very limited prior knowledge of model structure, the methods can detect changes or diferences in mean, in variance, in covariances and even in law. Although the theory of the stationarity test is worked out only for processes whose realizations are stationary over "epochs" separated by known change points, Monte Carlo evidence indicates that it can be useful also in detecting more general forms of nonstationarity. The test statistic is a quadratic form in diferences among epoch means of certain "sensing" functions, the choice of which governs sensitivity to specific forms of nonstationarity or inhomogeneity. The applicability of the general asymptotic theory of the test is verified for two specific forms of sensing function, and small-sample properties of tests of each form are studied by means of simulation.

1. Introduction. This paper introduces a class of chi-squared statistics which may be used to test the stationarity of a time series or the homogeneity of two or more independent processes. In either case we work with samples of equally spaced observations of processes which, at least under the null hypothesis, are assumed to be Gaussian. In developing the theory of the stationarity test we confine attention to processes which change abruptly at known points in time and are stationary in between, but Monte Carlo evidence indicates that the test will be useful in more general situations.

For the statistician who deals with i.i.d. samples there is an abundance of methods for studying the homogeneity of data, ranging from ANOVA to the various nonparametric two-sample tests. With time series the situation is quite different. One who has exact knowledge of the underlying model, including the values of the parameters, might hope to whiten the data and apply a method designed for use with random samples; but one rarely has such information. If at least the *model* is known, although not the parameters, the likelihood-ratio principle can be employed to test for stationarity. A related Bayesian procedure is described by Hsu (1984). Other methods which can be applied in special cases have been developed by Box and Tiao (1965), to test for changes in level, and Wichern, Miller and Hsu (1976), to test for changes in variance.

The usual situation, however, is one in which the statistician does *not* know the model structure, wants to estimate it, but must first confront the existence of suspected nonstationarities. In this situation the likelihood-ratio test is infeasible since there are too many unknown parameters, yet the range of other techniques to consider is narrow indeed. Quenouille (1958) has developed an approximate

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procedure to test for changes in autocovariance when the model structure is not fully known, and the test of Melard and Roy (1983, 1984)—a quadratic form in diferences among sample covariances—is sensitive to changes in variance as well. Picard (1985) also describes a test for changes in the autocovariance function, based on the spectral density functions of the two samples.

The present paper broadens considerably the collection of tools available for testing stationarity or homogeneity. It describes a class of chi-squared tests that utilize certain functionals of the data, called "sensing functions," which can be selected to enhance power against a variety of changes or diferences. With the appropriate choice of function one may look for changes in mean, variance, autocovariances and even in law—as when a Gaussian process becomes contaminated. A principal advantage of the procedure is that it requires very limited prior knowledge of model structure; namely, a single, weak condition on the summability of the autocovariances.

The general form of the test statistic is presented in Sections 2–4, where the necessary distribution theory is worked out under certain assumptions about the properties of the sensing functions. Two examples of sensing functions are given in Section 3 in order to motivate the procedure, and in Section 5 we show that these functions do have the necessary properties. Section 6 presents the results of a preliminary Monte Carlo study of the test's finite-sample behavior. We find that the test can give quite reliable indications of nonstationarities of various sorts for a variety of model structures.

To simplify the exposition, we confine our discussion to the problem of testing the stationarity of a single time series, but with minor changes in notation the results apply as well to testing the homogeneity of independent processes.

2. Assumptions. Let $\{X_t\}_{t=-\infty}^{\infty}$ be a discrete-parameter stochastic process, of which $\{X_t\}_{t=1}^{T}$ is a sample of equally spaced observations. The sample may be partitioned into $N \geq 2$ subsets, called "epochs," within each of which the process is known to be stationary, and we let $1 < \tau_1 < \tau_2 < \cdots < \tau_{N-1} < T$ represent the presumed change points. Taking $\tau_0 \equiv 0$ and $\tau_N \equiv T$, the length of the nth epoch is then $T_n \equiv \tau_n - \tau_{n-1}$, $n = 1, 2, \dots, N$. Although T_n is, of course, finite, it will be convenient to regard the nth epoch as the realization of an infinitely long process, denoted $\{X_t^n\}_{t=-\infty}^{\infty}$, with unspecified mean μ_n and covariance function $\{\sigma_n(r),\ r = 0, \pm 1, \pm 2, \dots\}$. We shall often write σ_n^2 for $\sigma_n(0)$. The null hypothesis to be tested is that the processes $\{X_t^1\}, \{X_t^2\}, \dots, \{X_t^N\}$ are identically distributed, with common mean μ and covariance function $\{\sigma(r)\}$.

As the maintained hypotehsis, we assume that, for each epoch n, $\{X_t^n\}$ is Gaussian and that there exists $\zeta > 0$ such that

$$\text{(A.1)} \qquad \sum_{r=-\infty}^{\infty} |r|^{\zeta} |\sigma_n(r)| < \infty.$$

Although stronger than absolute summability (the case $\zeta = 0$), (A.1) is an extremely mild condition, satisfied for *every* $\zeta > 0$ by all stationary ARMA models. Note that we need the process to satisfy (A.1) and to be stationary and

Gaussian within *each* epoch only for purposes of developing an asymptotic theory under local alternatives (Section 4). For the validity of the test it is enough that these conditions hold under H_0 for the process as a whole. Monte Carlo evidence (Section 6) confirms that the test does have power against more general alternatives, such as that $\{X_t^n\}$ is non-Gaussian or even nonstationary, for some n.

We now introduce the vector-valued sensing function

$$g^*(X_t, X_{t-1}, \ldots, X_{t-L}; \lambda), \qquad g^*: \mathbb{R}_{L+1} \times \mathbb{R}_d \to \mathbb{R}_J,$$

and certain conditions that it must satisfy. Here λ is a d-vector of real numbers, initially considered to be constants but later allowed to be sample dependent in specific versions of the test.

The first condition imposed on g^* is that all its elements be functions of the same linear combination of the observations $X_t, X_{t-1}, \ldots, X_{t-L}$, $L \geq 0$; that is, for certain real numbers $\{\alpha_l(\lambda)\}_{l=1}^{L}$ we have

(A.2) $$g^*(X_t, X_{t-1}, \ldots, X_{t-L}; \lambda) = g(X_t(L); \lambda), \qquad g: \mathbb{R}_1 \times \mathbb{R}_d \to \mathbb{R}_j,$$

where $X_t(L) \equiv X_t + \alpha_1(\lambda)X_{t-1} + \cdots + \alpha_L(\lambda)X_{t-L}$. Henceforth, we refer to both g^* and g as sensing functions.

The test statistic will depend on differences among sample means of g in diferent epochs, which are defined as

$$(2.1) \qquad \begin{aligned} g_{T_n}(\lambda) &\equiv T_n^{-1} \sum_t g^*(X_t, X_{t-1}, \ldots, X_{t-L}: \lambda) \\ &= T_n^{-1} \sum_{t=L+1}^{T_n} g(X_t^n(L); \lambda), \end{aligned}$$

the first sum running from $\tau_{n-1} + L + 1$ to τ_n. The sensing function must satisfy a second condition if we are to estimate consistently the asymptotic covariance matrices of these means. For epoch n let $\gamma_n(\lambda) \equiv Eg(X_t^n(L); \lambda) = Eg^*(X_t, X_{t-1}, \ldots, X_{t-L}; \lambda)$, $\tau_{n-1} + L < t \leq \tau_n$. Under H_0 we have $\gamma_n(\lambda) = \gamma_0(\lambda)$, say, for $n = 1, 2, \ldots, N$. Adopting momentarily a very compact notation, let $\tilde{g}^n(t) \equiv g(X_t^n(L); \lambda) - \gamma_n(\lambda)$, with $\tilde{g}_j^n(t)$ as jth element of this J-dimensional process. The fourth-order cumulants of g in epoch n may then be defined as

$$(2.2) \qquad \begin{aligned} \kappa_{jklm}^n(q, r, s; \lambda) \equiv{} & E\left[\tilde{g}_j^n(0)\tilde{g}_k^n(q)\tilde{g}_l^n(r)\tilde{g}_m^n(s)\right] \\ & - E\left[\tilde{g}_j^n(0)\tilde{g}_k^n(q)\right] E\left[\tilde{g}_l^n(r)\tilde{g}_m^n(s)\right] \\ & - E\left[\tilde{g}_j^n(0)\tilde{g}_l^n(r)\right] E\left[\tilde{g}_k^n(q)\tilde{g}_m^n(s)\right] \\ & - E\left[\tilde{g}_j^n(0)\tilde{g}_m^n(s)\right] E\left[\tilde{g}_k^n(q)\tilde{g}_l^n(r)\right], \end{aligned}$$

for $q, r, s = 0, \pm 1, \pm 2, \ldots,$ and $j, k, l, m \in \{1, 2, \ldots, J\}$. We require of our choice of g that, for each n,

$$(A.3) \qquad \sup_{-\infty < q < \infty} \sum_{r=-\infty}^{\infty} |\kappa_{jklm}^n(q, r, q + r; \lambda)| < \infty,$$

$$\text{for } j, k, l, m \in \{1, 2, \ldots, J\}.$$

In Section 5 we show for two specific sensing functions how this condition may actually be verified.

The asymptotic covariance matrix of the normalized mean vector $g_{T_n}(\lambda)$ is given by

$$\Gamma_n(\lambda) \equiv \sum_{r=-\infty}^{\infty} \mathrm{Cov}[g(X_r^n(L);\lambda), g(X_0^n(L);\lambda)].$$

That the elements of this matrix are finite will be shown (in the proof of Lemma 4.1) to follow from (A.1). We need the following assumption about $\Gamma_n(\lambda)$ in order to prove a central limit theorem for $g_{T_n}(\lambda)$:

(A.4) $\Gamma_n(\lambda)$ is positive definite.

3. Some examples. Before proceeding with the theory we give two examples of sensing functions that will be useful in testing stationarity or homogeneity. The first is designed to detect changes in mean and covariance structure. Taking $\lambda \equiv \{\mu_1, \mu_2, \dots, \mu_N, \lambda_1, \lambda_2, \dots, \lambda_L\}$, where L and the constants $\{\lambda_l\}$ are to be selected by the researcher, let

$$X_t^n(L) \equiv X_t^n + \lambda_1 X_{t-1}^n + \cdots + \lambda_L X_{t-L}^n, \tag{3.1}$$

$$\mu_n(L) \equiv \mu_n(1 + \lambda_1 + \cdots + \lambda_L) \tag{3.2}$$

and

$$g(X_t^n(L);\lambda) \equiv \{X_t^n(L), [X_t^n(L) - \mu_n(L)]^2\}'. \tag{3.3}$$

With this arrangement $g_{T_n}(\lambda)$ comprises the epoch mean of $\{X_t^n(L)\}$ and its sample second moment about $\mu_n(L)$. The case $L = 0$ gives a test based on the marginal mean and variance. Naturally, either component of (3.3) could be used alone to give special sensitivity to changes in mean or covariance structure. We show in Section 5 that (3.3) does satisfy (A.3) and that the epoch means $\mu_n(L)$, which are ordinarily unknown, can be replaced by their sample counterparts.

The second example affords a broad-spectrum test for changes in the process that do not involve just the first two moments, such as the contamination or truncation of a Gaussian process. Taking $L \geq 0$ and J an even integer, let $\lambda_0 \equiv \{1, \lambda_{01}, \dots, \lambda_{0L}\}$ be an $L + 1$-vector of real numbers; and let $\lambda_1, \lambda_2, \dots, \lambda_{J/2}$ be distinct, positive scalars. With $\lambda \equiv \{\lambda_0, \lambda_1, \dots, \lambda_{J/2}\} \in \mathbb{R}_{L+1+J/2}$ and $X_t^n(L) \equiv X_t^n + \lambda_{01} X_{t-1}^n + \cdots \lambda_{0L} X_{t-L}^n$, define

$$\begin{aligned} g(X_t^n(L);\lambda) \equiv \{&\cos \lambda_1 X_t^n(L), \sin \lambda_1 X_t^n(L), \dots, \\ &\cos \lambda_{J/2} X_t^n(L), \sin \lambda_{J/2} X_t^n(L)\}'. \end{aligned} \tag{3.4}$$

The mean vector $g_{T_n}(\lambda)$ now consists of real and imaginary parts of the joint empirical characteristic function of $\{X_t, X_{t-1}, \dots, X_{t-L}\}$ in epoch n, evaluated at points $\lambda_1\lambda_0, \lambda_2\lambda_0, \dots, \lambda_{J/2}\lambda_0$. Since these statistics are consistent estimators of the corresponding components of the true joint characteristic function of $\{X_t^n, X_{t-1}^n, \ldots, X_{t-L}^n\}$, they can register quite general changes in the joint distribution. We show in Section 5 that (3.4) does satisfy (A.3) and that certain useful forms of data dependence in λ are permissible.

4. Theory of the test. The test statistic to be developed is a quadratic form in differences among the epoch mean vectors $\{g_{T_n}(\lambda)\}$, which will be shown to be distributed asymptotically as chi-squared. The main tasks in demonstrating this are to prove the asymptotic joint normality of $T_1^{-1/2}[g_{T_1}(\lambda) - \gamma_1(\lambda)], \ldots, T_N^{-1/2}[g_{T_N}(\lambda) - \gamma_N(\lambda)]$ and to develop a consistent estimator of the covariance matrix of these vectors.

The first step is to prove a central limit theorem for each $g_{T_n}(\lambda)$, and this much can be done with weaker assumptions than (A.1) and (A.3). Condition (A.1) implies

$$\sum_{r=-\infty}^{\infty} |\sigma_n(r)| < \infty; \tag{4.1}$$

and (A.3) trivially guarantees that

$$\operatorname{Var} g_j(X_t^n(L); \lambda) < \infty, \qquad j \in \{1, 2, \ldots, J\}. \tag{4.2}$$

LEMMA 4.1. *Let $\{X_t^n\}$ be a stationary, Gaussian process satisfying* (4.1) *and choose g so as to satisfy* (A.2), (4.2) *and* (A.4). *Then as $T_n \to \infty$,*

$$T_n^{1/2}\left[g_{T_n}(\lambda) - \gamma_n(\lambda)\right] \to_D N(0, \Gamma_n(\lambda)). \tag{4.3}$$

The proof is based on the central limit theorem for functions of Gaussian processes given by Giraitis and Surgailis (1985). We will also need the following lemma.

LEMMA 4.2. *Let $\{X_t\}_{t=-\infty}^{\infty}$ be a stationary process satisfying* (A.1) *for some $\zeta \geq 0$, and define $X_t(L) \equiv X_t + \alpha_1 X_{t-1} + \cdots + \alpha_L X_{t-L}$, where the α_l are arbitrary real constants. Then $\{X_t(L)\}$ is a stationary process, and $\delta(r) \equiv$ $\operatorname{Cov}[X_0(L), X_r(L)]$, $r = 0, \pm 1, \pm 2, \ldots$, satisfies*

$$\sum_{r=-\infty}^{\infty} |r|^{\zeta} |\delta(r)| < \infty. \tag{4.4}$$

PROOF. Write $X_t(L) = \alpha(B)X_t \equiv (1 + \alpha_1 B + \cdots + \alpha_L B^L)X_t$, where B is the backshift operator, and let $B^r\sigma \equiv \sigma(-r) = \sigma(r)$. Then

$$\begin{aligned}\delta(r) &= \operatorname{Cov}\left[\alpha(B)X_0, B^r\alpha(B)X_0\right] = \alpha(B)\alpha(B^{-1})B^r\sigma \\ &= \sum_{j=0}^{L}\sum_{k=0}^{L} \alpha_j \alpha_k B^{r-(j-k)}\sigma,\end{aligned}$$

where $\alpha_0 \equiv 1$. For $\zeta \geq 0$ satisfying (A.1) we have

$$\sum_{r=-\infty}^{\infty} |r|^{\zeta}|\delta(r)| \leq \sum_{r=-\infty}^{\infty} |r|^{\zeta} \sum_{j=0}^{L} \sum_{k=0}^{L} |\alpha_j \alpha_k| \, |B^{r-(j-k)}\sigma|$$

$$= \sum_{j=0}^{L} \sum_{k=0}^{L} |\alpha_j \alpha_k| \Bigg\{ \sum_{\substack{r=-\infty \\ r \neq (j-k)}}^{\infty} |r-(j-k)|^{\zeta} |\sigma(r-(j-k))| \times |r/[r-(j-k)]|^{\zeta} + |j-k|^{\zeta} \Bigg\}$$

$$\leq \sum_{j=0}^{L} \sum_{k=0}^{L} |\alpha_j \alpha_k| \Bigg\{ [|j-k|+1]^{\zeta} \sum_{r=-\infty}^{\infty} |r-(j-k)|^{\zeta} \times |\sigma(r-(j-k))| + |j-k|^{\zeta} \Bigg\}$$

$$= \sum_{j=0}^{L} \sum_{k=0}^{L} |\alpha_j \alpha_k| \Bigg\{ [|j-k|+1]^{\zeta} \sum_{r=-\infty}^{\infty} |r|^{\zeta} |\sigma(r)| + |j-k|^{\zeta} \Bigg\}$$

$$< \infty,$$

the next-to-last inequality following from $\max_{-\infty < r < \infty} |r/[r-(j-k)]| \leq |j-k| + 1$. □

PROOF OF LEMMA 4.1. Let $Y_t^n \equiv X_t^n(L)$ and $\delta_n(r) \equiv \mathrm{Cov}(Y_0^n, Y_r^n)$. For $u \in \mathbb{R}_j$, $u \neq 0$, let $h(Y_t^n) \equiv u'g(Y_t^n; \lambda)$. By (A.4)

$$\sum_{r=-\infty}^{\infty} \mathrm{Cov}[h(Y_0^n), h(Y_r^n)] = u'\Gamma_n(\lambda)u > 0, \tag{4.5}$$

and it follows from Gebelein's lemma (1941) [Rozanov (1967), page 182] that

$$|\mathrm{Cov}[h(Y_0^n), h(Y_r^n)]| \leq |\delta_n(r)| \, \mathrm{Var}\, h(Y_0^n)/\sigma(0).$$

With (4.2) and Lemma 4.2 this implies

$$\sum_{r=-\infty}^{\infty} |\mathrm{Cov}[h(Y_0^n), h(Y_r^n)]| < \infty, \tag{4.6}$$

which establishes the finiteness of the elements of $\Gamma_n(\lambda)$. Inequalities (4.5) and (4.6) are the conditions needed for Theorem 5 of Giraitis and Surgailis (1985), page 200, which implies that $T_n^{-1/2}\Sigma_t[h(Y_t^n) - Eh(Y_0^n)] \to_D N(0, V_h)$, where [Anderson (1971), Theorem 8.3.1, page 459] $V_h = u'\Gamma_n(\lambda)u$. The conclusion now follows from the Cramér–Wold theorem [Billingsley (1968), page 49]. □

We remark for later purposes that the stronger version of (4.1) given by (A.1) implies the correspondingly stronger version of (4.6), namely,

$$\sum_{r=-\infty}^{\infty} |r|^{\zeta} |\mathrm{Cov}[h(Y_0^n), h(Y_r^n)]| < \infty, \quad \text{some } \zeta > 0. \tag{4.7}$$

Having established the limiting marginal distribution of each $g_{T_n}(\lambda)$, we must now deal with the joint distribution of these statistics for the N epochs. Assume that

$$\lim_{T\to\infty} T_n/T = c_n \in (0,1), \qquad n = 1,2,\dots, N. \tag{4.8}$$

Let $\mathscr{G}(\lambda)$ be a block-diagonal, square matrix of order $J \cdot N$ with matrices $c_1^{-1}\Gamma_1(\lambda),\dots, c_N^{-1}\Gamma_N(\lambda)$ on the diagonal; and define

$$G_T(\lambda) \equiv \{g_{T_1}(\lambda)',\dots, g_{T_N}(\lambda)'\}' \in \mathbb{R}_{J\cdot N}.$$

Theorem 4.1. *If the assumptions of Lemma* 4.1 *hold for each n, then as* $T \to \infty$,

$$T^{1/2}[G_T(\lambda) - EG_T(\lambda)] \to_D N(0,\mathscr{G}(\lambda)).$$

Proof. Let $Z_{T_n} \equiv T^{1/2}[g_{T_n}(\lambda) - \gamma_n(\lambda)]$. Using the definition (2.1), we can write

$$\begin{aligned} Z_{T_n} &= T^{1/2}T_n^{-1} \sum_t [g^*(X_t, X_{t-1},\dots, X_{t-L}; \lambda) - \gamma_n(\lambda)] - T^{1/2}T_n^{-1}L\gamma_n(\lambda) \\ &= T^{1/2}T_n^{-1} \sum_{t=L+1}^{T_n} [g(X_t^n(L); \lambda) - \gamma_n(\lambda)] - T^{1/2}T_n^{-1}L\gamma_n(\lambda), \end{aligned}$$

the first sum running from $\tau_{n-1} + L + 1$ to τ_n. Clearly, the last term is $o(1)$ as $T \to \infty$. Now choose a sequence of integers $\{q_T\}$, with $0 < q_T < \min\{T_1, T_2,\dots, T_n\} - L$ and such that

$$\lim_{T\to\infty} q_T = +\infty, \qquad \lim_{T\to\infty} T^{-1}q_T = 0. \tag{4.9}$$

Discarding the first q_T observations in each epoch, let

$$Z_{T_n}^* \equiv T^{1/2}T_n^{-1} \sum_{t=L+1+q_T}^{T_n} [g(X_t^n(L); \lambda) - \gamma_n(\lambda)].$$

Then for $n = 1,2,\dots, N$,

$$\begin{aligned} |Z_{T_n} - Z_{T_n}^*| &\le T^{1/2}T_n^{-1}\left|\sum_t [g(X_t^n(L); \lambda) - \gamma_n(\lambda)]\right| + o(1) \\ &= (T/T_n)(q_T/T)^{1/2}\left|q_T^{-1/2} \sum_t [g(X_t^n(L); \lambda) - \gamma_n(\lambda)]\right| + o(1), \end{aligned}$$

where the summations are from $L + 1$ to $L + q_T$. Lemma 4.1, (4.8) and (4.9) imply that the right side is $o_P(1)$, from which it follows that the limiting joint distributions of $\{Z_{T_n}\}_{n=1}^N$ and $\{Z_{T_n}^*\}_{n=1}^N$ coincide. By (4.8) and Lemma 4.1 the limiting marginal distribution of $Z_{T_n}^*$ is $N(0, c_n^{-1}\Gamma_n(\lambda))$, $n = 1,2,\dots, N$. The conclusion of the theorem will follow if we can show that $\{Z_{T_n}^*\}_{n=1}^N$ are asymptotically independent as $T \to \infty$.

Let E_{T_n} represent the index set $\{\tau_{n-1} + L + 1 + q_T, \dots, \tau_n\}$, and let $\sigma(E_{T_n})$ be the σ-field generated by $\{X_t, t \in E_{T_n}\}$. $Z^*_{T_n}$ is then measurable $\sigma(E_{T_n})$. Pick $1 \le n' < n \le N$. Since $\{X_t^n\}$ and $\{X_t^{n'}\}$ are Gaussian, any two σ-fields $\sigma(E_{T_n})$ and $\sigma(E_{T_{n'}})$ are asymptotically independent iff

$$\lim_{T\to\infty} \sup_{s\in E_{T_n},\, t\in E_{T_{n'}}} |\mathrm{Cov}(X_s, X_t)| = 0.$$

Now

$$\begin{aligned}
&\limsup_T \left\{|\mathrm{Cov}(X_s, X_t)| : s \in E_{T_n}, t \in E_{T_{n'}}\right\} \\
&\quad = \limsup_T \left\{|\sigma_n(s-t)| : \tau_{n'-1} + L + 1 + q_T \le t \le \tau_{n'}, \tau_{n-1} \right. \\
&\qquad\qquad \left. + L + 1 + q_T \le s \le \tau_n\right\} \\
&\quad \le \limsup_T \left\{|\sigma_n(s-t)| : -\infty < t \le \tau_{n'}, \tau_{n'} + q_T \le s < \infty\right\} \\
&\quad = \limsup_T \left\{|\sigma_n(r)| : q_T \le r < \infty\right\} \\
&\quad \le \lim_T \sum_{r=q_T}^{\infty} |\sigma_n(r)| = 0,
\end{aligned}$$

the value of the final limit following from (4.1) and (4.9). This establishes the asymptotic pairwise independence of $\sigma(E_{T_n})$ and $\sigma(E_{T_{n'}})$ and, hence, of $Z^*_{T_n}$ and $Z^*_{T_{n'}}$. Mutual independence of $\{Z^*_{T_n}\}_{n=1}^N$ is established by showing, in a similar way, that linear combinations of all disjoint subsets of $\{Z^*_{T_n}\}_{n=1}^N$ are asymptotically independent. □

Notice that the conclusion of the theorem applies both under H_0, when $\gamma_n(\lambda)$ and $\Gamma_n(\lambda)$ take the same values in each epoch, and under alternatives to H_0 in which the process remains stationary and Gaussian within epochs. In fact, it is clear that an "asymptotic" stationarity within epochs is sufficient, allowing for the dependence among observations to cause protracted transitions from one form to the next.

We now show that there exists for the stationary, Gaussian process in each epoch a consistent estimator of $\Gamma_n(\lambda)$, the asymptotic covariance matrix of $g_{T_n}(\lambda)$.

LEMMA 4.3. *Define the sequence of integers*

$$M_{T_n} \equiv \max\{[a_1 T_n^{a_2}], T_n - L - 1\}, \qquad a_1 > 0,\ a_2 \in (0,1), \tag{4.10}$$

where "$[\cdot]$" *denotes* "*greatest integer*," *and the function*

$$\begin{aligned}
w(y) &= 1, && y = 0,\\
&= 2(1-y), && 0 < y \le 1,\\
&= 0, && \textit{elsewhere}.
\end{aligned} \tag{4.11}$$

Let $\hat{g}(X_t^n(L); \lambda) \equiv g(X_t^n(L); \lambda) - g_{T_n}(\lambda)$, *where* g *satisfies* (A.2). *If* $\{X_t^n\}$ *is a stationary, Gaussian process satisfying* (A.1), *and if* (A.3) *holds, then as*

$T_n \to \infty$ *the statistic*

$$(4.12)\quad \Gamma_{T_n, n}(\lambda) \equiv \sum_{r=0}^{M_{T_n}} w(r/M_{T_n}) T_n^{-1} \sum_{t=L+1}^{T_n - r} \hat{g}(X_t^n(L); \lambda) \hat{g}(X_{t+r}^n(L); \lambda)'$$

converges almost surely to $\Gamma_n(\lambda)$.

PROOF. (4.6) and Theorem 8.3.1 of Anderson (1971), page 459, imply that $\Gamma_n(\lambda)$ equals 2π times the spectral density matrix of $g(X_t^n(L); \lambda)$, denoted $f_n(\nu; \lambda)$, evaluated at frequency $\nu = 0$. Apart from a factor 2π, the right side of (4.12) is a particular form of Gaposhkin's (1980) strongly consistent estimator of $f_n(0; \lambda)$. The lag window function w and the truncation point M_{T_n} satisfy Gaposhkin's conditions I and III′, and his conditions II′(a) and (b) are given by (A.3) and implied by (4.7), respectively. The conclusion follows from Gaposhkin's Theorems 4 and 5. Details are given in Epps (1987). □

Taking $\mathscr{G}_T(\lambda)$ to be the block-diagonal matrix with $(T/T_n)\Gamma_{T_n, n}(\lambda)$ as the nth diagonal element, it follows from Lemma 4.3 and (4.8) that $\mathscr{G}_T(\lambda) \to \mathscr{G}(\lambda)$ a.s.

The main ingredients of the distribution theory of the test are now assembled. The statistic which follows is a quadratic form in differences among means of the sensing functions in different epochs, the form of differencing depending on the researcher's beliefs about the type of nonstationarity that may be present. We allow for a variety of possibilities by introducing a $JK \times JN$ matrix Δ, $1 \leq K < N$, of the form $\Delta = D \otimes I_J$. Here I_J is the identity matrix, and each row of the $K \times N$ matrix D comprises $N - 2$ zeroes and the integers $\{-1, 1\}$ arranged so that $\operatorname{rank}(D) = K \leq N - 1$; e.g.,

$$(4.13)\quad D = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The product of Δ and $G_T(\lambda)$ is then a vector of differences among epoch means, such as, with (4.13) for D,

$$\Delta G_T(\lambda) = \{g_{T_2}(\lambda)' - g_{T_1}(\lambda)', \ldots, g_{T_N}(\lambda)' - g_{T_1}(\lambda)'\}'.$$

The test statistic may now be defined compactly as

$$(4.14)\quad Q_T(\lambda) \equiv T[\Delta G_T(\lambda)]'[\Delta \mathscr{G}_T(\lambda) \Delta']^+ [\Delta G_T(\lambda)],$$

where "+" signifies generalized inverse. In the case $N = 2$ this takes the simple form

$$(4.15)\quad \begin{aligned} Q_T(\lambda) = {} & [g_{T_2}(\lambda) - g_{T_1}(\lambda)]'[T_1^{-1}\Gamma_{T_1, 1}(\lambda) + T_2^{-1}\Gamma_{T_2, 2}(\lambda)]^+ \\ & [g_{T_2}(\lambda) - g_{T_1}(\lambda)]. \end{aligned}$$

We introduce the following sequence of "local" alternatives in order to analyze the limiting distribution of $Q_T(\lambda)$ when H_0 is false.

$$(4.16)\quad \begin{aligned} H_T\colon \gamma_n(\lambda) &= \gamma_n^{(T)}(\lambda) \equiv \gamma_0(\lambda) + T^{-1/2}\phi_n(\lambda), \qquad \phi_n(\lambda) \neq 0,\\ \Gamma_n(\lambda) &= \Gamma_n^{(T)}(\lambda) \equiv \Gamma_0(\lambda) + T^{-1/2}\psi_n(\lambda), \qquad n = 1,2,\dots,N. \end{aligned}$$

(4.16) allows the means and covariance matrices of the sensing functions to differ across epochs, but it requires the differences to vanish, as T increases, at a rate that assures a limiting distribution for the test statistic. This limiting behavior is expressed by the following theorem.

THEOREM 4.2. *Under the conditions for Theorem* 4.1 *and Lemma* 4.3 $Q_T(\lambda)$ *converges in distribution under* $\{H_T\}$ *to noncentral* $\chi^2(J \cdot K)$, *with noncentrality parameter*

$$(4.17)\qquad \xi(\lambda) \equiv [\Delta\Phi(\lambda)]'\{\Delta[I_N \otimes \Gamma_0(\lambda)]\Delta'\}^{-1}[\Delta\Phi(\lambda)] > 0,$$

where $\Phi(\lambda) \equiv \{\phi_1(\lambda)', \phi_2(\lambda)', \dots, \phi_N(\lambda)'\}'$. *Under* H_0, $Q_T(\lambda)$ *converges weakly to central* $\chi^2(J \cdot K)$.

PROOF. Under the sequence of alternatives $\{H_T\}$ we have $\mathscr{G}_T(\lambda) \to I_N \otimes \Gamma_0(\lambda)$ a.s. and $\lim_{T\to\infty} T^{1/2}\Delta E G_T(\lambda) = \Delta\Phi(\lambda)$. Theorem 4.1 and the Mann–Wald theorem imply that

$$T[\Delta G_T(\lambda)]'\{\Delta[I_N \otimes \Gamma_0(\lambda)]\Delta'\}^{-1}[\Delta G_T(\lambda)]$$

converges weakly to noncentral $\chi^2(J \cdot K)$, with noncentrality parameter $\xi(\lambda)$ given by (4.17). Under H_0, $\xi(\lambda) = 0$. The corresponding result for $Q_T(\lambda)$ follows from Lemma 4.3 and another application of Mann–Wald. □

5. The special cases. Here we develop some specific versions of the test by showing that sensing functions (3.3) and (3.4) satisfy condition (A.3), and we extend Theorem 4.2 to allow data-dependent λ in these special cases.

Consider first the mean-covariance test, based on

$$(3.3)\qquad g(X_t^n(L);\lambda) = \{X_t^n(L), [X_t^n(L) - \mu_n(L)]^2\}',$$

where $X_t^n(L)$ and $\mu_n(L)$ are given by (3.1) and (3.2). Since $X_t^n(L)$ is itself a stationary, Gaussian process, the cumulants $\kappa_{1111}^n(q, r, s; \lambda)$, which involve the first element of g only, are identically 0; (A.3) is, therefore, trivially met for $j = k = l = m = 1$. We will show that (A.3) holds also for $j = k = l = m = 2$; the general case can be handled similarly. Since, by Lemma 4.2, $X_t^n(L)$ satisfies (A.1), the necessary summability condition for the cumulants $\kappa_{2222}^n(q, r, s; \lambda)$ comes immediately from

LEMMA 5.1. *If* $\{X_t\}$ *is a stationary, Gaussian process satisfying* (A.1), *and if* $\{\kappa(q, r, s)\colon\ q, r, s = 0, \pm 1, \pm 2, \dots\}$ *represent the fourth-order cumulants of* $(X_t - \mu)^2$, *where* $\mu \equiv EX_0$, *then* $\sup_q \sum_r |\kappa(q, r, q + r)| < \infty$.

PROOF. Setting $\mu = 0$ and $\sigma(0) = 1$ without loss of generality, we have

$$\begin{aligned}\kappa(q,r,s) = {} & E\left[\left(X_0^2-1\right)\left(X_q^2-1\right)\left(X_r^2-1\right)\left(X_s^2-1\right)\right] \\ & -E\left[\left(X_0^2-1\right)\left(X_q^2-1\right)\right]E\left[\left(X_r^2-1\right)\left(X_s^2-1\right)\right] \\ & -E\left[\left(X_0^2-1\right)\left(X_r^2-1\right)\right]E\left[\left(X_q^2-1\right)\left(X_s^2-1\right)\right] \\ & -E\left[\left(X_0^2-1\right)\left(X_s^2-1\right)\right]E\left[\left(X_q^2-1\right)\left(X_r^2-1\right)\right].\end{aligned}$$

Multiplying out the expressions in brackets and evaluating moments of the form $E[X_0^{2j}X_q^{2k}X_r^{2l}X_s^{2m}]$, $j, k, l, m \in \{0,1\}$, by differentiating the joint moment generating function of X_0, X_q, X_r, X_s, we obtain

$$\begin{aligned}\kappa(q,r,q+r) = 16\big[& \sigma(q)^2\sigma(q+r)\sigma(q-r) \\ & +\sigma(r)^2\sigma(q+r)\sigma(q-r)+\sigma(q)^2\sigma(r)^2\big].\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_q\sum_r|\kappa(q,r,q+r)| & \le 16\sup_q\sum_r\left[|\sigma(q+r)|+|\sigma(q+r)|+|\sigma(r)|\right] \\ & = 48\sum_r|\sigma(r)| < \infty.\end{aligned}$$

□

With (A.3) verified, Lemma 4.3 furnishes a consistent estimator of $\Gamma_n(\lambda)$ in each epoch n; and if (A.4) holds, Theorem 4.2 gives the limiting distribution of $Q_T(\lambda)$ as $\chi^2(2K)$, $K \le N-1$. Since $\mu_n(L)$ in (3.2) depends on μ_n, which is ordinarily unknown, it will be necessary in practice to replace it by a statistic, $\hat{\mu}_n(L)$, such as $T_n^{-1}\sum_{t=L+1}^{T_n}X_t(L)$. We now show, under one further condition on the covariance estimator, that this change does not alter the large-sample theory of the test.

THEOREM 5.1. *Let $\Gamma_{T_n,n}$ be given by* (4.12) *but with the sequence of integers $\{M_{T_n}\}$ further restricted as*

$$M_{T_n} \equiv \max\{[a_1T_n^{a_2}], T_n-L-1\}, \qquad a_1>0,\ a_2\in(0,1/2). \tag{5.1}$$

Let $Q_T(\lambda)$ and $Q_T(\lambda_T)$ be constructed from, respectively, $g(X_t^n(L);\lambda)$ in (3.3) *and $g(X_t^n(L);\lambda_T) \equiv \{X_t^n(L),[X_t^n(L)-\hat{\mu}_n(L)]^2\}'$, where $\hat{\mu}_n(L) \equiv T_n^{-1}\sum_{t=L+1}^{T_n}X_t(L)$. Then under the remaining conditions for Theorem* 4.2, *the statistics $Q_T(\lambda_T)$ and $Q_T(\lambda)$ have the same limiting distributions as $T\to\infty$.*

The proof will follow easily from the next lemma.

LEMMA 5.2. *Under the conditions of Theorem* 5.1 $\Gamma_{T_n,n}(\lambda_T) \to_P \Gamma_n(\lambda)$.

PROOF. We show that when (5.1) holds each element of $R_{T_n} \equiv \Gamma_{T_n, n}(\lambda_T) - \Gamma_{T_n, n}(\lambda)$ converges in probability to 0. From (4.12) we have

$$R_{T_n} = \sum_{r=0}^{M_{T_n}} w(r/M_{T_n}) T_n^{-1} \sum_t \left[\hat{g}(X_t^n(L); \lambda_T)\hat{g}(X_{t+r}^n(L); \lambda_T)' - \hat{g}(X_t^n(L); \lambda)\hat{g}(X_{t+r}^n(L); \lambda)'\right].$$

Letting $R_{T_n}(j, k)$ denote the jkth element, for $j, k \in \{1, 2\}$, routine calculations give

$$R_{T_n}(1,1) = 0,$$

$$R_{T_n}(1,2) = R_{T_n}(2,1) = -2[\hat{\mu}_n(L) - \mu_n(L)] \sum_{r=0}^{M_{T_n}} S_n(r) + O_P(T_n^{-1}),$$

$$R_{T_n}(2,2) = -2[\hat{\mu}_n(L) - \mu_n(L)] \sum_{r=0}^{M_{T_n}} T_n^{-1} \times \sum_t \left\{[X_t^n(L) - \hat{\mu}_n(L)][X_{t+r}^n(L) - \hat{\mu}_n(L)]^2 + [X_t^n(L) - \mu_n(L)]^2[X_{t+r}^n(L) - \hat{\mu}_n(L)]\right\} + O_P(T_n^{-1}),$$

where $S_n(r)$ is the rth-order sample autocovariance of $X_t(L)$ in epoch n. Now $|R_{T_n}(1,2)| \le 2T_n^{-1/2}|\hat{\mu}_n(L) - \mu_n(L)|T_n^{1/2}M_{T_n}S_n(0) + o_P(1)$. Since $S_n(0)$ and the factor in brackets are $O_P(1)$, whereas $T_n^{1/2}M_{T_n} \to 0$ as $T_n \to \infty$ by (5.1), we conclude that $R_{T_n}(1,2) = o_P(1)$. The same conclusion holds a fortiori for $R_{T_n}(2,2)$, since $1/T_n$ times the second sum converges in probability to

$$E[X_t^n(L) - \mu_n(L)][X_{t+r}^n(L) - \mu_n(L)]^2 + E[X_t^n(L) - \mu_n(L)]^2[X_{t+r}^n(L) - \mu_n(L)] = 0. \qquad \square$$

PROOF OF THEOREM 5.1. We show that $|Q_T(\lambda_T) - Q_T(\lambda)| \to_P 0$. For brevity let $A_T \equiv \Delta G_T(\lambda_T)$, $A \equiv \Delta G_T(\lambda)$, $B_T \equiv [\Delta \mathscr{G}_T(\lambda_T)\Delta']^+$, $B \equiv [\Delta \mathscr{G}_T(\lambda)\Delta']^+$ and $\beta \equiv \{\Delta[I_N \otimes \Gamma_0(\lambda)]\Delta'\}^{-1}$. Then

$$\begin{aligned}|Q_T(\lambda_T) - Q_T(\lambda)| &= T|A_T'B_TA_T - A'BA| \\ &\le T|A_T'\beta A_T - A'\beta A| + T|A_T'(B_T - \beta)A_T| + T|A'(B - \beta)A|.\end{aligned}$$

Under the sequence of alternatives $\{H_T\}$ we have $T^{1/2}A = O_P(1)$ and $T^{1/2}A_T = O_P(1)$. Since $B \to \beta$ a.s. and $B_T \to_P \beta$ under Lemmas 4.3 and 5.2, respectively, it follows that the last two terms are $O_P(1)$. For the first term we have

$$T|A_T'\beta A_T - A'\beta A| \le T|(A_T - A)'\beta A_T| + T|A'\beta(A_T - A)|.$$

That this converges in probability to 0 will follow at once if we can show that

$T^{1/2}(A_T - A) \equiv T^{1/2}\Delta[G_T(\lambda_T) - G_T(\lambda)] = o_P(1)$. Since the nth component of $T^{1/2}[G_T(\lambda_T) - G_T(\lambda)]$ is

$$\begin{aligned} &T^{1/2}\left[g_{T_n}(\lambda_T) - g_{T_n}(\lambda)\right] \\ &= T^{1/2}\left\{0, T_n^{-1}\sum_t\left[X_t(L) - \hat{\mu}_n(L)\right]^2 - T_n^{-1}\sum_t\left[X_t(L) - \mu_n(L)\right]^2\right\} \\ &= \left\{0, -T^{1/2}\left[\hat{\mu}_n(L) - \mu_n(L)\right]^2\right\} \to_P 0, \end{aligned}$$

the conclusion does follow. □

We turn now to the test involving differences among sample characteristic functions of the various epochs, which is based on the sensing function

$$g(X_t^n(L);\lambda) = \left\{\cos\lambda_1 X_t^n(L), \sin\lambda_1 X_t^n(L),\ldots, \cos\lambda_{J/2}X_t^n(L), \sin\lambda_{J/2}X_t^n(L)\right\}'. \tag{3.4}$$

Here J is an even, positive integer; $X_T^n(L) \equiv X_t^n + \lambda_{01}X_{t-1}^n + \cdots + \lambda_{0L}X_{t-L}^n$; and $\lambda \in \mathbb{R}_{L+1+J/2}$ comprises the vector $\lambda_0 \equiv \{1, \lambda_{01}, \ldots, \lambda_{0L}\}$ and the scalars $\{\lambda_j\}_{j=1}^{J/2}$. To rule out obvious violations of (A.4), we require that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{J/2}. \tag{5.2}$$

Under these conditions the following lemma, proved in Epps (1987), establishes (A.3).

LEMMA 5.3. *If* $\{X_t\}$ *is a stationary, Gaussian process satisfying* (A.1) *and if* $g(X_t(L);\lambda)$ *is given by* (3.4), *then the fourth-order cumulants* $\{\kappa_{jklm}(q,r,s;\lambda),\ j,k,l,m \in \{1,2,\ldots,J\},\ q,r,s = 0,\pm1,\pm2,\ldots\}$ *satisfy*

$$\sup_{-\infty<q<\infty}\textstyle\sum_{r=-\infty}^{\infty}|\kappa_{jklm}(q,r,q+r;\lambda)| < \infty.$$

Assuming that (A.4) holds, the limiting distribution of $Q_T(\lambda)$ now follows from Theorem 4.2.

Similar quadratic forms in elements of the empirical characteristic function have been widely applied in estimation and, especially, in testing goodness of fit; see, for example, Feuerverger and McDunnough (1981), Koutrouvelis (1980), Koutrouvelis and Kellermeier (1981), Epps and Singleton (1986) and Epps (1987). Although these procedures are relatively simple and have considerable intuitive appeal, their obvious disadvantage is the need to select a priori the arguments of the characteristic function. In the present context these arguments are the points in $\mathbb{R}_{L+1}$ represented by products of the vector λ_0 with the scale factors $\{\lambda_j\}_{j=1}^{J/2}$. Practical considerations do, however, eliminate some of the ambiguity in choosing J and the elements of λ. For one thing, J must be fairly small if the matrices $\{\Gamma_{T_n,n}(\lambda)\}$ are to be inverted; and, fortunately, sampling experiments in other applications have shown that good results can usually be obtained with J as small as 4, and often as small as 2. For the same reason the distance $|\lambda_0|\,|\lambda_j - \lambda'_j|$ between any pair of arguments must not be too small, relative to the scale of the data. Of course, since the scale of the data can usually be determined only from the sample, the last requirement complicates the distribution theory of the test by making the arguments data dependent.

The next result shows that, under certain conditions, such dependence on the sample does not alter the limiting distribution of $Q_T(\lambda)$. Since the main interest is in scaling the arguments of the characteristic function, we allow only the scale factors $\lambda_1, \lambda_2, \ldots, \lambda_{J/2}$ to be sample dependent.

THEOREM 5.2. *Let* $\lambda \equiv \{\lambda_0, \lambda_1, \ldots, \lambda_{J/2}\}$ *be a vector of constants satisfying* (5.2) *and choose* $\lambda_T \equiv \{\lambda_0, \hat{\lambda}_1, \ldots, \hat{\lambda}_{J/2}\}$, *where the statistics* $\{\hat{\lambda}_j\}$ *satisfy*

$$(5.3) \qquad |\hat{\lambda}_j - \lambda_j| = O_P(T^{-1/2}) \quad \text{as } T \to \infty,\ j \in \{1, 2, \ldots, J/2\},$$

$$(5.4) \qquad P\left\{0 < \hat{\lambda}_1 < \hat{\lambda}_2 < \cdots < \hat{\lambda}_{J/2} \le C\right\} = 1, \qquad T = 1, 2, \ldots,$$

for some finite C. If $Q_T(\lambda)$ *and* $Q_T(\lambda_T)$ *are constructed from* (3.4) *with* $\Gamma_{T_n, n}$ *and* M_{T_n} *as in Theorem* 5.1 *and if the remaining conditions of Theorem* 4.2 *hold, then* $Q_T(\lambda_T)$ *and* $Q_T(\lambda)$ *have the same limiting distribution as* $T \to \infty$.

PROOF. Conditions (5.3) and (5.4) satisfy assumptions (A.8) in Epps (1987). Lemmas 3.2 and 3.3 in Epps (1987) then assure the consistency of $\Gamma_{T_n, n}(\lambda_T)$ and the weak convergence of the normalized vectors $g_{T_n}(\lambda)$ on the space of continuous functions on $[0, C]$. With these results the extension of the present Theorem 4.2 to λ_T follows along the lines of the proof of Theorem 3.1 in Epps (1987). □

As we have seen, the elements of $g_{T_n}(\lambda)$ in each epoch n are components of the empirical characteristic function of $X_t^n(L)$. The purpose of the a priori bound C in (5.4) is to limit the domain of the characteristic function to a bounded interval in $\mathbb{R}_1$, which is required for the weak convergence of the normalized $g_{T_n}(\lambda)$ [Epps (1987), Lemma 3.3, Remark 2].

As an example of a choice of λ_T which satisfies (5.3) and (5.4) and provides the desired scale adjustment, pick constants $\{\lambda'_j\}_{j=1}^{J/2}$ satisfying (5.2), and some small $\varepsilon > 0$. With S_1 representing the sample standard deviation of the data from epoch 1, let

$$\hat{\lambda}_j \equiv \min\{\lambda'_j/\varepsilon, \lambda'_j/S_1\}, \qquad j \in \{1, 2, \ldots, J/2\}.$$

The $\hat{\lambda}_j$ are now ordered and bounded with probability 1, thus satisfying (5.4) with $C = \lambda'_{j/2}/\varepsilon$. Letting

$$\lambda_j \equiv \min\{\lambda'_j/\varepsilon, \lambda'_j/\sigma_1\},$$

(5.3) follows from (A.1) and Theorem 8.3.3 of Anderson (1971), page 465 ff.

6. Monte Carlo experiments. As a preliminary study of the finite-sample performance of the stationarity test, simulation was used to estimate type I errors and powers of tests based on two versions of each of sensing functions (3.3) and (3.4). These are

1. marginal mean-variance test (MMV), based on (3.3) with $L = 0$ and $\hat{\mu}_n(L) = \hat{\mu}_n \equiv \overline{X}_n = T_n^{-1}\Sigma_t X_t^n$;
2. bivariate mean-variance test (BMV), based on (3.3) with $L = 1$, $X_t^n(L) = X_t^n - 1.25X_{t-1}^n$, $\hat{\mu}_n(L) = -0.25\overline{X}_n$;
3. marginal characteristic-function test (MCF), based on (3.4) with $L = 0$, $J = 4$, $\hat{\lambda}_1 = 0.5/S_1$, $\hat{\lambda}_2 = 1.0/S_1$;
4. bivariate characteristic-function test (BCF), based on (3.4) with $L = 1$, $X_t^n(L) = X_t^n - 1.25X_{t-1}^n$ and J and $\{\hat{\lambda}_1, \hat{\lambda}_2\}$ as in the MCF test.

The choice of $X_t^n(L)$ in the two bivariate tests was made experimentally. "Prewhitening" the data with the linear filter $X_t - 1.25X_{t-1}$ turns out to improve the accuracy of the test while preserving some sensitivity to a change in mean. The values of J and $\{\hat{\lambda}_1, \hat{\lambda}_2\}$ in the MCF and BCF tests are based on extensive simulations reported in Epps and Singleton (1986), which introduced a characteristic-function test for the classical two-sample problem with i.i.d. samples. Considering both power and computation, the best overall results were obtained in the i.i.d. case with the characteristic function evaluated at $J/2 = 2$ points, corresponding approximately to the values of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ given previously.

The following results are for $N = 2$ epochs. Parameters a_1 and a_2 in the truncation function M_{T_n} in (4.10) were set equal to 1.0 and 0.4, respectively. The present study is "preliminary" in the sense that we have not examined extensively the effects of varying these and other parameters. Results were obtained for epochs of length $(T_1, T_2) = (50, 50)$, $(50, 100)$, $(100, 100)$, $(100, 200)$ and $(200, 200)$, but for brevity only those for $(50, 50)$ and $(200, 200)$ are reported. For the same reason estimates of type I errors and powers are given for level 0.05 only.

Table 1 presents the estimates of type I errors from simulations of 500 trials, with standard errors of roughly 1%. Nine different stationary models were used

TABLE 1

Monte Carlo estimates (500 *replications*) *of* 5%*-level type I errors of* MMV, BMV, MCF *and* BCF *tests* (*in percent*) *for epochs of equal length*, 50 *and* 200

Model for epochs	MMV		BMV		MCF		BCF	
	50	200	50	200	50	200	50	200
IID	6.2	6.2	2.6	1.0	9.4	6.0	2.6	1.6
AR(0.2)	9.6	6.8	3.6	1.6	11.2	8.0	2.8	3.0
AR(0.8)	34.2	19.4	6.2	5.6	27.4	16.2	5.4	5.4
AR(−0.2)	5.2	5.2	2.6	1.0	8.0	5.0	2.2	1.8
AR(−0.8)	10.8	6.6	7.8	5.0	9.2	5.8	4.4	3.4
MA(0.2)	8.8	7.0	3.8	1.6	10.2	8.2	2.4	3.2
MA(0.8)	10.8	8.2	3.6	2.2	9.6	8.4	3.2	3.8
MA(−0.2)	5.2	5.4	2.8	1.0	7.8	5.2	2.4	1.8
MA(−0.8)	4.0	1.2	2.4	1.0	5.0	1.8	2.6	1.4

TABLE 2
Monte Carlo estimates (100 replications) of 5%-level powers of BMV *and* BCF *tests (in percent) for epochs of equal length,* 50 *and* 200

Models for epochs		BMV		BCF	
Type of nonstationarity	**Common features**	**50**	**200**	**50**	**200**
Mean shift					
$\mu \| \mu + \sigma$	AR(0.8), σ	35	82	21	72
$\mu \| \mu + \sigma$	MA(0.8), σ	13	100	9	98
Variance shift					
$\sigma \| 1.5\sigma$	AR(0.8), μ	64	100	49	100
$\sigma \| 1.5\sigma$	MA(0.8), μ	56	99	41	98
ARMA structure					
AR(0.8)|AR(0.2)	μ	100	100	91	100
AR(0.8)|MA(0.8)	μ	100	100	94	100
MA(0.2)|MA(0.8)	μ	3	4	0	6
Law of errors[a]					
$N(0,1)$|Student $t(3)$	AR(0.8), μ, σ	16	16	20	54
$N(0,1)$|Student $t(3)$	MA(0.8), μ, σ	19	11	14	47
$N(0,1)$|beta (2, 2)	AR(0.8), μ, σ	7	3	4	6
$N(0,1)$|beta (2, 2)	MA(0.8), μ, σ	2	2	2	3
Unit roots					
	AR1(1.0)	67	85	51	79
	AR2(2/3, 1/3)	57	83	50	79
Mean variance drift[b]					
$\mu \to \mu + 2\sigma$	AR(0.8), σ	10	80	6	71
$\mu \to \mu + 2\sigma$	MA(0.8), σ	4	100	1	96
$\sigma \to 3\sigma$	AR(0.8), μ	17	100	10	100
$\sigma \to 3\sigma$	MA(0.8), μ	9	100	6	100

[a] Student and beta variates standardized to have zero mean and unit variance.
[b] Linear drift over observations $t = 21, 22, \ldots, 420$: $\mu_t = 2\sigma(t - 21)/400$, $\sigma_t = [1 + 2(t - 21)/400]$.

to generate the epoch samples. Letting $\{U_t\}$ represent a sequence of i.i.d. standard-normal variates, the models are (1) IID: $X_t = U_t$; (2) AR(p): $X_t = pX_{t-1} + U_t$, $p = \pm 0.2, \pm 0.8$ and (3) MA(p): $X_t = pU_{t-1} + U_t$, $p = \pm 0.2, \pm 0.8$. In each case 420 observations were generated and the first 20 discarded. Epoch 1 comprises the T_1 observations ending with number 220; epoch 2, the T_2 observations beginning with number 221.

Table 1 shows that the MMV and MCF tests can be badly excessive, even with samples as large as 200. This is particularly noteworthy when the data are autoregressive with p near 1.0. By contrast, the two "bivariate" tests appear to be reasonably accurate, if somewhat conservative, even for samples as small as 50. The same conclusions hold for other epoch lengths and significance levels not reported here. It is clear that the accuracy of the tests is improved by the "prewhitening" that takes place in the bivariate versions.

Table 2 presents 100 trial estimates of the powers of the two bivariate tests to detect six different forms of nonstationarity: (1) shifts in mean; (2) shifts in variance; (3) changes in ARMA structure; (4) changes in the probability law of the ARMA errors $\{U_t\}$; (5) unit roots in AR models and (6) drift in mean or variance. The last three of these violate the assumptions under which the large-sample theory shows the tests to be consistent. Nevertheless, both tests are rather sensitive to the presence of unit roots and to drifts in mean or variance. Although the BCF test has some ability to detect a change in law from the normal to the thick-tailed Student $t(3)$, it seems unable to distinguish Gaussian from thin-tailed processes, such as beta $(2,2)$ and the truncated normal (not shown). In detecting shifts in mean or variance and changes in model structure the BMV test is generally more sensitive than the BCF test, but both seem generally quite satisfactory. In the form tested here neither has power to distinguish MA(0.2) from MA(0.8), because the marginal distributions of the filtered processes $\{X_t^n - 1.25X_{t-1}^n\}$, $n = 1, 2$, are nearly identical: $N(0, 2.1650)$ and $N(0, 2.2025)$.

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