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A NOTE ON THE VARIANCE OF A STOPPING TIME[1]

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Let $\{S_n = \Sigma_1^n X_i\}_{n \geq 0}$ be a random walk with positive drift $\mu = EX_1 > 0$ and finite variance $\sigma^2 = \operatorname{Var} X_1$. Let $\tau(b) = \inf\{n \geq 1\colon\ S_n > b\}$, $R_b = S_{\tau(b)} - b$, $M = \min_{n \geq 0} S_n$, $\tau^+ = \tau(0)$ and $H = S_{\tau^+}$. Lai and Siegmund show that $\operatorname{Var} \tau(b) = b\sigma^2/\mu^3 + K/\mu^2 + o(1)$ as $b \to \infty$, but give an unpleasant expression for the constant K. Using the identity $\int Eh(R_{-y})\, dP(M \leq y) = E\tau^+ h(H)/E\tau^+$, the expression for K can be simplified to a form that depends only on moments of ladder variables.

Let $X, X_1, X_2, \dots,$ be i.i.d. with mean $\mu > 0$ and finite positive variance σ^2. Let $S_0 = 0$, $S_n = \Sigma_{i=1}^n X_i$ and define $\tau(b) = \inf\{n\colon S_n > b\}$, $R_b = S_{\tau(b)} - b$, $M = \min_{n \geq 0} S_n$, $\tau^+ = \tau(0)$ and $H = S_{\tau^+}$. Lai and Siegmund [(1979), Theorem 5] show that if X is nonlattice and $E(X^+)^3 < \infty$, then

$$\operatorname{Var} \tau(b) = \frac{b\sigma^2}{\mu^3} + \frac{K}{\mu^2} + o(1)$$

as $b \to \infty$. The expression they give for the key constant K is

$$K = \frac{\sigma^2 EH^2}{2\mu EH} + \frac{3}{4}\left(\frac{EH^2}{EH}\right)^2 - \frac{2}{3}\frac{EH^3}{EH} - \frac{EH^2 EM}{EH} - 2\int_0^\infty ER_x P(M \leq -x)\, dx.$$

Similar constants arise in the expansions for the variance of stopping times for curved boundaries given by Zhang (1984). Since moments of the ladder height H can be obtained from recursions or quadrature formulas derived from Spitzer's (1966) identity [see Chapter 2 of Woodroofe (1982)] and $EM = EH^2/(2EH) - EX^2/(2\mu)$, the most troublesome term when computing K is the integral involving ER_x. Using the following theorem, this integral can be expressed in terms of ladder variables and EM as

$$\text{(1)} \qquad \int_0^\infty ER_x P(M \leq -x)\, dx = -\frac{EMEH^2}{2EH} + \frac{EH^2 EH\tau^+}{2EHE\tau^+} - \frac{EH^2\tau^+}{2E\tau^+}$$

and the constant K then simplifies to

$$K = \frac{\sigma^2 EH^2}{2\mu EH} + \frac{3}{4}\left(\frac{EH^2}{EH}\right)^2 - \frac{2}{3}\frac{EH^3}{EH} - \frac{EH^2 EH\tau^+}{EHE\tau^+} + \frac{EH^2\tau^+}{E\tau^+}.$$

Let G denote the distribution of M.

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THEOREM 1. *If $h \geq 0$, then*

$$\int Eh(R_{-y})\,dG(y) = \frac{E\tau^{+}h(H)}{E\tau^{+}}.$$

This identity in the special case where $h(x) = x^p$ appears as equation (23) of Siegmund (1979) or equation (10.34) of Siegmund (1985). The proof given here uses the same argument.

To obtain (1) from this result, define the ladder epochs $\tau_0^+ = 0$, $\tau_1^+ = \tau^+$ and $\tau_{n+1}^+ = \inf\{k > \tau_n^+ : S_k > S_{\tau_n^+}\}$ for $n \geq 1$. For $n \geq 0$, let $H_{n+1} = S_{\tau_{n+1}^+} - S_{\tau_n^+}$, so $H_1, H_2, \dots$ are i.i.d. and $H_1 = H$. Let $\tilde{\tau}(z) = \inf\{n: \sum_{i=1}^n H_i > z\}$, so $\tau(z) = \tau_{\tilde{\tau}(z)}^+$ and $\sum_{n=1}^{\tilde{\tau}(z)} H_n = z + R_z$. The following identity is due to Lorden (1970):

$$2\int_0^z R_x\,dx = -R_z^2 + \sum_{n=1}^{\tilde{\tau}(z)} H_n^2.$$

By Wald's identities,

$$E\sum_{n=1}^{\tilde{\tau}(z)} H_n^2 = E\tilde{\tau}(z)EH^2$$

and

$$E\tilde{\tau}(z) = \frac{z + ER_z}{EH}.$$

So

$$\int_0^z ER_x\,dx = \frac{zEH^2}{2EH} + \frac{EH^2ER_z}{2EH} - \frac{1}{2}ER_z^2.$$

By Fubini's theorem and Theorem 1,

$$\begin{aligned}\int_0^\infty ER_xP(M \leq -x)\,dx &= \iint_{0 \leq x \leq -y} ER_x\,dG(y)\,dx\\ &= \int\left[-\frac{yEH^2}{2EH} + \frac{EH^2ER_{-y}}{2EH} - \frac{1}{2}ER_{-y}^2\right]dG(y)\\ &= -\frac{EMEH^2}{2EH} + \frac{EH^2EH\tau^{+}}{2EHE\tau^{+}} - \frac{EH^2\tau^{+}}{2E\tau^{+}}.\end{aligned}$$

Theorem 1 will be proved using the following lemma. The proof is based on the duality principle, and the lemma is a corollary of various standard results, such as Lemma 18.4.1 of Feller (1971) or Theorem 2.7 of Woodroofe (1982). $E[Y; A]$ denotes $EY1_A$.

LEMMA 2. *If $h \geq 0$, then*

$$Eh(M) = \frac{1}{E\tau^{+}}\sum_{n=0}^{\infty} E[h(S_n); S_1 \leq 0, \dots, S_n \leq 0].$$

PROOF. Let $J = \sup\{n \geq 0: S_n = M\}$. Then

$$\{J = j\} = \{S_k \geq S_j, 0 \leq k < j, S_k > S_j, k > j\}$$

and

$$\begin{aligned} E[h(M); J = j] &= E\big[h(S_j); S_k \geq S_j, 0 \leq k < j, S_k - S_j > 0, k > j\big] \\ &= E\big[h(S_j); S_k \geq S_j, 0 \leq k < j\big] P(S_n > 0, n \geq 1) \\ &= E\big[h(S_j); S_k \leq 0, 1 \leq k \leq j\big] P(\tau^- = \infty), \end{aligned}$$

where $\tau^- = \inf\{n: S_n \leq 0\}$ and the last equality follows by reversing $X_1, \dots, X_n$. Lemma 2 follows by summing over j and using the identity $P(\tau^- = \infty) = 1/E\tau^+$. □

PROOF OF THEOREM 1. First note that for any $n \geq 0$ and $y \leq 0$,

$$\begin{aligned} Eh(R_{-y}) &= \sum_{m=1}^{\infty} E\big[h(R_{-y}); \tau(-y) = m\big] \\ &= \sum_{m=1}^{\infty} E[h(S_m + y); S_k \leq -y, 1 \leq k < m, S_m > -y] \\ &= \sum_{m=1}^{\infty} E[h(S_{m+n} - S_n + y); \\ &\qquad S_k \leq -y + S_n, n < k < n + m, S_{n+m} > -y + S_n]. \end{aligned}$$

Using Lemma 2,

$$\begin{aligned} \int_{-\infty}^{0} Eh(R_{-y})\, dG(y) &= \frac{1}{E\tau^+} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} E[h(S_{m+n}); \\ &\qquad S_k \leq 0, 1 \leq k < m + n, S_{m+n} > 0] \\ &= \frac{1}{E\tau^+} \sum_{n=0}^{\infty} nE[h(S_n); S_k \leq 0, 1 \leq k < n, S_n > 0] \\ &= \frac{1}{E\tau^+} E\tau^+ h(H). \end{aligned}$$

□

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