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A NOTE ON THE BUYER'S PROBLEM

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Bather considered a buyer's problem and reduced it to an optimal stopping problem. In this note it is shown that this optimal stopping problem can be related to the sequential testing of the sign of the drift parameter of a Wiener process problem considered by Chernoff. Hence the asymptotic expansions for the sequential testing problem can be used in the buyer's problem.

1. Introduction. Bather (1983) considered a buyer's problem that he reduced to the following optimal stopping problem. One observes a standard Wiener process $Y(s)$ in $-s$ scale, i.e., $E(dY(s)) = 0$ and $V(dY(s)) = -ds$. If one stops at $Y(s) = y$, one's reward function is

$$r(y, s) = |y| - s^{-1}. \tag{1.1}$$

One has to find a stopping rule τ_0 that maximizes the expected reward, i.e.,

$$\hat{r}(y, s, \tau_0) = \sup_{\tau} Er(y + Y(\tau), s - \tau). \tag{1.2}$$

Bather showed that the optimal boundary $\tilde{y}(s)$ corresponding to τ_0 is monotone and gave an approximation for $\tilde{y}(s)$ when $s \to 0$ and $s \to \infty$.

In this note it is shown that the preceding optimal stopping problem can be related to the Bayes sequential testing problem for the sign of the drift parameter of a Wiener process considered by Chernoff (1961, 1964, 1965, 1972). Hence the asymptotic expansions of the optimal boundary $\tilde{y}(s)$ and the optimal reward $\hat{r}(y, s)$ are obtained by using the asymptotic expansions for the Bayes sequential testing problem in Breakwell and Chernoff (1964) and Chernoff (1965).

2. Relation between the Bayes sequential testing problem and the buyer's problem. Chernoff (1961) considered a Bayes sequential problem for testing the sign of the drift parameter of a Wiener process. He showed that this problem can be reduced to the following optimal stopping problem. We observe a standard Wiener process $Y(s)$ in $-s$ scale. If we stop at $Y(s) = y$, our loss is

$$d(y, s) = s^{-1} + s^{1/2}[\phi(\alpha) - |\alpha|(1 - \Phi(|\alpha|))], \tag{2.1}$$

where

$$\alpha = \frac{y}{s^{1/2}}, \qquad \phi(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2}, \qquad \Phi(\alpha) = \int_{-\infty}^{\alpha} \phi(x)\, dx.$$

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We want to find a stopping time τ_0 such that it minimizes the expected loss, i.e.,

$$\hat{d}(y, s) = \inf_{\mu_n} E(d(y + Y(\tau), s - \tau)). \tag{2.2}$$

Breakwell and Chernoff (1964) and Chernoff (1965) obtained the asymptotic expansions for the optimal boundary and minimum expected loss for $s \to 0$ and $s \to \infty$. Let us take the symmetric solution of the heat equation,

$$U(y, s) = s^{1/2}\left[\phi(\alpha) - \tfrac{1}{2}\alpha + \alpha\Phi(\alpha)\right]. \tag{2.3}$$

If we subtract $U(y, s)$ from $d(y, s)$, the optimal boundary will not change. For $y > 0$,

$$d(y, s) - U(y, s) = s^{-1} - \tfrac{1}{2}y. \tag{2.4}$$

For $y < 0$,

$$d(y, s) - U(y, s) = s^{-1} + \tfrac{1}{2}y. \tag{2.5}$$

Hence from (2.4) and (2.5) we have

$$d(y, s) - U(y, s) = s^{-1} - \tfrac{1}{2}|y|. \tag{2.6}$$

Let

$$P(y, s) = d(y, s) - U(y, s), \tag{2.7}$$

$$Y^*(s^*) = ay(s), \tag{2.8}$$

$$s^* = a^2 s. \tag{2.9}$$

It is well known that $Y^*(s^*)$ is a standard Wiener process in $-s^*$ scale. From (2.6)–(2.9) and choosing $a = 2^{-1/3}$ we get

$$P(y^*, s^*) = s^{-2/3}P^*(y^*, s^*), \tag{2.10}$$

where

$$P^*(y^*, s^*) = s^{*-1} - |y|^*.$$

Bather's (1983) reward function $r(y, s)$ in (1.1) is the same as $-P^*(y^*, s^*)$. Hence from the optimal stopping boundary for the Bayes sequential problem in Breakwell and Chernoff (1964), Chernoff (1965, 1972) and (2.6)–(2.10), one can get the following asymptotic expansions for $\tilde{y}(s)$ and $\hat{r}(y, s)$:

$$\tilde{\alpha}(s) = \tilde{y}(s)/s^{1/2} \approx \left[\log s^3 - \log 2\pi - 6\left(\log(4s^3)\right)^{-1} + \cdots\right]^{-1}, \tag{2.11}$$

$$\tilde{\alpha}(s) = \tilde{y}(s)/s^{1/2} \approx \tfrac{1}{2}s^{3/2}\left[1 - \tfrac{1}{3}s^3 + \tfrac{7}{15}s^6 - \cdots\right] \quad \text{as } s \to 0, \tag{2.12}$$

$$\hat{r}(y, s) \approx Ks^{-1/2}\phi(y/s^{1/2}) - 2^{2/3}U(y, s) \quad \text{as } s \to \infty, \tag{2.13}$$

$$\hat{r}(y, s) \approx s^{2/3}\left(s^{-1} - U(y, s)\right) \quad \text{as } s \to 0, \tag{2.14}$$

where $U(y, s)$ is as in (2.3).

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