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CONVERGENCE RATES FOR THE BOOTSTRAPPED PRODUCT-LIMIT PROCESS

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We establish rates for strong approximations of the bootstrapped product-limit process and the corresponding quantile process. These results are used to show weak convergence of bootstrapped total time on test and Lorenz curve processes to the same limiting Gaussian processes as for the unbootstrapped versions. We develop fully nonparametric confidence bands and tests for these curves and apply these results to prostate cancer. We also present almost sure results for the bootstrapped product-limit estimator.

1. Introduction. We determine convergence rates for strong approximations of the bootstrapped product-limit process and the uniform product-limit quantile process and present almost sure results for the bootstrapped product-limit estimator. We prove the weak convergence of bootstrapped versions of the total time on test and Lorenz curve processes to the same limiting Gaussian processes as were found by Csörgő, Csörgő and Horváth (1986b) for the unbootstrapped case. This allows us to develop fully nonparametric confidence bands and tests for these curves. We apply these results to data from studies of prostate cancer and pacemakers.

Asymptotic properties of bootstrapped (uncensored) empirical processes were first studied by Bickel and Freedman (1981). Horváth and Yandell (1985) derived strong approximations of the bootstrapped multidimensional product-limit process. Akritas (1986) independently developed asymptotic results for the bootstrapped product-limit estimator using martingale theory.

In this section, we briefly review results for the product-limit process and the uniform product-limit quantile function. Section 2 presents the bootstrapped versions of the processes and our main results. The total time on test transform and the Lorenz curve are introduced in Section 3. Bootstrapped confidence bands are illustrated for both these curves in Section 4, for data on prostate cancer. Section 5 establishes preliminary results which are used in Sections 6–8 for the proofs.

Let $\{X_i^0,\ i \geq 1\}$ be a sequence of independent, identically distributed random variables (iidrv's) with continuous survival function $S^0(t) = P\{X_1^0 > t\}$ and distribution function $F^0(t) = 1 - S^0(t)$. Denote the quantile of F^0 by $Q^0(u) =$

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$\inf\{t\colon F^0(t) \geq u\}$. Let $\{Y_i, i \geq 1\}$, be an iid sequence of rv's, independent of $\{X_i^0, i \geq 1\}$, with "censoring" survival function $G(t) = P\{Y_1 > t\}$. We can only observe the random vectors $\{X_i, \delta_i\}_{i=1}^n$, where $X_i = \min\{X_i^0, Y\}$ and $\delta_i = 1$ if $X_i = X_i^0$, $\delta_i = 0$ if $X_i < X_i^0$. By the independence of $\{X_i^0\}$ and $\{Y_i\}$ $S(t) = 1 - F(t) = P\{X_1 > t\} = S^0(t)G(t)$.

Kaplan and Meier (1958) proposed the product-limit estimator of S^0,

$$\hat{S}_n^0(t) = 1 - \hat{F}_n^0(t) = \prod_{\{i\colon X_{i,n} \leq t\}} \left(\frac{n-i}{n-i+1}\right)^{\delta_{i,n}}, \quad \text{if } t < X_{n,n},$$

and $\hat{S}_n^0(t) = 0$ if $t \geq X_{n,n}$, where $X_{1,n} \leq \cdots \leq X_{n,n}$ are the order statistics of $\{X_i\}_{i=1}^n$, and $\{\delta_{i,n}\}_{i=1}^n$ are the induced order statistics of $\{\delta_i\}_{i=1}^n$. Denote the product-limit process by

$$\gamma_n(t) = n^{1/2}\big(\hat{S}_n^0(t) - S^0(t)\big).$$

Aalen (1976) and Breslow and Crowley (1974) proved the weak convergence of $\gamma_n(t)$ in $\mathscr{D}(-\infty, T]$, $S(T) > 0$. Burke, Csörgő and Horváth (1981, 1986) studied the speed of the weak convergence of γ_n. Csörgő, Csörgő and Horváth (1986b) showed the weak convergence of γ_n in weighted metrics. They considered the integral

$$I(q,c) = \int_0^p t^{-1}\exp\big(-cq^2(t)/t\big)\,dt,$$

in which $q \in K(p)$, the class of weight functions on $(0, p]$ such that $\inf_{\varepsilon \leq t \leq p} q(t) > 0$ for all $0 < \varepsilon < p$ and q is nondecreasing in a neighborhood of zero. Without loss of generality, we can assume that the underlying probability space is so rich that it accommodates all the random variables and processes introduced so far and later on. We formulate the strong approximation results of Burke, Csörgő and Horváth (1981, 1986) and Csörgő, Csörgő and Horváth (1986b) in the following:

THEOREM A. *Let $S(T) > 0$. We can define a sequence of Wiener processes $\{W_n(t), t \geq 0\}_{n=1}^\infty$ such that*

$$(1.1) \qquad P\Big\{\sup_{-\infty < t \leq T} \big|\gamma_n(t) - S^0(t)W_n(d(t))\big| \geq A_1 n^{-1/2}\log n\Big\} \leq B_1 n^{-\varepsilon},$$

with

$$d(t) = \int_{-\infty}^t [S(s)]^{-2}\,dF^0(s),$$

for all $\varepsilon > 0$, where $A_1 = A_1(\varepsilon)$ and B_1 are constants. Further, if $q \in K(F^0(T))$, then

$$(1.2) \qquad \sup_{-\infty < t \leq T} \big|\gamma_n(t) - S^0(t)W_n(d(t))\big|\big/q\big(F^0(t)\big) \to_{\mathbf{P}} 0,$$

if and only if $I(q,c) < \infty$ for all $c > 0$.

We can transform $X_{1,n}^0 \le \cdots \le X_{r_n,n}^0$, the r_n ordered uncensored observations, into censored uniform order statistics. Let

$$(1.3)\quad E_n(u) = \begin{cases} F^0(X_{1,n}^0), & \text{if } 0 \le u \le \hat{F}_n^0(X_{1,n}^0), \\ F^0(X_{i,n}^0), & \text{if } \hat{F}_n^0(X_{i-1,n}^0) < u \le \hat{F}_n^0(X_{i,n}^0), \quad 2 \le i \le r_n, \\ F^0(X_{r_n,n}^0), & \text{if } \hat{F}_n^0(X_{r_n,n}^0) < u \le 1 \end{cases}$$

denote the uniform product-limit quantile function and let

$$\beta_n(u) = n^{1/2}(E_n(u) - u)$$

be the uniform product-limit quantile process. Aly, Csörgő and Horváth (1985) proved the following:

THEOREM B. *Let $S(Q^0(p)) > 0$. Then we have that*

$$(1.4)\quad P\left\{ \sup_{0 \le u \le p} \left|\beta_n(u) - (1-u)W_n(d(Q^0(u)))\right| \ge A_2 n^{-1/4}(\log n)^{3/4} \right\} \le B_2 n^{-\varepsilon},$$

for all $\varepsilon > 0$, where $A_2 = A_2(\varepsilon)$ and B_2 are constants. Further, if $q \in K(p)$ and $I(q,c) < \infty$ for all $c > 0$, then

$$(1.5)\quad \sup_{1/(n+1) \le u \le p} \left|\beta_n(u) - (1-u)W_n(d(Q^0(u)))\right| \Big/ q(u) \to_{\mathbf{P}} 0.$$

In the applications of Theorems A and B it is very essential that we have used the *same* Wiener processes. We can show that the limit of a transformation of γ_n or β_n is a time-transformed Wiener process, or Brownian bridge. The distributions of the supremum and square-integral of the obtained time-transformed processes could be calculated. This idea was introduced by Efron (1967) and later used by Aalen (1976), Nair (1981), Gillespie and Fisher (1979) and Hall and Wellner (1980). A very general form of this transformation method can be found in Csörgő and Horváth (1986).

The weak convergence of γ_n and β_n in weighted metrics was used by Csörgő, Csörgő and Horváth (1986b) to study the asymptotic behavior of the total time on test transforms and Lorenz curves under random censorship. They proved the weak convergence of these processes to Gaussian processes, but the limits depend on the unknown S^0 and G. Therefore, their results cannot be applied immediately in practice. The bootstrap offers a solution to these problems.

2. The main results. We introduce the bootstrapped processes and state the main results. Efron (1981) proposed the following bootstrap for censored data. Draw m iid random vectors $\{(Z_j, \mu_j)\}_{j=1}^m$ with common distribution function

$$(2.1)\quad n^{-1}\#\{1 \le i \le n\colon X_i \le x,\ \delta_i \le y\}.$$

Using the bootstrap data, the bootstrapped product-limit estimator is defined by

$$\hat{S}^0_{m,n}(t) = 1 - \hat{F}^0_{m,n}(t) = \prod_{\{j:\, Z_{j,m} \le t\}} \left(\frac{m-j}{m-j+1} \right)^{\mu_{j,m}}, \quad \text{if } t < Z_{m,m},$$

and $\hat{S}^0_{m,n}(t) = 0$ if $t \ge Z_{m,m}$, where $Z_{1,m} \le \cdots \le Z_{m,m}$ are the order statistics of $\{Z_j\}_{j=1}^m$ and $\{\mu_{j,m}\}_{j=1}^m$ are the induced order statistics of $\{\mu_j\}_{j=1}^m$. Denote the bootstrapped product-limit process by

$$\gamma_{m,n}(t) = m^{1/2}\big(\hat{S}^0_{m,n}(t) - \hat{S}^0_n(t)\big).$$

We can define the bootstrapped uniform product-limit quantile function in a manner similar to (1.3). Let $Z^0_{1,m} \le \cdots \le Z^0_{s_m,m}$ denote the s_m uncensored bootstrapped data. Let

$$E_{m,n}(u) = \begin{cases} F^0\big(Z^0_{1,m}\big), & \text{if } 0 \le u \le \hat{F}^0_{m,n}\big(Z^0_{1,m}\big), \\ F^0\big(Z^0_{j,m}\big), & \text{if } \hat{F}^0_{m,n}\big(Y^0_{j-1,m}\big) < u \le \hat{F}^0_{m,n}\big(Z^0_{j,n}\big), \quad 2 \le j \le s_m, \\ F^0\big(Z^0_{s_m,m}\big), & \text{if } \hat{F}^0_{m,n}\big(Z^0_{s_m,m}\big) < u \le 1, \end{cases}$$

and

$$\beta_{m,n}(u) = m^{1/2}\big(E_{m,n}(u) - E_n(u)\big).$$

The main results of this paper are contained in the following bootstrapped versions of Theorems A and B.

THEOREM 2.1. *Assume that*

$$\text{(2.2)} \qquad 0 < \liminf_{n\to\infty} m/n \le \limsup_{n\to\infty} m/n < \infty$$

and $S(T) > 0$. *We can define a sequence of Wiener processes* $\{\hat{W}_m(t),\ t \ge 0\}_{m=1}^{\infty}$ *such that*

$$\text{(2.3)} \qquad P\Big\{ \sup_{-\infty < t \le T} \big|\gamma_{m,n}(t) - S^0(t)\hat{W}_m(d(t))\big| \ge A_3 n^{-1/4}(\log n)^{5/4} \Big\} \le B_3 n^{-\varepsilon},$$

for all $\varepsilon > 0$, *where* $A_3 = A_3(\varepsilon)$ *and* B_3 *are constants. Further, if* $q \in K(F^0(T))$ *and* $I(q, c) < \infty$ *for all* $c > 0$, *then*

$$\text{(2.4)} \qquad \sup_{-\infty < t \le T} \big|\gamma_{m,n}(t) - S^0(t)\hat{W}_m(d(t))\big| / q\big(F^0(t)\big) \to_{\mathbf{P}} 0.$$

THEOREM 2.2. *Assume* (2.2) *and let* $S(Q^0(p)) > 0$. *Then*

$$\text{(2.5)} \qquad P\Big\{ \sup_{0 \le u \le p} \big|\beta_{m,n}(u) - (1-u)\hat{W}_m\big(d(Q^0(u))\big)\big| \ge A_4 n^{-1/4}(\log n)^{5/4} \Big\} \le B_4 n^{-\varepsilon},$$

for all $\varepsilon > 0$, *where* $A_4 = A_4(\varepsilon)$ *and* B_4 *are constants. Further, if* $q \in K(p)$ *and*

$I(q, c) < \infty$ *for all* $c > 0$, *then*

$$\sup_{1/(m+1) \le u \le p} \left|\beta_{m,n}(u) - (1-u)\hat{W}_m(d(Q^0(u)))\right| \Big/ q(u) \to_{\mathbb{P}} 0. \tag{2.6}$$

REMARK 2.1. We used the *same* Wiener processes $\hat{W}_m$ in Theorems 2.1 and 2.2. It will follow from the proof that $\{\hat{W}_m(t),\ t \ge 0\}_{m=1}^{\infty}$ are independent of $\{(X_i, \delta_i)\}_{i=1}^{\infty}$ and $\{W_n(t),\ t \ge 0\}_{n=1}^{\infty}$. Results of Falk (1986) show that the rates in (2.3) and (2.5) are optimal except for the log terms.

REMARK 2.2. Bickel and Freedman (1981) showed the weak convergence of the bootstrapped uniform empirical and quantile processes in the uncensored case. Their results were extended into a weak convergence in weighted metrics by Csörgő, Csörgő and Horváth (1986a) and Csörgő, Csörgő, Horváth, Mason and Yandell (1986). Theorems 2.1 and 2.2 imply the weak convergence of $\gamma_{m,n}$ and $\beta_{m,n}$ in the usual function spaces $\mathscr{D}(-\infty, T]$ and $\mathscr{D}[0, p]$. The weak convergence of $\gamma_{m,n}$ is also a consequence of a more general multidimensional result of Horváth and Yandell (1985). Akritas (1986) and Lo and Singh (1986) proved that $\gamma_{m,n}$ and $\beta_{m,n}$ converge weakly for almost all realizations of $\{(X_i, \delta_i)\}_{i=1}^{\infty}$; the results of the first paper are also true for discontinuous F^0. Their results do not immediately imply the usual unconditional weak convergence of $\gamma_{m,n}$ and $\beta_{m,n}$. However, by Remark 2.1 we can reformulate our results to conditional ones.

Next, we consider two corollaries of our results. Let P_n and $P_{m,n}$ be the measures induced by the processes $\gamma_n(Q^0(t))$ and $\gamma_{m,n}(Q^0(t))$, $0 \le t \le p$. These measures are defined on $\mathscr{D}$, where $(\mathbb{D}, \mathscr{D})$ is the Skorohod space of functions defined on $[0,1]$. Let ρ be the Prohorov–Lévy distance of two measures.

COROLLARY 2.1. *Assume that the conditions of Theorem* 2.1 *are satisfied with* $T = Q^0(p)$. *Then*

$$\rho(P_n, P_{m,n}) = O\left(m^{-1/4}(\log m)^{5/4}\right).$$

A similar result holds for the measures induced by the uniform quantiles. Next we consider an unconditional almost sure result for the bootstrapped product-limit estimator.

COROLLARY 2.2. *We assume that the conditions of Theorems* 2.1 *and* 2.2 *are satisfied. Then*

$$\limsup_{m \to \infty} (m/\log m)^{1/2} \sup_{-\infty < t \le T} \left|\hat{S}^0_{m,n}(t) - \hat{S}^0_n(t)\right| \le A_5 \quad a.s., \tag{2.7}$$

$$\limsup_{m \to \infty} (m/\log m)^{1/2} \sup_{-\infty < t \le T} \left|\hat{S}^0_{m,n}(t) - S^0(t)\right| \le A_6 \quad a.s. \tag{2.8}$$

and

$$\limsup_{m \to \infty} m^{3/4} (\log m)^{-5/4} \sup_{0 \le u \le p} \left|\hat{F}^0_{m,n}(Q^0(u)) + E_{m,n}(u) - 2u\right| \le A_7 \quad a.s., \tag{2.9}$$

where A_5, A_6 *and* A_7 *are nonnegative constants.*

Aly, Csörgő and Horváth (1985) established the Bahadur–Kiefer representation for the product-limit estimator and its quantile function. By (2.9) a similar representation holds for the bootstrapped versions except that we proved (2.9) with rate $m^{3/4}(\log m)^{-5/4}$ instead of the usual $m^{3/4}(\log m)^{-1/2}(\log\log m)^{-1/4}$.

3. Implications for test transform and the Lorenz curve. We briefly review results on the total time on test transform and the Lorenz curve for censored data, and present our bootstrapped versions of these results. In this section we assume that F^0 is a life distribution function, i.e., $Q^0(0) = 0$, and Q^0 is continuous on $(0,1)$. The total time on test transform of F^0 is defined as

$$H_{F^0}^{-1}(u) = \int_0^u (1-s)\,dQ^0(s), \qquad 0 \le u \le 1,$$

and the unscaled Lorenz curve of F^0 is

$$M_{F^0}(u) = \int_0^u Q^0(s)\,ds, \qquad 0 \le u \le 1.$$

Using the continuity of Q^0 we can write

$$H_{F^0}^{-1}(u) = \int_0^{Q^0(u)} S^0(s)\,ds \quad \text{and} \quad M_{F^0}(u) = \int_0^{Q^0(u)} s\,dF^0(s). \tag{3.1}$$

The function $H_{F^0}^{-1}$ has been useful to characterize various aging properties of life distributions. There is a one-to-one correspondence between life distributions and their total time on test transforms. Recent surveys of the characterizations can be found in Klefsjő (1982) and Bergman (1985). For statistical applications of $H_{F^0}^{-1}$ we refer to Doksum and Yandell (1984) and Csörgő, Csörgő and Horváth (1986a). The unscaled Lorenz curve is used in economics. For its many-sided usefulness and applications we refer to Goldie (1977) and Csörgő, Csörgő and Horváth (1986a).

Define the empirical quantile function

$$\hat{Q}_n^0(u) = \inf\{s\colon \hat{F}_n^0(s) \ge u\} = Q^0(E_n(u)).$$

Csörgő, Csörgő and Horváth (1986b) proposed the following estimators for $H_{F^0}^{-1}$ and M_{F^0} based on (3.1) when data are censored:

$$H_n^{-1}(u) = \int_0^{\hat{Q}_n^0(u)} \hat{S}_n^0(s)\,ds \quad \text{and} \quad M_n(u) = \int_0^{\hat{Q}_n^0(u)} s\,d\hat{F}_n^0(s).$$

Define the processes

$$h_n(u) = n^{1/2}\big(H_n^{-1}(u) - H_{F^0}^{-1}(u)\big) \quad \text{and} \quad m_n(u) = n^{1/2}\big(M_n(u) - M_{F^0}(u)\big).$$

Although Csörgő, Csörgő and Horváth (1986b) proved that (1.2) and (1.4) imply the weak convergence of h_n and m_n, the limiting distributions depend on the unknown survival F^0 and censoring G. It is natural to consider the bootstrap. Let

$$\hat{Q}_{m,n}^0(u) = \inf\{s\colon \hat{F}_{m,n}^0(s) \ge u\} = Q^0(E_{m,n}(u))$$

and define the bootstrapped total time on test $H_{m,n}^{-1}$ and Lorenz $M_{m,n}$ curves in a similar fashion to H_n^{-1} and M_n. The corresponding bootstrapped processes are

$$h_{m,n}(u) = m^{1/2}\big(H_{m,n}^{-1}(u) - H_n^{-1}(u)\big)$$

and

$$m_{m,n}(u) = m^{1/2}\big(M_{m,n}(u) - M_n(u)\big).$$

Define the Gaussian processes

$$\Theta(u) = \int_0^u (1-s)W\big(d(Q^0(s))\big)\,dQ^0(s),$$

$$\Delta(u) = \Theta(u) + W\big(d(Q^0(u))\big)(1-u)^2/f^0(Q^0(u)),$$

where $\{W(t),\ t \geq 0\}$ is a Wiener process and f^0 is the derivative of F^0. If we replace (1.2) and (1.4) by (2.4) and (2.6) in the proofs of Csörgő, Csörgő and Horváth (1986b), we obtain the following two results:

THEOREM 3.1. *Let* $F(Q^0(p)) > 0$, *and assume that* (2.2) *holds and that* f^0 *is continuous and positive on* $(0, Q^0(p)]$. *If there exists a* $q \in K(p)$ *such that* $I(q,c) < \infty$ *for all* $c > 0$ *and*

$$\sup_{0 \leq u \leq p} q(u)(1-u)/f^0\big(Q^0(u)\big) < \infty,$$

then

$$h_{m,n}(u) \to_{\mathscr{D}[0,p]} \Delta(u), \qquad m \wedge n \to \infty.$$

THEOREM 3.2. *Let* $F(Q^0(p)) > 0$, *and assume* (2.2) *holds. If* Q^0 *is continuous on* $(0, p]$, *then*

$$m_{m,n}(u) \to_{\mathscr{D}[0,p]} \Theta(u), \qquad m \wedge n \to \infty.$$

We can now develop Kolmogorov–Smirnov type procedures for the total time on test and Lorenz curves. For convenience, we focus on the total time on test transform. Theorem 4.1 of Csörgő, Csörgő and Horváth (1986b) and Theorem 3.1 imply that $\sup_{[0,p]}|h_n|$ and $\sup_{[0,p]}|h_{m,n}|$ both converge to $\sup_{[0,p]}|\Delta|$. In other words, the corresponding distributions U_n and $U_{m,n}$, respectively, converge to $U(u) = P\{\sup_{0 \leq s \leq p}|\Delta(s)| \leq u\}$ at every continuity point of U. It can be shown, using Theorem 1 of Tsirel'son (1975) and the tightness of $\{\Delta\}$, that U is continuous. The distribution of U can be approximated by the empirical distribution based on a large number of bootstrap samples. Let $U_{N,m,n}$ denote the empirical distribution based on N independent realizations of the bootstrap process based on samples of size m drawn from (2.1). By the Glivenko–Cantelli theorem, $U_{N,m,n} - U_{m,n}$ converges to 0 uniformly, a.s., as $N \to \infty$ for each m and n. Hence, $U_{N,m,n} - U$ and $U_{N,m,n} - U_n$ converge to 0, as $N \to \infty$ and $m \wedge n \to \infty$, at every continuity point of U. This suggests approximating the critical value of U_n by the empirical critical value of $U_{N,m,n}$,

$$c_{N,m,n}(1-\alpha) = \inf\{u \geq 0\colon U_{N,m,n}(u) \geq 1-\alpha\}, \qquad 0 < \alpha < 1,$$

which converges a.s. to $c(1-\alpha) = \inf\{u \geq 0\colon U(u) \geq 1-\alpha\}$, $0 < \alpha < 1$. This bootstrap-based test will have asymptotic power 1 and asymptotic size α. The asymptotic consistency of statistical procedures based on other functionals can be discussed in a similar way.

As was pointed out by a referee, asymptotically distribution-free confidence bands can be constructed without resorting to the machinery of the present paper. For instance, noting that $F_1 \le F_2$ implies $H_{F_1}^{-1} \le H_{F_2}^{-1}$ and $M_{F_1} \le M_{F_2}$, one could use simultaneous confidence bands for S^0 to construct bands for the total time on test and Lorenz curves. However, such bands would be very conservative.

4. Simulations and example. In this section we present simulation results and graphical analysis of data on prostate cancer. We focus on graphical summaries of the data using the biometric tools of the previous section to develop simultaneous confidence bands. For each set, we constructed $N = 1000$ bootstrap samples of size m and developed the empirical distribution functions of $h_{m,n}$ and $m_{m,n}$ simultaneously. That is, the same bootstrap samples were used for the total time on test and Lorenz curves. We focused on 90% critical values read from the empirical distribution functions, using these to construct 90% simultaneous confidence bands for the total time on test transform and the Lorenz curve.

Akritas (1986) showed empirically that the bootstrapped confidence bands for the survival curve seem to have the right level for censored data. Simulation results presented by Csörgő, Csörgő, Horváth, Mason and Yandell (1986) showed that the uncensored bootstrapped empirical distribution function of maximal deviations for the scaled total time on test transform with true exponential survival (the one known theoretical case) is approximately correct.

We present results of some simulations with survival being exponential (1) and censoring being either uniform (0, c) or exponential (c), with c chosen to induce 0%, 20% or 50% censoring. We considered sample sizes $n = 20$, 50, 100 and 200, with $N = 1000$ and $m = n$. Table 1 contains the means and SD's for the bootstrapped statistics of the 90% critical values from 30 independent replications of the bootstrapped empirical distribution. Note that the SD's are large for sizes below 100, but the mean values remain fairly constant. The bandwidth is considerably narrower for the Lorenz curve when there is 50% uniform censoring, presumably reflecting the heavy censoring of large values. This is true to a lesser degree for the total time on test transform. We investigated 80% and 95% critical values and found similar results, the 80% being more benign and the 95% more pathological.

Common sense and a closer examination of the variance process for $h_{m,n}$ and $m_{m,n}$ suggested standardized versions of the processes. Bands based on such statistics would have variable widths but equal probability, asymptotically, at all points. Arguing in a similar manner to Csörgő, Csörgő, Horváth, Mason and Yandell (1986) we examine $\sup|h_{m,n}/\hat{\sigma}_{Hn}|$ and $\sup|m_{m,n}/\hat{\sigma}_{Mn}|$, with sup over an interval $[a, b]$, with $0 < a \le b < 1$ and $\hat{\sigma}_{Hn}$ and $\hat{\sigma}_{Mn}$ being uniformly consistent estimators (with probability 1) of the standard errors σ_H and σ_M, respectively, of h_n and m_n. Provided σ_H and σ_M are bounded away from 0 on $[a, b]$, the

TABLE 1
Critical values for constant width 90% simultaneous bands

	Total time on test					Lorenz curve				
Censoring		Uniform		Exponential			Uniform		Exponential	
Size	0%	20%	50%	20%	50%	0%	20%	50%	20%	50%
20	2.01	1.95	1.54	2.00	1.91	1.55	1.49	1.18	1.48	1.51
	(0.43)	(0.50)	(0.46)	(0.67)	(0.73)	(0.41)	(0.45)	(0.38)	(0.61)	(0.56)
50	2.10	2.08	1.73	2.18	2.47	1.63	1.58	1.05	1.61	2.02
	(0.50)	(0.46)	(0.34)	(0.39)	(0.83)	(0.39)	(0.36)	(0.13)	(0.29)	(0.89)
100	2.13	2.31	2.00	2.27	2.65	1.64	1.83	1.11	1.70	2.00
	(0.27)	(0.44)	(0.29)	(0.37)	(0.51)	(0.22)	(0.45)	(0.10)	(0.32)	(0.54)
200	2.09	2.36	2.13	2.43	2.83	1.65	1.77	1.13	1.83	2.34
	(0.18)	(0.27)	(0.25)	(0.26)	(0.53)	(0.15)	(0.22)	(0.06)	(0.23)	(0.64)

empirical distributions of maximal deviations have the same limiting distribution as $\sup|h_n/\sigma_H|$ and $\sup|m_n/\sigma_M|$, respectively. We used the natural extensions of the variance estimators of Csörgő, Csörgő and Horváth (1986b),

$$\hat{\sigma}_{Mn}^2(u) = 2\int_0^u V_H(s)\, dH_n^{-1}(s),$$

$$\hat{\sigma}_{Hn}^2(u) = \hat{\sigma}_{Mn}^2(u) + u\left[2V_H(u) + uV_S(u)/\hat{f}_n^0(\hat{Q}_n^0(u))\right]/\hat{f}_n^0(\hat{Q}_n^0(u)),$$

with

$$V_H(u) = \int_0^u V_S(s)\, dH_n^{-1}(s),$$

$$V_S(u) = \int_0^{\hat{Q}_n^0(u)} [F_n(s)F_n(s-)]^{-1}\, d\hat{F}_n^0(s)$$

and $\hat{f}_n^0(\hat{Q}_n^0(u))$, a uniform kernel density estimate with bandwidth $n^{-1/4}$.

Table 2 contains the means and SD's for the standardized bootstrapped statistics of the 90% critical values from 30 independent replications of the bootstrapped empirical distribution, restricting to the interval [0.1, 0.9]. Note that the mean values do not settle down as quickly as in Table 1, and that the SD's are quite large for the Lorenz curve when n is small. The variable width critical values are not directly comparable to any tabled distribution to our knowledge.

We examined prostate cancer data on 211 patients who were treated with estrogen [see Hollander and Proschan (1979)]. Ninety patients died of prostate cancer during the 10-year study, 105 died of other diseases and 16 were alive at the end of the study. These data have been examined by Koziol and Green (1976), Hollander and Proschan (1979), Csörgő and Horváth (1981) and Doksum and Yandell (1984). Doksum and Yandell (1984) constructed (asymptotic) simultaneous confidence bands for the hazard rate and the survival curve, finding some evidence for deviation from exponentiality. They also plotted the total time on test transform, but their definition of the total time on test differs

TABLE 2
Critical values for variable width 90% simultaneous bands

	Total time on test					Lorenz curve				
Censoring		Uniform		Exponential			Uniform		Exponential	
Size	0%	20%	50%	20%	50%	0%	20%	50%	20%	50%
20	2.43	2.28	1.98	2.26	1.75	5.39	3.86	9.08	3.66	3.08
	(0.33)	(0.37)	(0.51)	(0.30)	(0.38)	(9.92)	(2.00)	(31.60)	(3.45)	(1.09)
50	2.62	2.63	2.39	2.61	2.29	2.64	2.88	2.63	2.69	2.73
	(0.23)	(0.26)	(0.34)	(0.24)	(0.28)	(0.48)	(0.88)	(0.53)	(0.57)	(0.81)
100	2.87	2.84	2.52	2.76	2.51	2.47	2.49	2.30	2.42	2.38
	(0.29)	(0.26)	(0.18)	(0.23)	(0.22)	(0.22)	(0.26)	(0.19)	(0.23)	(0.26)
200	2.83	2.87	2.60	2.80	2.61	2.28	2.33	2.19	2.27	2.22
	(0.17)	(0.29)	(0.17)	(0.18)	(0.20)	(0.089)	(0.13)	(0.076)	(0.087)	(0.12)

slightly from ours. We present the total time on test transform and Lorenz curves with simultaneous 90% confidence bands on the interval $[0, 0.6]$, based on $N = 1000$ trials with bootstrap samples of size $m = 211$ (Figure 1). Simultaneous 90% bands based on the standardized statistic are presented in Figure 2 on the interval $[0.1, 0.6]$, along with pointwise 90% confidence intervals at selected times.

The critical values for pointwise intervals were determined directly from the bootstrap distributions using trials which were independent of those used for the simultaneous bands. Note in Table 3 that the critical values for constant width bands are about the same as those for the pointwise interval at $u = 0.6$. In fact the difference may be due simply to chance variation, since the variance increases over time for each curve, and the constant width band often picks up the maximal deviation at the upper endpoint. The variable width critical values

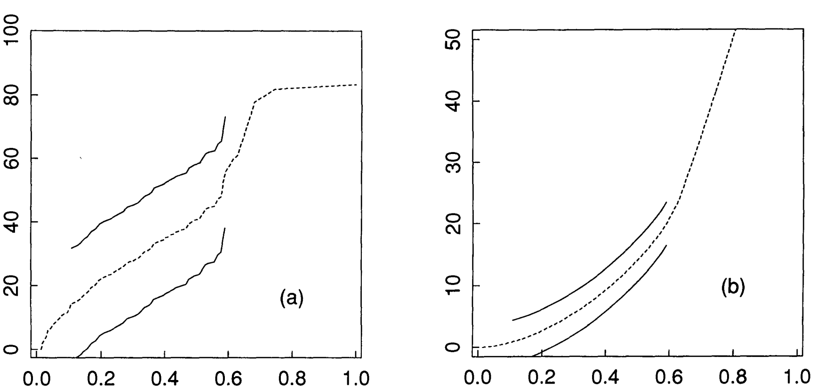

FIG. 1. *Prostate cancer data with constant width bands* ($n = 211$). *Solid line = confidence band, dashed line = curve.* (a) *Total time on test transform with* 90% *confidence bands on* $[0, 0.6]$, (b) *Lorenz curve with* 90% *confidence bands on* $[0, 0.6]$.

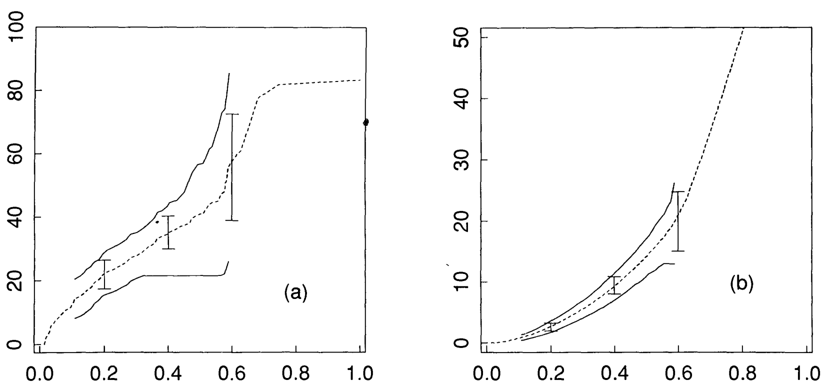

FIG. 2. *Prostate cancer data with variable width bands* ($n = 211$). *Solid line = confidence band, dashed line = curve, vertical bar = pointwise confidence interval.* (a) *Total time on test transform with* 90% *confidence bands on* [0.1, 0.6], (b) *Lorenz curve with* 90% *confidence bands on* [0.1, 0.6].

are similar to those of Table 2 with 50% censoring, though they are not directly comparable. Note that the variable width band in Figure 2 is wider than the pointwise intervals, as expected.

5. Preliminaries. We present a result on the rates of strong approximation of the bootstrapped subdistribution functions which is useful for the proofs of the main results. We use a representation similar to Bickel and Freedman (1981). For convenience, let $f(\infty)$ stand for $\lim_{t\to\infty} f(t)$. Consider the subdistributions $F^{(i)}(t) = P\{X_1 \le t \text{ and } \delta_1 = 2 - i\}$, $i = 1, 2$, and their empirical and bootstrapped counterparts $F_n^{(i)}(t) = n^{-1}\#\{1 \le i \le n\colon X_i \le t \text{ and } \delta_i = 2 - i\}$, $i = 1, 2$, and $F_{m,n}^{(i)}(t) = m^{-1}\#\{1 \le j \le m\colon Z_j \le t \text{ and } \mu_j = 2 - i\}$, $i = 1, 2$. For convenience let $F^{(3)}(t) = F(t)$, and similarly for $F_n^{(3)}$ and $F_{m,n}^{(3)}$. Let $\{\xi_i\}_{i=1}^{\infty}$ and $\{\eta_j\}_{j=1}^{\infty}$ be independent sequences of uniform [0, 1] iidrv's, with empirical distributions, respectively,

$$K_n(u) = n^{-1}\#\{1 \le i \le n\colon \xi_i \le u\} \quad \text{and} \quad L_m(u) = m^{-1}\#\{1 \le j \le m\colon \eta_j \le u\}.$$

TABLE 3
Bootstrapped critical values for prostate data

	Total time on test			Lorenz		
	80%	**90%**	**95%**	**80%**	**90%**	**95%**
Pointwise						
0.2	49.59	65.82	74.62	7.364	9.300	11.03
0.4	56.17	75.20	93.62	16.64	20.97	25.44
0.6	206.2	243.7	285.4	49.30	70.66	96.22
Constant	204.3	254.4	285.1	37.73	50.55	61.60
Variable	1.825	2.220	2.630	1.602	2.017	2.343

The lemma below follows by lengthy elementary calculations, using the conditional probabilities with respect to $X_1, \dots, X_n$ and the conditional independence of $\{Z_j\}_{j=1}^m$ from $\{X_i\}_{i=1}^n$.

LEMMA 5.1.

$$\begin{aligned}&\{F_n^{(1)}(s), F_n^{(2)}(t), F_{m,n}^{(1)}(x), F_{m,n}^{(2)}(y), -\infty < s, t, x, y < \infty\}\\ &\quad =_{\mathbf{D}} \{K_n(F^{(1)}(s)), K_n(F^{(2)}(t) + F^{(1)}(\infty)) - K_n(F^{(1)}(\infty)),\\ &\qquad L_m(K_n(F^{(1)}(x))), L_m(K_n(F^{(2)}(y) + F^{(1)}(\infty))) - L_m(K_n(F^{(1)}(\infty))),\\ &\qquad -\infty < s, t, x, y < \infty\}.\end{aligned}$$

We also will use the following immediate consequence of the tail behavior of the Brownian bridge $\{B^0(u), 0 \le u \le 1\}$.

LEMMA 5.2. *For all $\varepsilon > 0$ and some $C = C(\varepsilon)$ and D constants,*

$$P\Big\{\sup_{0 \le u \le 1} |B^0(u)| > C(\log m)^{1/2}\Big\} \le Dm^{-\varepsilon}.$$

The following result is a generalization of Theorem 3.1 of Burke, Csörgő and Horváth (1981). Define the processes

$$\alpha_n^{(i)}(t) = n^{1/2}(F_n^{(i)}(t) - F^{(i)}(t)), \qquad i = 1, 2, 3,$$

and their bootstrapped counterparts $\alpha_{m,n}^{(i)}(t)$, $i = 1, 2, 3$.

THEOREM 5.1. *Assume that* (2.2) *holds. We can define two independent sequences of Gaussian processes $\{B_n^{(1)}, B_n^{(2)}, B_n^{(3)}\}_{n=1}^\infty$ and $\{C_m^{(1)}, C_m^{(2)}, C_m^{(3)}\}_{m=1}^\infty$ such that*

$$(5.1)\qquad P\Big\{\sup_{-\infty < t < \infty} |\alpha_n^{(i)}(t) - B_n^{(i)}(t)| > A_8 n^{-1/2} \log n\Big\} \le B_8 n^{-\varepsilon},$$

$i = 1, 2, 3$, *and*

$$(5.2)\qquad P\Big\{\sup_{-\infty < t < \infty} |\alpha_{m,n}^{(i)}(t) - C_m^{(i)}(t)| > A_9 m^{-1/4} (\log m)^{3/4}\Big\} \le B_9 m^{-\varepsilon},$$

$i = 1, 2, 3$, *for all $\varepsilon > 0$, where $A_8 = A_8(\varepsilon)$, $A_9 = A_9(\varepsilon)$, B_8 and B_9 are constants. Moreover, $EB_n^{(i)}(t) = EC_m^{(i)}(t) = 0$, $i = 1, 2, 3$, and*

$$\begin{aligned}&EB_n^{(i)}(t)B_n^{(i)}(s) = EC_m^{(i)}(t)C_m^{(i)}(s) = F^{(i)}(t \wedge s) - F^{(i)}(t)F^{(i)}(s), \qquad i = 1, 2, 3,\\ &EB_n^{(i)}(t)B_n^{(3)}(s) = EC_m^{(i)}(t)C_m^{(3)}(s) = F^{(i)}(t)F(s) - F^{(i)}(t \wedge s), \qquad i = 1, 2,\\ &EB_n^{(1)}(t)B_n^{(2)}(s) = EC_m^{(1)}(t)C_m^{(2)}(s) = -F^{(1)}(t)F^{(2)}(s).\end{aligned}$$

PROOF. We sketch the proof only for $\alpha_{m,n}^{(1)}$, using the representation of Lemma 5.1. Komlós, Major and Tusnády (1975) allow the approximations

$$(5.3)\qquad \begin{aligned}&P\Big\{\sup_{0 \le u \le 1} |n^{1/2}(K_n(u) - u) - \tilde{B}_n(u)| > A_{9,1} n^{-1/2} \log n\Big\} \le B_{9,1} n^{-\varepsilon},\\ &P\Big\{\sup_{0 \le u \le 1} |m^{1/2}(L_m(u) - u) - \hat{B}_m(u)| > A_{9,1} m^{-1/2} \log m\Big\} \le B_{9,1} m^{-\varepsilon},\end{aligned}$$

for all $\varepsilon > 0$, where $A_{9,1} = A_{9,1}(\varepsilon)$ and $B_{9,1}$ are constants and $\{\tilde{B}_n\}_{n=1}^{\infty}$ and $\{\hat{B}_m\}_{m=1}^{\infty}$ are two independent sequences of Brownian bridges. Using (2.2), Lemma 5.2 and Theorem 2.C of Burke, Csörgő and Horváth (1981) [cf. Lemma 1.1.1 of Csörgő and Révész (1981)], we obtain

$$P\left\{\sup_{-\infty<t<\infty}\left|\hat{B}_m\big(K_n(F^{(1)}(t))\big) - \hat{B}_m\big(F^{(1)}(t)\big)\right| > A_{9,2}m^{-1/4}(\log m)^{3/4}\right\} \le B_{9,2}m^{-\varepsilon},$$

for all $\varepsilon > 0$, where $A_{9,2} = A_{9,2}(\varepsilon)$ and $B_{9,2}$ are constants. Combining this with (5.3), with u replaced by $K_n(F^{(1)}(t))$, and Lemmas 3.1.1–3.1.3 of Csörgő (1983) yields (5.2). □

6. Proof of Theorem 2.1. We proceed along similar lines to Burke, Csörgő and Horváth (1981) by constructing a strong approximation, with rate, to the cumulative hazard process. We proceed to the proof of Theorem 2.1 by using the same trick as in Breslow and Crowley (1974). Finally, we show the convergence of the weighted process. Denote the empirical and bootstrapped cumulative hazard curves by

$$\Lambda_n(t) = \int_{-\infty}^{t}[1 - F_n(s)]^{-1}\,dF_n^{(1)}(s),$$

$$\Lambda_{m,n}(t) = \int_{-\infty}^{t}[1 - F_{m,n}(s)]^{-1}\,dF_{m,n}^{(1)}(s)$$

and the bootstrapped cumulative hazard process by $\lambda_{m,n}(t) = m^{1/2}(\Lambda_{m,n}(t) - \Lambda_n(t))$.

LEMMA 6.1. *Under the conditions of Theorem* 2.1, *for all* $\varepsilon > 0$ *and some constants* $A_{10} = A_{10}(\varepsilon)$ *and* B_{10},

$$P\left\{\sup_{-\infty<t\le T}|\lambda_{m,n}(t) - \Gamma_m(t)| > A_{10}m^{-1/4}(\log m)^{5/4}\right\} \le B_{10}m^{-\varepsilon},$$

with

$$\Gamma_m(t) = \int_{-\infty}^{t}[S(s)]^{-1}\,dC_m^{(1)}(s) - \int_{-\infty}^{t}C_m^{(3)}(s)[S(s)]^{-2}\,dF^{(1)}(s).$$

PROOF. We decompose $\lambda_{m,n}$ and argue as in the proof of Theorem 4.2 of Burke, Csörgő and Horváth (1981):

$$\lambda_{m,n}(t) = -\int_{-\infty}^{t}\alpha_{m,n}^{(3)}(s)[S(s)]^{-2}\,dF^{(1)}(s) + \int_{-\infty}^{t}[S(s)]^{-1}\,d\alpha_{m,n}^{(1)}(s) + A_m^{(1)}(t) + A_m^{(2)}(t) + A_m^{(3)}(t),$$

with

$$A_m^{(1)}(t) = \int_{-\infty}^{t}\alpha_{m,n}^{(3)}(s)\big([S(s)]^{-2} - [S_{m,n}(s)S_n(s)]^{-1}\big)\,dF_{m,n}^{(1)}(s),$$

$$A_m^{(2)}(t) = \int_{-\infty}^{t}\big([S_n(s)]^{-1} - [S(s)]^{-1}\big)\,d\alpha_{m,n}^{(1)}(s),$$

$$A_m^{(3)}(t) = \int_{-\infty}^{t}\alpha_{m,n}^{(3)}(s)[S(s)]^{-2}\,d\big(F^{(1)}(s) - F_{m,n}^{(1)}(s)\big).$$

By Theorem 5.1 we need only show that $A_m^{(i)}$, $i = 1, 2, 3$, are negligible, which is established later.

LEMMA 6.2. *Under the contditions of Theorem* 2.1, *for all* $\varepsilon > 0$ *and some constants* $A_k = A_k(\varepsilon)$ *and* B_k, $k = 11, 12, 13$,

$$P\left\{\sup_{-\infty<t\le T}|A_m^{(1)}(t)| > A_{11}m^{-1/2}\log m\right\} \le B_{11}m^{-\varepsilon},$$

$$P\left\{\sup_{-\infty<t\le T}|A_m^{(2)}(t)| > A_{12}m^{-1/4}(\log m)^{5/4}\right\} \le B_{12}m^{-\varepsilon},$$

$$P\left\{\sup_{-\infty<t\le T}|A_m^{(3)}(t)| > A_{13}m^{-1/4}(\log m)^{5/4}\right\} \le B_{13}m^{-\varepsilon}.$$

PROOF. Theorem 5.1, (2.2) and Lemma 5.2 give

$$P\left\{\sup_{-\infty<t<\infty}|\alpha_n^{(i)}(t)| > A_{11,1}m^{-1/2}(\log m)^{1/2}\right\} \le B_{11,1}m^{-\varepsilon},$$

for $i = 1, 2, 3$. Thus, if $m, n \ge N_0 = N_0(\varepsilon)$,

$$P\left\{\sup_{-\infty<t\le T}S(t)/S_n(t) \ge 2\right\} \le B_{11,2}m^{-\varepsilon}. \tag{6.1}$$

Similar statements obtain for $\alpha_{m,n}^{(i)}$, $i = 1, 2, 3$, and $S_{m,n}$. The result for $A_m^{(1)}$ follows from

$$\sup_{-\infty<t\le T}|A_m^{(1)}(t)| \le \frac{\sup_{-\infty<t\le T}|\alpha_{m,n}^{(3)}(t)|\left[2\sup_{-\infty<t\le T}|\alpha_n^{(3)}(t)| + \sup_{-\infty<t\le T}|\alpha_{m,n}^{(3)}(t)|\right]}{[S(T)]^2 S_{m,n}(T)S_n(T)}.$$

The last two terms are more difficult, but can be handled in a similar manner to the proof of Theorem 4.2 of Burke, Csörgő and Horváth (1981). We consider only $A_m^{(2)}$. By (6.1) it is enough to deal with

$$m^{-1/2}\int_{-\infty}^{t}\alpha_n^{(3)}(s)[S(s)]^{-2}\,d\alpha_{m,n}^{(1)}(s).$$

It follows from Theorem 5.1 that

$$P\left\{\sup_{-\infty<t\le T}\left|\int_{-\infty}^{t}\left(\alpha_n^{(3)}(s) - B_n^{(3)}(s)\right)[S(s)]^{-2}\,d\alpha_{m,n}^{(1)}(s)\right| > A_{12,1}\log m\right\}$$
$$\le B_{12,1}m^{-\varepsilon}.$$

We can divide the half-line $(-\infty, T]$ into $k = [m^{1/2}]$ parts with the points $-\infty = t_1 \le t_2 \le \cdots \le t_k = T$ such that $F(t_{i+1}) - F(t_i-) \le 2/k$. We define the jump process

$$B_n^*(t) = B_n^{(3)}(t_{i+1})[S(t_{i+1})]^{-2}, \quad \text{if } t_i < t \le t_{i+1},$$

and proceed by repeated use of Theorem 2.C of Burke, Csörgő and Horváth

(1981) to show

$$(6.2)\quad P\left\{\sup_{-\infty<t\le T}\left|B_n^*(t)-B_n^{(3)}(t)\right|>2A_{12,2}m^{-1/4}(\log m)^{1/2}\right\}\le B_{12,2}m^{-\varepsilon}$$

and

$$(6.3)\quad \begin{aligned}&P\left\{\max_{1\le i\le k-1}\left|\alpha_{m,n}^{(1)}(t_{i+1})-\alpha_{m,n}^{(1)}(t_i-)\right|>2A_{12,3}m^{-1/4}(\log m)^{3/4}\right\}\\&\quad\le 2B_{12,3}m^{-\varepsilon}.\end{aligned}$$

From (6.2),

$$\begin{aligned}&P\left\{\sup_{-\infty<t\le T}\left|\int_{-\infty}^{t}(B_n^*(s)-B_n^{(3)}(s))[S(s)]^{-2}\,d\alpha_{m,n}^{(1)}(s)\right|>A_{12,2}m^{1/4}(\log m)^{1/2}\right\}\\&\quad\le B_{12,2}m^{-\varepsilon}.\end{aligned}$$

By the definition of B_n^*,

$$\sup_{-\infty<t\le T}\left|\int_{-\infty}^{t}B_n^*(s)\,d\alpha_{m,n}^{(1)}(s)\right|\le\sum_{i=1}^{k-1}\left|B_n^{(3)}(t_i)\left(\alpha_{m,n}^{(1)}(t_{i+1})-\alpha_{m,n}^{(1)}(t_i-)\right)\right|.$$

Combining this with (6.3) and Lemma 5.2 [for $B_n^{(3)}(Q^0)$] yields

$$\begin{aligned}&P\left\{m^{-1/2}\sup_{-\infty<t\le T}\left|\int_{-\infty}^{t}B_n^*(s)\,d\alpha_{m,n}^{(1)}(s)\right|\ge 2A_{12,2}A_{12,3}m^{-1/4}(\log m)^{5/4}\right\}\\&\quad\le(B_{12,2}+2B_{12,3})m^{-\varepsilon},\end{aligned}$$

which completes the proof of Lemma 6.2. □

Now we can turn to the proof of the main results of Theorem 2.1 concerning the bootstrapped product-limit process. The product-limit curves can be approximated in a standard fashion [cf. Lemma 1 of Breslow and Crowley (1974) and Horváth (1980)] using (6.1), via

$$(6.4)\quad \begin{aligned}&0\le-\log\hat{S}_n^0(t)-\Lambda_n(t)\le n^{-1}\int_{-\infty}^{t}[S_n(s)-1/n]^{-2}\,dF_n^{(1)}(s),\\&P\left\{\sup_{-\infty<t\le T}\left|\hat{S}_n^0(t)-\exp(-\Lambda_n(t))\right|>A_{3,1}m^{-1}\right\}\le B_{3,1}m^{-\varepsilon}\end{aligned}$$

and analogously for $\hat{S}_{m,n}^0$, with the same rate. Thus, to prove (2.3), we need only show that

$$\begin{aligned}&P\left\{\sup_{-\infty<t\le T}\left|m^{1/2}(\exp(-\Lambda_n(t))-\exp(-\Lambda_{m,n}(t)))-S^0(t)\Gamma_m(t)\right|\right.\\&\qquad\qquad\left.>A_{3,2}m^{-1/4}(\log m)^{5/4}\right\}\\&\quad\le B_{3,2}m^{-\varepsilon},\end{aligned}$$

but this follows from Lemma 6.1, Lemma 5.2 [for $\Gamma_m(Q^0)$] and

$$\begin{aligned}(6.5)\qquad &\left|m^{1/2}(\exp(-\Lambda_n(t)) - \exp(-\Lambda_{m,n}(t))) - S^0(t)\lambda_{m,n}(t)\right| \\ &\quad \le m^{-1/2}[\lambda_{m,n}(t)]^2.\end{aligned}$$

For the proof of (2.4), let $T_0 < K < T$, with $T_0 = \sup\{t\colon F^0(t) = 0\}$. Csörgő, Csörgő and Horváth (1986b) showed that

$$(6.6)\qquad \lim_{K\to T_0} P\left\{\sup_{-\infty<t\le K} \left|S^0(t)\hat{W}_m(d(t))\right|/q(F^0(t)) > \varepsilon\right\} = 0,$$

for all $\varepsilon > 0$ and $m \ge 1$. Therefore, by (2.3) we need only show

$$(6.7)\qquad \lim_{K\to T_0} \limsup_{m\wedge n\to\infty} P\left\{\sup_{-\infty<t\le K} |\gamma_{m,n}(t)|/q(F^0(t)) > \varepsilon\right\} = 0,$$

for all $\varepsilon > 0$. In fact, it is enough to deal with the limit as $K \to T_1 = \sup\{t\colon F^{(1)}(t) = 0\}$. We leave out elementary calculations and application of $O_p(1)$ rates for inverses of S_n and $S_{m,n}$, which follow from Theorem 5.1, (6.1), Lemma 5.1 and Wellner's (1978) inequality. Csörgő, Csörgő, Horváth and Mason (1986) proved that under the conditions of Theorem 2.1,

$$\begin{aligned}(6.8)\qquad &q(u)u^{-1/2} \to \infty, \qquad u \to 0, \\ &\lim_{K\to T_i} \limsup_{n\to\infty} P\left\{\sup_{-\infty<t\le K} |\alpha_n^{(i)}(t)|/q(F^{(i)}(t)) > \varepsilon\right\} = 0,\end{aligned}$$

for all $\varepsilon > 0$, where $T_i = \sup\{t\colon F^{(i)}(t) = 0\}$, $i = 1,2,3$. Theorem 17.11 in Csörgő, Csörgő and Horváth (1986a) implies the analogous result for $\alpha_{m,n}^{(i)}$, $i = 1,2,3$. Thus, we have

$$\lim_{K\to T_0} \limsup_{m\wedge n\to\infty} P\left\{\sup_{-\infty<t\le K} |\gamma_{m,n}(t)|/q(F^{(1)}(t)) > \varepsilon\right\} = 0.$$

Using (6.5), we obtain

$$\begin{aligned}(6.9)\qquad &\lim_{K\to T_0} \limsup_{m\wedge n\to\infty} P\left\{\sup_{-\infty<t\le K} m^{1/2}|\exp(-\Lambda_{m,n}(t))\right. \\ &\qquad \left. -\exp(-\Lambda_n(t))|/q(F^{(1)}(t)) > \varepsilon\right\} = 0.\end{aligned}$$

Some elementary calculations and (6.4) lead to

$$\begin{aligned}(6.10)\qquad &\lim_{K\to T_0} \limsup_{n\to\infty} P\left\{n^{1/2} \sup_{-\infty<t\le K} |\hat{S}_n^0(t) - \exp(-\Lambda_n(t))|/q(F^{(1)}(t)) > \varepsilon\right\} \\ &= 0\end{aligned}$$

and a similar expression for $\hat{S}_{m,n}^0$. It is immediate that (6.9) and (6.10) imply (6.7), which completes the proof of (2.4) and Theorem 2.1.

7. Proof of Theorem 2.2. The proof follows similar lines to that of Aly, Csörgő and Horváth (1985). The bootstrapped uniform product-limit quantile function can be written as

$$\begin{aligned}(7.1)\qquad \beta_{m,n}(u) &= \gamma_{m,n}(Q^0(E_{m,n}(u))) + m^{1/2}(E_{m,n}(u) - \hat{F}_n^0(Q^0(E_{m,n}(u)))) \\ &\quad + m^{1/2}(\hat{F}_{m,n}^0(Q^0(E_{m,n}(u))) - u) + m^{1/2}(u - E_n(u)).\end{aligned}$$

In order to prove (2.5), it is enough to show

$$P\Big\{\sup_{0\le u\le p}|\gamma_{m,n}(Q^0(E_{m,n}(u))) - (1-u)\hat{W}_m(d(Q^0(u)))| > A_{4,1}m^{-1/4}(\log m)^{5/4}\Big\} \le B_{4,1}m^{-\varepsilon} \tag{7.2}$$

and that the other terms are negligible. Theorem 2.1 and Theorem A and arguments as in the proof of Theorem 3.1 of Aly, Csörgő and Horváth (1985), with the help of Lemma 1.1.1 of Csörgő and Révész (1981) at Step 4, establish the negligibility. Further application of Lemma 1.1.1 of Csörgő and Révész (1981) yields

$$P\Big\{\sup_{0\le u\le p}|(1-E_{m,n}(u))\hat{W}_m(d(Q^0(E_{m,n}(u)))) - (1-u)\hat{W}_m(d(Q^0(u)))| > A_{4,2}m^{-1/4}(\log m)^{3/4}\Big\} \le B_{4,2}m^{-\varepsilon}.$$

Now applying Theorem 2.1 with t replaced by $Q^0(E_{m,n}(u))$ completes the proof of (2.5).

Let $0 < \delta < p$. In order to prove (2.6) we need only show

$$\lim_{\delta\to 0}\limsup_{m\wedge n\to\infty} P\Big\{\sup_{1/m\le u\le\delta}|\beta_{m,n}(u)|/q(u) > \varepsilon\Big\} = 0, \tag{7.3}$$

for all $\varepsilon > 0$, since (2.5) implies

$$\sup_{\delta\le u\le p}|\beta_{m,n}(u) - (1-u)\hat{W}_m(d(Q^0(u)))|/q(u) \to_{\mathbb{P}} 0$$

and (6.6) yields

$$\lim_{\delta\to 0} P\Big\{\sup_{0\le u\le\delta}|(1-u)\hat{W}_m(d(Q^0(u)))|/q(u) > \varepsilon\Big\} = 0,$$

for all $\varepsilon > 0$ and $m \ge 1$.

Theorem A [cf. Theorem 3.3 in Csörgő, Csörgő and Horváth (1986b)] states the analogous result to (7.3) for the unbootstrapped β_n. If q satisfies $I(q, c) < \infty$ for all $c < 0$, then so does $q_\lambda(u) = q(\lambda u)$, $\lambda > 0$. Thus, by Theorem A and $O_p(1)$ results for $E_n(u)/u$, $E_{m,n}(u)/u$ and their inverses [cf. Csörgő, Csörgő and Horváth (1986a, b)] and (6.4), we have

$$\lim_{\delta\to 0}\limsup_{m\wedge n\to\infty} P\Big\{m^{1/2}\sup_{1/m\le u\le\delta}|\gamma_n(Q^0(E_{m,n}(u)))|/q(u) > \varepsilon\Big\} = 0. \tag{7.4}$$

A similar result obtains for $\gamma_{m,n}$ via Theorem 2.1. We can show using (6.8) and intermediate steps in the proof of (2.5) that

$$\lim_{\delta\to 0}\limsup_{m\wedge n\to\infty} P\Big\{m^{1/2}\sup_{1/m\le u\le\delta}|\hat{F}^0_{m,n}(Q^0(E_{m,n}(u))) - u|/q(u) > \varepsilon\Big\} = 0.$$

Combining these and recalling (7.1), we establish (7.3) which completes the proof of (2.6).

8. Proof of the corollaries. Corollary 2.1 follows from Theorems 2.1 and 2.2 and [cf. Komlós, Major and Tusnády (1974)]

$$\rho(P_n, P_{m,n}) \le \inf_{\varepsilon>0}\left(\varepsilon + P\left\{\sup_{0\le u\le p}{}^{*}|\gamma_n(Q^0(u)) - (1-u)W_n(d(Q^0(u)))| > \varepsilon\right\}\right)$$

$$+ \inf_{\varepsilon>0}\left(\varepsilon - P\left\{\sup_{0\le u\le p}|\gamma_{m,n}(Q^0(u)) - (1-u)\hat{W}_m(d(Q^0(u)))| > \varepsilon\right\}\right).$$

For Corollary 2.2, we use Borel–Cantelli and

$$P\left\{\sup_{-\infty<t\le T}|\hat{F}^0_{m,n}(t) - \hat{F}^0_n(t)| > A_5 m^{-1/2}(\log m)^{1/2}\right\} \le B_5 m^{-2},$$

for some A_5 and B_5, which follows from Theorem 2.1 and Lemma 5.2, to get (2.7). We obtain (2.8) in a similar way. Theorems 2.1 and 2.2 imply that

$$\limsup_{m\to\infty} m^{3/4}(\log m)^{-5/4}\sup_{0\le u\le p}|\hat{F}^0_{m,n}(Q^0(u)) + E_{m,n}(u) - \hat{F}^0_n(u) - E_n(u)|$$
$$\le A_7 \quad \text{a.s.}$$

Combining this with (2.2) and the unbootstrapped version of (2.9) [cf. Aly, Csörgő and Horváth (1985)] completes the proof of (2.9).

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REFERENCES

AALEN, O. (1976). Nonparametric inferences in connection with multiple decrement models. *Scand. J. Statist.* **3** 15–27.

AKRITAS, M. G. (1986). Bootstrapping the Kaplan–Meier estimator. *J. Amer. Statist. Assoc.* **81** 1032–1038.

ALY, E.-E. A. A., CSÖRGŐ, M. and HORVÁTH, L. (1985). Strong approximations of the quantile process of the product-limit estimator. *J. Multivariate Anal.* **15** 185–210.

BERGMAN, B. (1985). On reliability theory and its applications. *Scand. J. Statist.* **12** 1–42.

BICKEL, P. J. and FREEDMAN, D. A. (1981). Some asymptotic theory for the bootstrap. *Ann. Statist.* **9** 1196–1217.

BRESLOW, N. R. and CROWLEY, J. (1974). A large sample study of the life table and product limit estimates under random censorship. *Ann. Statist.* **2** 437–453.

BURKE, M. D., CSÖRGŐ, S. and HORVÁTH, L. (1981). Strong approximation of some biometric estimates under random censorship. *Z. Wahrsch. verw. Gebiete* **56** 87–112.

BURKE, M. D., CSÖRGŐ, S. and HORVÁTH, L. (1986). An improved approximation of the product-limit process. Preprint.

CSÖRGŐ, M. (1983). *Quantile Processes with Statistical Applications.* SIAM, Philadelphia.

CSÖRGŐ, M., CSÖRGŐ, S. and HORVÁTH, L. (1986a). *An Asymptotic Theory for Empirical Reliability and Concentration Processes. Lecture Notes in Statist.* **33**. Springer, New York.

CSÖRGŐ, M., CSÖRGŐ, S. and HORVÁTH, L. (1986b). Estimation of total time on test transforms and Lorenz curves under random censorship. *Statistics* **18** 77–97.

CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986). Weighted empirical and quantile processes. *Ann. Probab.* **14** 31–85.

CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L., MASON, D. M. and YANDELL, B. S. (1986). Asymptotic theory of some bootstrapped empirical processes. Preprint.

CSÖRGŐ, M. and RÉVÉSZ, P. (1981). *Strong Approximations in Probability and Statistics*. Akadèmiai Kiadó, Budapest/Academic, New York.

CSÖRGŐ, S. and HORVÁTH, L. (1981). On the Koziol–Green model for random censorship. *Biometrika* **68** 391–401.

CSÖRGŐ, S. and HORVÁTH, L. (1986). Confidence bands for censored samples. *Canad. J. Statist.* **14** 131–144.

DOKSUM, K. A. and YANDELL, B. S. (1984). Tests for exponentiality. In *Handbook of Statistics* (P. R. Krishnaiah and P. K. Sen, eds.) **4** 579–611. North Holland, Amsterdam.

EFRON, B. (1967). The two-sample problem with censored data. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **4** 831–853. Univ. California Press.

EFRON, B. (1981). Censored data and the bootstrap. *J. Amer. Statist. Assoc.* **76** 312–319.

FALK, M. (1986). On the accuracy of the bootstrap approximation of the joint distribution of sample quantiles. *Comm. Statist. A—Theory Methods* **15** 2867–2876.

GILLESPIE, M. J. and FISHER, L. (1979). Confidence bands for the Kaplan–Meier survival curve estimate. *Ann. Statist.* **7** 920–924.

GOLDIE, C. M. (1977). Convergence theorems for empirical Lorenz curves and their inverses. *Adv. in Appl. Probab.* **9** 765–791.

HALL, W. J. and WELLNER, J. A. (1980). Confidence bands for a survival curve from censored data. *Biometrika* **67** 113–143.

HOLLANDER, M. and PROSCHAN, F. (1979). Testing to determine the underlying distribution using randomly censored data. *Biometrics* **35** 393–401.

HORVÁTH, L. (1980). Dropping continuity and independence assumptions in random censorship models. *Studia Sci. Math. Hungar.* **15** 381–389.

HORVÁTH, L. and YANDELL, B. S. (1985). Bootstrapped multi-dimensional product limit process. Technical Report 71, Lab. Res. Statist. Probab., Carleton Univ.

KAPLAN, E. L. and MEIER, P. (1958). Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.* **53** 457–481.

KLEFSJŐ, B. (1982). On aging properties and total time on test transforms. *Scand. J. Statist.* **9** 37–41.

KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1974). Weak convergence and embedding. In *Limit Theorems of Probability Theory* (P. Révész, ed.) 149–165. North Holland, Amsterdam.

KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent RV's and the sample DF. I. *Z. Wahrsch. verw. Gebiete* **32** 111–131.

KOZIOL, J. A. and GREEN, S. B. (1976). A Cramér–von Mises statistic for randomly censored data. *Biometrika* **63** 465–474.

LO, S.-H. and SINGH, K. (1986). The product-limit estimator and the bootstrap: Some asymptotic representations. *Probab. Theory Related Fields* **71** 455–465.

NAIR, V. N. (1981). Plots and tests for goodness of fit with randomly censored data. *Biometrika* **68** 99–104.

TSIREL'SON, V. S. (1975). The density of the distribution of the maximum of a Gaussian process. *Theory Probab. Appl.* **20** 847–856.

WELLNER, J. A. (1978). Limit theorems for the ratio of the empirical distribution to the true distribution function. *Z. Wahrsch. verw. Gebiete* **45** 37–88.

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