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ON TESTING WHETHER NEW IS BETTER THAN USED USING RANDOMLY CENSORED DATA

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Under a model of random censorship, we consider the test H_0: a life distribution is exponential, versus H_1: it is new better than used, but not exponential. This paper introduces a class of tests by using the Kaplan–Meier estimator for the sample distribution in the uncensored model. Under some regularity conditions, the asymptotic normality of statistics is derived by an application of von Mises' method, and asymptotically valid tests are obtained by using estimators for the null standard deviations. The efficiency loss in the proportional censoring model is studied and a Monte Càrlo study of power is performed.

1. Introduction. A life distribution $F(x)$ is said to be new better than used (NBU) if

$$\bar{F}(x+y) \le \bar{F}(x)\bar{F}(y) \quad \text{for all } x,\ y \ge 0,$$

where $\bar{F}(x) = 1 - F(x)$. This criterion was first introduced by Hollander and Proschan (1972), and the NBU class of life distributions has proved to be very useful in reliability studies.

We consider the problem of testing the null hypothesis H_0: $F(x) = 1 - \exp(-x/\mu)$, $x \ge 0$ (μ unspecified), versus the alternative H_1: $F(x)$ is NBU, but not exponential. In the uncensored model, where we get a complete sample, the class of statistics

$$\int_0^\infty \int_0^\infty \psi\{1 - F_n(x+y)\}\, dF_n(x)\, dF_n(y) \tag{1.1}$$

was proposed for the above testing problem by Koul (1978), where $F_n(x)$ denotes the distribution function (d.f.) of the sample and ψ is a weight function. The purpose of this paper is to construct the optimal test that minimizes the efficiency loss and has good power performance against NBU alternatives in the proportional censoring model.

To describe a randomly censored model, let $X_1, X_2, \dots, X_n$ denote independent identically distributed (i.i.d.) random variables (r.v.'s) having a common continuous life d.f. $F(x)$. Independent of the X_i's, let $Y_1, Y_2, \dots, Y_n$ also denote i.i.d. r.v.'s having an unknown continuous d.f. $H(x)$. The Y_i's are treated as the random times to the right censorship. Finally, let $Z_i = \min\{X_i, Y_i\}$ and $\delta_i = I[X_i \le Y_i]$ for $1 \le i \le n$, where $I[A]$ denotes the indicator function of the set A. We are interested in testing H_0 against H_1 on the basis of an incomplete sample $(X_1, \delta_1), (X_2, \delta_2), \dots, (X_n, \delta_n)$.

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Now we introduce a class of statistics

$$S_n(\psi) = \int_0^\infty \int_0^\infty \psi\{1 - \hat{F}_n(x+y)\}\, d\hat{F}_n(x)\, d\hat{F}_n(y),$$

where $\hat{F}_n(x)$ is the Kaplan–Meier estimator defined by

$$\hat{F}_n(x) = 1 - \prod_{\{i:\, Z_{(i)} \le x\}} \{(n-i)/(n-i+1)\}^{\delta_{(i)}}$$

and ψ is a suitable weight function. Here $Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$ denote the ordered values of the Z_i's and $\delta_{(1)}, \delta_{(2)}, \dots, \delta_{(n)}$ are the δ_i's corresponding to $Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}$, respectively. The test based on $S_n(\psi)$ rejects H_0 in favor of H_1 for small values. Chen, Hollander and Langberg's statistic (1983) is obtained by putting $\psi(t) = t$. Koziol (1980) introduced versions of goodness-of-fit statistics with randomly censored data and performed a Monte Carlo study of power comparison.

In Section 2 we state the asymptotic normality of a suitably normalized version of $S_n(\psi)$ under some regularity conditions on the d.f.'s F and H and the function ψ, and give a proof by applying the von Mises' statistical functional method. In Section 3 the efficiency loss based on Pitman efficacy is computed in the proportional censoring model and the optimal tests are discussed based on a Monte Carlo power comparison against the Weibull alternative in the proportional censoring model.

2. Asymptotic normality. We establish the asymptotic normality of $S_n(\psi)$ defined in Section 1 by the application of results on the Kaplan–Meier estimator and von Mises' calculus. Let $\{\phi(t): 0 \le t < \infty\}$ be a Gaussian process with mean zero and covariance kernel defined by

$$E[\phi(s)\phi(t)] = \bar{F}(s)\bar{F}(t)\int_0^s \{\bar{H}(u)\bar{F}(u)^2\}^{-1}\, dF(u) \quad \text{for } 0 \le s \le t < \infty.$$

We impose the following conditions on the d.f.'s F and H:

(A.1) Both F and H have support $[0, \infty)$,
(A.2) $\int_0^\infty \{\bar{H}(u)\}^{-1}\, dF(u) < \infty$.

Then Theorem 2.1 of Gill (1983) implies that the weak convergence of $\phi_n(x) = n^{1/2}\{\hat{F}_n(x \wedge T_n) - F(x \wedge T_n)\}$ to $\phi(x)$ on $[0, \infty)$ holds as $n \to \infty$, where $T_n = \max_i Z_i$. We can state the following result.

THEOREM 2.1. *Assume that the function ψ is continuous, nondecreasing and piecewise differentiable with bounded derivative on* $[0,1]$. *Define* $T(F) = \int_0^\infty \int_0^\infty \psi\{\bar{F}(x+y)\}\, dF(x)\, dF(y)$. *Then under Conditions* (A.1) *and* (A.2), *$n^{1/2}\{S_n(\psi) - T(F)\}$ converges in distribution to a normal r.v. with mean zero* *and finite variance* σ^2 *as* $n \to \infty$, *where*

$$\sigma^2 = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E\big[\{\phi(x+y)\psi'\{\bar{F}(x+y)\} - 2\phi(y-x)\psi'\{\bar{F}(y)\}I[y \ge x]\}$$
$$\times \{\phi(s+t)\psi'\{\bar{F}(s+t)\} - 2\phi(t-s)\psi'\{\bar{F}(t)\}I[t \ge s]\}\big]$$
$$\times dF(x)\, dF(y)\, dF(s)\, dF(t).$$

COROLLARY 2.2. *Let* $\psi(u) = u^\alpha$, $\alpha \ge 1$. *Under the null hypothesis* H_0 *and Conditions* (A.1) *and* (A.2), $n^{1/2}\{S_n(u^\alpha) - (\alpha+1)^{-2}\}$ *converges in distribution to a normal r.v. with mean zero and variance*

$$\sigma_\alpha^2 = \int_0^\infty f_\alpha\{\bar{F}(x)\}\{K(x)\}^{-1}\, dF(x), \tag{2.1}$$

where $f_\alpha(s) = \alpha^2(\alpha+1)^{-4}\{(\alpha+1)\log s + 1\}^2 s^{2\alpha+1}$ *and* $\bar{K}(x) = \bar{F}(x)\bar{H}(x)$.

We first show the Hadamard differentiability of the statistical functional $T(F)$. For a continuous d.f. F, we have

$$T(F) = \int_0^1 \int_0^1 \psi[1 - F\{F^{-1}(s) + F^{-1}(u)\}]\, ds\, du,$$

where $F^{-1}(u) = \inf\{x\colon F(x) \ge u\}$. Let $D[0,1]$ be the space of functions on $[0,1]$ that are right continuous and have left-hand limits.

LEMMA 2.3. *Let* τ *be the functional induced on* $D[0,1]$ *by* $\tau(G) = T(G \circ F)$ *for* G *in* $D[0,1]$. *Then* τ *is Hadamard differentiable at* I *with derivative*

$$\tau_I'(G) = -\int_0^1 \int_0^1 \psi'\{1 - F(F^{-1}(s) + F^{-1}(t))\}G \circ F\{F^{-1}(s) + F^{-1}(t)\}\, ds\, dt$$
$$+2\int_0^1 \int_0^t G \circ F\{F^{-1}(t) - F^{-1}(s)\}\, ds\, \psi'(1-t)\, dt,$$

for G *in* $D[0,1]$, *where* $I(u) = u$ *for* $0 \le u \le 1$.

PROOF. Following Fernholz (1983), it suffices to show that τ can be expressed as a composition of Hadamard differentiable transformations. For fixed F and ψ, we define

$$\gamma_1(G)(s) = F^{-1} \circ G^{-1}(s),$$
$$\gamma_2(V)(s,u) = V(s) + V(u),$$
$$\gamma_3(U,G)(s,u) = \psi\{1 - G \circ F[U(s,u)]\},$$

and

$$\gamma_4(U) = \int_0^1 \int_0^1 U(x,y)\, dx\, dy,$$

for G in $D[0,1]$, V in $L^1[0,1]$, U in $L^1[0,1] \times [0,1]$ and $0 \le s, u \le 1$. Then from Propositions 6.1.1, 6.1.2 and 6.1.6 of Fernholz (1983) it can be seen that the preceding transformations are Hadamard differentiable. Since $\tau(G) = \gamma_4 \circ \gamma_3(\gamma_2 \circ \gamma_1(G), G)$, τ is Hadamard differentiable at I and some calculation yields the derivative $\tau_I'(G)$. □

PROOF OF THEOREM 2.1. We first note that

$$n^{1/2}|\{T(F_n(\cdot)) - T(F(\cdot))\} - \{T(F_n(\cdot \wedge T_n)) - T(F(\cdot \wedge T_n))\}|$$
$$= n^{1/2}\left|\int_{T_n}^{\infty}\int_{T_n - x}^{\infty} \psi\{\bar{F}(x+y)\}\, dF(x)\, dF(y)\right|$$
$$\leq n^{1/2}\psi\{\bar{F}(T_n)\}\bar{F}(T_n) \to_p 0 \quad \text{as } n \to \infty.$$

Hence, we consider the asymptotic behavior of $n^{1/2}\{\tau[F_n(F^{-1}(\cdot) \wedge T_n) - \tau[F(F^{-1}(\cdot) \wedge T_n)]\}$. Since Proposition 4.3.3 of Fernholz (1983) also holds for the stochastic process $\{\phi_n(F^{-1}(t))\colon 0 \leq t \leq 1\}$, the asymptotic distribution of $\tau_I'\{\phi(F^{-1})\}$ can be derived from the continuous mapping theorem and the fact that $\tau_I'(\cdot)$ is a bounded functional on $D[0,1]$. □

Corollary 2.2 shows that the asymptotic variance of $S_n(u^\alpha)$ depends on the unknowns μ and H. In order to perform the test based on $S_n(u^\alpha)$, we must estimate the variance σ_α^2 from the observations (Z_i, δ_i). To this end, we set

$$\hat{\sigma}_n^2 = n\int_0^{\infty} f_\alpha\{1 - \hat{F}_n(t-)\}\{1 - \hat{F}_n(t-)\}\{Y^{(n)}(t)\}^{-2}\, dN^{(n)}(t),$$

where $Y^{(n)}(t) = \sum_{i=1}^n I[Z_i \geq t]$ and $N^{(n)}(t) = \sum_{i=1}^n I[Z_i \leq t, \delta_i = 1]$. Then we have the following result.

LEMMA 2.4. *Under H_0 and Conditions* (A.1) *and* (A.2), *the estimator $\hat{\sigma}_n^2$ consistently estimates the variance σ_α^2.*

PROOF. We define $\{H^{(n)}(s)\}^2 = nf_\alpha\{1 - \hat{F}_n(s-)\}\{1 - \hat{F}_n(s-)\}\{Y^{(n)}(s)\}^{-2}$. Since $H^{(n)}(s)$ is a predictable process and $Y^{(n)}(s)/n$ converges in probability to $\bar{K}(s)$ uniformly in s as $n \to \infty$, Theorem 4.1.1 of Gill (1980) implies

$$\int_0^t \{H^{(n)}(s)\}^2\, dN^{(n)}(s) \to_p \int_0^t f_\alpha\{\bar{F}(s)\}\{\bar{K}(s)\}^{-1}\, dF(s) \quad \text{as } n \to \infty$$

for any t, $0 \leq t < \infty$. To prove the case $t = \infty$, it suffices from Corollary 2.3 of Andersen, Borgan, Gill and Keiding (1982) to show that for any $\varepsilon > 0$,

$$\lim_{u \uparrow \infty} \limsup_{n \to \infty} P\left(\int_u^{\infty} \{H^{(n)}(s)\}^2 Y^{(n)}(s)\{\bar{F}(s)\}^{-1}\, dF(s) > \varepsilon\right) = 0.$$

Then Theorem 1.1 of Van Zuijlen (1978) and Theorem 3.2.1 of Gill (1980) yield that

$$P\left(\{H^{(n)}(s)\}^2 Y^{(n)}(s) \leq \beta^{-3} M\bar{F}(s)^2\{\bar{K}(s)\}^{-1} \text{ for } 0 \leq s < \infty\right) = 1 - o(\beta^0)$$

as $\beta \downarrow 0$ uniformly in n for some constant M such that $f_\alpha(t) \leq Mt$. Hence the desired result follows from Condition (A.2). □

Corollary 2.2 and Lemma 2.4 show that the test rejecting H_0 in favor of H_1 for small values of $J_n(\alpha) = n^{1/2}\{S_n(u^\alpha) - (\alpha+1)^{-2}\}\hat{\sigma}_n^{-1}$ is consistent against all continuous NBU alternatives.

TABLE 1
Monte Carlo properties of $\hat{\sigma}_n^2$ and the normal approximation for $J_n(\alpha)$ when F is exponential (1), H is exponential (λ), and $n = 50$ (25).

Test	σ_α^2	Ave $\hat{\sigma}_n^2$	SD($\hat{\sigma}_n^2$)	$\hat{P}(J_n(\alpha) \le -z_{0.10})$	$\hat{P}(J_n(\alpha) \le -z_{0.05})$
			$\lambda = \frac{1}{4}$		
$J_n(1)$	0.014712	0.01566	0.004349	0.028	0.024
		(0.01561)	(0.004177)	(0.063)	(0.045)
$J_n(\frac{5}{4})$	0.010767	0.01216	0.002570	0.026	0.018
		(0.01267)	(0.002952)	(0.062)	(0.034)
$J_n(\frac{3}{2})$	0.008533	0.00949	0.001457	0.021	0.017
		(0.01018)	(0.001992)	(0.057)	(0.030)
$J_n(1.874)$	0.006163	0.00675	0.000606	0.030	0.015
		(0.00744)	(0.001058)	(0.052)	(0.029)
			$\lambda = \frac{4}{5}$		
$J_n(1)$	0.023713	0.01346	0.009897	0.515	0.471
		(0.01275)	(0.006160)	(0.523)	(0.452)
$J_n(\frac{5}{4})$	0.016307	0.01221	0.008866	0.293	0.229
		(0.01148)	(0.005551)	(0.334)	(0.267)
$J_n(\frac{3}{2})$	0.018470	0.01046	0.007457	0.141	0.114
		(0.00999)	(0.004715)	(0.191)	(0.137)
$J_n(1.874)$	0.007873	0.00799	0.005368	0.056	0.031
		(0.00788)	(0.003472)	(0.080)	(0.047)

Table 1 investigates the performance of $\hat{\sigma}_n^2$ and the accuracy of the normal approximation to $J_n(\alpha)$ in the case where $F(x) = 1 - e^{-x}$ and $H(x) = 1 - e^{-\lambda x}$, for the choices $\lambda = \frac{1}{4}$ and $\lambda = \frac{4}{5}$. Column 2 of Table 1 gives the values of σ_α^2 computed from Corollary 2.2. Column 3 gives the average value of $\hat{\sigma}_n^2$ over 1000 Monte Carlo replications. Column 4 gives the sample standard deviation of the 1000 $\hat{\sigma}_n^2$ values. Columns 4 and 5 give the frequency of the event $\{J_n(\alpha) \le -z_\eta\}$, where z_η is the η percentile of a standard normal distribution. The normal approximation of $J_n(\alpha)$ with small values of α is not good for $\lambda = \frac{4}{5}$ and the $J_n(\alpha)$ test with large values of α is conservative for the random censorship model.

3. Efficiency loss and power comparison. We shall compare the performance of some NBU tests $J_n(\alpha)$ in the proportional censoring model. To this end we consider the Pitman efficacy of the statistic V_n defined by

$$\text{(3.1)} \qquad \operatorname{eff}(V_n) = \lim_{n\to\infty} \{dE_\theta[V_n]/d\theta|_{\theta=0}\}^2/(n\operatorname{Var}_0[V_n]),$$

where $E_\theta[\cdot]$ is the expectation under some alternative F_θ with exponential for $\theta = 0$ and $\operatorname{Var}_0[\cdot]$ is the null variance. Then the efficiency loss due to censoring is defined by $\operatorname{eff}(V_n^c)/\operatorname{eff}(V_n)$, where V_n^c denotes the statistic constructed for the censored model and V_n the one for the uncensored. The discussion of the use of the Pitman efficacy is given in Chen, Hollander and Langberg (1983). For the

TABLE 2
Monte Carlo power comparison for the Weibull alternative $F_\theta(x) = 1 - \exp(-x^{\theta+1})$, $H(x) = 1 - (1 - F_\theta(x))^\lambda$ *and* $n = 50$ (25). *Entries are frequencies of samples declared significant with* 5% *significance level.*

Test	$\lambda = \frac{1}{4}$		$\lambda = \frac{8}{10}$	
	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.25$	$\theta = 0.50$
$J_n(1)$	0.232	0.728	0.588	0.746
	(0.228)	(0.572)	(0.560)	(0.690)
$J_n(\frac{5}{4})$	0.210	0.702	0.372	0.548
	(0.200)	(0.508)	(0.374)	(0.502)
$J_n(\frac{3}{2})$	0.216	0.708	0.186	0.388
	(0.186)	(0.478)	(0.218)	(0.344)
$J_n(1.874)$	0.238	0.718	0.064	0.234
	(0.172)	(0.556)	(0.102)	(0.204)

censoring d.f. $H(x) = 1 - e^{-\lambda x}$, $0 < \lambda < 1$, Theorem 4.1 of Koul (1978) and Corollary 2.2 yield that the efficiency loss of $S_n(u^\alpha)$, $\alpha \geq 1$, with respect to Koul's statistic is given by

$$(2\alpha - \lambda + 1)^3(2\alpha^2 + 2\alpha + 1)/\left[(2\alpha + 1)^3\left\{(\alpha + 1)^2 + (\alpha - \lambda)^2\right\}\right].$$

This expression reveals that the efficiency loss increases with α and decreases with λ. Note that, as is to be expected, this value tends to 1 as λ tends to 0 (corresponding to the case of no censoring).

Next we compute the efficacy (3.1) for the Weibull alternative $F_\theta(x) = 1 - \exp(-x^{1+\theta})$. The efficacy of $S_n(u^\alpha)$ is $(2\alpha - \lambda + 1)^3(\alpha + 1)^{-2}\{(\alpha + 1)^2 + (\alpha - \lambda)^2\}^{-1}$, which reaches its maximum given by a positive solution of $2\alpha(\alpha + 1)(\alpha - 2\lambda) = 1 + \lambda + \lambda^2 + \lambda^3$. For $\lambda = \frac{1}{4}$ and $\frac{4}{5}$, the approximate solution is given by $\alpha = 0.8928$ and $\alpha = 1.874$, respectively, but Corollary 2.2 implies that when $\lambda = \frac{1}{4}$ the optimal statistic is $S_n(u)$. It can be seen that the optimal value α increases with λ and such a result holds for the other NBU alternatives.

Table 2 gives results on 1000 Monte Carlo samples drawn from Weibull alternative F_θ in the proportional censoring model and each entry represents the percentage of samples declared significant by $J_n(\alpha)$ with a 5% significance level for some values of α. The $J_n(\alpha)$ test with small values of α has good performance, but has a large probability of a type I error. We recommend the $J_n(1.5)$ test for this testing problem from the preceding results.

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