The Annals of Statistics
1987, Vol. 15, No. 1, 376–384

A COMPARISON OF CLASSES OF SINGLE REPLICATE FACTORIAL DESIGNS

BY D. T. VOSS AND A. M. DEAN

Wright State University and Ohio State University

The classes of designs produced by well-known confounding methods are compared for single replicate factorial experiments involving a single blocking factor. It is shown that several of these methods produce equivalent classes of designs, whilst others produce a subset of this common class. It is shown that in certain experimental settings the use of pseudofactors does not enlarge the size of the common class of designs.

1. Introduction. The problem of confounding in factorial designs is well known. A variety of methods exists in the literature for the construction of single replicate designs for factorial experiments. The earliest method is by Bose (1947) and is only applicable for symmetrical prime-powered experiments. A number of authors have given generalizations of this method; see, for example, White and Hultquist (1965), Raktoe (1969, 1970), Worthley and Banerjee (1974) and Sihota and Banerjee (1981). All but the latest of these are reviewed and discussed by Raktoe, Rayner and Chalton (1978). Voss (1986) investigates the relationships between the various methods and shows that, with the exception of the method of Raktoe (1970) for experiments involving factors with number of levels being different powers of the same prime, they each produce classes of designs which form a subset of a more general class, namely the *general classical* class of designs.

More recently, authors have used a variety of techniques in order to provide confounding methods applicable in more general settings. Cotter (1974) presents a method applicable for general symmetrical factorial experiments based on orthogonal arrays, which is essentially a generalization of the method of Das (1964). A confounding method involving abelian groups produces the class of single replicate generalized cyclic designs [see John and Dean (1975) and Dean and John (1975)]. Patterson (1976) and Patterson and Bailey (1978) describe an algorithm called DSIGN which can be used for constructing factorial designs with a wide variety of blocking structures. Bailey, Gilchrist and Patterson (1977) and Bailey (1977) give a method of identifying confounding patterns in a large subclass of the designs produced by DSIGN. The latter method, which can also be used for construction, will be called the bilinear classical method in this paper. An alternative, but related, method is given by Collings (1984).

Comparative illustrations of the above construction methods and of various methods of determining the confounded degrees of freedom are given by Voss and Dean (1985).

Received August 1985; revised February 1986.

AMS 1980 *subject classifications.* Primary 62K15; secondary 62K10.

Key words and phrases. Confounding, DSIGN algorithm, factorial experiments, generalized cyclic designs, pseudofactors, single replicate designs.

The purpose of this paper is to show that the seemingly different construction methods, in most cases, produce equivalent classes of designs. In particular, it is shown in Section 2 that (i) the bilinear classical method, when restricted to the single replicate, single blocking-factor setting, is equivalent to the generalized cyclic method, (ii) for factor levels being equal or relatively prime, the general classical method is equivalent to the generalized cyclic method and (iii) the designs of Cotter (1974) form a subclass of the generalized cyclic class.

In Section 3, the use of pseudofactors is discussed. If prime-leveled pseudofactors are used, the generalized cyclic, bilinear classical and general classical methods generate the same class of designs, referred to in this paper as the pseudofactor class of designs. For experiments in which the numbers of levels of each factor has no prime-powered divisor, the classes of generalized cyclic, bilinear classical and general classical designs are each equivalent to the pseudofactor class in the sense of producing designs which confound the same number of degrees of freedom from the same factorial spaces.

2. Comparison of classes of designs. Throughout this paper we will consider a general $s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_k^{n_k}$ factorial experiment with $n = \sum_{i=1}^k n_i$ factors, F_{ij}, where F_{ij} has s_i levels, $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, k$. A treatment combination will be denoted by $t = t_{11}t_{12} \cdots t_{1n_1}t_{21} \cdots t_{kn_k}$, where $0 \le t_{ij} \le s_i - 1$ for $i = 1, 2, \ldots, k$. The factorial experiment will be arranged in b blocks of size k, where $bk = \prod_{i=1}^k s_i^{n_i}$. The hth block of a design will be denoted by $B_h = \{t\colon t \text{ occurs in the } h\text{th block}\}$. The block containing the zero treatment combination will be called the principal block and denoted by B_0.

The set of treatment combinations forms a group G_t with group operation *addition componentwise modulo* s_i defined as follows:

$$t_{11} \cdots t_{kn_k} + t'_{11} \cdots t'_{kn_k} = t^*_{11} \cdots t^*_{kn_k},$$

where $t^*_{ij} = (t_{ij} + t'_{ij}) \pmod{s_i}$, $j = 1, \ldots, n_i$, $i = 1, \ldots, k$.

Note that G_t can be represented as the direct sum $(\oplus_{i=1}^k \oplus_{j=1}^{n_i} Z(s_i))$ [see, for example, Herstein (1964), Chapter 4.5], where $Z(s_i)$ is the ring of integers modulo s_i, $i = 1, \ldots, k$.

REMARK. If s_i is a prime-power, then the ring $Z(s_i)$ may be replaced by the Galois field $\mathrm{GF}(s_i)$ and the group operation of G_t is addition componentwise $\mathrm{GF}(s_i)$. A similar remark applies to G_a below.

2.1. *Equivalence of generalized cyclic and bilinear classical designs.* Patterson (1965, 1976) presents an algorithm, called DSIGN, for the computer generation of factorial designs. The algorithm is applicable to any of the simple block structures described by Nelder (1965). The class of designs generated by the algorithm is large and includes the class of generalized cyclic designs as a special case. The use of prime-leveled pseudofactors in the DSIGN algorithm for both treatment and plot factors is described by Patterson and Bailey (1978), who give several examples.

In this paper, we restrict attention to single replicate experiments and only one blocking category. In addition, we consider only those designs for which each confounded degree of freedom corresponds to a contrast belonging to a single factorial space. Bailey, Gilchrist and Patterson (1977) discuss a theoretical framework which is applicable to such designs and which does not require the use of pseudofactors.

The DSIGN algorithm requires the specification of a key matrix, $K = [B', D']'$, where the rows of B define the generators of the subgroup of treatment combinations in the principal block, B_0. The other blocks of the design are the cosets of B_0, that is, the quotient group G_t/B_0.

Let T be a diagonal matrix with elements γs_i^{-1}, where γ is the lowest common multiple of $s_1, s_2, \dots, s_k$, and let P be a diagonal matrix with elements δp_i^{-1}, where δ is the lowest common multiple of the levels $p_1, p_2, \dots, p_q$ of the plot factors. Define $K_* = [B'_*, D'_*]' = P^{-1}KT$. If the inverse of K_* exists, then $K_*^{-1} = [E, C]$ and the columns of C specify the generators of the subgroup of degrees of freedom which are confounded [see Bailey, Gilchrist and Patterson, (1977)].

Now $B_*C = 0$ since $K_*K_*^{-1} = I$, and hence the rows of B_* generate the annihilator subgroup of the group of degrees of freedom generated by the columns of C. It is therefore possible to obtain the principal block corresponding to a given set, A, of confounded degrees of freedom by computing the annihilators of the generators of A. This method, which was introduced by Bailey (1977, 1982), is more general than that of Bailey, Gilchrist and Patterson (1977) since it does not require that K_* be invertible. For a general method of computing the annihilators of a subgroup see El Mossadeq, Kobilinsky and Collombier (1985). Collings (1984) also gives a method of determining K from C without inverting K_*^{-1}, but his method is only applicable to prime $s_1, s_2, \dots, s_k$ or to prime-leveled pseudofactors.

Since the annihilator method of Bailey (1977) is the most general of the DSIGN related construction methods mentioned above, it is the only approach that will be considered in this paper. When restricted to single replicate experiments involving only one blocking category, the resulting class of designs will be referred to as the class of bilinear classical designs.

Let G_t be defined as above. Let $G_a = \{a\colon a = a_{11}a_{12} \cdots a_{1n_1}a_{21} \cdots a_{kn_k};\ 0 \le a_{ij} \le s_i - 1;\ j = 1, 2, \dots, n_i;\ i = 1, 2, \dots, k\}$, and define the group operation of G_a to be addition componentwise modulo s_i. Then G_a and G_t are isomorphic groups.

For each pair of elements $t \in G_t$, $a \in G_a$, define the integer $[a, t]$ as

$$[a, t] = \sum_i \sum_j a_{ij} t_{ij} \gamma s_i^{-1} \pmod{\gamma},$$

where γ is the lowest common multiple of $s_1, s_2, \dots, s_k$.

DEFINITION (Bailey, 1977). A *bilinear classical design* is a design such that, for a fixed subgroup A of G_a, the principal block is the subgroup $B_0 = \{t\colon t \in G_t,\ [a, t] = 0 \text{ for all } a \in A\}$, and the other blocks are cosets of B_0.

Following Bailey, Gilchrist and Patterson (1977), each $a = a_{11} \cdots a_{1n_1}a_{21} \cdots a_{kn_k} \in G_a$ denotes one degree of freedom belonging to the interaction between those factors for which $a_{ij} \neq 0$. The set A represents the subgroup of degrees of freedom which are confounded with blocks.

Bailey (1977) states that the bilinear classical class of designs includes the generalized cyclic designs of John and Dean (1975) and Dean and John (1975), as well as the classes generated by the methods of White and Hultquist (1965) and Bose (1947). In Theorem 1 it is shown that the classes of single replicate generalized cyclic and bilinear classical designs are, in fact, identical.

DEFINITION (Dean and John, 1975). A single replicate *generalized cyclic design* is a design whose principal block, B_0, forms a subgroup of G_t, the other blocks being the cosets of B_0.

THEOREM 1. *The classes of single replicate generalized cyclic designs and bilinear classical designs are identical.*

PROOF. It follows from the definition of Bailey (1977) that any bilinear classical design is a generalized cyclic design.

Conversely, consider the generalized cyclic design with principal block B_0. Let $B_0^0 = \{a\colon a \in G_a, [a, t] = 0 \text{ for all } t \in B_0\}$ denote the annihilator of B_0. Since B_0 is a subgroup of G_t, the annihilator of B_0^0, namely $(B_0^0)^0 = \{t\colon t \in G_t, [a, t] = 0 \text{ for all } a \in B_0^0\}$, is B_0 itself. Thus $A = B_0^0$ generates the bilinear classical design with principal block B_0. □

REMARK. The DSIGN algorithm is capable of producing a wider class of single replicate designs, but the confounded degrees of freedom span more than one factorial space [see Patterson (1976)].

2.2. *Equivalence of the generalized cyclic (bilinear classical) and general classical designs.* For $i = 1, \dots, k$, let $G_{t_i} = \{t_i\colon t_i = t_{i1}t_{i2} \cdots t_{in_i}, 0 \le t_{ij} \le s_i - 1, j = 1, \dots, n_i\}$ and $G_{a_i} = \{a_i\colon a_i = a_{i1}a_{i2} \cdots a_{in_i}, 0 \le a_{ij} \le s_i - 1, j = 1, \dots, n_i\}$. Thus, $t = t_1t_2 \cdots t_k$ denotes a treatment combination in G_t and $a = a_1a_2 \cdots a_k$ an element of G_a. Then G_{t_i} and G_{a_i} form isomorphic groups under componentwise addition modulo s_i.

DEFINITION (Voss, 1986). A *general classical design* is a design such that, for fixed subgroups $A_i \subset G_{a_i}$, $i = 1, 2, \dots, k$, the principal block is the subgroup $B_0 = \{t\colon t = t_1t_2 \cdots t_k \in G_t, t_i \in G_{t_i}, \text{and } \sum_{j=1}^{n_i} a_{ij}t_{ij} = 0 \pmod{s_i} \text{ for all } a_i \in A_i, i = 1, 2, \dots, k\}$, and the other blocks are obtained as cosets of B_0. If s_i is a prime power, $GF(s_i)$ can be used instead of $Z(s_i)$.

The general classical method is equivalent to applying the classical method of Bose (1947) separately within symmetric subexperiments using either $Z(s_i)$ or $GF(s_i)$. As such, the resulting class of designs encompasses the classes of designs generated by the method of Bose (1947) and the generalizations of White and Hultquist (1965), Raktoe (1969), Worthley and Banerjee (1974) and Sihota and Banerjee (1981), and that subclass of the designs of Raktoe (1970) for which any two factors have either the same or relatively prime numbers of levels. That each of these methods generates a subclass of the general classical designs is a consequence of the existence of a unique decomposition of each element of the superalgebra employed by each author into the sum of elements, one mapped from each subring or subfield employed [see Voss (1986)].

THEOREM 2. *The class of general classical designs using the rings* $Z(s_i)$, $i = 1, 2, \dots, k$, *forms a subclass of the generalized cyclic (bilinear classical) designs.*

PROOF. From the definition of a general classical design, B_0 is a subgroup of G_t and the other blocks are cosets, so the result follows. □

For the general classical method, the smallest block size possible in an $s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_k^{n_k}$ experiment keeping all main effects unconfounded is $\Pi_{i=1}^{k} s_i$ [see Voss (1986)]. The generalized cyclic design generated by $11 \cdots 1$ has all main effects unconfounded and a block size equal to the lowest common multiple of $s_1, s_2, \dots, s_k$ [see Dean and John (1975)]. Hence, if $s_1, s_2, \dots, s_k$ are not all relatively prime, the general classical designs form a proper subclass of the generalized cyclic designs. However, when $k = 1$ or $s_1, s_2, \dots, s_k$ are relatively prime, the two classes of designs coincide as shown by Theorems 3 and 4.

THEOREM 3. *For an* $s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_k^{n_k}$ *experiment, the generalized cyclic (bilinear classical) designs form a subclass of the general classical designs if* $s_1, s_2, \dots, s_k$ *are relatively prime.*

PROOF. From Theorem 1, the principal block of a generalized cyclic design can be expressed in the form $B_0 = \{t\colon [a, t] = 0$ for all $a \in A\}$ for some subgroup $A \subset G_a$, where $[a, t] = \gamma \Sigma_i s_i^{-1} \Sigma_j a_{ij} t_{ij} \pmod{\gamma} = \gamma \Sigma_i s_i^{-1} m_i \pmod{\gamma}$ for $m_i = \Sigma_j a_{ij} t_{ij}$ and where $\gamma = \Pi_i s_i$ since $s_1, s_2, \dots, s_k$ are relatively prime. Consider $t \in B_0$. Since $[a, t] = 0 \pmod{\gamma}$, $\Sigma_i s_i^{-1} m_i$ is an integer. For $s_1, s_2, \dots, s_k$ relatively prime, it can then be shown that $m_i = q_i s_i$ for some integer q_i, that is, $m_i = \Sigma_j a_{ij} t_{ij} = 0 \pmod{s_i}$, $i = 1, 2, \dots, k$. Writing $a_i = a_{i1} a_{i2} \cdots a_{in_i}$ and $a = a_1 a_2 \cdots a_k$, let $A_i = \{a_i\colon a_1 a_2 \cdots a_k \in A\}$ Then $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, where $\oplus$ denotes external direct sum. Hence,

$$B_0 \subset B_0^* = \left\{t\colon \Sigma_j a_{ij} t_{ij} = 0 \pmod{s_i} \text{ for all } a_i \in A_i,\ i = 1, \dots, k\right\}.$$

Also, if $t \in B_0^*$, then s_i divides m_i, so $[a, t] = \gamma \Sigma_i s_i^{-1} m_i = 0 \pmod{\gamma}$ and $t \in B_0$. Hence, $B_0 = B_0^*$, which is the principal block of a general classical design. Other blocks are cosets. □

THEOREM 4. *The classes of generalized cyclic, bilinear classical and general classical designs are equivalent using the rings $Z(s_i)$, $i = 1, 2, \ldots, k$, for $s_1, s_2, \ldots, s_k$ relatively prime.*

PROOF. Follows directly from Theorems 1, 2 and 3. □

REMARK. For any s_i which is a prime-power, one could use GF(s_i) in place of $Z(s_i)$ in the construction of both generalized cyclic and general classical designs, in which case the resulting classes are still equivalent.

2.3. *Equivalence of orthogonal array designs and a subclass of generalized cyclic designs.* The symmetrical orthogonal array designs of Cotter (1974) form a subclass of the generalized cyclic designs as shown in Theorem 5.

DEFINITION (Cotter, 1974). For an s^n single replicate experiment, an *orthogonal array design* consists of s^{n-m} blocks of size s^m where the principal block $B_0 \subset G_t$ is given by $B_0 = \{t = t_1 \cdots t_m t_{m+1} \cdots t_n\colon t_i \in Z(s),$ $i = 1, \ldots, m,$ and $t_j = \sum_{i=1}^m c_{ij} t_i \pmod{s},\ c_{ij} \in Z(s),\ j = m+1, \ldots, n\}$. The c_{ij} are chosen so that the elements of B_0, when written as an $n \times s^m$ array, form an orthogonal array of strength $p, 1 \le p \le m$. The hth block of the design, $h = 0, \ldots, s^{n-m} - 1$, is of the form

$$B_h = \Big\{t'\colon t' = t + b_h,\ t \in B_0 \text{ and } b_h = \big(0, \ldots, 0, b_{(m+1)h}, \ldots, b_{nh}\big),$$
$$\text{for fixed } b_{jh} \in Z(s),\ j = m+1, \ldots, n\Big\}.$$

THEOREM 5. *The orthogonal array designs are equivalent to a subclass of the class of generalized cyclic designs.*

PROOF. Consider an orthogonal array design with principal block B_0. Let $t = t_1 \cdots t_n$, $t' = t'_1 \cdots t'_n \in B_0$, and let $t + t' = t'' = t''_1 \cdots t''_n$ with addition defined componentwise modulo s. Then $t''_j = t_j + t'_j \pmod{s}$, $j = 1, \ldots, m$. For $j = m+1, \ldots, n$, $t''_j = \sum_i c_{ij} t_i + \sum_i c_{ij} t'_i \pmod{s} = \sum_i c_{ij}(t_i + t'_i) \pmod{s}$ $= \sum c_{ij} t''_i \pmod{s}$. Hence $t''_j \in B_0$ and B_0 is a group. From the definition B_h is a coset of B_0. □

REMARK. The method of Cotter (1974) is essentially a generalization of the method of Das (1964) which was only applicable for s being a power of a prime.

2.4. *Other classes of designs.* Mukerjee (1981) and Gupta (1983) considered the construction of Kronecker product designs, a design being obtained by taking Kronecker products of incidence matrices of varietal designs. In the single replicate case, each design produced by these methods has incidence matrix of the form $N = N_1 \otimes N_2 \otimes \cdots \otimes N_n$, where N_i is the incidence matrix of a single replicate varietal design in the levels of F_i. As a consequence, some main effect degree(s) of freedom must be confounded in any single replicate block design.

Hence, the corresponding classes of designs will not be considered further in this paper.

The designs of Raktoe (1970) are shown by Voss (1986) to be general classical designs if, for each pair of factors, the respective numbers of levels are either equal or relatively prime. Otherwise, the method of Raktoe (1970) does not seem in general to generate a large class. For example, in a 2×4 experiment, the only block sizes possible are 1, 2 and 8, whilst the methods considered in this paper also allow blocks of size 4. This class of designs will not be considered further in this paper.

3. Use of pseudofactors. Several authors have discussed the use of pseudofactors; see, for example, Kempthorne (1952, page 343), Giovagnoli (1977) and Collings (1984). Pseudofactors are commonly employed to enlarge the class of designs generated by a given method, and it is stated by Giovagnoli (1977) that the use of prime-leveled pseudofactors gives the largest class of designs. The purpose of this section is to identify the cases in which the use of pseudofactors does not enlarge the class of designs under study.

By virtue of Theorem 4, all the standard construction methods yield exactly the same class of designs when prime-leveled pseudofactors are used. This common class of designs will be called the *pseudofactor class of designs*. In Theorem 6 we show that, under certain restrictions, the class of single replicate generalized cyclic (bilinear classical) designs using $Z(s_i)$ and the class of pseudofactor designs are equivalent in the sense of the following definition.

DEFINITION. Two classes of designs will be called *d.f. equivalent* if for each design in one class there exists a design in the other class with the same number of degrees of freedom confounded in each factorial space.

THEOREM 6. *Consider an $s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_k^{n_k}$ experiment such that s_i has no prime-powered divisor, for $i = 1, 2, \ldots, k$. Then the classes of generalized cyclic (bilinear classical) and pseudofactor designs are d.f. equivalent.*

PROOF. Let γ denote the lowest common multiple of $s_1, \ldots, s_k$. Since each s_i has no prime-powered divisor, γ factors into distinct primes $p_1, \ldots, p_m$, say. Thus, for $i = 1, \ldots, k$, $s_i = p_{j_1} \times \cdots \times p_{j_{m(i)}}$ for some $\{j_1, \ldots, j_{m(i)}\} \subset \{1, \ldots, m\}$. Let $R_i = \{(r_{j_1}, \ldots, r_{j_{m(i)}})\colon r_{j_q} \in Z(p_{j_q}),\ q = 1, \ldots, m(i)\}$. Let $\phi_i\colon Z(s_i) \to R_i$ be the function defined by $\phi_i(z) = z \times 1'_{m(i)}$, where $1'_{m(i)}$ denotes the $1 \times m(i)$ vector of unit elements and where the qth component is reduced modulo p_{j_q}, $q = 1, \ldots, m(i)$. Note that ϕ_i is a one-to-one function and $\phi_i(z) = 0 \times 1'_{m(i)}$ if and only if $z = 0 \pmod{s_i}$. Defining addition of vectors in R_i to be componentwise addition modulo p_{j_q}, it follows that ϕ_i is an isomorphism.

Now, for each generalized cyclic (bilinear classical) design, the degrees of freedom confounded with blocks correspond to a fixed subgroup $A \subset G_a$, where $G_a = \{a = a_{11} \cdots a_{1n_1}a_{21} \cdots a_{kn_k}\colon a_{ij} \in Z(s_i),\ i = 1, \ldots, k,\ j = 1, \ldots, n_i\}$.

Similarly, for each pseudofactor design, the degrees of freedom confounded with blocks correspond to a fixed subgroup $A^* \subset G_a^*$, where

$$G_a^* = \{\phi(a) = \phi_1(a_{11}) \cdots \phi_1(a_{1n_1})\phi_2(a_{21}) \cdots \phi_k(a_{kn_k}) \colon a \in G_a\}.$$

Defining addition of elements of G_a^* to be addition componentwise modulo p_{j_q}, it follows that $\phi\colon G_a \to G_a^*$ is an isomorphism and $\phi_i(a_{ij}) = 0 \times 1'_{m(i)}$ if and only if $a_{ij} = 0 \pmod{s_i}$. Hence, given the subgroup A for a generalized cyclic design, $\phi(A) = \{\phi(a)\colon a \in A\}$ determines a d.f. equivalent pseudofactor design, and similarly, given the subgroup A^* for a pseudofactor design, $\phi^{-1}(A^*) = \{a\colon a \in A, \phi(a) \in A^*\}$ determines a d.f. equivalent generalized cyclic design. □

A result corresponding to that of Theorem 6 does not hold if any s_i has a prime-powered divisor. For example, Giovagnoli (1977) gives a $2 \times 3 \times 4 \times 6$ single replicate pseudofactor design in 12 blocks of size 12 confounding $F_1F_3(1)$ (that is, one degree of freedom from the interaction between F_1 and F_3), $F_3F_4(1)$, $F_2F_4(2)$, $F_1F_3F_4(1)$, $F_2F_3F_4(2)$ and $F_1F_2F_3F_4(4)$. Consider a $2 \times 3 \times 4 \times 6$ generalized cyclic design with principal block denoted by B_0 and the same confounding pattern as above, and let $t = t_1t_2t_3t_4$ denote a treatment combination. Then, the requirement that F_2, F_3 and F_2F_3 are unconfounded implies that each of the 12 subtreatment combinations t_2t_3 occurs in B_0, including $t_2t_3 = 11$; Since $|B_0| = 12$ and $t_1 11 t_4$ generates a subgroup of size 12 for any choice of t_1 and t_4, B_0 is a cyclic group. However, B_0 cyclic with F_1 at 2 levels and F_4 at 6 levels implies at most 6 subtreatment combinations t_1t_4 occur in B_0, and hence a degree of freedom from F_1, F_4 or F_1F_4 is confounded. Hence, no generalized cyclic design with the above confounding pattern exists.

REMARK. Since this paper was first written, a result similar to that of Theorem 6 has been published by Bailey (1985).

REFERENCES

BAILEY, R. A. (1977). Patterns of confounding in factorial designs. *Biometrika* **64** 597–603.

BAILEY, R. A. (1982). Dual abelian groups in the design of experiments. In *Algebraic Structures and Applications* (P. Schultz, C. E. Praeger and R. P. Sullivan, eds.). Dekker, New York.

BAILEY, R. A. (1985). Factorial design and abelian groups. *Linear Algebra Appl.* **70** 349–368.

BAILEY, R. A., GILCHRIST, F. H. L. and PATTERSON, H. D. (1977). Identification of effects and confounding patterns in factorial designs. *Biometrika* **64** 347–354.

BOSE, R. C. (1947). Mathematical theory of the symmetrical factorial design. *Sankhyā* **8** 107–166.

COLLINGS, B. J. (1984). Generating the intrablock and interblock subgroups for confounding in general factorial experiments. *Ann. Statist.* **12** 1500–1509.

COTTER, S. C. (1974). A general method of confounding for symmetrical factorial experiments. *J. Roy. Statist. Soc. Ser. B* **36** 267–276.

DAS, M. N. (1964). A somewhat alternative approach for construction of symmetrical factorial designs and obtaining maximum number of factors. *Calcutta Statist. Assoc. Bull.* **13** 1–17.

DEAN, A. M. and JOHN, J. A. (1975). Single replicate factorial experiments in generalized cyclic designs: II. Asymmetrical arrangements. *J. Roy. Statist. Soc. Ser. B* **37** 72–76.

EL MOSSADEQ, A., KOBILINSKY, A. and COLLOMBIER, D. (1985). Construction d'orthogonaux dans les groupes abéliens finis et confusions d'effects dans les plans factoriels. *Linear Algebra Appl.* **70** 303–320.

GIOVAGNOLI, A. (1977). On the construction of factorial designs using abelian group theory. *Rend. Sem. Mat. Univ. Padova* **58** 195–206.

GUPTA, S. C. (1983). Some new methods for constructing block designs having orthogonal factorial structure. *J. Roy. Statist. Soc. Ser. B* **45** 297–307.

HERSTEIN, I. N. (1964). *Topics in Algebra*. Ginn, Waltham, Mass.

JOHN, J. A. and DEAN, A. M. (1975). Single replicate factorial experiments in generalized cyclic designs: I. Symmetrical arrangements. *J. Roy. Statist. Soc. Ser. B* **37** 63–71.

KEMPTHORNE, O. (1952). *The Design and Analysis of Experiments*. Wiley, New York.

MUKERJEE, R. (1981). Construction of effect-wise orthogonal factorial designs. *J. Statist. Plann. Inference* **5** 221–229.

NELDER, J. A. (1965). The analysis of randomized experiments with orthogonal block structure. 1. Block structure and the null analysis of variance. *Proc. Roy. Soc. London A* **283** 147–162.

PATTERSON, H. D. (1965). Factorial combination of treatments in rotation experiments. *J. Agric. Sci.* **65** 171–182.

PATTERSON, H. D. (1976). Generation of factorial designs. *J. Roy. Statist. Soc. Ser. B* **38** 175–179.

PATTERSON, H. D. and BAILEY, R. A. (1978). Design keys for factorial experiments. *J. Roy. Statist. Soc. Ser. C* **27** 335–343.

RAKTOE, B. L. (1969). Combining elements from distinct finite fields in mixed factorials. *Ann. Math. Statist.* **40** 498–504.

RAKTOE, B. L. (1970). Generalized combining of elements from finite fields. *Ann. Math. Statist.* **41** 1763–1767.

RAKTOE, B. L., RAYNER, A. A. and CHALTON, D. O. (1978). On construction of confounded mixed factorial and lattice designs. *Austral. J. Statist.* **20** 209–218.

SIHOTA, S. S. and BANERJEE, K. S. (1981). On the algebraic structures in the construction of confounding plans in mixed factorial designs on the lines of White and Hultquist. *J. Amer. Statist. Assoc.* **76** 996–1001.

VOSS, D. T. (1986). On generalizations of the classical method of confounding to asymmetric factorial experiments. *Comm. Statist. A—Theory Methods* **15** 1299–1314.

VOSS, D. T. and DEAN, A. M. (1985). Methods of confounding in single replicate factorial designs. Technical Report 318, Dept. of Statistics, Ohio State Univ.

WHITE, D. and HULTQUIST, R. A. (1965). Construction of confounding plans for mixed factorial designs. *Ann. Math. Statist.* **36** 1256–1271.

WORTHLEY, R. and BANERJEE, K. S. (1974). A general approach to confounding plans in mixed factorial experiments when the number of levels of a factor is any positive integer. *Ann. Statist.* **2** 579–585.

DEPARTMENT OF MATHEMATICS
AND STATISTICS
WRIGHT STATE UNIVERSITY
DAYTON, OHIO 45435

DEPARTMENT OF STATISTICS
OHIO STATE UNIVERSITY
1958 NEIL AVENUE
COLUMBUS, OHIO 43210