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ADMISSIBLE ESTIMATION OF THE BINOMIAL PARAMETER n

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Suppose that X has a binomial distribution $B(n, p)$, with known $p \in (0,1)$ and unknown $n \in \{1, 2, \dots\}$. A natural estimator for n is given by $T(0) = 1$, $T(x) = x/p$, $x = 1, 2, \dots$. This estimator is shown to be inadmissible under quadratic loss. It is shown that modifying $T(0)$ to $T(0) = -(1-p)/(p \ln p)$ results in an admissible estimator. For $p \geq \frac{1}{2}$ it is further shown that this is the only admissible modification of $T(0)$. A partial result is also obtained for $p < \frac{1}{2}$.

1. Introduction. Suppose that X has a binomial distribution $B(n, p)$, with known $p \in (0,1)$, and consider estimating the unknown parameter n. The problem has an obvious interpretation: estimate the number of tosses of a (biased) coin on the basis of the observed number of heads [hence the title of the paper by Ghosh and Meeden (1975)]. For a practical application (an animal counting problem), see Rukhin (1975). Estimation of n, when several independent observations are available, is considered by Feldman and Fox (1968), who develop some asymptotic results, and, more recently, by Olkin, Petkau, and Zidek (1981), who compare the stability of various estimators. Draper and Guttman (1971) present a Bayesian treatment. We shall be concerned with the decision theoretic aspects of the problem.

When the parameter space is $N = \{0, 1, 2, \dots\}$, Rukhin (1975) has shown that the estimator T^0, given by

$$T^0(x) = x/p, \qquad x = 0, 1, 2, \dots,$$

is (i) a variant of the maximum likelihood estimator, (ii) the only unbiased estimator, and (iii) minimax under a weighted square error loss function. Ghosh and Meeden (1975) then showed that T^0 is admissible (and minimax) under quadratic loss.

In this paper, we consider the case when the parameter space is $N^+ = \{1, 2, \dots\}$; i.e., when the possibility $n = 0$ is excluded. This may be a more natural parameter space in some contexts, e.g., in the context of tossing a coin. The class of estimators to be considered is not restricted to that of range-preserving (integer-valued) estimators. Also, quadratic loss is assumed throughout.

2. Preliminary analysis. When the parameter space is restricted to N^+, the family of distributions $\{B(n, p),\ n \in N^+\}$ is not complete $(E\{-(1-p)/p\}^X = 0$ for all $n \in N^+)$, and T^0 is no longer the only unbiased

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estimator. In fact, it is known that a UMVU estimator of $n \in N^+$ does not exist. Also, since $T^0(0) < 1$, T^0 is clearly inadmissible for $n \in N^+$ [see the remark by Ghosh and Meeden (1975), page 524]. The obvious modification is to increase $T^0(0)$ to 1, which is in fact the maximum likelihood value. The resulting estimator is denoted by T^1:

$$T^1(0) = 1; \qquad T^1(x) = x/p, \qquad x = 1, 2, \dots .$$

However, as will be shown in Section 4, T^1 is not admissible either. Since the dominating estimator constructed in Section 4 is rather complicated, we illustrate this result by giving a simple dominating estimator for the special case $p = \frac{1}{2}$. For $p = \frac{1}{2}$, it can be shown that the estimator T^1,

$$T^1(0) = 1, \qquad T^1(x) = 2x, \qquad x = 1, 2, \dots ,$$

is dominated by the estimator T, given by

$$T(0) = 1.1, \qquad T(1) = 1.98, \qquad T(2) = 4.01,$$
$$T(x) = 2x, \qquad x = 3, 4, \dots .$$

The source of inadmissibility of T^1 is apparently the value of $T^1(0) = 1$. [Intuitively, $T(0)$ should be a (decreasing) function of p—and not a constant.] The question considered in this paper is how to modify the value of $T^1(0)$ so that the resulting estimator is admissible. The motivation for the answer (which, for $p \geq \frac{1}{2}$ turns out to be unique) comes from the observation, made by Rukhin (1975) and also by Ghosh and Meeden (1975), that, for $x \geq 1$, x/p is the generalized Bayes estimator of n with respect to the improper prior

$$\pi(n) = \frac{1}{n}, \qquad n = 1, 2, 3, \dots .$$

(Note that this prior excludes the possibility $n = 0$; therefore it is more natural in the present case.) Now with respect to this prior, the estimate of n, when $x = 0$, is

$$\left(\sum_{n=1}^{\infty} q^n\right) \Big/ \left(\sum_{n=1}^{\infty} q^n/n\right) = -\frac{q}{p \ln p},$$

where $q = 1 - p$. It will be shown in the next section that the resulting estimator, denoted by T^*,

$$T^*(0) = -q/(p \ln p), \qquad T^*(x) = x/p, \qquad x = 1, 2, \dots ,$$

is admissible. In Section 4 we further show that, for $p \geq \frac{1}{2}$, this is the only admissible modification (for $p < \frac{1}{2}$ we obtain a partial result).

The following simplifying notation is used:

$$q = 1 - p, \qquad r = q/p, \qquad c^* = -q/(p \ln p).$$

Note that

$$T^*(0) = c^* = r/\ln(1 + r).$$

Also, T^c denotes the estimator

$$T^c(0) = c, \qquad T^c(x) = x/p, \qquad x = 1, 2, \dots. \tag{2.1}$$

The estimators T^0, T^1 and T^* then correspond to $c = 0$, 1 and c^*, respectively.

3. Admissibility of T^*. In this section we show that

THEOREM 1. *The estimator T^* is admissible for $n \in N^+$ under quadratic loss.*

The method of proof is a well-known method [see Blyth and Roberts (1972) and Blyth (1974)], originally due to Hodges and Lehmann (1951). [This is one of the two methods used by Ghosh and Meeden (1975) to prove the admissibility of T^0 when $n \in N$.]

Consider the estimator T^c, defined by (2.1). T^c is inadmissible if there exists another estimator T such that

$$E_n(T - n)^2 \le E_n(T^c - n)^2 \tag{3.1}$$

for all $n \in N^+$, with strict inequality for some n [E_n denotes expectation with respect to $B(n, p)$]. Letting $Z = T - T^c$, (3.1) may be restated as

$$E_n(Z^2) + 2E_n\{Z(T^c - n)\} \le 0. \tag{3.2}$$

As in Ghosh and Meeden (1975), it may be shown that

$$E_n\{Z(T^c - n)\} = nr\{E_n(Z) - E_{n-1}(Z)\} + cq^n Z(0), \tag{3.3}$$

for all $n \in N^+$, where we define $E_0(Z) = Z(0)$. Therefore (3.2) may be restated as

$$E_n(Z^2) + 2nr\{E_n(Z) - E_{n-1}(Z)\} + 2cq^n Z(0) \le 0. \tag{3.4}$$

Clearly, (3.4) implies the weaker inequality

$$\{E_n(Z)\}^2 + 2nr\{E_n(Z) - E_{n-1}(Z)\} + 2cq^n Z(0) \le 0. \tag{3.5}$$

Letting

$$m_0 = Z(0); \qquad m_n = E_n(Z), \qquad n = 1, 2, \dots,$$

inequality (3.5) may be restated as

$$m_n^2 + 2nr(m_n - m_{n-1}) + 2cm_0 q^n \le 0. \tag{3.6}$$

Therefore, to prove the admissibility of T^c, it is sufficient to show that the only solution $\{m_n, n \in N\}$ to the system of inequalities (3.6) is the trivial solution $m_n \equiv 0$; since this would then imply that there is no Z satisfying (3.2) except the trivial solution $Z \equiv 0$, i.e., there is no T satisfying (3.1).

In this and the next section, we show that the system of inequalities (3.6) has no nontrivial solution only if $c = c^*$. In the proof we use the following result, which is based on the admissibility proof of Ghosh and Meeden (1975) for the case $n \in N$.

LEMMA 1. *Suppose that the sequence $\{b_n,\ n \in N\}$ satisfies*

$$b_n^2 + an(b_n - b_{n-1}) \le 0,$$

for all $n \in N^+$, where $a > 0$. If $b_k \le 0$ for some $k \in N$, then $b_n = 0$ for all $n \ge k$.

PROOF OF THEOREM 1. It is sufficient to show that the inequalities

$$m_n^2 + 2nr(m_n - m_{n-1}) + 2c^* m_0 q^n \le 0, \qquad n \in N^+ \tag{3.7}$$

imply that $m_n \equiv 0$. If $m_0 = 0$, this immediately follows from Lemma 1. We show that the other possibilities $m_0 < 0$ and $m_0 > 0$ result in contradictions. First, we note that (3.7) may be restated as $(m_n^2/n) + 2r(m_n - m_{n-1}) + 2c^* m_0 (q^n/n) \le 0$. Hence, for all $n \in N^+$,

$$\sum_{i=1}^{n} (m_i^2/i) + 2r(m_n - m_0) + 2c^* m_0 \sum_{i=1}^{n} (q^i/i) \le 0,$$

which may be rewritten as

$$t_n + 2rm_n - 2c^* m_0 d_n \le 0, \tag{3.8}$$

where

$$t_n = \sum_{i=1}^{n} (m_i^2/i) \quad \text{and} \quad d_n = -\ln p - \sum_{i=1}^{n} (q^i/i).$$

It is convenient to define $t_0 = 0$, so that $m_n^2 = n(t_n - t_{n-1})$ holds for all $n \in N^+$. Note that $d_n \downarrow 0$ as $n \to \infty$.

(i) Suppose that $m_0 < 0$. Then (3.8) implies that, for all $n \in N^+$, $t_n + 2rm_n \le 0$. Since $t_n \ge 0$, it follows that $t_n^2 \le 4r^2 m_n^2$. But $m_n^2 = n(t_n - t_{n-1})$. Therefore $t_n^2 - 4r^2 n(t_n - t_{n-1}) \le 0$, for all $n \in N^+$. Since $t_0 = 0$, it follows from Lemma 1 (with $b_n = -t_n$) that $t_n \equiv 0$. Hence $m_n = 0$, for all $n \in N^+$. But then (3.8), with $n = 1$, gives $m_0 c^* d_1 \ge 0$. Since $c^* d_1 > 0$, this contradicts the assumption $m_0 < 0$.

(ii) Suppose that $m_0 > 0$. Then (3.7) implies that

$$m_n^2 + 2rn(m_n - m_{n-1}) \le 0, \qquad n \in N^+.$$

If $m_k \le 0$ for some $k \ge 1$, then by Lemma 1, $m_n = 0$ for all $n \ge k$. But in that case, (3.7) (with $n = k + 1$, say) implies that $c^* m_0 < 0$, which contradicts $m_0 > 0$. Therefore $m_n > 0$ for all n. But then (3.8) implies that $t_n \le 2c^* m_0 d_n$ for all $n \in N^+$. Since $t_n \ge m_1^2$ and $d_n \downarrow 0$ as $n \to \infty$, it follows that $m_1 = 0$. This contradiction completes the proof of Theorem 1. □

4. Inadmissibility results. The question considered in this section is whether there is any other value of $c \ne c^*$ for which T^c is admissible. For $p \ge \frac{1}{2}$ we have a complete answer in

THEOREM 2. *For $p \ge \frac{1}{2}$, the estimator T^c is inadmissible if*

$$c \ne c^* = -(1 - p)/(p \ln p).$$

For $p < \frac{1}{2}$ we have a partial answer in

THEOREM 3. *For $p < \frac{1}{2}$, the estimator T^c is inadmissible if either $c < 1/(\ln 2)$ or $c > (1-p)/(p \ln 2)$.*

COROLLARY. *The estimator T^1 is inadmissible.*

REMARK. As regards the values of c not covered by Theorem 3, we observe that even if for some $c \neq c^*$ T^c is admissible, the method used in Section 3 fails to prove it (this follows from Lemma 2 below). Also, this remains true of a more refined version of the method [see Blyth and Roberts (1972)], wherein instead of (3.5) the stronger inequality

$$\{\mathrm{Cov}(T^c, Z)\}^2/\mathrm{Var}(T^c) + \{E(Z)\}^2 + 2E\{Z(T^c - n)\} \le 0$$

is used [it can be shown that, for δ sufficiently small, Z, as defined by (4.1) below, satisfies this inequality as well].

To prove the inadmissibility of T^c, it is sufficient to show that inequality (3.4) has a nontrivial solution for Z; since then $T = T^c + Z$ would dominate T^c. But first we consider the weaker inequality (3.5), and show that, for any $c \neq c^*$, this has a nontrivial solution.

LEMMA 2. *If $c \neq c^*$, then Z, defined by*

$$Z(x) = \delta(-r)^x \int_0^1 \frac{t^x}{1+rt}\,dt, \qquad x = 0, 1, \dots, \tag{4.1}$$

satisfies (3.5), *provided δ is between* 0 *and* $2(1 - c/c^*)$.

(A proof of this lemma is given in the Appendix.) However, it can be shown that Z, as defined by (4.1), does not satisfy the stronger inequality (3.4). A suitable modification is given in the following lemma, which is proved in the Appendix.

LEMMA 3. *Let Z be defined by*

$$Z(x) = \delta\{\varepsilon + (-1)^x\} r^x \int_0^\theta \frac{t^x}{1+rt}\,dt, \qquad x = 0, 1, \dots.$$

If $\varepsilon\delta > 0$, $0 < \theta < \min\{1, 1/r\}$, and

$$|\delta| < |\varepsilon|(1 - r\theta)/(1 + |\varepsilon|)^2, \tag{4.2}$$

then

$$E_n(Z^2) + 2nr\{E_n(Z) - E_{n-1}(Z)\} + 2cZ(0)q^n \le 2\delta q^n g_n(\theta, \varepsilon), \tag{4.3}$$

where

$$g_n(\theta, \varepsilon) = \{\varepsilon(1 - r\theta)/2\} + (1 - \theta)^n + \{(c/r)(1 + \varepsilon)\ln(1 + r\theta)\} - 1.$$

PROOF OF THEOREM 2. First note that $p \geq \frac{1}{2}$ corresponds to $r \leq 1$.

(i) Suppose that $c < c^*$. Since

$$g_1(\theta, \varepsilon) \to (c/r)\ln(1 + r) - 1 = (c/c^*) - 1 < 0, \quad \text{as } \varepsilon \to 0 \text{ and } \theta \to 1,$$

there exist $\varepsilon^* > 0$ and $0 < \theta^* < 1$ such that $g_1(\theta^*, \varepsilon^*) < 0$. Then, since $(1 - \theta^*)^n \leq (1 - \theta^*)$, we have $g_n(\theta^*, \varepsilon^*) < 0$, for all $n \in N^+$. Now choose δ^* between 0 and $\varepsilon^*(1 - r\theta^*)/(1 + \varepsilon^*)^2$. Applying Lemma 3, it follows that (3.4) has a nontrivial solution.

(ii) Suppose that $c < c^*$. Since

$$\varepsilon(1 - r\theta)/2 + (c/r)(1 + \varepsilon)\ln(1 + r\theta) \to (c/c^*) - 1 > 0, \quad \text{as } \varepsilon \to 0 \text{ and } \theta \to 1,$$

there exist $\varepsilon^* < 0$ and $0 < \theta^* < 1$ such that

$$\varepsilon^*(1 - r\theta^*)/2 + (c/r)(1 + \varepsilon^*)\ln(1 + r\theta^*) - 1 > 0.$$

Since $(1 - \theta^*)^n \geq 0$, it follows that $g_n(\theta^*, \varepsilon^*) > 0$, for all $n \in N^+$. Now choose $\delta^* < 0$ such that condition (4.2) is satisfied. It then follows from Lemma 3 that (3.4) has a nontrivial solution. □

PROOF OF THEOREM 3. The proof is similar to the proof of Theorem 2, but with $\theta \to 1/r$ (note that in this case $r > 1$). □

APPENDIX

PROOF OF LEMMA 2. It is easily shown that

$$Z(0) = (\delta/r)\ln(1 + r) = \delta/c^*,$$

$$E_n(Z) = \delta q^n \int_0^1 \frac{(1 - t)^n}{1 + rt}\, dt,$$

and

$$nr\{E_n(Z) - E_{n-1}(Z)\} = -\delta q^n$$

(see the proof of Lemma 3). Since $0 < \{(1 - t)^n/(1 + rt)\} < 1$ for $0 < t < 1$, we have

$$\{E_n(Z)\}^2 \leq \{|\delta| q^n\}^2 \leq \delta^2 q^n.$$

Therefore

$$\{E_n(Z)\}^2 + 2nr\{E_n(Z) - E_{n-1}(Z)\} + 2cZ(0)q^n \leq q^n\delta\{\delta - 2(1 - c/c^*)\} \leq 0,$$

for any δ between 0 and $2(1 - c/c^*)$. □

PROOF OF LEMMA 3. We have

$$E_n(Z) = \delta\varepsilon A_n + \delta B_n,$$

where

$$A_n = \sum_{x=0}^{n} \left\{ \binom{n}{x} p^x q^{n-x} r^x \int_0^\theta \frac{t^x}{1+rt}\, dt \right\}$$

and

$$B_n = \sum_{x=0}^{n} \left\{ \binom{n}{x} p^x q^{n-x} (-r)^x \int_0^\theta \frac{t^x}{1+rt}\, dt \right\}.$$

It is easily shown that

$$A_n = q^n \int_0^\theta \frac{(1+t)^n}{1+rt}\, dt \quad \text{and} \quad B_n = q^n \int_0^\theta \frac{(1-t)^n}{1+rt}\, dt.$$

Hence

$$B_n - B_{n-1} = -pq^{n-1} \int_0^\theta (1-t)^{n-1}\, dt = \frac{pq^{n-1}}{n} \{(1-\theta)^n - 1\}$$

and

$$A_n - A_{n-1} = -pq^{n-1} \int_0^\theta \frac{1-rt}{1+rt} (1+t)^{n-1}\, dt.$$

Since $0 < \theta < 1/r$, we have $\{(1-rt)/(1+rt)\} > \{(1-r\theta)/2\} > 0$ for $0 < t < \theta$. Therefore

$$\begin{aligned} A_n - A_{n-1} &\le -(pq^{n-1}) \frac{1-r\theta}{2} \int_0^\theta (1+t)^{n-1}\, dt \\ &= \frac{pq^{n-1}(1-r\theta)}{2n} \{1 - (1+\theta)^n\}. \end{aligned}$$

Noting that $\delta\varepsilon > 0$ and $rp = q$, it follows that

$$\begin{aligned} 2nr\{E_n(Z) - E_{n-1}(Z)\} &= 2nr\delta\varepsilon(A_n - A_{n-1}) + 2nr\delta(B_n - B_{n-1}) \\ &\le -\varepsilon\delta q^n(1-r\theta)(1+\theta)^n + \varepsilon\delta q^n(1-r\theta) \\ &\quad + 2\delta q^n(1-\theta)^n - 2\delta q^n. \end{aligned} \tag{A.1}$$

We now consider $E_n(Z^2)$. Since $0 < \theta < 1$, we have

$$\int_0^\theta \frac{t^x}{1+rt}\, dt \le \theta^x, \qquad x = 0, 1, \dots .$$

Therefore

$$\{Z(x)\}^2 \le \delta^2(1+|\varepsilon|)^2 (r^2\theta^2)^x \le \delta^2(1+|\varepsilon|)^2 (r\theta)^x,$$

since $r\theta < 1$. Hence

$$E_n(Z^2) \le \delta^2(1+|\varepsilon|)^2 E_n\{(r\theta)^X\} = \delta^2(1+|\varepsilon|)^2 q^n(1+\theta)^n.$$

Using condition (4.2), it follows that

$$E_n(Z^2) \le \varepsilon\delta(1-r\theta)q^n(1+\theta)^n. \tag{A.2}$$

Finally, it is easily verified that

$$Z(0) = \frac{\delta(1+\varepsilon)}{r}\ln(1+r\theta). \tag{A.3}$$

Inequality (4.3) now follows from (A.1), (A.2) and (A.3). □

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