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A PROBABILITY INEQUALITY FOR ELLIPTICALLY CONTOURED DENSITIES WITH APPLICATIONS IN ORDER RESTRICTED INFERENCE[1]

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Anderson (1955) established the monotonicity of the integral of a symmetric, unimodal density over translates of a symmetric, convex set. A similar result is developed for integrals of elliptically contoured, unimodal densities over translates of an asymmetric, convex set in certain directions related to the set. This result is used to establish some monotonicity properties of the power functions of the likelihood ratio tests for determining whether or not a vector of normal means satisfies a specified ordering.

1. Introduction. Anderson (1955) studied the monotonicity of the integral of a symmetric, unimodal density over translates of a symmetric, convex set. In particular, he showed that if $f(\cdot)$ is a unimodal probability density with respect to Lebesgue measure on R^k, A is a convex subset of R^k, and $f(\cdot)$ and A are symmetric about the origin, then for all $\mu \in R^k$,

$$h(\delta) = \int_{A - \delta\mu} f(x)\, dx$$

is nonincreasing in $|\delta|$. We give a related result for asymmetric, convex sets and particular directions, μ. This result is useful in the study of the monotonicity properties of power functions of certain tests in order restricted inference. For a fixed direction μ, we give conditions on the convex set A, which imply that $h(\delta)$ is nonincreasing in $\delta \geq 0$. (Results for $h(\delta)$ with $\delta < 0$ are obtained from those with $\delta > 0$ with direction $-\mu$.)

In the case $k = 1$ with $\mu > 0$, it is clear that if the reflection of $A \cap (0, \infty)$ through the origin is contained in the closure of A, then $h(\delta)$ is nonincreasing in $\delta \geq 0$ for any symmetric, unimodal density $f(\cdot)$. In fact, the converse is true. In Section 2 this result is extended to $k > 1$. In particular, suppose $f(\cdot)$ is unimodal and elliptically contoured (defined in Section 2) and let A^+ denote the positive part of A in the direction μ (defined by (2.3)). If the reflection of A^+ through the orthogonal complement of the subspace generated by μ is contained in the closure of A, then $h(\delta)$ is nonincreasing in $\delta \geq 0$. Also, the members of a large

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collection of convex sets are shown to satisfy this condition. This collection contains the convex cones in R^k.

Although the results of Section 2 hold for arbitrary closed, convex cones such as those considered by Kudô (1963) and Perlman (1969), we focus on some quasiordered cones for which the estimates (i.e., projections) are easier to compute and the distributions of related test statistics are more tractable.

In Section 3, our result is applied to a problem in isotonic inference that motivated this generalization of Anderson's inequality. Let $\ll$ be a quasiorder on $\{1, 2, \dots, k\}$ (i.e., it is reflexive and transitive) and let $H_1 = \{x \in R^k;\ x_i \le x_j$ for all $i \ll j\}$. The likelihood ratio tests of $\mu \in H_1$ versus $\mu \notin H_1$ with $\mu_1, \mu_2, \dots, \mu_k$ normal means are considered, both when the variances are known and when they are unknown but equal. The monotonicity properties of the power functions of these tests are studied, assuming independent random samples. By conditioning on the variance estimate, monotonicity properties of the power function in the unknown variance case are implied by the case of known variances.

Robertson and Wegman (1978) proved that homogeneity, $\mu_1 = \mu_2 = \cdots = \mu_k$, is least favorable within H_1, and the results in Robertson and Wright (1982; 1984) imply the monotonicity in certain directions, of the power functions of such tests. The ideas involved in their arguments are given by Perlman (1969), who considered, with C a fixed cone, the likelihood ratio test (LRT) of $\mu \in C$ versus $\mu \notin C$ based upon a random sample from a multivariate normal population with unknown covariance matrix. He also studied the limits of the power function as the distance between μ and the cone becomes infinite. The monotonicity properties given in the above references are based upon containment arguments, and the geometric approach used here extends the monotonicity of the power functions of these tests to other directions. In Section 4 we discuss the bias of these tests.

2. A probability inequality.

In this section, we study the monotonicity of

$$h(\delta) = \int_{A - \delta\mu} f(x)\, dx$$

for a fixed direction $\mu \in R^k$, a convex set A with restrictions on its asymmetry in the μ direction and a unimodal density f with elliptical contours. Das Gupta et al. (1972) studied elliptically contoured distributions. A probability density function of the form $f(x) = |\Sigma|^{-1/2} g(x'\Sigma^{-1}x)$, where Σ is a positive definite $k \times k$ matrix, is said to be elliptically contoured. We assume that f is an elliptically contoured density that is unimodal, that is, g is nonincreasing on $[0, \infty)$.

Since the "symmetry" assumption of A needs to be matched to the distribution, we consider the inner product $(x, y)_\Sigma = x'\Sigma^{-1}y$ and the corresponding norm $\|x\|_\Sigma = ((x, x)_\Sigma)^{1/2}$. For C a closed, convex cone in R^k, $E_\Sigma(x|C)$ denotes the unique projection of x onto C, i.e., $E_\Sigma(x|C)$ is the unique element in C that minimizes $\|x - y\|_\Sigma$ for $y \in C$. Theorem 7.8 of Barlow et al. (1972) characterizes

$E_\Sigma(x|C)$ as follows:

$$(2.1)\qquad \begin{aligned} &E_\Sigma(x|C) \in C, \qquad (x - E_\Sigma(x|C), E_\Sigma(x|C))_\Sigma = 0,\\ &\text{and for all } y \in C, \qquad (x - E_\Sigma(x|C), y)_\Sigma \le 0. \end{aligned}$$

It follows from (2.1) that for $x \in R^k$ and $a \ge 0$, $E_\Sigma(ax|C) = aE_\Sigma(x|C)$ and that for $x \in R^k$,

$$(2.2)\qquad (x, E_\Sigma(x|C))_\Sigma = (E_\Sigma(x|C), E_\Sigma(x|C))_\Sigma = \|E_\Sigma(x|C)\|_\Sigma^2.$$

For $\mu \in R^k$, $S_\mu = \{b\mu\colon -\infty < b < \infty\}$ is the subspace generated by μ, and for $A \subset R^k$, the positive part of A, in the μ direction, is defined by

$$(2.3)\qquad A^+ = \{x \in A\colon E_\Sigma(x|S_\mu) = b\mu \text{ with } b > 0\}.$$

Since the boundary of A has Lebesgue measure zero, we may without loss of generality, assume that A is closed.

THEOREM 2.1. *Let $\mu \in R^k$, let f be a unimodal density that has elliptical contours determined by Σ and let A be a convex subset of R^k. If*

$$(2.4)\qquad x - 2E_\Sigma(x|S_\mu) \in A \quad \textit{for each } x \in A^+,$$

then $h(\delta)$ is nonincreasing for $\delta \ge 0$.

PROOF. Let V be the positive, symmetric square root of Σ^{-1} and make the change of variable $y = Vx$ to obtain (recall, $f(x) = |\Sigma|^{-1/2}g(x'\Sigma^{-1}x)$)

$$h(\delta) = \int_{VA - \delta V\mu} f(V^{-1}y)\,dy = \int_{B - \delta\eta} g(y'y)\,dy/|V|,$$

with $B = VA$ and $\eta = V\mu$. With I the $k \times k$ identity, it is easy to show that $VE_\Sigma(x|S_\mu) = E_I(Vx|VS_\mu)$; $VS_\mu = S_\eta$; $VA^+ = \{Vx\colon E_\Sigma(x|S_\mu) = b\mu \text{ with } b > 0\} = \{Vx\colon E_I(Vx|S_\eta) = b\eta \text{ with } b > 0\} = B^+$; and for $y \in B^+$, there is $x \in A^+$ with $y = Vx$ and

$$y - 2E_I(y|S_\eta) = Vx - 2E_I(Vx|VS_\mu) = V(x - 2E_\Sigma(x|S_\mu)) \in VA = B.$$

Hence, it suffices to prove the result for $\Sigma = I$.

Without loss of generality we assume $\|\mu\| = 1$ (we omit the subscript on $\|\cdot\|$, $(\cdot,\cdot)$ and $E(\cdot|\cdot)$ when $\Sigma = I$) and let O be an orthogonal transformation with $O\mu = (1, 0, \dots, 0)$. The transformed distribution has elliptical contours with $\Sigma = I$, $OS_\mu = S_{O\mu}$, $OA^+ = (OA)^+$ and for $y \in OA^+$, there is an $x \in A^+$ with $y = Ox$ and

$$y - 2E(y|S_{O\mu}) = O(x - 2E(x|S_\mu)) \in OA.$$

Hence, we assume that $\mu = (1, 0, \dots, 0)$ and denote the $(x_2, \dots, x_k)$ section of a set D by $D_{(x_2,\dots,x_k)} = \{x_1\colon (x_1, x_2, \dots, x_k) \in D\}$. Since $(A - \delta\mu)_{(x_2,\dots,x_k)} = A_{(x_2,\dots,x_k)} - \delta$ and A is convex,

$$h(\delta) = \int \cdots \int\int_{A_{(x_2,\dots,x_k)} - \delta} f(x_1, x_2, \dots, x_k)\,dx_1\,dx_2 \cdots dx_k$$

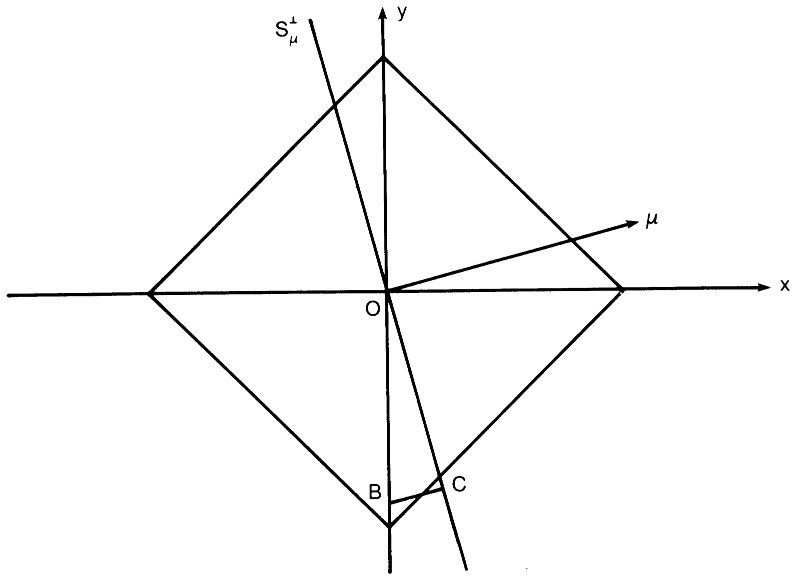

FIG. 1.

and $A_{(x_2,\dots,x_k)}$ is an interval. Furthermore, if $x \in A^+$, then $x - 2E(x|S_\mu) = (-x_1, x_2, \dots, x_k) \in A$ or $x_1 \in (A_{(x_2,\dots,x_k)})^+$ implies that $-x_1 \in A_{(x_2,\dots,x_k)}$. For $(x_2,\dots,x_k)$ fixed, $h(x_1) = g(x_1^2 + \cdots + x_k^2)$ is symmetric and unimodal. The desired result follows from the case $k = 1$, discussed in the introduction. □

REMARK 1. One wonders if the assumption that f is elliptically contoured could be weakened. However, the matrix, Σ, in the definition of elliptically contoured densities, is used to compute the projection in (2.4). We give an example of a convex set A and a density with square contours for which the conclusion of the theorem does not hold. Since the density has square contours, we take $\Sigma = I$ and note that (2.4) holds with $\Sigma = I$. Let $k = 2$, f be uniform on the square S, with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$; let $\mu = (1, \tan\theta)$ with $0 < \theta < \pi/4$ ($S_\mu^\perp$ is the line perpendicular to μ); let $\overline{BC}$ be a line segment parallel to μ with B a point on the negative part of the y axis inside of S and C a point on $S_\mu^\perp$ outside of S; and let A be the triangle OBC with O the origin (see Figure 1). For small positive δ, $S \cap A$ has smaller area than $S \cap (A - \delta\mu)$ which shows the conclusion of the theorem does not hold. Hence, Theorem 2.1 does not hold for an arbitrary symmetric unimodal density.

REMARK 2. The "reflection–inclusion" argument used in the proof of Theorem 2.1 has also been used in the proof of Lemma 5.1 in Das Gupta et al. (1972).

Next, we study a collection of convex sets of interest in isotonic inference, which satisfy (2.4). For C a closed, convex cone in R^k and $t \geq 0$, set $A(C, t) =$

$\{x \in R^k\colon \|E_\Sigma(x|C)\|_\Sigma^2 \le t\}$. Theorem 2.4, to be given, states that for certain directions, μ, the set $A(C, t)$ satisfies (2.4).

The dual of C, which is defined by $C^{*\Sigma} = \{y \in R^k\colon (x, y)_\Sigma \le 0$ for each $x \in C\}$, is of interest. Clearly, $C^{*\Sigma}$ is a closed, convex cone and using its definition together with (2.1) and (2.2) we see that

$$E_\Sigma(x|C^{*\Sigma}) = x - E_\Sigma(x|C) \quad \text{and} \qquad (2.5)$$
$$\|E_\Sigma(x|C^{*\Sigma})\|_\Sigma^2 = \|x\|_\Sigma^2 - \|E_\Sigma(x|C)\|_\Sigma^2.$$

It is also well known that

$$(C^{*\Sigma})^{*\Sigma} = C. \qquad (2.6)$$

If $t = 0$, then since $E_\Sigma(x|C) = x - E_\Sigma(x|C^{*\Sigma}) = 0$ if and only if $x \in C^{*\Sigma}$, it follows that $A(C,0) = C^{*\Sigma}$. Hence, the collection of sets of the form $A(C, t)$ contains the closed convex cones.

For $A = A(C, t)$ and certain directions μ, the monotonicity of $h(\delta)$ follows from a containment argument. The following lemma is given implicitly in the proof of Lemma 8.2 in Perlman (1969) and explicitly in Robertson and Wegman (1978).

LEMMA 2.2. *If* $x \in C^{*\Sigma}$, *then* $\|E_\Sigma(x + y|C)\|_\Sigma \le \|E_\Sigma(y|C)\|_\Sigma$.

Applying this lemma, we see that if $\mu \in -C^{*\Sigma}(-D = \{-x\colon x \in D\})$ and $a \ge 0$, then $A - a\mu \subset A$. Hence, $A - \mu_0 - \delta\mu \supset A - \mu_0 - \delta'\mu$ for $\mu_0 \in R^k$, $\delta \le \delta'$ and $\mu \in -C^{*\Sigma}$. So, $h(\delta)$ is nonincreasing in $\delta \in (-\infty, \infty)$ for $\mu \in -C^{*\Sigma}$.

Because $E_\Sigma(\cdot|C)$ is not necessarily linear, the following lemma is used to bound $\|E_\Sigma(x + y|C)\|_\Sigma$.

LEMMA 2.3. *For* $x, y \in R^k$,

$$\|E_\Sigma(x + y|C)\|_\Sigma \le \|E_\Sigma(x|C) + E_\Sigma(y|C)\|_\Sigma \le \|E_\Sigma(x|C)\|_\Sigma + \|E_\Sigma(y|C)\|_\Sigma.$$

PROOF. We need only establish the first inequality, because the second follows from the triangular inequality for norms. Because

$$x + y = E_\Sigma(x|C) + E_\Sigma(x|C^{*\Sigma}) + E_\Sigma(y|C) + E_\Sigma(y|C^{*\Sigma})$$
$$= E_\Sigma(x|C) + E_\Sigma(y|C) + z,$$

with $z = E_\Sigma(x|C^{*\Sigma}) + E_\Sigma(y|C^{*\Sigma}) \in C^{*\Sigma}$,

$$\|E_\Sigma(x + y|C)\|_\Sigma = \|E_\Sigma(E_\Sigma(x|C) + E_\Sigma(y|C) + z|C)\|_\Sigma.$$

Applying Lemma 2.2, the last term is bounded above by

$$\|E_\Sigma(E_\Sigma(x|C) + E_\Sigma(y|C)|C)\|_\Sigma = \|E_\Sigma(x|C) + E_\Sigma(y|C)\|_\Sigma.$$

The proof is completed. □

If $x, y \in A(C, t)$ and $0 < \alpha < 1$, then

$$\|E_\Sigma(\alpha x + (1-\alpha)y|C)\|_\Sigma \le \|\alpha E_\Sigma(x|C) + (1-\alpha)E_\Sigma(y|C)\|_\Sigma$$
$$\le \alpha\|E_\Sigma(x|C)\|_\Sigma + (1-\alpha)\|E_\Sigma(y|C)\|_\Sigma \le \sqrt{t}.$$

So $A(C, t)$ is convex. We now establish

THEOREM 2.4. *Let C be a closed, convex cone, $t \ge 0$, μ, $\mu_0 \in R^k$ and $A = A(C, t) - \mu_0$. The subset A is convex. If $\mu \in C$ and $(\mu, \mu_0)_\Sigma = 0$, then for each $x \in A^+$, $x - 2E_\Sigma(x|S_\mu) \in A$.*

PROOF. The convexity of A is immediate. Since $E_\Sigma(y - \mu_0|S_\mu) = E_\Sigma(y|S_\mu)$ and $y - \mu_0 - 2E_\Sigma(y|S_\mu) \in A$ if and only if $y - 2E_\Sigma(y|S_\mu) \in A(C, t)$, we may assume $\mu_0 = 0$. Let $x \in A^+$. Applying (2.5),

$$\|E_\Sigma(x - 2E_\Sigma(x|S_\mu)|C)\|_\Sigma^2 = \|x - 2E_\Sigma(x|S_\mu)\|_\Sigma^2 - \|E_\Sigma(x - 2E_\Sigma(x|S_\mu)|C^{*\Sigma})\|_\Sigma^2$$
$$= x'\Sigma^{-1}x - 4(x - E_\Sigma(x|S_\mu))'\Sigma^{-1}E_\Sigma(x|S_\mu)$$
$$-\|E_\Sigma(x - 2b\mu|C^{*\Sigma})\|_\Sigma^2,$$

with $b > 0$. Using Lemma 2.2 with C replaced by its dual, we see that $\|E_\Sigma(x - 2b\mu|C^{*\Sigma})\|_\Sigma^2 \ge \|E_\Sigma(x|C^{*\Sigma})\|_\Sigma^2$, and using (2.1), the characterization of a projection, we see that $(x - E_\Sigma(x|S_\mu), E_\Sigma(x|S_\mu))_\Sigma = 0$. Hence,

$$\|E_\Sigma(x - 2E_\Sigma(x|S_\mu)|C)\|_\Sigma^2 \le \|x\|_\Sigma^2 - \|E_\Sigma(x|C^{*\Sigma})\|_\Sigma^2 = \|E_\Sigma(x|C)\|_\Sigma^2 \le t.$$

As required, $x - 2E_\Sigma(x|S_\mu) \in A$. □

3. Monotonicity of power functions. We assume independent random samples from each of k normal populations. Denote the sample items by X_{ij}, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n_i$; the sample means by $\bar{X}_i$, $i = 1, 2, \ldots, k$; and the population variances by σ_i^2, $i = 1, 2, \ldots, k$, so that $\bar{X}_i \sim n(\mu_i, \omega_i^{-1})$ with $\omega_i = n_i/\sigma_i^2$, $i = 1, 2, \ldots, k$. Let $\ll$ be a quasiorder on $\{1, 2, \ldots, k\}$ and assume that H_1 denotes the closed, convex cone $\{x \in R^k;\ x_i \le x_j$ whenever $i \ll j\}$ as well as the hypothesis that $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in H_1$. We consider LRT's of the null hypothesis H_1 against the alternative that $\mu \notin H_1$ (see Perlman (1969) and Robertson and Wegman (1978)). If the population variances are known, then the LRT rejects H_1 for large values of the statistic

$$T_{12} = \|\bar{X} - E_\Sigma(\bar{X}|H_1)\|_\Sigma^2 = \|E_\Sigma(\bar{X}|H_1^{*\Sigma})\|_\Sigma^2,$$

where $\bar{X} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k)$ and Σ denotes the diagonal matrix whose ith diagonal element is ω_i for $i = 1, 2, \ldots, k$. If the population variances are unknown but assumed to be equal, the LRT rejects H_1 for large values of

$$S_{12} = \|\bar{X} - E_\Sigma(\bar{X}|H_1)\|_\Sigma^2/[(N-k)(\hat{\sigma}^2/\sigma^2) + \|\bar{X} - E_\Sigma(\bar{X}|H_1)\|_\Sigma^2]$$

or, equivalently, for large values of

$$L_{12} = (N-k)S_{12}/(1 - S_{12}) = \sigma^2 T_{12}/\hat{\sigma}^2,$$

where $N = \Sigma_{i=1}^k n_i$ and $\hat{\sigma}^2 = (N-k)^{-1}\Sigma_{i=1}^k \Sigma_{j=1}^{n_i}(X_{ij} - \overline{X}_i)^2$. We will apply the results from Section 2 to obtain results regarding the monotonicity of the power functions of T_{12} and L_{12}. Robertson and Wright (1982; 1984) obtained analogous results for the LRT's of H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$ against the alternative $H_1 - H_0$ (H_1 but not H_0) using cone orderings and containment arguments like those mentioned in the introduction. These containment arguments can be used to show monotonicity of the power function of the test developed by Kudô (1963) and the geometric approach used here provides no new information in these cases.

The null hypothesis distribution of T_{12} and S_{12} are given in Robertson and Wegman (1978). The statistic, L_{12}, is an increasing function of S_{12} so that critical values for L_{12} can be found from those for S_{12}. Assume that we have specified a significance level and found critical values t for T_{12} and l for L_{12} for these tests. We denote the power functions of T_{12} and L_{12} by $\pi_{12}(\mu) = P_\mu[T_{12} > t]$ and $\pi_{12}'(\mu) = P_\mu[L_{12} > l]$. If we let $f_N(\cdot)$ denote the density of $\hat{\sigma}^2/\sigma^2$ then, because $\overline{X}$ and $\hat{\sigma}^2$ are independent,

$$\pi_{12}'(\mu) = \int_0^\infty P_\mu[T_{12} > ly]\, f_N(y)\, dy.$$

Hence, the monotonicity of $\pi_{12}'(\cdot)$ will follow from that of $\pi_{12}(\cdot)$. Furthermore, since $\pi_{12}(\cdot)$ is invariant under translations by elements of H_0, we need only study $\pi_{12}(\cdot)$ on the subspace $S = \{\mu \in R^k;\ \Sigma_{i=1}^k \omega_i \mu_i = 0\}$.

For $k = 2$ and $H_1 = \{(\mu_1, \mu_2)\colon\ \mu_1 \le \mu_2\}$, T_{12} rejects for large values of $\overline{X}_1 - \overline{X}_2$. This test is known to be unbiased, uniformly most powerful, and $\pi_{12}(\cdot)$ is known to be nondecreasing in $\mu_1 - \mu_2$.

For $k = 3$, $H_1 = \{\mu \in R^3;\ \mu_1 \le \mu_2 \le \mu_3\}$, and $\omega_1 = \omega_2 = \omega_3$, one can employ the techniques used by Bartholomew (1961) to obtain an expression for $\pi_{12}(\cdot)$. For $\mu \in S$, Bartholomew considered the parameterization

$$(\mu_2 - \mu_1)/\sqrt{2} = \Delta \sin \beta, \qquad (2\mu_3 - \mu_1 - \mu_2)/\sqrt{6} = \Delta \cos \beta.$$

With this notation,

$$\begin{aligned}\pi_{12}(\mu) = {} & \frac{\exp(-\Delta^2/2)}{2\pi} \int_{\beta-\pi}^{\beta-\pi/3} \psi(\Delta \sin \theta, t)\, d\theta \\ & + \Phi(-\Delta \sin \beta - \sqrt{t})\Phi(\Delta \cos \beta) \\ & + \Phi(-\Delta \cos(\beta + \pi/6) - \sqrt{t})\Phi(\Delta \sin(\beta + \pi/6)),\end{aligned}$$

where $\psi(x, t) = (x\Phi(x - \sqrt{t}) + \varphi(x - \sqrt{t}))/\varphi(x)$ and Φ and φ are the c.d.f. and p.d.f. of the standard normal distribution. Straightforward calculations yield

$$\left.\frac{\partial \pi_{12}}{\partial \Delta}\right|_{\Delta=0} = -\varphi(\sqrt{t})\left(\frac{3t}{(2t)^{1/2}} + \frac{1}{2}\right)\cos\left(\beta - \frac{\pi}{6}\right),$$

which is negative for $\beta \in (-\pi/3, 2\pi/3)$ and positive for $\beta \in (2\pi/3, 5\pi/3)$.

Figure 2 shows $H_1' = H_1 \cap S$, $H_1^{*\Sigma}$ and $A' = A(H_1^{*\Sigma}, t) \cap S$ where $A(H_1^{*\Sigma}, t)$ is the acceptance region for T_{12}. It follows from the discussion following Lemma 2.2 that $\pi_{12}(\delta\mu)$ is nondecreasing for $\delta \in (-\infty, \infty)$ provided $-\mu \in H_1$. Applying

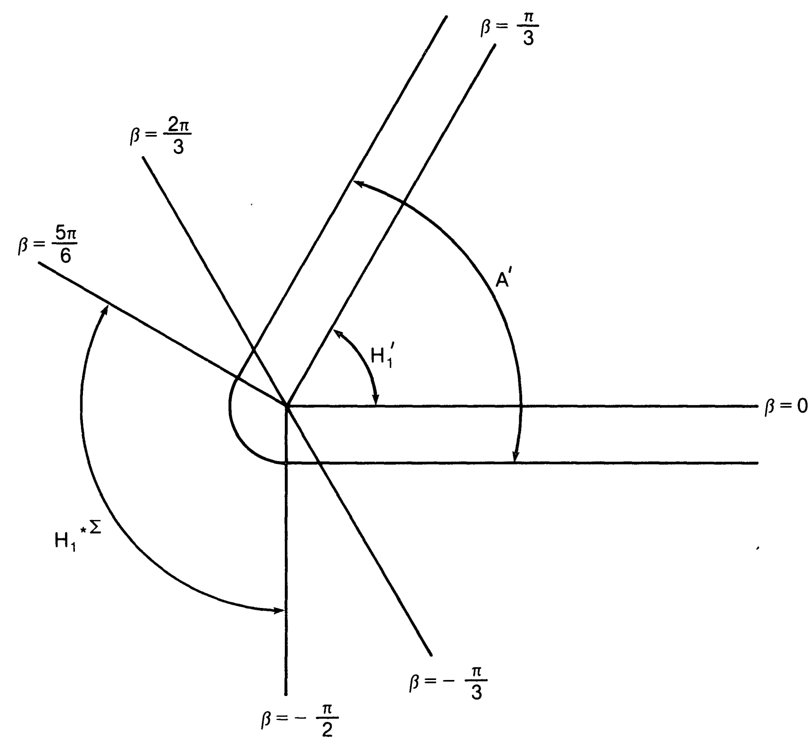

FIG. 2.

Theorems 2.1 and 2.4, it follows that if μ is in the larger set, $H_1^{*\Sigma} \oplus H_0$, then $\pi_{12}(\delta\mu)$ is nondecreasing for $\delta \geq 0$. Because A' is symmetric for $k = 3$, one can apply Theorem 2.1 to extend this result to $2\pi/3 \leq \beta \leq 5\pi/3$. However, this type of reasoning cannot be used for $k > 3$ since A' is not symmetric for such k. Even for $k = 3$ this result cannot be extended beyond $[2\pi/3, 5\pi/3]$ for the following reasons: Using arguments like those given in the proofs of Theorem 5.3 of Perlman (1969) and Theorem 3.4 of Barlow et al. (1972), it can be shown that $\pi_{12}(\delta\mu) \to 1$ as $\delta \to \infty$ for $\mu \notin H_1$. Also recall that for $\beta \in (-\pi/3, 2\pi/3)$, $\partial\pi_{12}/\partial\Delta|_{\Delta=0}$ is negative so that $\pi_{12}(\delta\mu)$ decreases for small but positive δ. Thus $\pi_{12}(\cdot)$ is not monotonic for rays beginning at the origin for $\beta \in (-\pi/3, 0) \cup (\pi/3, 2\pi/3)$. We conjecture that in higher dimensions there are also directions of monotonicity not given by these results, but because of the complexity of the power function this cannot be verified by direct computations.

We now return to the general case (i.e., arbitrary k, quasiorder, $\ll$, and weights, $\omega_1, \omega_2, \ldots, \omega_k$). The radial monotonicity of $\pi_{12}(\cdot)$ is a corollary to the following theorem.

THEOREM 3.1. *If $v \in R^k$ and $\mu \in H_1$ then, as a function of δ, $\pi_{12}(v + \delta\mu)$ is nonincreasing for $\delta \in (-\infty, \infty)$ and $\pi_{12}(v + \delta E_\Sigma(v|H_1^{*\Sigma}))$ is nondecreasing for $\delta \geq -1$.*

PROOF. The first conclusion is a consequence of the discussion following Lemma 2.2 by setting $C = H_1^{*\Sigma}$, $\mu_0 = v$ and $\mu = -\mu$. For the second conclusion, write $v + \delta E_\Sigma(v|H_1^{*\Sigma}) = E_\Sigma(v|H_1) + (\delta + 1)E_\Sigma(v|H_1^{*\Sigma})$, note that $(E_\Sigma(v|H_1^{*\Sigma}), E_\Sigma(v|H_1))_\Sigma = 0$ and invoke Theorem 2.4 with $C = H_1^{*\Sigma}$, $\mu_0 = E_\Sigma(v|H_1)$ and $\mu = E_\Sigma(v|H_1^{*\Sigma})$. The proof is completed by applying Theorem 2.1. □

Because of the translation invariance of $\pi_{12}(\cdot)$, the following result is an immediate consequence of Theorem 3.1.

COROLLARY 3.2. *If* $\mu \in -H_1$ *then* $\pi_{12}(\delta\mu)$ *is nondecreasing for* $\delta \in (-\infty, \infty)$ *and if* $\mu \in H_1^{*\Sigma} \oplus H_0$, *then* $\pi_{12}(\delta\mu)$ *is nondecreasing for* $\delta \geq 0$.

It follows immediately from Corollary 3.2 that H_0 is least favorable within H_1. Also, the containment arguments show that $\pi_{12}(\cdot)$ is nondecreasing along rays in $-H_1$ and the geometric approach shows that $\pi_{12}(\cdot)$ is nondecreasing along rays in $H_1^{*\Sigma} \oplus H_0$. It is of interest to compare the sizes of $-H_1$ and $H_1^{*\Sigma} \oplus H_0$ for various partial orders. For the total order, $1 \ll 2 \ll \cdots \ll k$, $H_1^{*\Sigma} = \{x \in R^k; \sum_{j=1}^{i}\omega_j x_j \geq 0;\ i = 1, 2, \ldots, k-1 \text{ and } \sum_{j=1}^{k}\omega_j x_j = 0\}$ (cf. Barlow et al. (1972), page 49). Thus, $-H_1 \subset H_1^{*\Sigma} \oplus H_0$ and the containment is easily seen to be proper. On the other hand, for some partial orders, the containment is reversed. For example, for $k = 3$ and the simple tree ordering (i.e., $1 \ll 2$ and $1 \ll 3$) and $\omega_1 = \omega_2 = \omega_3$, H_1^* is the region with $\beta \in [0, 2\pi/3]$ and $H_1^{*\Sigma}$ is the region with $\beta \in [7\pi/6, 3\pi/2]$ so that $-H_1 \supset H_1^{*\Sigma} \oplus H_0$. It is interesting to note that in some cases, the containment arguments, which make no assumption on the underlying densities, provide stronger results than the geometric techniques employed here.

4. Bias of the T_{12} test.

We now study the bias of the T_{12} test for the total order $1 \ll 2 \ll \cdots \ll k$. As we have seen, $\sup_{\mu \in H_1}\pi_{12}(\mu) = \pi_{12}(0)$ and from Theorem 8.4 of Perlman (1969), we see that $\inf_{\mu \in H_1}\pi_{12}(\mu) = 0$. Hence, we consider the behavior of $\pi_{12}(\mu)$ for $\mu \notin H_1$. As noted previously, it can be shown that $\lim_{\delta \to \infty}\pi_{12}(\delta\mu) = 1$ for $\mu \notin H_1$. We will partition the complement of H_1 into several sets and examine the behavior of $\pi_{12}(\cdot)$ on each of these sets. It follows from the above observation that the supremum of $\pi_{12}(\cdot)$ over each of these sets is one.

The set H_1 has $2^{k-1} - 1$ faces and the complement of H_1 can be partitioned into sets, each of which is paired with one of these faces. The infimum of $\pi_{12}(\cdot)$ varies over the members of this partition. The idea is best explained by considering an example. Suppose $k = 3$ and $H_1 = \{\mu \in R^3;\ \mu_1 \leq \mu_2 \leq \mu_3\}$ and keep Figure 2 in mind. The complement of H_1 is partitioned into the following three sets:

$$E_1 = \{x \notin H_1;\ E_\Sigma(x|H_1)_1 = E_\Sigma(x|H_1)_2 = E_\Sigma(x|H_1)_3\} = (H_1^{*\Sigma} - \{0\}) \oplus H_0,$$
$$E_2 = \{x \notin H_1;\ E_\Sigma(x|H_1)_1 < E_\Sigma(x|H_1)_2 = E_\Sigma(x|H_1)_3\},$$
$$E_3 = \{x \notin H_1;\ E_\Sigma(x|H_1)_1 = E_\Sigma(x|H_1)_2 < E_\Sigma(x|H_1)_3\}.$$

E_1 is paired with H_0, E_2 is paired with the face $\beta = \pi/3$ and E_3 with the face $\beta = 0$. If $\mu \in E_1$, then letting $v = \mu$ in the second part of Theorem 3.1 we find that $\inf_{\mu \in E_1} \pi_{12}(\mu) = \pi_{12}(0)$, which is the significance level of the test.

Suppose $\mu \in E_2$. Using (2.1) and a straightforward argument, it is easy to see that for $\delta \geq -1$, $E_\Sigma(\mu + \delta E_\Sigma(\mu|H_1)|H_1) = (1 + \delta)E_\Sigma(\mu|H_1)$ and thus the points $\mu + \delta E_\Sigma(\mu|H_1)$ with $\delta \geq -1$ are all in E_2. Incidentally, this also implies that they are all equidistant from H_1. Then, using the first part of Theorem 3.1 it follows that $\pi_{12}(\mu + \delta E_\Sigma(\mu|H_1))$ is nonincreasing in δ. On the other hand, using the second part of Theorem 3.1, it follows that for $0 < \delta < 1$ and $v \in R^k$, $\pi_{12}(E_\Sigma(v|H_1)) \leq \pi_{12}(v - \delta E_\Sigma(v|H_1^{*\Sigma})) \leq \pi_{12}(v)$. Applying the second observation with $v = \mu + \delta E_\Sigma(\mu|H_1)$, it follows from the first observation that $\inf_{\mu \in E_2} \pi_{12}(\mu)$ can be obtained by taking the infimum of $\lim_{\delta \to \infty} \pi_{12}(\delta\mu)$ over all μ in the face of H_1 corresponding to E_2 (i.e., $\mu_1 < \mu_2 = \mu_3$). However, for such a μ, as $\delta \to \infty$ the probability that $\bar{X}_1 < \bar{X}_2 \wedge \bar{X}_3$ converges to one so that

$$\lim_{\delta \to \infty} \pi_{12}(\delta\mu) = P\left[\|\bar{X} - E_\Sigma(\bar{X}|C_2)\|_\Sigma^2 > t\right],$$

where $C_2 = \{x \in R^3;\ x_2 \leq x_3\}$ and the probability on the right-hand side is computed under the assumption that $\mu_2 = \mu_3$. It then follows from Corollary 2.6 in Robertson and Wegman (1978) that $\inf_{\mu \in E_2} \pi_{12}(\mu) = \frac{1}{2}P[\chi_1^2 > t]$, where χ_1^2 denotes a standard chi-squared random variable with one degree of freedom. A similar result holds for E_3.

For arbitrary k the infimum over each member of the partition of the complement of H_1 is equal to $\lim_{\delta \to \infty} \pi_{12}(\delta\mu)$, where μ is an element of the face of H_1 corresponding to the set under consideration. Each of these limits is equal to $P[\|\bar{X} - E_\Sigma(\bar{X}|C)\|_\Sigma^2 > t]$, where C imposes the order restriction only between adjacent pairs that are equal in the face of H_1. For example, if $k = 4$ and the face is $\{x;\ x_1 < x_2 = x_3 = x_4\}$ then $C = \{x;\ x_2 \leq x_3 \leq x_4\}$. Each of these limits is a weighted average of chi-squared variables from Corollary 2.6 in Robertson and Wegman (1978). The smallest of these limits corresponds to a C that imposes the fewest number of restrictions (i.e., only one). Thus, for any k, $\inf_{\mu \notin H_1} \pi_{12}(\mu) = \frac{1}{2}P[\chi_1^2 > t]$ where t is the critical value of the test.

This infimum is not zero for any k, but does get small fairly rapidly as k grows large. For various α, the critical values of the test are given in Table 2.1 of Robertson and Wegman (1978). The values of $\inf_{\mu \notin H_1} \pi_{12}(\mu)$ for $\alpha = 0.05$ are

TABLE 1

k	**Critical Value**	$\frac{1}{2}P[\chi_1^2 > t]$
3	4.578	0.01620
4	6.175	0.00648
5	7.665	0.00281
6	9.095	0.00128
7	10.485	0.00060

given in Table 1 for $k = 3, 4, \ldots, 7$. Thus, even though $\inf_{\mu \notin H_1} \pi_{12}(\mu) > 0$, the amount of bias in the test based upon T_{12} is sizable even for moderate values of k.

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