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THE TOTAL TIME ON TEST PLOT AND THE CUMULATIVE TOTAL TIME ON TEST STATISTIC FOR A COUNTING PROCESS

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Results on the total time on test plot are usually obtained on the assumption that the number of events to be observed is fixed in advance. Here it is shown that the same large sample results hold when the number of events is random if a simple condition is satisfied.

Consider a univariate counting process on the time interval $\mathscr{T} = [0, \tau)$ or $[0, \tau] \subseteq [0, \infty]$ with continuous compensator A. Suppose we are interested in testing the hypothesis: $A = cA_0$ for some unknown constant $c > 0$ and a given observed process A_0. For example, suppose N has intensity process Λ with $\Lambda(t) = \lambda(t)Y(t)$ for some observable process Y and some unknown function λ, and take $A_0(t) = \int_0^t Y(s)\,ds$. The hypothesis $A = cA_0$ corresponds to the interesting hypothesis $\lambda =$ constant. The total time on test plot and the cumulative total time on test statistic are two common techniques for investigating this hypothesis when the alternatives of special interest are that dA/dA_0 is monotone (i.e., in our example, λ is monotone). They are based on the observation that in the new time scale measured by A_0, N is transformed into the process $N \circ A_0^{-1}$ which, under the null hypothesis, is a counting process with constant intensity c on the (random) time interval $[0, A_0(\tau)]$; i.e., a randomly stopped Poisson process with constant intensity. Under the alternative it is a counting process with monotone intensity $(dA/dA_0) \circ A_0^{-1}$. (See Aalen and Hoem, 1978.)

The total time on test plot is usually carried out as follows: Choose some number of events R (in classical applications R is a fixed number r say, but this is not in general possible[1]) such that $T_R < \tau$ almost surely (here $0 < T_1 < T_2 < \cdots$ are the jump times of N; say $T_l = \tau$ for all $l > N(\tau)$), and make a plot of $A_0(T_i)/A_0(T_R)$ versus i/R, $i = 0, 1, \ldots, R$. Under the null hypothesis the plot should approximate the straight line $y = x$, $x \in [0, 1]$. Under the alternative it tends to be concave or convex depending on whether dA/dA_0 is decreasing or increasing. The cumulative total time on test statistic is the quantity $\sum_{i=1}^{R-1} A_0(T_i)/A_0(T_R)$. The standardized version of the statistic corresponds to the (signed) area between total time on test plot and the line $y = x$ (using an appropriate interpolation convention between consecutive plotting positions).

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[1] This point is often overlooked; cf. Barlow and Proschan (1969) and many later authors.

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More generally one could think of choosing some random time T (not necessarily even a stopping time, e.g., the last jump time before some fixed time t) and plotting $A_0(t)/A_0(T)$ against $N(t)/N(T)$, $t \in [0, T]$. Taking $T = T_R$ gives the previous plot with a particular interpolation convention. We shall develop some asymptotic null-hypothesis distribution theory for this plot. From this corresponding results for the statistic follow immediately.

Now if $N(\tau) \geq r$ with probability 1, and we take $T = T_r$, exact distributional results are available for plot and statistic since $N \circ A_0^{-1}$, stopped at $A_0(T_r)$, is simply (under the null hypothesis) a Poisson process with constant intensity c stopped at the rth event. The plot has the same distribution as the empirical d.f. based on $r - 1$ i.i.d. uniform [0, 1] r.v.'s. Large sample results (i.e., as $r \to \infty$) for plot and statistic are now immediately available: we have a Brownian bridge and the signed area beneath a Brownian bridge, respectively.

So we shall consider here the case when T is chosen as arbitrarily as possible, not even necessarily as a stopping time. Consider a sequence of situations indexed by n, all under the null hypothesis with $A = A^n = cA_0^n$ for a fixed constant c. Our results are obtained under the following simple assumption on the times $T = T^n$.

Assumption. Suppose T^n is such that there exists a sequence of constants $a_n \to \infty$ as $n \to \infty$ such that

$$A_0^n(T^n)/a_n \to_{\mathscr{P}} \alpha \in (0, \infty) \quad \text{as } n \to \infty.$$

Consider the processes $N^n \circ (A_0^n/a_n)^{-1}$. These have intensity ca_n on $[0, A_0^n(\tau)/a_n]$. At this final time instant $A_0^n(\tau)/a_n$ (if it is finite) start up an independent Poisson process with constant intensity ca_n and fasten it onto $N^n \circ (A_0^n/a_n)^{-1}$. In this way we obtain a process U^n say, coinciding with $N^n \circ (A_0^n/a_n)^{-1}$ on $[0, A_0^n(\tau)/a_n]$, with intensity ca_n on the *whole* line. So U^n is a Poisson process and we have easily

$$(a_n)^{1/2}\left(\frac{U^n}{a_n} - cI\right) \to_{\mathscr{D}} c^{1/2}W \quad \text{in } D[0, \infty),$$

where W is a standard Wiener process and I is the identity function $I(x) = x$. Now look at the process V^n defined by

$$V^n(x) = U^n\left(x\frac{A_0^n(T^n)}{a_n}\right), \qquad x \in [0, 1],$$

i.e., on [0,1]

$$V^n = N^n \circ \left(\frac{A_0^n}{a_n}\right)^{-1} \circ \left(\frac{A_0^n(T^n)}{a_n} \cdot I\right)$$

$$= N^n \circ (A_0^n)^{-1} \circ (A_0^n(T^n)I)$$

since $A_0^n(T^n) \leq A_0^n(\tau)$ a.s.; i.e., we do not run into the appended Poisson process.

Recall that $(A_0^n(T^n))/a_n \to_{\mathscr{P}} \alpha$. Since

$$\frac{U^n}{a_n} - cI \to_{\mathscr{D}} 0,$$

we obtain $N^n(T^n)/a_n \to_{\mathscr{P}} c\alpha$.

By a change of time argument (see Appendix) we have in $D[0,1]$

$$(a_n)^{1/2}\left(\frac{U^n}{a_n} - cI\right) \circ \left(\frac{A_0^n(T^n)}{a_n} \cdot I\right) \to_{\mathscr{D}} c^{1/2} W \circ (\alpha \cdot I) =_{\mathscr{D}} W(c\alpha \cdot I)$$

i.e.,

$$(a_n)^{1/2}\left(\frac{V^n}{a_n} - c\frac{A_0^n(T^n)}{a_n} \cdot I\right) \to_{\mathscr{D}} W(c\alpha \cdot I).$$

Since W has continuous paths, the process obtained from the process on the left-hand side by subtracting the straight line connecting its end points (at $x = 0$ and $x = 1$) also converges in distribution; i.e.,

$$(a_n)^{1/2}\left(\frac{V^n}{a_n} - \frac{V^n(1)}{a_n} \cdot I\right) \to_{\mathscr{D}} W(c\alpha \cdot I) - W(c\alpha) \cdot I$$

$$=_{\mathscr{D}} \sqrt{c\alpha}\, B^0,$$

where B^0 is a Brownian bridge on [0,1]. We have obtained, therefore,

$$(a_n)^{-1/2}\left(N^n \circ (A_0^n)^{-1} \circ (A_0^n(T^n) \cdot I) - N^n(T^n) \cdot I\right) \to_{\mathscr{D}} (c\alpha)^{1/2} B^0,$$

from which it easily follows that

$$N^n(T^n)^{1/2}\left(\frac{N^n \circ (A_0^n)^{-1} \circ (A_0^n(T^n) \cdot I)}{N^n(T^n)} - I\right) \to_{\mathscr{D}} B^0.$$

This is the required result since a plot of

$$\frac{N^n \circ (A_0^n)^{-1} \circ (A_0^n(T^n) \cdot x)}{N^n(T^n)} \quad \text{against } x$$

is a plot of $N^n(t)/N^n(T^n)$ against $A_0^n(t)/A_0^n(T^n)$ (replace x by this last quantity).

Thus the total time on test plot (under the null hypothesis) has the same asymptotic distribution as the uniform empirical d.f., taking $N^n(T^n)$ as the number of observations.

We obtain immediately that the asymptotic distribution of the signed area between total time on test plot and diagonal, times $N^n(T^n)^{1/2}$, is the same as that of $\int_0^1 B^0\, dx =_{\mathscr{D}} \mathscr{N}(0, \frac{1}{12})$, which is the required result on the cumulative total time on test statistic.

EXAMPLES. Suppose there exist $a_n \to \infty$ as $n \to \infty$ such that $\mathscr{P}(N^n(\tau) \geq a_n) \to 1$. Then we can consider the total time on test plot for the first a_n events;

i.e., take $T^n = T^n_{a_n}$. We show how we can recover the classical asymptotic results on the total time on test statistic from our general result. To simplify the discussion suppose actually $N^n(\tau) \geq a_n$ almost surely for each n. We have $N^n(T^n) = a_n$ almost surely. By the properties of a compensator of a counting process, we have

$$\mathscr{E}\Big(\big(N^n(T^n) - cA^n_0(T^n)\big)^2\Big) = \mathscr{E}\big(cA^n_0(T^n)\big) = \mathscr{E}\big(N^n(T^n)\big) = a_n.$$

Thus $\mathscr{E}(((A^n_0(T^n))/a_n - 1/c)^2) = 1/a_n c^2 \to 0$ as $n \to \infty$ and our condition $A^n_0(T^n)/a_n \to_{\mathscr{P}} \alpha = c^{-1}$ is satisfied.

More generally, one can check that if T^n is a stopping time for each n and $a_n \to \infty$ satisfies both $N^n(T^n)/a_n \to_{\mathscr{P}} 1$, and

$$\mathscr{E}\big(N^n(T^n)\big)/(a_n)^2 \to 0 \quad \text{as } n \to \infty,$$

then

$$A^n_0(T^n)/a_n \to_{\mathscr{P}} c^{-1} \quad \text{as } n \to \infty.$$

For a second example, consider the classical random censorship model $(\tilde{X}_i, \delta_i) = (\min(X_i, C_i), I\{X_i \leq C_i\})$, $i = 1, \ldots, n$, where $X_1, \ldots, X_n$ and $C_1, \ldots, C_n$ are all independent and nonnegative, the X_i's with absolutely continuous distribution function F and the C_i's with distribution function G. Define $N^n(t) = \#\{i\colon\ \tilde{X}_i \leq t, \delta_i = 1\}$ and $Y^n(t) = \#\{i\colon\ \tilde{X}_i \geq t\}$. Then N^n is a counting process with intensity $\Lambda^n = Y^n \cdot \lambda$ where λ is the hazard rate of F. Taking $A^n_0(t) = \int_0^t Y^n(s)\, ds$, we obtain a plot and a test statistic for testing exponentiality of F versus alternatives of an increasing or decreasing hazard rate.

If we use all the observations, i.e., take $T^n = \infty$, we see that

$$A^n_0(T^n)/n \to_{\mathscr{P}} \int_0^\infty (1 - F(s))(1 - G(s))\, ds = \mathscr{E}(\tilde{X}_i) \leq \mathscr{E}(X_i) < \infty$$

when λ is constant, so our conditions are satisfied with $a_n = n$.

This example brings up the question as to whether the total time on test plot described here is the appropriate generalization from the uncensored to the censored case. As $n \to \infty$ the plot converges to the curve obtained by plotting

$$\frac{\int_0^t (1 - G(s-))\, dF(s)}{\int_0^\infty (1 - G(s-))\, dF(s)}$$

against

$$\frac{\int_0^t (1 - G(s-))(1 - F(s))\, ds}{\int_0^\infty (1 - G(s-))(1 - F(s))\, ds}, \qquad t \in [0, \infty),$$

which depends heavily on the censoring distribution G (though to be sure it is convex or concave according to whether λ is decreasing or increasing). If one is really more interested in estimating the curve obtained when $G \equiv 0$, then one would do this by replacing F by the product-limit estimator and deleting G. Unfortunately the asymptotic distribution theory becomes rather more complicated then (see Gill, 1983).

REMARKS. One might hope that $N^n(T^n) \to_{\mathscr{P}} \infty$ or $A_0^n(T^n) \to_{\mathscr{P}} \infty$ would be sufficient to prove our weak convergence result. However, this is easily seen not to be the case. Suppose for instance N^n is a standard Poisson process, $A_0^n = I$, $c = 1$, and let T^n be the stopping time, the first time after the nth event that the cumulative total time on test statistic takes a positive value. One can show that $T^n < \infty$ with probability one for each n. Obviously we cannot now have weak convergence to a Brownian bridge.

Two-sample versions of the total time on test plot and statistic are introduced by Gill and Schumacher (1985). Other recent related work has been done by Arjas and Haara (1985) and Arjas (1985).

APPENDIX

A lemma on random time change.

LEMMA. *Suppose $X^n \to_{\mathscr{D}} X$ in $D[0, \sigma]$ and $T^n \to_{\mathscr{P}} \tau < \sigma$ as $n \to \infty$ where X has continuous sample paths and $T^n \in [0, \sigma]$ for all n almost surely. Then $Y^n = X^n \circ (T^n \cdot I)$ satisfies $Y^n \to_{\mathscr{D}} X \circ (\tau \cdot I)$ in $D[0, 1]$.*

PROOF. By a Skorohod–Dudley construction (cf. Vervaat (1972) for a statement and nice application of this) and continuity of the paths of X, we may suppose that we have, on a single sample space,

$$\|X^n - X\|_\sigma \to 0 \quad \text{a.s.},$$
$$T^n \to \tau \quad \text{a.s.},$$

where $\|\cdot\|_a$ is the supremum norm on $[0, a]$. Immediately we have

$$\|X^n \circ (T^n \cdot I) - X \circ (T^n \cdot I)\|_1 \to 0 \quad \text{a.s.}$$

We must check

$$\|X \circ (T^n \cdot I) - X \circ (\tau \cdot I)\|_1 \to 0 \quad \text{a.s.}$$

But

$$\begin{aligned} &\|X \circ (T^n \cdot I) - X \circ (\tau \cdot I)\|_1 \\ &\quad = \sup_{x \in [0,1]} |X(T^n x) - X(\tau x)| \\ &\quad \le \sup_{u, v:\, |u - v| \le |T^n - \tau|} |X(u) - X(v)| \\ &\quad \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty, \end{aligned}$$

since the paths of X are continuous. We now have

$$\|X^n \circ (T^n \cdot I) - X \circ (\tau \cdot I)\|_1 \to 0 \quad \text{a.s.},$$

which implies

$$X^n \circ (T^n \cdot I) \to_{\mathscr{D}} X \circ (\tau \cdot I)$$

in $D[0, 1]$. □

NOTE. The lemma also holds when the closed interval $[0, \sigma]$ is replaced by the semiopen interval $[0, \sigma)$, $0 < \sigma \le \infty$. Just run through the above proof replacing σ by σ', $\tau < \sigma' < \sigma$.

REFERENCES

AALEN, O. O. and HOEM, J. M., (1978). Random time changes for multivariate counting processes. *Scand. Actuarial J.* 81–101.

ARJAS, E. (1985). On graphical methods for assessing goodness of fit in Cox's proportional hazards model. Preprint, Univ. of Oulu.

ARJAS, E. and HAARA, P. (1985). A note on the exponentiality of total hazards before failure. Preprint, Univ. of Oulu.

BARLOW, R. E. and PROSCHAN, F. (1969). A note on tests for monotone failure rate based on incomplete data. *Ann. Math. Statist.* **40** 595–600.

GILL, R. D. (1983). Large sample behaviour of the product-limit estimator on the whole line. *Ann. Statist.* **11** 49–58.

GILL, R. D. and SCHUMACHER, M. (1985). A simple test of the proportional hazards assumption. Report MS-R8504, Centre for Mathematics and Computer Science, Amsterdam.

VERVAAT, W. (1972). Functional central limit theorems for processes with positive drift and their inverses. *Z. Wahrsch. verw. Gebiete* **23** 245–253.

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