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ON THE ASYMPTOTIC FORMULA FOR THE PROBABILITY OF A TYPE I ERROR OF MIXTURE TYPE POWER ONE TESTS[1]

BY MOSHE POLLAK

The Hebrew University of Jerusalem

Let $X_1, X_2, \dots$ be iid with density f_y with respect to a sigma finite measure μ, where $\{f_y\}_{y \in \Omega}$, $\Omega \subseteq R$, is an exponential family. Let F be a probability measure on Ω and let $\theta_0 \in \Omega$. Define

$$T(B, F) = \min\left\{ n \middle| \int_\Omega \frac{f_y(X_1) \dots f_y(X_n)}{f_{\theta_0}(X_1) \dots f_{\theta_0}(X_n)} \, dF(y) \geq B \right\},$$

$T(B, F) = \infty$ if no such n exists. Previous studies have found that if F has a positive and continuous density with respect to Lebesgue measure on Ω, then

$$BP_{\theta_0}(T(B, F) < \infty) \to_{B \to \infty} \int_\Omega \int_0^\infty \exp\{-x\} \, dH_\theta(x) \, dF(\theta),$$

where H_θ are certain measures arising in a renewal-theoretic context.

Here we show that in a nonlattice context, this convergence holds for general probability measures F. We also show that the convergence is uniform for all probability measures F whose support is contained in an arbitrary interval $[a, b]$ interior to Ω, if the distribution of X_1 is strongly nonlattice for all $y \in \Omega$.

1. Introduction and summary. Let Ω be an open interval on the real line and let $\{f_y\}_{y \in \Omega}$ be the densities of a one-parameter exponential family with natural state space Ω with respect to a sigma-finite measure μ. Denote:

$$f_y(x) = \exp\{yx - \psi(y)\}, \qquad -\infty < x < \infty, \quad y \in \Omega.$$

Without loss of generality, assume that $0 = \psi(0) = \psi'(0)$. Let $X_1, X_2, \dots$ be a sequence of iid random variables, let P_θ be the probability measure (on R^∞) under which X_i have density f_θ with respect to μ, and let E_θ denote expectation under P_θ. Let F be a probability measure over Ω with $F(\{0\}) = 0$. Denote:

$$L(n, y) = \prod_{i=1}^{n} \left[f_y(X_i)/f_0(X_i) \right],$$

$$L(n, F) = \int_\Omega L(n, y) \, dF(y),$$

$$T(B, F) = \min\{n | L(n, F) \geq B\}$$
$$= \infty \quad \text{if no such } n \text{ exists.}$$

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The statistical test which stops at $T(B, F)$ and rejects H_0: $\theta = 0$ in favor of H_1: $\theta \neq 0$ has power one for certain values of θ [cf. Robbins (1970)]. It is known that [Lai and Siegmund (1977) and Woodroofe (1982), Section 6.1] the significance level of this test is

$$(1) \qquad P_0(T(B, F) < \infty) = \int_\Omega E_\theta[1/L(T(B, F), F)]\, dF(\theta).$$

Let $S_n^y = y\sum_{i=1}^n X_i - n\psi(y)$, let $\tau = \min\{n | S_n^y \geq A\}$, $\tau = \infty$ if no such n exists and let $\rho_y = S_\tau^y - A$ on $\{\tau < \infty\}$. If the P_0-distribution of $yX_i - \psi(y)$ is nonlattice, it follows from standard renewal theory [cf. Feller (1971)] that under P_y, ρ_y has (as $A \to \infty$) a limiting distribution H_y. If F has a positive continuous density with respect to Lebesgue measure on Ω, then by Lai and Siegmund (1977)

$$(2) \qquad BP_0(T(B, F) < \infty) \to_{B \to \infty} \int_\Omega \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta)$$

[see also Woodroofe (1982), Section 6.2]. The method involved in the proof of (2) is nonlinear renewal theory [developed by Woodroofe (1976) and Lai and Siegmund (1977); see Woodroofe (1982) for a survey]. Formula (2) yields an approximation for the significance level of the test associated with $T(B, F)$. The approximation is remarkably good, even for low values of B [Lai and Siegmund (1977)].

Let δ_θ denote the probability measure degenerate at θ. For testing H_0: $\theta = 0$ against an alternative H_1: $\theta = y$ with a power one test, $T(B, \delta_y)$ is optimal in the sense that it has least P_y-expected sample size among all power one tests with significance level $\alpha \leq P_0(T(B, \delta_y) < \infty)$. If the alternative H_1 is not simple, $T(B, \delta_y)$ may not yield a test of power one at every point in the alternative and is not efficient for values of θ other than y. One can maintain power one and asymptotic ($B \to \infty$) efficiency at every point in the alternative by employing a rule $T(B, F)$ with F having a positive continuous density for all points in the alternative [see Pollak and Siegmund (1975) and Pollak (1978)]. The asymptotic efficiency of $T(B, F)$ is manifest when B is large. When B is not large $T(B^*, \delta_y)$ will have a significantly smaller P_y-expected sample size than $T(B, F)$ [where B^* is such that $T(B^*, \delta_y)$ and $T(B, F)$ yield tests with power one at $\theta = y$ having the same level of significance]. For reasons of continuity, $E_\theta T(B^*, \delta_y)$ will be smaller than $E_\theta T(B, F)$ for a sizeable θ-neighborhood of y. As for practicality, the integration involved in computing $L(n, F)$ may make application of $T(B, F)$ cumbersome. Therefore, choosing a measure F^* concentrated at a single point or having atoms at a few points and employing $T(B, F^*)$ may be much more appealing than using $T(B, F)$ with continuous F. The range of values of B where this may be the case is large, and seems to include many of the "practical" cases. [For an indication of this, see Pollak and Siegmund (1985).] While the test associated with $T(B, F^*)$ may not have power one at all points in the alternative, averaging F^* with a continuous F will rectify this, at the same time retaining reasonably good efficiency for most points in the alternative. Therefore, there is an interest in establishing (2) for a wider range of measures F.

Other questions of interest concern the uniformity (in F) of the convergence in (2). These become of importance when several measures are being considered or when the measure F is random. [This may be the case, for instance, if a machine requires calibration at a value $\theta = 0$ at the start of each day, a power one test is daily employed on the products to check whether $\theta = 0$, and the measure F, which represents the values of θ when $\theta \neq 0$, is daily updated. For another example of a random F, arising in a different context—one where the uniformity of the convergence is crucial—see Pollak (1983).]

Here we show that in case $yX_i - \psi(y)$ are nonlattice, the convergence in (2) exists for general probability measures F. We show that this convergence is uniform for all probability measures F whose support is contained in an arbitrary interval $[a, b]$ interior to Ω, under the restriction that the P_y-distribution of X_i be strongly nonlattice [see Stone (1965)] for all $y \in \Omega$. (This requirement is fulfilled, for example, in case the observations are normal or exponential. It is not fulfilled if they are binomial or Poisson.) It should be noted that the uniformity result may not hold if the strongly nonlattice assumption is not satisfied (e.g., the Bernoulli case).

Even when the strongly nonlattice assumption is satisfied, the uniformity result is not transparent. The standard approach of decomposing a nonlinear renewal process calls for representing $L(n, F)$ via

$$\log L(n, F) = \theta \sum_{i=1}^{n} X_i - n\psi(\theta) + \xi(n, \theta, F), \tag{3}$$

where $\xi(n, \theta, F)$ are sequences which are slowly varying. If these are slowly varying uniformly in θ and F, letting $T = T(B, F)$, this representation would be applied to

$$BP_0(T(B, F) < \infty) = \int E_\theta \exp\{-[\log L(T, F) - \log B]\}\, dF(\theta) \tag{4}$$

[which is equivalent to (1)] to yield a uniform convergence in (2). The difficulty is that the representation (3) may fail, let alone $\xi(n, \theta, F)$ slowly vary uniformly [e.g., consider the $N(\theta, 1)$ case with $F = \frac{1}{2}\delta_{y-\varepsilon} + \frac{1}{2}\delta_{y+\varepsilon}$—the representation fails for $\theta = y$]. Looking at it differently, to each B, F there corresponds a (one-sided or two-sided) boundary $\gamma(t)$ via the relation

$$B = \int_\Omega \exp\{y\gamma(t) - t\psi(y)\}\, dF(y).$$

The stopping time $T(B, F)$ is equal to the first time n that the sequence of partial sums $\sum_{i=1}^{n} X_i$ crosses the boundary $\gamma(t)$, and one can try to get the asymptotics of the overshoot $\sum_{i=1}^{T(B,F)} X_i - \gamma(t)$ to account for uniform convergence in (2) via smoothness properties of $\gamma(t)$. However, it is generally not the case that for "neighboring" mixing measures F_1, F_2 the corresponding boundaries $\gamma_1(t), \gamma_2(t)$ are close uniformly in B. [For instance, consider $F_1 = F_a$, $F_2 = (1-\varepsilon)\delta_a + \varepsilon\delta_b$ where $0 < a < b$. An easy calculation shows that $\gamma_1(0) = (\log B)/a$, while $\gamma_2(0) = (\log B - \log \varepsilon)/b + o(1)$.]

Nevertheless, due to the following reasoning, the uniformity result is true. By virtue of (4) it is enough to show that the representation (3) and the uniformity in θ and F of the slowly varying characteristics of $\zeta(n, \theta, F)$ hold not for all θ, but for a θ-set $\Delta_B(F)$ having arbitrarily large F-probability—larger, say, than $1 - \varepsilon$, ε arbitrary. It does not matter if $\Delta_B(F)$ varies with B or F, as long as $1 - \varepsilon$ remains a lower bound for its probability and the slowly varying characteristics of $\xi(n, \theta, F)$ continue to hold for n in the vicinity of $T(B, F)$. The rigorous presentation of this reasoning is the content of the proof supplied in this article.

2. General convergence. We will use the notation of the previous section.

THEOREM 1. *Suppose that* $F\{0\} = 0$ *and* $F\{\{\theta | \theta X_1 - \psi(\theta)$ *has a lattice* P_0*-distribution*$\}\} = 0$. *Then*

$$BP_0\{T(B, F) < \infty\} \to_{B \to \infty} \int_\Omega \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta).$$

Essentially, the idea of the proof is to apply Lemmas 1 and 5 below and the nonlinear renewal theorem to (4) above.

Let $a \le b$ be interior points of Ω. Fix $\delta > 0$, $\beta = \frac{1}{20}$, $\rho = \frac{1}{2}$, and $\alpha = \frac{3}{4}$. Denote

$$T = T(B, F), \qquad \overline{X}_n = \textstyle\sum_{i=1}^n X_i/n, \qquad \varepsilon = \varepsilon(n) = n^{\beta - 1/2},$$

$$\kappa = \kappa(n, \theta) = \min\{\varepsilon, \tfrac{1}{2}|\theta|, |\theta| / \sqrt{\psi''(\theta)}\}, \qquad \sigma = \sigma(n) = n^{2\beta - 1/2},$$

$S[a, b]$ = set of all probability measures F whose support is contained in $[a, b]$, and which satisfy $F\{0\} = 0$ and $F\{\{\theta | \theta X_1 - \psi(\theta)$ has a lattice distribution$\}\} = 0$,

$$I(\theta) = \theta\psi'(\theta) - \psi(\theta), \qquad l_\theta = (\log B)/I(\theta),$$

m_θ = the integer value of $l_\theta - (l_\theta)^\alpha \rho/4$,

n_θ = the integer value of $l_\theta + (l_\theta)^\alpha \rho/4$,

$$\varepsilon_\theta = (n_\theta)^{\beta - 1/2}, \qquad \varepsilon_\theta^* = \min\{\varepsilon_\theta, \tfrac{1}{2}|\theta|, |\theta| / \sqrt{\psi''(\theta)}\}, \qquad \sigma_\theta = (m_\theta)^{2\beta - 1/2},$$

$$\Delta_B = \Delta_B(F) = \{\theta | \theta \ne 0,\ F\{[\theta - \tfrac{1}{2}\kappa, \theta + \tfrac{1}{2}\kappa]\} \ge \delta\kappa \text{ for all } m_\theta \le n \le n_\theta\},$$

$$R_B = R_B(F) = \Delta_B \cap \{\theta \,|\, |\theta| > (\log B)^{-1/10}\},$$

θ^* = is defined by $\psi'(\theta^*) = \overline{X}_n$,

$$\theta^\# = \theta^\#(\theta, B) = |\theta| + [(1 - \rho) l_\theta]^{2\beta - 1/2},$$

$$n^* = n^*(\theta, B) = \min\{|\theta|^{-2/(1 - 3\beta)}, 7 l_\theta\}.$$

The following lemmas are stated in somewhat greater generality than needed for Theorem 1 so as to enable their use for the proof of Theorem 2.

LEMMA 1. *There exists* $0 < B_\delta < \infty$ *such that if* $B \ge B_\delta$ *then*

$$F\{\Delta_B^{\text{complement}}\} \le 5(b - a)\delta \quad \textit{whenever } F \in S[a, b].$$

PROOF. Suppose $a < 0 < b$. Suppose $0 \ne \theta \in [a, b]$ and $F\{[\theta - \frac{1}{2}\kappa, \theta + \frac{1}{2}\kappa]\} < \delta\kappa$ for some $m_\theta \le n \le n_\theta$. There exists $0 < B_\delta < \infty$ (independent of

θ, F) such that if $B \geq B_\delta$ then, for such θ, $F\{[\theta - \frac{1}{2}\varepsilon_\theta^*, \theta + \frac{1}{2}\varepsilon_\theta^*]\} \leq 2\delta\varepsilon_\theta^*$ whenever $F \in S[a, b]$. Therefore

$$(\Delta_B)^{\text{complement}} \subseteq \left\{\theta | F\left\{\left(\theta - \tfrac{1}{2}\varepsilon_\theta^*, \theta + \tfrac{1}{2}\varepsilon_\theta^*\right)\right\} \leq 2\delta\varepsilon_\theta^*\right\}$$

if $B \geq B_\delta$.

Let $b_0 = b$, and define recursively $b_i = \max\{\theta | 0 < \theta \leq b_{i-1} - \frac{1}{2}\varepsilon_{b_{i-1}}^*$, $F\{(\theta - \frac{1}{2}\varepsilon_\theta^*, \theta + \frac{1}{2}\varepsilon_\theta^*)\} \leq 2\delta\varepsilon_\theta^*\}$, $i = 1, 2, \ldots$, and define

$$D_i = \left\{\theta | b_i - \tfrac{1}{2}\varepsilon_{b_i}^* < \theta < b_i + \tfrac{1}{2}\varepsilon_{b_i}^*\right\}.$$

Thus, $(0, b] \cap (\Delta_B)^{\text{complement}} \subseteq \bigcup_i D_i$. Clearly, $F\{D_i\} \leq 2\delta\varepsilon_{b_i}^* = 2\delta|D_i|$. Also $D_i \cap D_j = \phi$ if $|i - j| > 1$, so $F\{\bigcup_i D_i\} \leq F\{\bigcup_j D_{2j}\} + F(\bigcup_j D_{2j+1}\} \leq 2\delta b + 2\delta\varepsilon_b^* + 2\delta b \leq 5\delta b$. Consequently, $F\{(0, b] \cap (\Delta_b)^{\text{complement}}\} \leq 5\delta b$. A similar argument holds for $[a, 0)$, so that $F\{(\Delta_B)^{\text{complement}}\} \leq 5\delta(b - a)$. The argument for $0 \leq a \leq b$ and for $a \leq b \leq 0$ is analogous. □

LEMMA 2. *Let* $0 < \eta < \infty$ *and denote*

$$\zeta(n, \theta, F) = \log\left[L(n, F) \Big/ \int_{\theta - (1/2)\sigma_\theta}^{\theta + (1/2)\sigma_\theta} L(n, y)\, dF(y)\right].$$

Then

$$\sup_{F \in S[a, b]} \sup_{\theta \in \Delta_B} \max_{n \geq (1-\rho)l_B} P_\theta\left\{\max_{j \geq 1} |\zeta(n + j, \theta, F) - \zeta(n, \theta, F)| > \eta\right\} \to_{B \to \infty} 0.$$

PROOF. Let W_n^θ denote the event $\{|\theta^* - \theta| < \varepsilon\}$. There exists a constant $c > 0$ independent of n, θ, F such that if $a \leq \theta \leq b$, then

$$\left\{|\overline{X}_n - \psi'(\theta)| < c\varepsilon\right\} \subset W_n^\theta. \tag{5}$$

Following the proof of Lemma 2 of Pollak and Siegmund (1975), for $\Lambda > \theta$

$$\begin{aligned} &P_\theta\{\overline{X}_n - \psi'(\theta) > z\} \\ &\quad = \int_{\{\overline{X}_n - \psi'(\theta) > z\}} \exp\{n[(\theta - \Lambda)\overline{X}_n - (\psi(\theta) - \psi(\Lambda))]\}\, dP_\Lambda \\ &\quad \leq \exp\{-n[(\psi'(\theta) + z)(\Lambda - \theta) - (\psi(\Lambda) - \psi(\theta))]\} \\ &\qquad \times P_\Lambda\{\overline{X}_n - \psi'(\theta) > z\}. \end{aligned} \tag{6}$$

Setting $z = c\varepsilon$ and $\Lambda = \theta + n^{(1/2)\beta - 1/2}$ yields for large enough n

$$P_\theta\{\overline{X}_n - \psi'(\theta) > c\varepsilon\} < \exp\left\{-\tfrac{1}{2}cn^{3\beta/2}\right\} \tag{7}$$

uniformly for $a \leq \theta \leq b$. Hence

$$\sup_{a \leq \theta \leq b} \sum_{n=r}^{\infty} P_\theta\{\overline{X}_n - \psi'(\theta) > c\varepsilon\} \to_{r \to \infty} 0.$$

A similar analysis yields

$$\sup_{a \le \theta \le b} \sum_{n=r}^{\infty} P_\theta\{\bar{X}_n - \psi'(\theta) < -c\varepsilon\} \to_{r\to\infty} 0.$$

Hence, by (5)

$$\inf_{a \le \theta \le b} P_\theta\left\{\bigcap_{n=r}^{\infty} W_n^\theta\right\} \to_{r\to\infty} 1. \tag{8}$$

Note that

$$\begin{aligned}\log L(n, \theta + \Delta\theta)\\ &= n[(\theta + \Delta\theta)\bar{X}_n - \psi(\theta + \Delta\theta)]\\ &= n\left\{\theta\bar{X}_n - \psi(\theta) + \Delta\theta[\bar{x}_n - \psi'(\theta)] - \tfrac{1}{2}\psi''(\theta)(\Delta\theta)^2[1 + o(1)]\right\},\end{aligned}$$

where $o(1) \to 0$ as $\Delta\theta \to 0$ uniformly in $\theta \in [a - \frac{1}{2}\varepsilon, b + \frac{1}{2}\varepsilon]$ if $a - \frac{1}{2}\varepsilon, b + \frac{1}{2}\varepsilon \in \Omega$. For large enough n, for $\theta = \theta^*$, this becomes

$$\log L(n, \theta^* + \Delta\theta) = n\left\{\theta^*\bar{X}_n - \psi(\theta^*) - \tfrac{1}{2}\psi''(\theta^*)(\Delta\theta)^2[1 + o(1)]\right\}.$$

Therefore, there exists B_0 (independent of F) such that if $B > B_0$ and $\theta \in \Delta_B$, on W_n^θ for $n \ge (1 - \rho)l_\theta$, if $F \in S[a, b]$

$$\begin{aligned}&\frac{\int_a^b L(n, y)\,dF(y)}{\int_{\theta-(1/2)\sigma_\theta}^{\theta+(1/2)\sigma_\theta} L(n, y)\,dF(y)}\\
&\le 1 + \frac{\left(\int_{\theta+(1/2)\sigma_\theta}^{b} + \int_a^{\theta-(1/2)\sigma_\theta}\right)L(n, y)\,dF(y)}{\int_{\theta^*-\varepsilon_\theta}^{\theta^*+\varepsilon_\theta} L(n, y)\,dF(y)}\\
&\le 1 + \frac{L(n, \theta^* + \sigma_\theta/4) + L(n, \theta^* - \sigma_\theta/4)}{\delta\varepsilon_\theta^* \min\{L(n, \theta^* - \varepsilon_\theta), L(n, \theta^* + \varepsilon_\theta)\}}\\
&\le 1 + \frac{2\exp\left\{n\left[\theta^*\bar{X}_n - \psi(\theta^*) - \frac{1}{2}\psi''(\theta^*)(\sigma_\theta/4)^2[1 + o(1)]\right]\right\}}{\delta\varepsilon_\theta^* \exp\left\{n\left[\theta^*\bar{X}_n - \psi(\theta^*) - \frac{1}{2}\psi''(\theta^*)(\varepsilon_\theta)^2[1 + o(1)]\right]\right\}}\\
&\le 1 + \frac{2\exp\{-\frac{1}{64}[\min_{a\le\theta\le b}\psi''(\theta)]\sigma_\theta^2(1 - \rho)l_\theta\}}{\delta\varepsilon_\theta^*}\\
&\to_{B\to\infty} 1.\end{aligned}$$

This, together with (8), accounts for Lemma 2. □

LEMMA 3. *Let* $0 < \eta < \infty$. *Let*

$$K_n(\theta) = \tfrac{1}{2}n\psi''(\theta)\{[\bar{X}_n - \psi'(\theta)]/\psi''(\theta)\}^2,$$

$$\begin{aligned}J_n(y, \theta, \lambda) = \tfrac{1}{2}n\psi''(\theta)\{y - \theta - [\bar{X}_n - \psi'(\theta)]/\psi''(\theta)\}^2\\ + n(y - \theta)^3\psi'''(\lambda)/6.\end{aligned}$$

Let $\{\lambda_n(\theta)\}_{n=1}^{\infty}$ *be any random sequence such that* $\theta - \sigma \le \lambda_n(\theta) \le \theta + \sigma$. *Then*

$$\text{(i)} \quad \sup_{a \le \theta \le b} P_\theta\Big\{ \max_{j=1,\dots,\rho n^\alpha} |K_{n+j}(\theta) - K_n(\theta)| \ge \eta \Big\} \to_{n\to\infty} 0,$$

$$\text{(ii)} \quad \sup_{a \le \theta \le b} P_\theta\Big\{ \max_{\theta-\sigma \le y \le \theta+\sigma} \max_{j=1,\dots,\rho n^\alpha} \big| J_{n+j}(y, \theta, \lambda_{n+j}(\theta)) - J_n(y, \theta, \lambda_n(\theta)) \big| \ge \eta \Big\} \to_{n\to\infty} 0.$$

PROOF. Part (i) follows from Proposition 1 of Lai and Siegmund (1979), noting that the proof of this Proposition 1 can be carried through uniformly for $a \le \theta \le b$.

As for part (ii), note that

$$J_n(y, \theta, \lambda) = n(y-\theta)^3 \psi'''(\lambda)/6 + K_n(\theta) + \tfrac{1}{2}\psi''(\theta)(y-\theta)^2 n - \sum_{i=1}^{n} [X_i - \psi'(\theta)](y-\theta).$$

Therefore, for $\theta - \sigma \le y \le \theta + \sigma$, $1 \le j \le \rho n^\alpha$ and large enough n

$$\begin{aligned} &\big| J_{n+j}(y, \theta, \lambda_{n+j}(\theta)) - J_n(y, \theta, \lambda_n(\theta)) \big| \\ &\quad \le \rho n^\alpha \sigma^3 \sup_{a \le \theta \le b} |\psi'''(\theta)| + \sup_{a \le \theta \le b} |\psi''''(\theta)| \sigma^4 n \\ &\qquad + |K_{n+j}(\theta) - K_n(\theta)| + \psi''(\theta)\rho n^\alpha \sigma^2 + \left| \sum_{i=n+1}^{n+j} [X_i - \psi'(\theta)] \right| \sigma \end{aligned} \tag{9}$$

uniformly for $a \le \theta \le b$. By Kolmogorov's inequality,

$$\begin{aligned} &P_\theta\left\{ \max_{j=1,\dots,\rho n^\alpha} \left| \sum_{i=n+1}^{n+j} [X_i - \psi'(\theta)] \right| \sigma > \eta/4 \right\} \\ &\quad = P_\theta\left\{ \max_{j=1,\dots,\rho n^\alpha} \left(\sum_{i=n+1}^{n+j} [X_i - \psi'(\theta)] \right)^2 > (\eta/4)^2/\sigma^2 \right\} \\ &\quad \le \psi''(\theta)\rho n^\alpha \sigma^2/(\eta/4)^2 \to_{n\to\infty} 0 \end{aligned}$$

uniformly for $a \le \theta \le b$. Part (ii) now follows from (9) and part (i). □

LEMMA 4. *Using the notation of Lemma* 3, *denote*

$$Q_n(\theta, F) = \log \int_{\theta-(1/2)\sigma_\theta}^{\theta+(1/2)\sigma_\theta} \exp\{-J_n(y, \theta, \lambda_n(\theta))\}\, dF(y).$$

Let $0 < \eta < \infty$. *Then*

$$\sup_{F \in S[a,b]} \sup_{\theta \in \Delta_B} \max_{m_\theta \le n \le n_\theta} P_\theta\Big(\max_{j=1,\dots,\rho n^\alpha \wedge (n_\theta - n)} |Q_{n+j}(\theta, F) - Q_n(\theta, F)| > \eta\Big) \to_{B \to \infty} 0.$$

PROOF.

$$\begin{aligned} &Q_{n+j}(\theta, F) - Q_n(\theta, F) \\ &\quad = \log\bigg\{ \int_{\theta-(1/2)\sigma_\theta}^{\theta+(1/2)\sigma_\theta} \exp\big\{ -\big[J_{n+j}(y, \theta, \lambda_{n+j}(\theta)) - J_n(y, \theta, \lambda_n(\theta))\big]\big\} \\ &\qquad \times \exp\{ -J_n(y, \theta, \lambda_n(\theta))\}\, dF(y) / \\ &\qquad \int_{\theta-(1/2)\sigma_\theta}^{\theta+(1/2)\sigma_\theta} \exp\{ -J_n(y, \theta, \lambda_n(\theta))\}\, dF(y)\bigg\}. \end{aligned}$$

This is the logarithm of an expectation of $\exp\{-[J_{n+j}(y, \theta, \lambda_{n+j}(\theta)) - J_n(y, \theta, \lambda_n(\theta))]\}$. Lemma 4 is therefore a consequence of Lemma 3(ii). □

LEMMA 5.

$$\log L(n, F) = \theta \sum_{i=1}^{n} X_i - n\psi(\theta) + \xi(n, \theta, F),$$

where $\{\xi(n, \theta, F)\}_{n=1}^{\infty}$ *is a sequence of random variables which satisfies for any* $0 < \eta < \infty$

$$\sup_{F \in S[a,b]} \sup_{\theta \in \Delta_B} \max_{m_\theta \le n \le n_\theta} P_\theta\Big(\max_{j=1,\dots,\rho n^\alpha \wedge (n_\theta - n)} |\xi(n+j, \theta, F) - \xi(n, \theta, F)| > \eta\Big) \to_{B \to \infty} 0.$$

PROOF. Using the notation of Lemma 3 and Lemma 4, for $\theta \in \Delta_B$ and $m_\theta \le n \le n_\theta$ there exists $\lambda_n(\theta) \in (\theta - \sigma, \theta + \sigma)$ such that

$$\begin{aligned} &\log \int_{\theta-(1/2)\sigma_\theta}^{\theta+(1/2)\sigma_\theta} L(n, y)\, dF(y) \\ &\quad = \log \int_{\theta-(1/2)\sigma_\theta}^{\theta+(1/2)\sigma_\theta} \exp\{ n[y\overline{X}_n - \psi(y)]\}\, dF(y) \\ &\quad = \log \int_{\theta-(1/2)\sigma_\theta}^{\theta+(1/2)\sigma_\theta} \exp\Big\{ n\Big[\theta\overline{X}_n - \psi(\theta) + (y-\theta)(\overline{X}_n - \psi'(\theta)) \\ &\qquad\qquad - \tfrac{1}{2}(y-\theta)^2\psi''(\theta) - (y-\theta)^3\psi'''(\lambda_n(\theta))/6\Big]\Big\}\, dF(y) \\ &\quad = \theta \sum_{i=1}^{n} X_i - n\psi(\theta) + K_n(\theta) + Q_n(\theta, F). \end{aligned}$$

Lemma 5 now follows from Lemma 2, Lemma 3(i) and Lemma 4. □

The following two lemmas are needed in order to apply the nonlinear renewal theorem.

LEMMA 6.

$$\sup_{F\in S[a,b]}\ \sup_{\theta\in R_B} P_\theta\{T(B,F) > n_\theta\} \to_{B\to\infty} 0.$$

PROOF. Denote $\omega_n = \bar{X}_n - \psi'(\theta)$.

$$\begin{aligned} L(n,F) &= \int \exp\{n[y\bar{X}_n - \psi(y)]\}\,dF(y) \\ &\geq \int_{\theta-(1/2)\varepsilon}^{\theta+(1/2)\varepsilon} \exp\{n[y(\psi'(\theta)+\omega_n) - \psi(y)]\}\,dF(y) \\ &= \int_{\theta-(1/2)\varepsilon}^{\theta+(1/2)\varepsilon} \exp\Big\{n\Big[\theta(\psi'(\theta)+\omega_n) + (y-\theta)(\psi'(\theta)+\omega_n) \\ &\qquad -\Big[\psi(\theta) + (y-\theta)\psi'(\theta) + \tfrac{1}{2}(y-\theta)^2\psi''(\bar{\theta})\Big]\Big]\Big\}\,dF(y), \end{aligned}$$

where $|\bar{\theta} - \theta| < \frac{1}{2}\varepsilon$; so for $\theta \in \Delta_B$, $n \geq l_\theta$ and large enough B,

$$L(n,F) \geq \exp\{nI(\theta)\}\exp\{-n|\omega_n|(|\theta| + \tfrac{1}{2}\varepsilon)\}\exp\{-\psi''(\theta)n^{2\beta}\}\delta\kappa. \tag{10}$$

Denote $z = \{I(\theta)\rho/[8(|\theta| + \frac{1}{2}\varepsilon)]\}(l_\theta)^{-(1-\alpha)}$. By (6) above, setting $\Lambda = \theta + z/\psi''(\theta)$,

$$P_\theta\{\omega_{n_\theta} > z\} \leq \exp\{-n_\theta[(\psi'(\theta)+z)(\Lambda-\theta) - (\psi(\Lambda) - \psi(\theta))]\} \to_{B\to 0} 0$$

uniformly in $\theta \in R_B$, and similarly $P_\theta\{\omega_{n_\theta} < -z\} \to_{B\to\infty} 0$.

Therefore, inserting n_θ instead of n in (10), it follows that

$$\inf_{F\in S[a,b]}\ \inf_{\theta\in R_B} P_\theta\{L(n_\theta,F) \geq B\} \to_{B\to\infty} 1,$$

which is equivalent to the statement of Lemma 6. □

LEMMA 7.

$$\sup_{F\in S[a,b]}\ \sup_{\theta\in R_B} P_\theta\{T(B,F) < m_\theta\} \to_{B\to\infty} 0.$$

PROOF. Let $T = T(B,F)$ and denote $S_n = n\bar{X}_n$. Following the proof of Lemma 3 of Pollak and Siegmund (1975), for any $y > 0$,

$$\begin{aligned} P_\theta\{T \leq m_\theta\} &\leq P_\theta\Big\{S_{m_\theta} - \psi'(\theta)m_\theta \geq y[\psi''(\theta)m_\theta]^{1/2}\Big\} \\ &\quad + \int_{\{T\leq m_\theta,\, S_{m_\theta}-\psi'(\theta)m_\theta \leq y[\psi''(\theta)m_\theta]^{1/2}\}} \exp\{\theta S_{m_\theta} - m_\theta\psi(\theta)\}\,dP_0 \\ &\leq P_\theta\Big\{S_{m_\theta} - \psi'(\theta)m_\theta \geq y[\psi''(\theta)m_\theta]^{1/2}\Big\} \\ &\quad + \exp\Big\{I(\theta)m_\theta + \theta y[\psi''(\theta)m_\theta]^{1/2}\Big\}P_0\{T \leq m_\theta\}. \end{aligned} \tag{11}$$

Since $P_0\{T \leq m_\theta\} < P_0\{T < \infty\} \leq 1/B$ [cf. Robbins (1970)], for $y = (m_\theta)^{1/9}/\sqrt{\psi''(\theta)}$, the second term on the far right side of (11) is less than $\exp\{-(\log B)^{1/2}\}$ when B is large enough, for all $\theta \in R_B$. Also for large enough

B, for the same value of y, for all $\theta \in R_B$

$$P_\theta\left\{S_{m_\theta} - \psi'(\theta)m_\theta \geq y\left[\psi''(\theta)m_\theta\right]^{1/2}\right\}$$
$$\leq P_\theta\left\{\overline{X}_{m_\theta} - \psi'(\theta) \geq (m_\theta)^{(1/9)-1/2}\right\}$$
$$\leq \exp\left\{-\tfrac{1}{2}(m_\theta)^{1/6}\right\} \to_{B\to\infty} 0,$$

where the last inequality follows from (7) above and the convergence is uniform in $\theta \in [a, b] - \{0\}$. This completes the proof of Lemma 7. □

PROOF OF THEOREM 1. Let $a_1 < b_1 < 0 < a_2 < b_2$ be interior points of Ω and denote $\Gamma = [a_1, b_1] \cup [a_2, b_2]$. Suppose first that the support of F is contained in Γ. By virtue of (4) and the definition of T

$$\begin{aligned}(12)\qquad &\int_{\Delta_B} E_\theta \exp\{-[\log L(T, F) - \log B]\}\, dF(\theta) \leq BP_0\{T < \infty\} \\ &\leq \int_{\Delta_B} E_\theta \exp\{-[\log L(T, F) - \log B]\}\, dF(\theta) + F\left\{\Delta_B^{\text{complement}}\right\}.\end{aligned}$$

Letting $\Xi(A)$ denote the indicator function of the set A, define

$$\phi(\theta) = \left|E_\theta \exp\{-[\log L(T, F) - \log B]\} - \int_0^\infty \exp\{-x\}\, dH_\theta(x)\right| \Xi(\theta \in \Delta_B).$$

Lemmas 5, 6, and 7 ensure that the considerations of the proof of Theorem 1 of Lai and Siegmund (1977) carry through for $\theta \in \Delta_B$, so that $\phi(\theta) \to_{B\to\infty} 0$. Hence $\int_\Gamma \phi(\theta)\, dF(\theta) \to_{B\to\infty} 0$, which implies

$$\begin{aligned}(13)\qquad &\left|\int_{\Delta_B} E_\theta \exp\{-[\log L(T, F) - \log B]\}\, dF(\theta)\right. \\ &\left. - \int_{\Delta_B}\int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta)\right| \to_{B\to\infty} 0.\end{aligned}$$

Clearly

$$\begin{aligned}(14)\qquad &\left|\int_\Omega\int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) - \int_{\Delta_B}\int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta)\right| \\ &\leq F\left\{\Delta_B^{\text{complement}}\right\}.\end{aligned}$$

Since δ is arbitrary, (12), (13), and (14) in conjunction with Lemma 1 account for Theorem 1 for F whose support is contained in Γ.

For general F, suppose first that $-\infty = \inf\{x | x \in \Omega\}$ and $\sup\{x | x \in \Omega\} = \infty$. Let $\gamma > 0$ and choose $-\infty < a_1 < b_1 < 0 < a_2 < b_2 < \infty$ such that $F(\Gamma) \geq (1 - \gamma)$ where $\Gamma = [a_1, b_1] \cup [a_2, b_2]$. Let $B^{\#} = B/F\{\Gamma\}$ and let $dF^{\#}(x) = dF(x)/F\{\Gamma\}$ for $x \in \Gamma$; $dF^{\#}(x) = 0$ otherwise. Clearly $\{T(B, F) < \infty\} \supseteq$

$\{T(B^{\#}, F^{\#}) < \infty\}$. Since $F^{\#} \in S[a_1, b_2]$

$$\liminf_{B\to\infty} BP_0\{T(B, F) < \infty\} \geq \liminf_{B\to\infty} BP_0\{T(B^{\#}, F^{\#}) < \infty\}$$

$$\geq \int_\Gamma \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta)(1-\gamma).$$

On the other hand, from (4) it follows that

$$BP_0\{T(B, F) < \infty\} \leq \int_\Gamma \exp\{-[\log L(T, F) - \log B]\, dF(\theta) + \gamma$$

and so in a manner similar to the proof for $F \in S[a, b]$ it follows that

$$\limsup_{B\to\infty} BP_0\{T(B, F) < \infty\} \leq \int_\Gamma \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) + \gamma.$$

Decreasing γ towards zero—i.e., increasing b_1 to zero and b_2 to ∞, and decreasing a_1 to $-\infty$ and a_2 to 0—concludes the proof.

If $\inf(x|x \in \Omega) > -\infty$ or $\sup\{x|x \in \Omega) < \infty$, a similar proof is valid. The details are omitted. □

3. Uniform convergence. We will continue to use the notation of the previous sections. The distribution of X is said to be strongly nonlattice [Stone (1965)] if $\liminf_{|t|\to\infty}|E\exp\{itX\} - 1| > 0$.

THEOREM 2. *Suppose that the P_y-distribution of X_1 is strongly nonlattice for all $y \in \Omega$. Let $a \leq b$ be interior points of Ω. Then*

$$BP_0\{T(B, F) < \infty\} \to_{B\to\infty} \int_a^b \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta)$$

uniformly in $F \in S[a, b]$.

The proof breaks down into two parts: for $\theta \in R_B$, the proof is similar to that of Theorem 1, with the strongly nonlattice property ensuring uniform convergence. For $\theta \in \Delta_B - R_B$, the proof shows that $\theta X_\iota - \psi(\theta)$ is stochastically small enough to ensure that any "overshoot" is negligible. The details are spelled out in the following lemmas.

LEMMA 8. *If X is strongly nonlattice then so is $cX + d$ for any $c \neq 0$, $-\infty < d < \infty$.*

PROOF. It suffices to show that if X is not strongly nonlattice, then neither is $cX + d$. Without loss of generality, let $c = 1$.

Suppose there exists a sequence $\{t_j\}_{j=1}^\infty$, $|t_j| \to_{j\to\infty} \infty$ such that $Ee^{it_jX} \to_{j\to\infty} 1$. Then $e^{it_jX} \to_{j\to\infty} 1$ in probability. Therefore, for any integer k, $e^{ikt_jX} \to_{j\to\infty} 1$ in probability, and this convergence is uniform for any finite set of integers $k = 1, \dots, m$.

Let $\eta > 0$. If j is large enough, then $\max_{k=1,\dots,m}|Ee^{ikt_jX} - 1| < \eta$. Clearly, m can be chosen to be large enough so that $\sup_{j\geq 1}\min_{1\leq k\leq m}|e^{ikt_jd} - 1| < \eta$. Thus, for any large enough j, there exists $k_j \in \{1,\dots,m\}$ such that

$$\begin{aligned}|Ee^{ik_jt_j(X+d)} - 1| &= \left|(Ee^{ik_jt_jX} - 1) + (e^{ik_jt_jd} - 1) + (Ee^{ik_jt_jX} - 1)(e^{ik_jt_jd} - 1)\right| \\ &\leq 2\eta + \eta^2.\end{aligned}$$

Letting $\eta \to 0$, it follows that $X + d$ is not strongly nonlattice. □

LEMMA 9. *Let $Y_1, Y_2, \dots$ be strongly nonlattice iid random variables with $EY_i \geq 0$, $P(Y_i = 0) < 1$. Let*

$$V = \min\left\{n \,\middle|\, \sum_{i=1}^{n} Y_i > 0\right\}, \qquad Z = \sum_{i=1}^{V} Y_i.$$

Then Z is strongly nonlattice.

PROOF. The lemma is trivial if $P(Y_i \geq 0) = 1$. Consider the case that $P(Y_i \geq 0) < 1$. One must show that $\liminf_{|t|\to\infty}|Ee^{itZ} - 1| > 0$. Suppose this were not the case, but that there exists a sequence $\{t_j\}_{j=1}^{\infty}$, $|t_j| \to_{j\to\infty} \infty$ such that $Ee^{it_jZ} \to_{j\to\infty} 1$. Then

$$e^{it_jZ} \to_{j\to\infty} 1 \tag{15}$$

in probability. Since $Ee^{it_jZ} = EE(e^{it_jZ}|Y_1)$, it follows that on $\{Y_1 > 0\}$ (on which $Z = Y_1$)

$$P\left(e^{it_jY_1} \to_{j\to\infty} 1 | Y_1 > 0\right) = 1. \tag{16}$$

Let $u^* = \sup\{x | P(Y_1 \leq x) < 1\}$. Clearly, $u^* > 0$. Let $U_1 = (-u^*, 0]$. If $u^* < \infty$, let $U_k = (-ku^*, -(k-1)u^*]$, $k = 1, 2, \dots$.

Suppose $Y_1 \in U_k$. Then $P(V = k+1, Y_2 > 0, \dots, Y_{k+1} > 0 | Y_1) > 0$. On this event

$$e^{it_jZ} = e^{it_jY_1}\prod_{m=2}^{k+1} e^{it_jY_m}. \tag{17}$$

From (16) it follows that

$$P\left(\prod_{m=2}^{k+1} e^{it_jY_m} \to_{j\to\infty} 1 | Y_1, V = k+1, Y_2 > 0, \dots, Y_{k+1} > 0\right) = 1$$

a.s. on $Y_1 \in U_k$. From (15) and (17) it therefore follows that

$$P\left(e^{it_jY_1} \to_{j\to\infty} 1 | Y_1 \in U_k\right) = 1.$$

Therefore $P(e^{it_jY_1} \to_{j\to\infty} 1) = 1$ and so $Ee^{it_jY_1} \to_{j\to\infty} 1$, contradicting the assumption that Y_1 is strongly nonlattice. □

LEMMA 10. *Let $S_n^\theta = \theta\sum_{i=1}^n X_i - n\psi(\theta)$, let $\tau = \tau(\theta, A) = \min\{n | S_n^\theta > A\}$ and let H_θ be the P_θ-limiting distribution of $S_\tau^\theta - A$ as $A \to \infty$. Then the convergence in P_θ-distribution of $S_\tau^\theta - A$ is uniform in $\theta \in [a, b] - \{0\}$.*

PROOF. Let

$$Y_\iota = \theta X_\iota - \psi(\theta),$$

$$V_0 = 0, \qquad V_k = \min\left\{ n | n > V_{k-1}, \sum_{\iota = V_{k-1}+1}^{n} Y_\iota > 0 \right\}, \qquad k = 1, 2, \ldots,$$

and

$$Z_k = \sum_{i = V_{k-1}+1}^{V_k} Y_\iota, \qquad k = 1, 2, \ldots .$$

By Lemmas 8 and 9, the P_θ-distribution of Z_k is strongly nonlattice.

Let G_θ denote the renewal function defined by $G_\theta(x) = \sum_{n=0}^{\infty} G_\theta^{(n)}(x)$ where $G_\theta^{(n)}$ is the distribution of $\sum_{k=1}^{n} Z_k$. By Theorem (ii) of Stone (1965), there exists $r > 0$ ($r = r(\theta)$) such that

$$G_\theta(x) = x/\mu_1 + \mu_2/(2\mu_1^2) + Y_\theta(x), \tag{18}$$

where $\mu_1 = E_\theta Z_1$, $\mu_2 = E_\theta Z_1^2$ and

$$Y_\theta(x)\exp\{rx\} \to_{x\to\infty} 0. \tag{19}$$

Let $0 < \gamma$. A check of Stone's (1965) proof reveals that there exists a constant $r > 0$ independent of $\theta \in [a, b] - [-\gamma, \gamma]$ such that the convergence in (19) is uniform in $\theta \in [a, b] - [-\gamma, \gamma]$. Now

$$\begin{aligned} P_\theta(S_\tau^\theta - A > x) &= \sum_{n=1}^{\infty} P_\theta\left(\sum_{\iota=1}^{n-1} Z_\iota < A, \sum_{\iota=1}^{n} Z_\iota > A + x \right) \\ &= \sum_{n=1}^{\infty} \int_0^A \left[1 - G_\theta^{(1)}(A + x - s)\right] G_\theta^{(n-1)}(ds) \\ &= \int_0^A \left[1 - G_\theta^{(1)}(A + x - s)\right] G_\theta(ds). \end{aligned} \tag{20}$$

Note that

$$\begin{aligned} \int_0^A \left[1 - G_\theta^{(1)}(A + x - s)\right] ds &\to_{A\to\infty} \int_0^\infty \left[1 - G_\theta^{(1)}(x + s)\right] ds \\ &= \int_x^\infty \left[1 - G_\theta^{(1)}(s)\right] ds \end{aligned} \tag{21}$$

uniformly for $x \geq 0$, $\theta \in [a, b] - [-\gamma, \gamma]$. Also, letting Y_θ be as in (19)

$$\begin{aligned} &\int_0^A \left[1 - G_\theta^{(1)}(A + x - s)\right] Y_\theta(ds) \\ &= \left[1 - G_\theta^{(1)}(A + x - s)\right] Y_\theta(s)\Big|_0^A + \int_0^A Y_\theta(s) G_\theta^{(1)}(A + x - ds) \\ &= \left[1 - G_\theta^{(1)}(x)\right] Y_\theta(A) - \left[1 - G_\theta^{(1)}(A + x)\right] Y_\theta(0) \\ &\quad - \int_0^A Y_\theta(A - s) G_\theta^{(1)}(x + ds) \\ &\to_{A\to\infty} 0 \end{aligned} \tag{22}$$

uniformly for $x \geq 0$, $\theta \in [a, b] - [-\gamma, \gamma]$. Combining (18) and (20) together with

(21) and (22) proves that the convergence of the P_θ-distribution of $S_\tau^\theta - A$ to H_θ is uniform in $\theta \in [a, b] - [-\gamma, \gamma]$.

Now suppose $\theta \in [-\gamma, \gamma] - \{0\}$. By Theorem 3 of Lorden (1970), for any $A \geq 0$,

$$E_\theta\big(S_\tau^\theta - A\big)^3 \leq \tfrac{5}{4}E\big\{[\theta X_1 - \psi(\theta)]^+\big\}^4/I(\theta) \leq 2\theta^4 E_\theta X_1^4/I(\theta) \to_{\theta \to 0} 0.$$

Hence, $S_\tau^\theta - A \to_{\theta\to 0} 0$ in P_θ-probability, uniformly in $A \geq 0$, and thus also $H_\theta \to_{\theta \to 0} \delta_{\{0\}}$. Therefore, if $\eta > 0$, there exists $\gamma > 0$ such that the Lévy distance between H_θ and the P_θ-distribution of $S_\tau^\theta - A$ is bounded by η, uniformly for $A \geq 0$, $\theta \in [-\gamma, \gamma] - \{0\}$. By the above, there exists A_η such that if $A \geq A_\eta$, the Lévy distance between H_θ and the P_θ-distribution of $S_\tau^\theta - A$ is bounded by η uniformly for $\theta \in [a, b] - \{0\}$. Since η is arbitrary, the proof of Lemma 10 is complete. □

LEMMA 11.

$$\sup_{F \in S[a,b]} \sup_{\theta \in [a,b]-\{0\}} P_\theta\{T(B,F) \leq (1-\rho)l_\theta\} \to_{B\to\infty} 0.$$

PROOF. The proof is analogous to the proof of Lemma 7, replacing m_θ by $(1-\rho)l_\theta$ and y by $\rho[I(\theta)\log B]^{1/2}/[2|\theta|(\psi''(\theta))^{1/2}]$. The second term on the right side of (11) is less than $\exp\{-\frac{1}{2}\rho\log B\}$. There exists a constant $c_1 > 0$ such that the first term on the right side of (11) is less than $P_\theta\{\overline{X}_{(1-\rho)l_\theta} - \psi'(\theta) > c_1|\theta|\}$. By virtue of (6), there exists a constant $c_2 > 0$ such that this probability is uniformly less than

$$\exp\{-(1-\rho)l_\theta c_2\theta^2\} = \exp\{-c_2(1-\rho)[\theta^2/I(\theta)]\log B\} \to_{B\to\infty} 0$$

uniformly for $\theta \in [a, b] - \{0\}$. □

LEMMA 12. *Let*

$$n^* = \left(\frac{1}{|\theta|}\right)^{2/(1-3\beta)} \vee (7l_\theta).$$

Then

$$\sup_{F\in S[a,b]} \sup_{\theta \in \Delta_B - R_B} P_\theta\{T(B,F) > n^*\} \to_{B\to\infty} 0.$$

PROOF. Let $\omega_n = \overline{X}_n - \psi'(\theta)$. For large enough B, for $\theta \in \Delta_B - R_B$

$$\begin{aligned} L(n,F) &\geq \int_{\theta-(1/2)\varepsilon_\theta^*}^{\theta+(1/2)\varepsilon_\theta^*} \exp\left\{y\sum_{i=1}^n X_i - n\psi(y)\right\} dF(y) \\ &= e^{nI(\theta)} \int_{\theta-(1/2)\varepsilon_\theta^*}^{\theta+(1/2)\varepsilon_\theta^*} \exp\big(n[(y-\theta)\psi'(\theta) \\ &\qquad\qquad - [\psi(y) - \psi(\theta)] + y\omega_n]\big)\, dF(y) \\ &\geq e^{nI(\theta)} \exp\left(-\frac{\psi''(\theta)}{4}n(\varepsilon_\theta^*)^2\right) e^{-2|\theta||\omega_n|n}\,\delta\varepsilon_\theta^* \\ &\geq e^{n(I(\theta)-\theta^2/4)} e^{-2|\theta||\omega_n|n}\,\delta\varepsilon_\theta^*. \end{aligned}$$

Since $|\theta| \geq (n^*)^{-(1-3\beta)/2}$, it follows that $n^*\theta^2 \geq (n^*)^{3\beta}$; and, since also $l_\theta \leq n^*/7$, it follows that $\varepsilon_\theta^* \geq (n^*)^{\beta-1/2}$. Therefore, for large enough B (for $\theta \in \Delta_B - R_B$)

$$L(n^*, F) \geq \delta \exp\left(n^*\left[\tfrac{1}{5}\theta^2 - 2|\theta|\,|\omega_{n^*}| - \left[\tfrac{1}{2} - \beta\right](\log n^*)/n^*\right]\right)$$
$$\geq \delta \exp\left(\theta^2 n^*\left[\tfrac{1}{5} - 2|\omega_{n^*}|/|\theta| - \left[\tfrac{1}{2} - \beta\right](\log n^*)/(n^*)^{3\beta}\right]\right).$$

Following the notation and proof of Lemma 2, if $|\overline{X}_{n^*} - \psi'(\theta)| < c(n^*)^{\beta-1/2}$, then

$$2|\omega_{n^*}|/|\theta| \leq 2c(n^*)^{\beta-(1/2)+(1-3\beta)/2} = 2c(n^*)^{-\beta/2}$$

so that (for large enough B)

$$L(n^*, F) \geq \exp\{\theta^2 n^*/6\} \geq B.$$

Lemma 12 now follows from the fact that by (7)

$$\sup_{\theta\in\Delta_B - R_B} P_\theta\left\{|\overline{X}_{n^*} - \psi'(\theta)| > c(n^*)^{\beta-1/2}\right\} \to_{n^*\to\infty} 0. \square$$

LEMMA 13. *Let n^* be as in Lemma 12. Denote $\theta^\# = |\theta| + [(1-\rho)l_\theta]^{2\beta-1/2}$. Let $x > 0$. Then*

$$\sup_{F\in S[a,b]}\ \sup_{\theta\in\Delta_B - R_B} P_\theta\left\{\max_{i=1,\dots,n^*} \theta^\#|X_i| > x\right\} \to_{B\to\infty} 0.$$

PROOF. Let $(2\eta) > 0$ be in the interior of Ω. For all $\theta \in \Delta_B - R_B$, when B is large enough

$$\begin{aligned} P_\theta\left\{\max_{i=1,\dots,n^*} \theta^\# X_i > x\right\} &= 1 - \left[1 - P_\theta\{\theta^\# X_1 > x\}\right]^{n^*} \\ &= 1 - \left[1 - P_\theta\{\exp\{\eta X_1\} > \exp\{\eta x/\theta^\#\}\}\right]^{n^*} \\ &\leq 1 - \left[1 - E_\theta \exp\{\eta X_1\}/\exp\{\eta x/\theta^\#\}\right]^{n^*} \\ &\leq 1 - \left[1 - \exp\{\psi(2\eta)\}/\exp\{\eta x/\theta^\#\}\right]^{n^*} \\ &\to_{B\to\infty} 0. \end{aligned}$$

A similar analysis for $P_\theta\{\min_{i=1,\dots,n^*}\theta^\# X_i < -x\}$ completes the proof of Lemma 13. $\square$

PROOF OF THEOREM 2. By virtue of (1) and the definition of T

$$\begin{aligned} &\int_{\Delta_B} E_\theta \exp\{-[\log L(T,F) - \log B]\}\, dF(\theta) \leq BP_0\{T < \infty\} \\ (23)\qquad &\leq \int_{\Delta_B} E_\theta \exp\{-[\log L(T,F) - \log B]\}\, dF(\theta) + F\{\Delta_B^{\text{complement}}\}. \end{aligned}$$

Lemmas 5, 6, 7, and 10 ensure that the considerations of the proof of Theorem 1 of Lai and Siegmund (1977) carry through uniformly for $\theta \in R_B$, $F \in S[a, b]$, so that

$$\sup_{F \in S[a,b]} \left| \int_{R_B} E_\theta \exp\{-[\log L(T,F) - \log B]\}\, dF(\theta) \right. \tag{24}$$
$$\left. - \int_{R_B} \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) \right| \to_{B \to \infty} 0.$$

Let $\gamma > 0$. By (24) there exists B_1 such that if $B > B_1$

$$\sup_{F \in S[a,b]} \left| \int_{R_B} E_\theta \exp\{-[\log L(T,F) - \log B]\}\, dF(\theta) \right. \tag{25}$$
$$\left. - \int_{R_B} \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) \right| < \gamma/4.$$

CASE I. If $F\{\Delta_B - R_B\} \le \gamma/8$, then it clearly follows from (25) that if $B > B_1$

$$\left| \int_{\Delta_B} E_\theta \exp\{-[\log L(T,F) - \log B]\}\, dF(\theta) \right. \tag{26}$$
$$\left. - \int_{\Delta_B} \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) \right| < \gamma/2.$$

CASE II. If $F(\Delta_B - R_B\} > \gamma/8$, let $\theta^\#$ be as in Lemma 13. By virtue of Lemma 11 and Lemma 2, if $\eta > 0$, then

$$\sup_{F \in S[a,b]} \sup_{\theta \in \Delta_B - R_B} P_\theta \left\{ \left| \log L(T,F) - \log \int_{\theta - (\theta^\# - |\theta|)}^{\theta + (\theta^\# - |\theta|)} L(T,y)\, dF(y) \right| > \eta \right\}$$
$$\to_{B \to \infty} 0.$$

Clearly

$$\int_{\theta - (\theta^\# - |\theta|)}^{\theta + (\theta^\# - |\theta|)} L(T,y)\, dF(y) \le \exp\{\theta^\# |X_T|\} B.$$

Lemmas 12 and 13 therefore imply that $\log L(T,F) - \log B \to_{B \to \infty} 0$ in P_θ-probability uniformly for $\theta \in \Delta_B - R_B$. Since $H_\theta \to_{\theta \to 0} \delta_{\{0\}}$ (see the proof of Lemma 10), it follows that there exists B_2 such that if $B > B_2$, then

$$\left| \int_{\Delta_B - R_B} E_\theta \exp\{-[\log L(T,F) - \log B]\}\, dF(\theta) \right.$$
$$\left. - \int_{\Delta_B - R_B} \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) \right| < \gamma/4,$$

so that by (25) if $B > \max(B_1, B_2)$, then

$$\left| \int_{\Delta_B} E_\theta \exp\{-[\log L(T, F) - \log B]\}\, dF(\theta) - \int_{\Delta_B} \int_0^\infty \exp\{-x\}\, dH_\theta\, dF(\theta) \right| < \gamma/2. \tag{27}$$

Clearly

$$\left| \int_{\Delta_B} \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) - \int_a^b \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) \right| \le F\{\Delta_B^{\text{complement}}\}. \tag{28}$$

By virtue of Lemma 1, δ can be chosen so that

$$F\{\Delta_B^{\text{complement}}\} < \gamma/4. \tag{29}$$

Now (23), (25), (26)/(27), (28), and (29) imply that there exists B_3 such that if $B > B_3$ then

$$\sup_{F \in S[a,b]} \left| BP_0\{T < \infty\} - \int_a^b \int_0^\infty \exp\{-x\}\, dH_\theta(x)\, dF(\theta) \right| < \gamma.$$

Since γ is arbitrary, this completes the proof of Theorem 2. □

4. Remarks. The set $\{\theta | \theta X_1 - \psi(\theta)$ has a lattice P_θ-distribution$\}$ is at most countable [Woodroofe (1982), Section 6.2], so that the restriction in Theorem 1 is not prohibitive. If F gives positive probability to a value θ for which $\theta X_1 - \psi(\theta)$ has a lattice P_θ-distribution, then it can be shown that

$$\log L(n, F) = \theta \sum_{i=1}^n X_i - n\psi(\theta) + \log F\{\theta\} + o_p(1),$$

where $o_p(1) \to_{n \to \infty} 0$ in probability. Therefore, Theorem 1 will not hold as is; the lattice property is not asymptotically negligible.

The strongly nonlattice property is utilized to attain uniformity in the convergence of the distribution of $S_\tau^\theta - A$ (Lemma 10). If other means can be found to yield uniformity—such as regarding probability measures F whose support is a finite set of points—then the requirement of the strongly nonlattice property may be relaxed.

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DEPARTMENT OF STATISTICS
HEBREW UNIVERSITY
91905 JERUSALEM
ISRAEL