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RECTANGULAR LATTICE DESIGNS: EFFICIENCY FACTORS AND ANALYSIS

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Rectangular lattice designs are shown to be generally balanced with respect to a particular decomposition of the treatment space. Efficiency factors are calculated, and the analysis, including recovery of interblock information, is outlined. The ideas are extended to rectangular lattice designs with an extra blocking factor.

1. Introduction. The class of incomplete block designs known as *rectangular lattice* designs was introduced by Harshbarger (1946), with further details and extensions being given in a subsequent series of papers by Harshbarger (1947, 1949, 1951) and Harshbarger and Davis (1952). Apart from a contribution by Grundy (1950) concerning the efficient estimation of the stratum variances and the papers by Nair (1951, 1952, 1953) relating rectangular lattice designs to partially balanced designs, little further theoretical discussion of this class of designs seems to have occurred. Expositions of the basic results about rectangular lattice designs in two and three replicates, as well as tables of designs, can be found in Robinson and Watson (1949) and Cochran and Cox (1957). Discussions exist in other standard texts on the design and analysis of experiments, for example Kempthorne (1952), but, apart from recent contributions by Williams (1977) and Williams and Ratcliff (1980), the literature seems to end in the early 1950's. [In his recent note, Thompson (1983) uses the results in the present paper, as he acknowledges.] A possible explanation of this fact may be the observations of Nair (1951, 1953) that every 2-replicate rectangular lattice design is a partially balanced incomplete block design with four associate classes, whilst the obvious extension of the argument to r-replicate rectangular lattice designs for $r \geq 3$ fails in general, although the classes of rectangular lattice designs for $n(n-1)$ treatments in $n-1$ or n replicates again turn out to be partially balanced. Perhaps it was felt that, in not being partially balanced, rectangular lattice designs were rather too complicated.

In his fundamental papers on designed experiments with *simple orthogonal block structure* Nelder (1965a, b) introduced the notion of *general balance*, this being a relationship between the treatment structure and the block structure of the design. It is immediate from his definition that all block experiments (in the usual sense of the term) are generally balanced for some treatment structure [see Houtman and Speed (1983)], although here we might more properly use the term *treatment pseudo-structure*, and when this structure is elucidated for a given class of designs they can be regarded as understood and readily analysed. In a

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later paper, Nelder (1968) showed the importance of general balance in permitting the straightforward estimation of stratum variances, introducing a method equivalent to that which has come to be known as *restricted maximum likelihood estimation* of variances [see Patterson and Thompson (1971) and Harville (1977)]. The definition of general balance in block designs is intimately connected with the eigenspaces of a certain linear transformation, denoted by $\mathbf{L}_B$ in this paper, and in this form a number of other authors have recently emphasised the same concept [see, for example, Pearce, Caliński, and Marshall (1974), who called the eigenvectors of $\mathbf{L}_B$ *basic contrasts*, and Corsten (1976)].

In Sections 3 and 4 of this paper we obtain an orthogonal decomposition of the space of all treatment contrasts associated with a general r-replicate rectangular lattice design. In Section 5 we use this decomposition to identify all the eigenspaces of the linear transformation $\mathbf{L}_B$. An equivalent description of our results is that we determine the treatment pseudo-structure relative to which the designs are generally balanced; equivalently again, we describe the basic contrasts of the design. Using these results, a full analysis, modelled on Nelder's (1965b, 1968) general approach, of rectangular lattice designs is given in Section 6, involving the derivation of a fully orthogonal analysis of variance and estimates of the stratum variances, and the calculations of estimates of treatment contrasts, together with their standard errors. A recursive analysis along the lines of Wilkinson (1970) is most satisfactory, as the eigenspaces are orthogonal complements of subspaces each of which has a simple formula for its orthogonal projection in terms of averaging operators, and so these subspaces can be swept out successively in a quite straightforward manner. Our general approach to the analysis of designed experiments is framed in vector space terms, similar to that used by James and Wilkinson (1971) and Bailey (1981), but in the multistratum framework of Nelder's papers.

Finally, we use the foregoing ideas to sketch the design and analysis of an experiment in which an extra blocking factor was imposed on a rectangular lattice design. Two examples are used throughout the paper to illustrate the theory.

EXAMPLE 1. This is a rectangular lattice for 20 treatments in three replicates of five blocks of four plots. Although this is an entirely abstract example, there being no associated experiment, it illustrates the general theory well because it has no special features: the design is *not* partially balanced, and its construction does *not* use a complete set of mutually orthogonal Latin squares. Tables 1, 3–5, 7, and 12–15 refer to Example 1.

EXAMPLE 2. In an experiment into the digestibility of stubble, 12 feed treatments were applied to sheep. There were 12 sheep, in three rooms of four animals each. There were three test periods of four weeks each, separated by two-week recovery periods. Each sheep was fed three treatments, one in each test period. During the recovery periods all animals received their usual feed, so that they would return to normal conditions before being subjected to a new treatment.

TABLE 1
Transversal of a 5 × 5 *Latin square*

①	2	3	4	5
2	1	④	5	3
3	⑤	1	2	4
4	3	5	1	②
5	4	2	③	1

It was desired that each treatment should be fed once in each room and once in each period. If periods are ignored, a suitable design is a rectangular lattice design in which sheep are blocks and rooms are replicates. We shall ignore the periods until Section 7, where we show how to deal with this extra blocking factor. Tables 9–11 and 18–19 refer to Example 2.

2. Construction. In this section we review the construction of rectangular lattice designs, partly in order to establish our terminology and notation.

A rectangular lattice design is a resolvable incomplete block design for t treatments in r replicates of n blocks of size $n - 1$, where $t = n(n - 1)$ and $2 \leq r \leq n$, for some integer n. We write b for rn, the total number of blocks, and N for $b(n - 1)$, the total number of plots. The design has the property that any pair of treatments occur together in at most one block. The design is constructed from a set of $r - 2$ mutually orthogonal $n \times n$ Latin squares $\Lambda_1, \dots, \Lambda_{r-2}$.

A *transversal* of such a set of Latin squares is defined [see Dénes and Keedwell (1974), pages 28 and 331] to be a set of n cells with one cell in each row and one in each column, which between them have all the letters of all the squares $\Lambda_1, \dots, \Lambda_{r-2}$. In Table 1 a transversal of a single 5 × 5 Latin square is indicated with circles. Transversals do not always exist: Table 2 shows a 4 × 4 Latin square with no transversal. A sufficient condition for the existence of a transversal is the existence of a Latin square Λ_{r-1} orthogonal to each of $\Lambda_1, \dots, \Lambda_{r-2}$, for then each letter of Λ_{r-1} corresponds to a transversal. Such a set of mutually orthogonal $n \times n$ Latin squares $\Lambda_1, \dots, \Lambda_{r-1}$ exists whenever n is a prime or prime power and r is less than or equal to n [see Dénes and Keedwell (1974), page 165]. However, this condition is not necessary, because the square in Table 1 has no orthogonal mate.

It is convenient (although not essential) to permute the rows and columns of $\Lambda_1, \dots, \Lambda_{r-2}$ simultaneously so that the transversal lies down the main diagonal.

TABLE 2
A 4 × 4 *Latin square with no transversal*

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

TABLE 3a
Table 1 with rows permuted

1	2	3	4	5
3	5	1	2	4
2	1	4	5	3
5	4	2	3	1
4	3	5	1	2

TABLE 3b
Table 3a with letters permuted

1	5	4	3	2
4	2	1	5	3
5	1	3	2	4
2	3	5	4	1
3	4	2	1	5

This is achieved by moving the ith row to the jth row if the unique transversal cell in row i is in column j. It is also convenient to rename the "letters" of each square independently so that the letters on the main diagonal are in natural order. Tables 3a and 3b show the results of applying these processes to the square in Table 1.

An $n \times n$ square array is drawn. The diagonal cells are left blank, and the t treatments are allocated to the remaining cells, as in Table 4. In this example we have labelled the treatments $A, B, \ldots, T$, but we shall usually use ω to denote a general treatment, to avoid confusion with other symbols. We denote the n diagonal cells by $i, j, \ldots$ and the r *classifications* (that is, rows, columns, letters of $\Lambda_1, \ldots$, letters of Λ_{r-2}) by $a, b, \ldots$.

We define subsets of the treatments called *spokes* and *fans*. A *1-spoke* is the set of $n-1$ treatments in any row; a *2-spoke* is the set of $n-1$ treatments in any column. For $a = 3, \ldots, r$, an *a-spoke* is the set of $n-1$ treatments in the positions of any one letter of square Λ_{a-2}. For $a = 1, \ldots, r$ and $i = 1, \ldots, n$ we denote by $\mathscr{S}_{ai}$ the unique a-spoke which would naturally go through the ith diagonal cell if the diagonal cells were not excluded. For each fixed i, the *fan* $\mathscr{F}_i$ through the ith diagonal cell is defined to be the union of all spokes through that

TABLE 4
Treatment array for Example 1

*	*A*	*B*	*C*	*D*
E	*	*F*	*G*	*H*
I	*J*	*	*K*	*L*
M	*N*	*O*	*	*P*
Q	*R*	*S*	*T*	*

TABLE 5
Rectangular lattice block design (Example 1)
(blocks are columns)

replicate 1					replicate 2					replicate 3				
A	E	I	M	Q	E	A	B	C	D	F	D	C	B	A
B	F	J	N	R	I	J	F	G	H	J	K	H	E	G
C	G	K	O	S	M	N	O	K	L	P	M	N	L	I
D	H	L	P	T	Q	R	S	T	P	T	S	Q	R	O

diagonal cell; that is,

$$\mathscr{F}_i = \mathscr{S}_{1i} \cup \mathscr{S}_{2i} \cup \cdots \cup \mathscr{S}_{ri}.$$

The terminology is suggested by the fact that all spokes in a fan have the corresponding diagonal cell in common, while no two spokes in the same fan have any further cells in common. In the example given by Tables 3b and 4, we have

$$\begin{aligned}\mathscr{S}_{11} &= \{A, B, C, D\},\\ \mathscr{S}_{24} &= \{C, G, K, T\},\\ \mathscr{S}_{32} &= \{D, K, M, S\},\\ \mathscr{F}_5 &= \{Q, R, S, T, D, H, L, P, A, G, I, O\}.\end{aligned}$$

The design is now constructed very easily. For $a = 1, \ldots, r$, the blocks of the ath replicate are just the a-spokes. Table 5 shows the (unrandomized) design which emerges in this way from Tables 3b and 4. Thus spokes have a genuine statistical meaning, as each spoke gives a block of the design. Fans have no direct statistical meaning, but they are a combinatorial consequence of the spokes which prove useful for the analysis of the design.

Orthogonal cyclic Latin squares may be constructed by the automorphism method of Mann (1942), which is described in Section 7.2 of Dénes and Keedwell (1974). If p is the smallest prime divisor of n then $p - 1$ orthogonal squares are obtained, and hence rectangular lattice designs may be constructed for $r \le p$ (reserving one of the squares for the transversal). The same designs may also be constructed as α-designs [Patterson and Williams (1976)]. Let $q_1, q_2, \ldots, q_{r-1}$ be any integers such that no two are congruent modulo p and none is divisible by p. Without loss of generality we may take $q_1 = 1$. The generating α-array is in Table 6, in the format used by Patterson and Williams (1976), whose series I, II, and IV are all examples of the array shown here.

3. Decomposition of the treatment space. Let $\mathbb{R}^t$ be the real vector space of vectors indexed by the t treatments. We need to find an orthogonal decomposition of $\mathbb{R}^t$ that will enable us to analyse data from experiments with the rectangular lattice design. To this end, we define certain special vectors in and subspaces of $\mathbb{R}^t$.

Let $\mathbf{u}$ be the vector $(1, 1, \ldots, 1)$. For $a = 1, \ldots, r$ and $i = 1, \ldots, n$ let $\mathbf{v}_{ai}$ be the characteristic vector of the spoke $\mathscr{S}_{ai}$; that is, the ω-entry $(\mathbf{v}_{ai})_\omega$ of $\mathbf{v}_{ai}$ is 1 if

TABLE 6
Generators for α-designs which are also rectangular lattice designs (entries in the array should be reduced modulo n)

0	0	0	$\cdots$	0
0	1	q_2	$\cdots$	q_{r-1}
0	2	$2q_2$	$\cdots$	$2q_{r-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
0	$n-2$	$(n-2)q_2$	$\cdots$	$(n-2)q_{r-1}$
0	$n-1$	$(n-1)q_2$	$\cdots$	$(n-1)q_{r-1}$

$\omega \in \mathscr{S}_{ai}$ and 0 otherwise. Similarly, for $i = 1, \dots, n$, let $\mathbf{w}_i$ be the characteristic vector of the fan $\mathscr{F}_i$, so that

$$\mathbf{w}_i = \mathbf{v}_{1i} + \mathbf{v}_{2i} + \cdots + \mathbf{v}_{ri}.$$

Let U_μ be the subspace spanned by $\mathbf{u}$; let U_f be the subspace spanned by the fan vectors $\mathbf{w}_i$; let U_s be the subspace spanned by the spoke vectors $\mathbf{v}_{ai}$; and let U_ε be the whole space $\mathbb{R}^t$. [Our conventions for labelling the first and last of these spaces agree with those used by Throckmorton (1961) and Kempthorne (1982).] Then

$$U_\mu \subseteq U_f \subseteq U_s \subseteq U_\varepsilon.$$

For Example 1 we display each vector in $\mathbb{R}^{20}$ in a two-dimensional array corresponding to Table 4. Tables 7a and 7b give examples of vectors in $U_s \setminus U_f$ and in U_f respectively.

The dimension of U_μ is 1. The space $\mathbb{R}^t$ has an inner product $\langle\, ,\rangle$ on it defined by

$$\langle \mathbf{z}, \mathbf{z}' \rangle = \sum_{\omega=1}^{t} z_\omega z'_\omega.$$

TABLE 7a
The vector $\mathbf{v}_{11} - 2\mathbf{v}_{24} + 5\mathbf{v}_{32}$

*	1	1	−1	6
0	*	0	−2	0
0	0	*	3	0
5	0	0	*	0
0	0	5	−2	*

TABLE 7b
The vector $\mathbf{w}_1 + 3\mathbf{w}_5$

*	4	1	1	4
1	*	1	3	3
4	1	*	0	3
1	0	3	*	4
4	3	3	4	*

We use this to find the dimensions of the spaces U_f and U_s. Note that

$$(3.1)\qquad \langle \mathbf{v}_{a\iota}, \mathbf{v}_{bj} \rangle = |\mathscr{S}_{ai} \cap \mathscr{S}_{bj}| = \begin{cases} n-1 & \text{if } a=b \text{ and } i=j, \\ 0 & \text{if } a=b \text{ and } i\neq j, \\ 0 & \text{if } a\neq b \text{ and } i=j, \\ 1 & \text{if } a\neq b \text{ and } i\neq j, \end{cases}$$

so that

$$(3.2)\qquad \langle \mathbf{w}_i, \mathbf{w}_j \rangle = |\mathscr{F}_\iota \cap \mathscr{F}_j| = \begin{cases} r(n-1) & \text{if } i=j, \\ r(r-1) & \text{if } i\neq j. \end{cases}$$

Moreover, $\Sigma_\iota \mathbf{w}_i = r\mathbf{u}$. Suppose that $\Sigma_i \lambda_i \mathbf{w}_i = \mathbf{0}$ for some real numbers λ_ι. If $r \neq n$, taking inner products with individual $\mathbf{w}_i$ shows that $\lambda_1 = \cdots = \lambda_n$, and hence that $\lambda_1 = \cdots = \lambda_n = 0$: thus the fan vectors are linearly independent and so U_f has dimension n. On the other hand, if $r = n$ then $\mathbf{w}_i = \mathbf{u}$ for $i, \ldots, n$: thus $U_f = U_\mu$. Now suppose that $\Sigma_a \Sigma_i \lambda_{ai} \mathbf{v}_{a\iota} = \mathbf{0}$ for some real numbers λ_{ai}. Taking inner products with individual $\mathbf{v}_{a\iota}$ shows that there are real numbers θ_a and ϕ_i such that $\lambda_{ai} = \theta_a + \phi_i$ for all a and i. Since

$$\mathbf{v}_{a1} + \mathbf{v}_{a2} + \cdots + \mathbf{v}_{an} = \mathbf{u}$$

for $a = 1, \ldots, r$, this implies that $(\Sigma_a \theta_a)\mathbf{u} + \Sigma_i \phi_i \mathbf{w}_i = \mathbf{0}$. Hence U_s has dimension $nr - (r-1)$ if $r \neq n$, and $nr - (r-1) - (n-1)$ if $r = n$.

For Example 1, Equations (3.1) and (3.2) are demonstrated in Tables 7a and 7b, respectively. For example, the six entries equal to 4 in Table 7b correspond to the elements of $\mathscr{F}_1 \cap \mathscr{F}_5$. In this case the five fan vectors form a basis for U_f; while a basis of U_s consists of $\mathbf{u}$ and all but three spoke vectors, one being omitted for each classification.

We can form the orthogonal complements of the U-subspaces, and thus obtain the subspaces that really interest us. Specifically, we put

$$V_\mu = U_\mu,$$

$$V_f = \text{the orthogonal complement of } U_\mu \text{ in } U_f,$$

$$V_s = \text{the orthogonal complement of } U_f \text{ in } U_s,$$

$$V_\varepsilon = \text{the orthogonal complement of } U_s \text{ in } U_\varepsilon.$$

Then V_f is spanned by vectors of the form $\mathbf{w}_i - \mathbf{w}_j$; while V_s is spanned by vectors of the form $\mathbf{v}_{ai} - \mathbf{v}_{b\iota}$. Now $\mathbb{R}^t$ is the orthogonal direct sum

$$\mathbb{R}^t = V_\mu \oplus V_f \oplus V_s \oplus V_\varepsilon.$$

We record the important facts about this decomposition in Table 8.

In two special cases this decomposition can be described in simpler terms. If $r = n$ then the set $\{\Lambda_1, \ldots, \Lambda_{r-2}\}$ is only one square short of a complete set of mutually orthogonal Latin squares. Thus there exists a (unique) Latin square

TABLE 8
Decomposition of the treatment subspace

subspace	V_μ	V_f	V_s	V_ε
description	mean	contrasts between fans	contrasts between spokes within fans	orthogonal to spokes
dimension ($r < n$)	1	$n - 1$	$(n - 1)(r - 1)$	$(n - r)(n - 1) - 1$
dimension ($r = n$)	1	0	$(n - 1)^2$	$n - 2$

Λ_{n-1} orthogonal to all the others, by Theorem 1.6.1 of Rhagavarao (1971). One letter of Λ_{n-1} must correspond to the transversal. Each other letter of Λ_{n-1} occurs just once in each a-spoke, for each classification a. Hence the contrasts between these $n - 1$ other letters are orthogonal to spokes, and so they form the whole space V_ε. Since V_f is null in this case, V_s must consist of all treatment contrasts which are orthogonal to the letters of Λ_{n-1}. Thus the treatments have the simple nested structure $(n - 1) \to n$ [in the notation of Nelder (1965a)], and the treatment space decomposition is the familiar one into mean, between letters of Λ_{n-1} and within letters.

If $r = n - 1$ and $n \neq 4$, the results of Shrikhande (1961) and Bruck (1963) show that there is a unique complete orthogonal set $\{\Lambda_1, \dots, \Lambda_{n-1}\}$ containing the original set $\{\Lambda_1, \dots, \Lambda_{n-3}\}$ and that the original transversal corresponds to a letter of one of the two extra squares, say Λ_{n-2}. The same result is true even when $n = 4$, because the existence of the original transversal prevents Λ_1 from being isotopic to the square in Table 2, which is the only 4×4 Latin square (up to isotopy) which is not uniquely embeddable in a complete set of mutually orthogonal Latin squares [isotopy classes are also called transformation sets (see Fisher and Yates (1934))]. The treatments now have the simple crossed factorial structure $Q_1 \times Q_2$, where the levels of Q_1 are the $n - 1$ other letters of Λ_{n-2} and the levels of Q_2 are the n letters of Λ_{n-1}. Now V_ε is the main effect of Q_1; while V_f is the main effect of Q_2 and V_s is the Q_1Q_2 interaction.

Example 2 has $r = n - 1 = 3$. The rectangular lattice design is constructed from the set of mutually orthogonal 4×4 Latin squares in Table 9 : the rows, columns, and letters of Λ_1 are the three classifications; letter 1 of Λ_2 gives the transversal; the remaining letters of Λ_2 and Λ_3 give the 3×4 factorial treatment structure described above and shown in Table 10. The design is that shown in Table 11, ignoring periods.

In both these special cases the factorial treatment decomposition has no direct statistical meaning, but is merely an aid to the analysis. The factors Q_1 and Q_2 are entirely analogous to the pseudo-factors used in the construction and analysis of square lattice designs [Yates (1936)].

4. Treatment projection. Let $\mathbf{z}$ be a vector in $\mathbb{R}^t$. In order to use the spaces V_μ, V_f, V_s, and V_ε in the analysis of an experiment we need to know how to calculate the projections of $\mathbf{z}$ onto these spaces. This is done in terms of the

TABLE 9a
Three mutually orthogonal 4 × 4 *Latin squares*

Λ_1 (gives 3rd replicate)				**Λ_2 ("1" gives transversal; other letters are levels of Q_1)**				**Λ_3 (letters are levels of Q_2)**			
1	4	2	3	1	2	3	4	1	2	3	4
3	2	4	1	2	1	4	3	3	4	1	2
4	1	3	2	3	4	1	2	4	3	2	1
2	3	1	4	4	3	2	1	2	1	4	3

TABLE 9b
Array of twelve treatments for Example 2

*	*A*	*B*	*C*
D	*	*E*	*F*
G	*H*	*	*I*
J	*K*	*L*	*

TABLE 10
3 × 4 *factorial structure for Example* 2

treatment	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
level of Q_1	2	3	4	2	4	3	3	4	2	4	3	2
level of Q_2	2	3	4	3	1	2	4	3	1	2	1	4

TABLE 11
Design which is not generally balanced

room		1				2				3			
sheep		1	2	3	4	5	6	7	8	9	10	11	12
time period	**1**	*B*	*D*	*I*	*L*	*K*	*E*	*F*	*G*	*A*	*J*	*C*	*H*
	2	*C*	*E*	*H*	*K*	*A*	*L*	*I*	*J*	*G*	*B*	*D*	*F*
	3	*A*	*F*	*G*	*J*	*H*	*B*	*C*	*D*	*E*	*I*	*K*	*L*

following totals:

$$\text{grand total } G(\mathbf{z}) = \sum_{\omega} \mathbf{z}_{\omega},$$

$$\text{spoke total } S_{ai}(\mathbf{z}) = \sum \{\mathbf{z}_{\omega}\colon \omega \in \mathscr{S}_{ai}\} = \langle \mathbf{z}, \mathbf{v}_{ai} \rangle,$$

$$\text{fan total } F_i(\mathbf{z}) = \sum \{\mathbf{z}_{\omega}\colon \omega \in \mathscr{F}_i\} = \langle \mathbf{z}, \mathbf{w}_i \rangle.$$

TABLE 12
A particular vector $\mathbf{z}$ *in* $\mathbb{R}^{20}$

*	7	3	2	3
6	*	5	9	4
5	2	*	6	7
4	5	8	*	1
2	4	2	5	*

It is immediate that

$$\sum_i S_{ai}(\mathbf{z}) = G(\mathbf{z}), \tag{4.1}$$

$$\sum_a S_{ai}(\mathbf{z}) = F_i(\mathbf{z}), \tag{4.2}$$

$$\sum_i F_i(\mathbf{z}) = rG(\mathbf{z}). \tag{4.3}$$

Define the *fan totals vector* $\mathbf{f}(\mathbf{z})$ and the *spoke totals vector* $\mathbf{s}(\mathbf{z})$ by

$$\mathbf{f}(\mathbf{z}) = \sum_i F_i(\mathbf{z})\mathbf{w}_i,$$

$$\mathbf{s}(\mathbf{z}) = \sum_a \sum_i S_{ai}(\mathbf{z})\mathbf{v}_{ai}.$$

We also need the *grand totals vector* $\mathbf{g}(\mathbf{z})$, all of whose entries are equal to $G(\mathbf{z})$.

Continuing our Example 1, a vector $\mathbf{z}$ is shown in Table 12. Its spoke totals are in Table 13: the column margins are the fan totals, and the row totals are all the grand total. The vectors $\mathbf{f}(\mathbf{z})$ and $\mathbf{s}(\mathbf{z})$ are shown in Table 14.

We aim to give the projections of $\mathbf{z}$ onto the spaces V_μ, V_f, V_s, and V_ε in terms of $\mathbf{f}(\mathbf{z})$, $\mathbf{s}(\mathbf{z})$, and $\mathbf{g}(\mathbf{z})$. The necessary calculations are contained in the following two lemmas.

LEMMA 1.

(i) $\langle \mathbf{s}(\mathbf{z}), \mathbf{v}_{ai}\rangle = nS_{ai}(\mathbf{z}) + (r-1)G(\mathbf{z}) - F_i(\mathbf{z}),$

(ii) $\langle \mathbf{f}(\mathbf{z}), \mathbf{v}_{ai}\rangle = (n-r)F_i(\mathbf{z}) + r(r-1)G(\mathbf{z}),$

(iii) $\langle \mathbf{f}(\mathbf{z}), \mathbf{w}_i\rangle = r(n-r)F_i(\mathbf{z}) + r^2(r-1)G(\mathbf{z}).$

TABLE 13
Spoke totals of $\mathbf{z}$

i	1	2	3	4	5	total
row ($a = 1$)	15	24	20	18	13	90
column ($a = 2$)	17	18	18	22	15	90
letter ($a = 3$)	13	15	13	20	29	90
fan totals	45	57	51	60	57	270

TABLE 14

fan totals vector f(z)					spoke totals vector s(z)				
*	159	156	156	159	*	62	53	50	45
162	*	153	174	165	61	*	55	75	52
153	153	*	168	168	66	51	*	57	55
162	168	168	*	162	50	49	65	*	46
153	174	165	162	*	43	51	46	48	*

PROOF. To simplify the expressions, we omit "(z)", the vector **z** being understood.

(i)
$$\begin{aligned}\langle \mathbf{s}, \mathbf{v}_{ai}\rangle &= \sum_b \sum_j S_{bj}\langle \mathbf{v}_{bj}, \mathbf{v}_{ai}\rangle \\ &= (n-1)S_{ai} + \sum_{b\neq a}\sum_{i\neq j} S_{bj} \quad \text{(by (3.1))} \\ &= (n-1)S_{ai} + \sum_{b\neq a}(G - S_{bi}) \quad \text{(by (4.1))} \\ &= nS_{ai} + (r-1)G - \sum_b S_{bi} \\ &= nS_{ai} + (r-1)G - F_i \quad \text{(by (4.2))}.\end{aligned}$$

(ii)
$$\begin{aligned}\langle \mathbf{f}, \mathbf{v}_{ai}\rangle &= \sum_j F_j\langle \mathbf{w}_j, \mathbf{v}_{ai}\rangle \\ &= (n-1)F_i + (r-1)\sum_{j\neq i} F_j \quad \text{(by (3.1))} \\ &= (n-r)F_i + (r-1)\sum_j F_j \\ &= (n-r)F_i + r(r-1)G \quad \text{(by (4.3))}.\end{aligned}$$

(iii) Summing the equation in (ii) over all the spokes in $\mathscr{F}_i$ gives

$$\langle \mathbf{f}, \mathbf{w}_i\rangle = r(n-r)F_i + r^2(r-1)G. \qquad \square$$

LEMMA 2. *The orthogonal projections of* **z** *onto* $U_\mu, U_f, U_s, U_\varepsilon$, *respectively, are*

$$\frac{\mathbf{g}(\mathbf{z})}{n(n-1)}, \qquad \frac{\mathbf{f}(\mathbf{z})}{r(n-r)} - \frac{(r-1)\mathbf{g}(\mathbf{z})}{(n-1)(n-r)},$$

$$\frac{\mathbf{s}(\mathbf{z})}{n} + \frac{\mathbf{f}(\mathbf{z})}{n(n-r)} - \frac{(r-1)\mathbf{g}(\mathbf{z})}{(n-1)(n-r)}, \qquad \mathbf{z}$$

when $r \neq n$. *When* $r = n$ *then* $U_f = U_\mu$ *and the orthogonal projection of* **z** *onto* U_s *is*

$$\frac{\mathbf{s}(\mathbf{z})}{n} - \frac{(n-2)\mathbf{g}(\mathbf{z})}{n(n-1)}.$$

PROOF. Put $\mathbf{x} = [r(n-r)]^{-1}\mathbf{f} - (r-1)[(n-1)(n-r)]^{-1}\mathbf{g}$ when $r \neq n$. Since $\mathbf{f}$ and $\mathbf{g}$ are both sums of fan vectors, $\mathbf{x} \in U_f$. Thus it suffices to show that $\mathbf{z} - \mathbf{x}$ is orthogonal to U_f. This is so if $\langle \mathbf{z} - \mathbf{x}, \mathbf{w}_i \rangle = 0$ for each fan $\mathscr{F}_i$. By Lemma 1(iii) and (3.2),

$$\langle \mathbf{x}, \mathbf{w}_i \rangle = \frac{r(n-r)F_i + r^2(r-1)G}{r(n-r)} - \frac{r(n-1)(r-1)G}{(n-1)(n-r)} = F_i = \langle \mathbf{z}, \mathbf{w}_i \rangle.$$

Similarly, put $\mathbf{y} = n^{-1}\mathbf{s} + [n(n-r)]^{-1}\mathbf{f} - (r-1)[(n-1)(n-r)]^{-1}\mathbf{g}$. Then $\mathbf{y} \in U_s$, because $\mathbf{s}$, $\mathbf{f}$, and $\mathbf{g}$ are all sums of spoke vectors, so it suffices to show that $\langle \mathbf{z} - \mathbf{y}, \mathbf{v}_{ai} \rangle = 0$ for all spokes $\mathscr{S}_{ai}$. Lemmas 1(i) and (ii) show that $\langle \mathbf{y}, \mathbf{v}_{ai} \rangle$ is equal to

$$\frac{nS_{ai} + (r-1)G - F_i}{n} + \frac{(n-r)F_i + r(r-1)G}{n(n-r)} - \frac{(n-1)(r-1)G}{(n-1)(n-r)},$$

which is S_{ai}, which is $\langle \mathbf{z}, \mathbf{v}_{ai} \rangle$.

Now let $r = n$ and put $\mathbf{y} = n^{-1}\mathbf{s} - (n-2)[n(n-1)]^{-1}\mathbf{g}$. Then

$$\langle \mathbf{y}, \mathbf{v}_{ai} \rangle = \frac{nS_{ai} + (n-2)G}{n} - \frac{(n-2)(n-1)G}{n(n-1)} = S_{ai} = \langle \mathbf{z}, \mathbf{v}_{ai} \rangle$$

so that $\mathbf{y} \in U_s$ and $\mathbf{z} - \mathbf{y}$ is orthogonal to U_s. □

Now subtraction gives the orthogonal projection of $\mathbf{z}$ onto $V_\mu, V_f, V_s, V_\varepsilon$.

THEOREM 1. *Let $\mathbf{T}_\mu, \mathbf{T}_f, \mathbf{T}_s, \mathbf{T}_\varepsilon$ be the operators of orthogonal projection from $\mathbb{R}^t$ onto $V_\mu, V_f, V_s, V_\varepsilon$, respectively. Then, for all $\mathbf{z}$ in $\mathbb{R}^t$,*

$$\mathbf{T}_\mu \mathbf{z} = \frac{\mathbf{g}(\mathbf{z})}{n(n-1)},$$

$$\mathbf{T}_f \mathbf{z} = \frac{\mathbf{f}(\mathbf{z})}{r(n-r)} - \frac{r\mathbf{g}(\mathbf{z})}{n(n-r)} \quad \textit{when } r \neq n \textit{ and zero otherwise},$$

$$\mathbf{T}_s \mathbf{z} = \frac{\mathbf{s}(\mathbf{z})}{n} - \frac{\mathbf{f}(\mathbf{z})}{rn},$$

$$\mathbf{T}_\varepsilon \mathbf{z} = \mathbf{z} - (\mathbf{T}_\mu \mathbf{z} + \mathbf{T}_f \mathbf{z} + \mathbf{T}_s \mathbf{z}).$$

In Example 1 we have $n = 5$ and $r = 3$, so $\mathbf{T}_\mu \mathbf{z} = \mathbf{g}(\mathbf{z})/20$; $\mathbf{T}_f(\mathbf{z}) = \mathbf{f}(\mathbf{z})/6 - 3\mathbf{g}(\mathbf{z})/10$; $\mathbf{T}_s \mathbf{z} = \mathbf{s}(\mathbf{z})/5 - \mathbf{f}(\mathbf{z})/15$, and $\mathbf{T}_\varepsilon \mathbf{z}$ is best obtained by subtraction. For the particular vector $\mathbf{z}$ shown in Table 12, these four components of $\mathbf{z}$ are shown in Table 15. The orthogonality of the decomposition may be verified by noting that

$$\|\mathbf{T}_\mu \mathbf{z}\|^2 + \|\mathbf{T}_f \mathbf{z}\|^2 + \|\mathbf{T}_s \mathbf{z}\|^2 + \|\mathbf{T}_\varepsilon \mathbf{z}\|^2$$
$$= 405 + 24 + 47.2 + 21.8 = 498 = \|\mathbf{z}\|^2.$$

TABLE 15

$\mathbf{T}_\mu \mathbf{z}$					$\mathbf{T}_f \mathbf{z}$				
*	4.5	4.5	4.5	4.5	*	−0.5	−1.0	−1.0	−0.5
4.5	*	4.5	4.5	4.5	0.0	*	−1.5	2.0	0.5
4.5	4.5	*	4.5	4.5	−1.5	−1.5	*	1.0	1.0
4.5	4.5	4.5	*	4.5	0.0	1.0	1.0	*	0.0
4.5	4.5	4.5	4.5	*	−1.5	2.0	0.5	0.0	*
$\mathbf{T}_s \mathbf{z}$					$\mathbf{T}_\varepsilon \mathbf{z}$				
*	1.8	0.2	−0.4	−1.6	*	1.2	−0.7	−1.1	0.6
1.4	*	0.8	3.4	−0.6	0.1	*	1.2	−0.9	−0.4
3.0	0.0	*	0.2	−0.2	−1.0	−1.0	*	0.3	1.7
−0.8	−1.4	1.8	*	−1.6	0.3	0.9	0.7	*	−1.9
−1.6	−1.4	−1.8	−1.2	*	0.6	−1.1	−1.2	1.7	*

5. General balance. The block structure of a rectangular lattice design is the double nested classification of plots within blocks within replicates. This is one of the *simple orthogonal block structures* defined by Nelder (1965a). In what follows we retain the notation of Nelder (1965a, b, 1968) and Bailey (1981) as far as possible.

Let $\mathbb{R}^N$ be the real vector space associated with the N plots. Each grouping of the plots according to the block structure defines an averaging operation $\mathbf{P}$ on $\mathbb{R}^N$. In our case there are four averaging operators: the grand mean averaging operator $\mathbf{P}_\mu = \mathbf{J}/N$, where $\mathbf{J}$ is the all-1's matrix; the replicates averaging operator $\mathbf{P}_R$; the blocks averaging operator $\mathbf{P}_B$; and the identity $\mathbf{P}_\varepsilon = \mathbf{I}$. Nelder (1965a) showed that there is an orthogonal direct sum decomposition $\oplus_\alpha W_\alpha$ of $\mathbb{R}^N$ such that each W_α is an eigenspace of every $\mathbf{P}$. Let $\mathbf{C}_\alpha$ be the operator of orthogonal projection from $\mathbb{R}^N$ onto W_α. Nelder (1965a) showed that each $\mathbf{C}_\alpha$ is a linear combination of the $\mathbf{P}$'s with integer coefficients: Speed and Bailey (1982) gave explicit formulae for these coefficients. In our case we have

$$\mathbf{C}_\mu = \mathbf{P}_\mu, \qquad \mathbf{C}_R = \mathbf{P}_R - \mathbf{P}_\mu,$$

$$\mathbf{C}_B = \mathbf{P}_B - \mathbf{P}_R, \qquad \mathbf{C}_\varepsilon = \mathbf{P}_\varepsilon - \mathbf{P}_B.$$

The spaces W_α are called *strata*: they play an important role in analysis of variance [see Nelder (1965b) and Bailey (1981)]. Our covariance model for the data vector $\mathbf{y}$ is

$$\text{(5.1)} \qquad \operatorname{Cov}(\mathbf{y}) = \xi_\mu \mathbf{C}_\mu + \xi_R \mathbf{C}_R + \xi_B \mathbf{C}_B + \xi_\varepsilon \mathbf{C}_\varepsilon$$

for unknown scalars ξ_μ, ξ_R, ξ_B, and ξ_ε.

Denote by $\mathbf{X}$ the $N \times t$ design matrix; that is, $\mathbf{X}_{p\omega}$ is 1 if plot p receives treatment ω and 0 otherwise. For each stratum W_α, the matrix $\mathbf{L}_\alpha$ defined by $\mathbf{L}_\alpha = \mathbf{X}'\mathbf{C}_\alpha\mathbf{X}$ is called the *information matrix* for that stratum. For designs with equal replication r, we have $\mathbf{L}_\mu = r\mathbf{T}_\mu$. If $\mathbf{L}_\alpha = \mathbf{0}$ there is no information about treatments in stratum W_α. Strata, other than W_μ, for which $\mathbf{L}_\alpha \neq \mathbf{0}$, are called *effective strata.*

Suppose that $\oplus_\theta V_\theta$ is an orthogonal direct sum decomposition of $\mathbb{R}^t$. Nelder (1965b) defined an equally replicated design to be *generally balanced* with respect to this treatment decomposition if each V_θ is an eigenspace of every information matrix; that is, there are numbers $\lambda_{\alpha\theta}$ such that $\mathbf{L}_\alpha = \sum_\theta \lambda_{\alpha\theta}\mathbf{T}_\theta$, where $\mathbf{T}_\theta$ denotes orthogonal projection onto V_θ. We have $0 \le \lambda_{\alpha\theta} \le r$ for all α and θ; and $\sum_\alpha \lambda_{\alpha\theta} = r$ for all θ. The quantity $\lambda_{\alpha\theta}/r$ is the *efficiency factor* for treatment term V_θ in the stratum W_α. In a simple block design with blocks stratum W_B, examination of the trace of $\mathbf{L}_B$ shows that $\sum_\theta \lambda_{B\theta}\dim(V_\theta)/r = b/r - 1$, the so-called *loss of information due to blocks.*

Houtman and Speed (1983) have shown that in any design with only two effective strata there must be *some* decomposition $\oplus V_\theta$ of $\mathbb{R}^t$ with respect to which the design is generally balanced. However, the decomposition may not be easy to find, use or interpret. Our claim is that a rectangular lattice design is generally balanced with respect to the treatment decomposition given in Section 3.

LEMMA 3. *For $a = 1,\dots, r$ and $i = 1,\dots, n$,*

$$\mathbf{X}'\mathbf{P}_B\mathbf{X}\mathbf{v}_{ai} = \left(n\mathbf{v}_{ai} - \mathbf{w}_i + (r-1)\mathbf{u}\right)/(n-1).$$

PROOF. If $\mathscr{B}$ is any block and $\mathbf{v}$ is any vector in $\mathbb{R}^t$ then the entries of $\mathbf{P}_B\mathbf{X}\mathbf{v}$ for the plots in $\mathscr{B}$ are all equal to the average of the entries of $\mathbf{v}$ for those treatments which occur in $\mathscr{B}$. If $\mathbf{v} = \mathbf{v}_{ai}$ and $\mathscr{B}$ consists of $\mathscr{S}_{bj}$ then this average is equal to $\langle \mathbf{v}_{ai}, \mathbf{v}_{bj}\rangle/(n-1)$. Denote the characteristic vector of this block by $\mathbf{x}_{bj}$. Then

$$(n-1)\mathbf{P}_B\mathbf{X}\mathbf{v}_{ai} = \sum_b \sum_j \langle \mathbf{v}_{ai}, \mathbf{v}_{bj}\rangle \mathbf{x}_{bj}.$$

Since $\mathbf{X}'\mathbf{x}_{bj} = \mathbf{v}_{bj}$ we have

$$\begin{aligned}(n-1)\mathbf{X}'\mathbf{P}_B\mathbf{X}\mathbf{v}_{ai} &= \sum_b \sum_j \langle \mathbf{v}_{ai}, \mathbf{v}_{bj}\rangle \mathbf{v}_{bj}\\ &= (n-1)\mathbf{v}_{ai} + \sum_{b\neq a} (\mathbf{u} - \mathbf{v}_{bi}) \quad \text{(by (3.1))}\\ &= n\mathbf{v}_{ai} + (r-1)\mathbf{u} - \mathbf{w}_i. \end{aligned}$$ □

THEOREM 2. *Rectangular lattice designs are generally balanced with respect to the treatment decomposition given in Section* 3.

PROOF. We always have $\mathbf{L}_\mu\mathbf{u} = r\mathbf{u}$, and $\mathbf{L}_\mu\mathbf{z} = \mathbf{0}$ whenever $\mathbf{z}$ is orthogonal to $\mathbf{u}$. By definition of replicate, $\mathbf{X}'\mathbf{P}_R\mathbf{X}\mathbf{z} = r\mathbf{g}(\mathbf{z})/n(n-1) = \mathbf{X}'\mathbf{P}_\mu\mathbf{X}\mathbf{z}$, so $\mathbf{L}_R = \mathbf{0}$. Moreover, $\mathbf{L}_B = \mathbf{X}'\mathbf{P}_B\mathbf{X} - \mathbf{X}'\mathbf{P}_R\mathbf{X}$, and so

$$\mathbf{L}_B(\mathbf{v}_{ai} - \mathbf{v}_{bi}) = n(n-1)^{-1}(\mathbf{v}_{ai} - \mathbf{v}_{bi})$$

by Lemma 3. Since V_s is spanned by vectors of the form $\mathbf{v}_{ai} - \mathbf{v}_{bi}$, this shows that

TABLE 16
Efficiency factors of a rectangular lattice design

	treatment subspace			
	V_μ	V_f	V_s	V_ε
stratum				
mean W_μ	1	0	0	0
replicates W_R	0	0	0	0
blocks W_B	0	$\frac{n-r}{r(n-1)}$	$\frac{n}{r(n-1)}$	0
plots W_ε	0	$\frac{n(r-1)}{r(n-1)}$	$\frac{rn-r-n}{r(n-1)}$	1

V_s is an eigenspace of $\mathbf{L}_B$ with eigenvalue $\lambda_{Bs} = n/(n-1)$. Similarly, Lemma 3 shows that

$$\mathbf{L}_B(\mathbf{w}_i - \mathbf{w}_j) = (n-r)(n-1)^{-1}(\mathbf{w}_i - \mathbf{w}_j),$$

so V_f is an eigenspace of $\mathbf{L}_B$ with eigenvalue $\lambda_{Bf} = (n-r)/(n-1)$. Whether or not $r = n$, Table 8 now shows that $\lambda_{Bs}\dim(V_s) + \lambda_{Bf}\dim(V_f) = b - r$, so there can be no further nonzero eigenvalues in the blocks stratum. Thus V_ε must be an eigenspace of $\mathbf{L}_B$ with $\lambda_{B\varepsilon} = 0$.

By the result of Houtman and Speed (1983), the spaces V_f, V_s, V_ε are also eigenspaces of $\mathbf{L}_\varepsilon$. □

The eigenvalues in stratum W_ε are calculated by subtraction. Division by r gives the efficiency factors, which are shown in Table 16, which is laid out like the table in Section 4.2 of Nelder (1968).

Block designs are often classified by a single measure of efficiency: the harmonic mean of the efficiency factors (taking account of multiplicity) in stratum W_ε. It follows from Tables 8 and 16, that, whether $r = n$ or $r < n$, the harmonic mean efficiency factor for a rectangular lattice design is

$$\frac{n(r-1)(rn-r-n)(n^2-n-1)}{(r-1)^2 n^2(n^2-n-1) - r^2(n-1)^2 + rn(r-1)}.$$

This efficiency factor is proportional to the reciprocal of the average variance of the intrablock estimates of simple treatment differences, and so may also be obtained from this average variance, which is given by Williams (1977, page 413).

6. Analysis. Since rectangular lattice designs are generally balanced, their analysis follows the pattern described by Nelder (1965b, 1968), Wilkinson (1970), and James and Wilkinson (1971). In this section we specialize their results to rectangular lattice designs, retaining most of Nelder's notation. We outline the procedure for fitting the model, deriving a complete analysis of variance, estimating the stratum variances ξ_R, ξ_B, and ξ_ε, and obtaining minimum variance unbiased linear estimates (with estimated weights) of arbitrary treatment contrasts, together with their estimated variances.

Let $\mathbf{t}$ be the $t \times 1$ vector of individual treatment effects and let $\mathbf{y}$ be the $N \times 1$ vector of observations. If $\lambda_{\alpha\theta} \neq 0$, the treatment *effect* $\mathbf{T}_\theta \mathbf{t}$ is estimated in stratum W_α by $\mathbf{h}_{\alpha\theta}$, where $\mathbf{h}_{\alpha\theta} = \mathbf{T}_\theta \mathbf{X}'\mathbf{C}_\alpha \mathbf{y}/\lambda_{\alpha\theta}$. The contribution of treatment term V_θ to the *fitted value* in stratum W_α is $\mathbf{C}_\alpha \mathbf{X}\mathbf{h}_{\alpha\theta}$, with the sum of squares $\lambda_{\alpha\theta}\|\mathbf{h}_{\alpha\theta}\|^2$. Thus the overall fitted value in stratum W_α is $\Sigma'_\theta \mathbf{C}_\alpha \mathbf{X}\mathbf{h}_{\alpha\theta}$, where Σ'_θ denotes summation over those θ for which $\lambda_{\alpha\theta} \neq 0$. The residual sum of squares, RSS_α, in stratum W_α, and its number of degrees of freedom, d_α, are obtained by subtraction:

$$\mathrm{RSS}_\alpha = \mathbf{y}'\mathbf{C}_\alpha \mathbf{y} - \sum_\theta{}' \lambda_{\alpha\theta}\|\mathbf{h}_{\alpha\theta}\|^2, \tag{6.1}$$

$$d_\alpha = \dim(W_\alpha) - \sum_\theta{}' \dim(V_\theta). \tag{6.2}$$

Thus we obtain the analysis of variance shown in Tables 17a ($r < n$) and 17b ($r = n$).

If the stratum variances ξ_α are known, we put $w_\theta = \Sigma_\alpha \lambda_{\alpha\theta}/\xi_\alpha$ and define *weights* $w_{\alpha\theta}$ by $w_{\alpha\theta} = \lambda_{\alpha\theta}/\xi_\alpha w_\theta$. The *weighted effect* corresponding to treatment term V_θ is $\Sigma_\alpha w_{\alpha\theta}\mathbf{h}_{\alpha\theta}$, and the overall *weighted fitted value* $\hat{\mathbf{t}}$ is $\Sigma_\theta \Sigma_\alpha w_{\alpha\theta}\mathbf{h}_{\alpha\theta}$. If $\mathbf{x}$ is any treatment contrast (that is, $\mathbf{x} \in \mathbb{R}^t$ and $\langle \mathbf{x}, \mathbf{u}\rangle = 0$) then the minimum variance unbiased linear estimate of $\langle \mathbf{x}, \mathbf{t}\rangle$ is $\langle \mathbf{x}, \hat{\mathbf{t}}\rangle$, with variance $\Sigma_\theta \|\mathbf{T}_\theta \mathbf{x}\|^2/w_\theta$.

TABLE 17a
Analysis of variance when $r < n$

stratum	source of variation	df	SS	EMS
mean		1	$\mathbf{y}'\mathbf{C}_\mu\mathbf{y}$	$r\|\mathbf{T}_\mu\mathbf{t}\|^2 + \xi_\mu$
replicates		$r-1$	$\mathbf{y}'\mathbf{C}_R\mathbf{y}$	ξ_R
blocks	V_f	$n-1$	$\lambda_{Bf}\|\mathbf{h}_{Bf}\|^2$	$\dfrac{\lambda_{Bf}\|\mathbf{T}_f\mathbf{t}\|^2}{n-1} + \xi_B$
	V_s	$(n-1)(r-1)$	$\lambda_{Bs}\|\mathbf{h}_{Bs}\|^2$	$\dfrac{\lambda_{Bs}\|\mathbf{T}_s\mathbf{t}\|^2}{(n-1)(r-1)} + \xi_B$
	total	$r(n-1)$	$\mathbf{y}'\mathbf{C}_B\mathbf{y}$	
plots	V_f	$n-1$	$\lambda_{\varepsilon f}\|\mathbf{h}_{\varepsilon f}\|^2$	$\dfrac{\lambda_{\varepsilon f}\|\mathbf{T}_f\mathbf{t}\|^2}{n-1} + \xi_\varepsilon$
	V_s	$(n-1)(r-1)$	$\lambda_{\varepsilon s}\|\mathbf{h}_{\varepsilon s}\|^2$	$\dfrac{\lambda_{\varepsilon s}\|\mathbf{T}_s\mathbf{t}\|^2}{(n-1)(r-1)} + \xi_\varepsilon$
	V_ε	$(n-r)(n-1)-1$	$\lambda_{\varepsilon\varepsilon}\|\mathbf{h}_{\varepsilon\varepsilon}\|^2$	$\dfrac{\lambda_{\varepsilon\varepsilon}\|\mathbf{T}_\varepsilon\mathbf{t}\|^2}{(n-r)(n-1)-1} + \xi_\varepsilon$
	error	$n(rn-2r-n+1)+1$	RSS_ε	ξ_ε
	total	$rn(n-2)$	$\mathbf{y}'\mathbf{C}_\varepsilon\mathbf{y}$	

TABLE 17b
Analysis of variance when $r = n$

stratum	source of variation	df	SS	EMS
mean		1	$\mathbf{y}'\mathbf{C}_\mu\mathbf{y}$	$r\|\mathbf{T}_\mu\mathbf{t}\|^2 + \xi_\mu$
replicates		$r-1$	$\mathbf{y}'\mathbf{C}_R\mathbf{y}$	ξ_R
blocks	V_s	$(n-1)^2$	$\lambda_{Bs}\|\mathbf{h}_{Bs}\|^2$	$\dfrac{\lambda_{Bs}\|\mathbf{T}_s\mathbf{t}\|^2}{(n-1)^2} + \xi_B$
	error	$n-1$	RSS_B	ξ_B
	total	$n(n-1)$	$\mathbf{y}'\mathbf{C}_B\mathbf{y}$	
plots	V_s	$(n-1)^2$	$\lambda_{\varepsilon s}\|\mathbf{h}_{\varepsilon s}\|^2$	$\dfrac{\lambda_{\varepsilon s}\|\mathbf{T}_s\mathbf{t}\|^2}{(n-1)^2} + \xi_\varepsilon$
	V_ε	$n-2$	$\lambda_{\varepsilon\varepsilon}\|\mathbf{h}_{\varepsilon\varepsilon}\|^2$	$\dfrac{\lambda_{\varepsilon\varepsilon}\|\mathbf{T}_\varepsilon\mathbf{t}\|^2}{n-2} + \xi_\varepsilon$
	error	$(n-1)(n^2-2n-1)$	RSS_ε	ξ_ε
	total	$n^2(n-2)$	$\mathbf{y}'\mathbf{C}_\varepsilon\mathbf{y}$	

Usually the stratum variances ξ_α are not known. If $d_\alpha \neq 0$ then $\mathrm{RSS}_\alpha/d_\alpha$ provides an unbiased estimate of ξ_α, but in general such estimates are based on too few degrees of freedom, because one or more treatment terms have been fitted and removed in more than one stratum. For a rectangular lattice design with $r < n$ there is no such estimate of ξ_B, because $d_B = 0$.

The solution to this difficulty is to estimate the stratum variances and the weights simultaneously. With the weighted fitted value $\hat{\mathbf{t}}$ given above, the sum of squares, R_α, for the residual in stratum W_α is given by

$$R_\alpha = \mathrm{RSS}_\alpha + \sum_\theta \lambda_{\alpha\theta} \sum_\beta \sum_\gamma w_{\beta\theta} w_{\gamma\theta} \langle \mathbf{h}_{\alpha\theta} - \mathbf{h}_{\beta\theta}, \mathbf{h}_{\alpha\theta} - \mathbf{h}_{\gamma\theta} \rangle, \tag{6.3}$$

with expected value $d'_\alpha \xi_\alpha$, where

$$d'_\alpha = \dim(W_\alpha) - \sum_\theta{}' w_{\alpha\theta} \dim(V_\theta). \tag{6.4}$$

Equating observed and expected values of the R_α gives a set of equations in the ξ_α. As Nelder (1968) observed, (6.3) simplifies considerably when there are only two effective strata. Thus for rectangular lattice designs we obtain the following equations for ξ_B and ξ_ε:

$$\mathrm{RSS}_B + \sum_\theta \lambda_{B\theta} w_{\varepsilon\theta}^2 \|\mathbf{h}_{B\theta} - \mathbf{h}_{\varepsilon\theta}\|^2 = \xi_B \left[r(n-1) - \sum_\theta{}' w_{B\theta} \dim(V_\theta) \right],$$

$$\mathrm{RSS}_\varepsilon + \sum_\theta \lambda_{\varepsilon\theta} w_{B\theta}^2 \|\mathbf{h}_{\varepsilon\theta} - \mathbf{h}_{B\theta}\|^2 = \xi_\varepsilon \left[rn(n-2) - \sum_\theta{}' w_{\varepsilon\theta} \dim(V_\theta) \right].$$

Note that RSS_B is zero when $r < n$, and that the weights $w_{\alpha\theta}$ also involve the unknown ξ_α. However, these equations may be solved, iteratively if necessary, to give us estimates ξ_B and ξ_ε, which, under normality, correspond to the so-called *restricted maximum likelihood estimates*, and these may be used to give the best available estimates of linear combinations $\langle \mathbf{x}, \mathbf{t} \rangle$ and the estimated variances of those estimates.

It is clear that the analysis depends on the availability of the projection operators $\mathbf{C}_\alpha$ and $\mathbf{T}_\theta$. The former are quite standard, and correspond to fitting and removing the grand mean, replicate means, and block means. The latter are given by the fan and spoke totals, and so are straightforward to calculate, even by hand. If the statistical programming language GENSTAT is used, spoke totals are automatically calculated if r treatment pseudo-factors are declared, one for each classification: the levels of the ath pseudo-factor are the a-spokes. An alternative strategy is to input r copies of the data and use just two treatment pseudo-factors, FAN and SPOKE. In the ath copy, treatments in spoke $\mathscr{S}_{ai}$ are declared to have level i of FAN and level a of SPOKE. The treatment declaration FAN/SPOKE ensures that all the correct major calculations are done, using the *sweeps* of Wilkinson (1970), although minor adjustments have to be made to the output to allow for the multiple copies. Thompson (1983) explains this method, and its difficulties, in more detail, using the general methods of Thompson (1984), and shows that this type of pseudo-factorial structure is also useful for diallel experiments.

Thus, apart from the use of estimated weights because the stratum variances are in general not known, a completely satisfactory analysis of any rectangular lattice design can be made once the operators $\mathbf{T}_\theta$ are available. Given these, the analysis is analogous to that of a balanced incomplete block design with recovery of interblock information.

Williams and Ratcliff (1980) gave a procedure for the analysis of rectangular lattice designs which differs from ours in two respects. In the first place, their covariance model is of the form

$$\mathrm{Cov}[(\mathbf{I} - \mathbf{P}_R)\mathbf{y}] = \gamma_B \mathbf{P}_B + \gamma_\varepsilon \mathbf{I},$$

which differs from our equation (5.1). Secondly, our iterative analysis ensures that the final estimates of ξ_B, ξ_ε and the treatment effects are consistent with each other, while the Williams–Ratcliff procedure, which is based on that given by Yates (1940) and Cochran and Cox (1957, Section 1.3), is, roughly speaking, only the first cycle of the restricted maximum likelihood analysis of Patterson and Thompson (1971). The differences between these methods, which apply not only to rectangular lattice designs, will be discussed in more detail elsewhere.

7. Rectangular lattices with cross-blocking. The foregoing ideas may be extended to a more complicated block structure.

In Example 2 we have so far ignored the periods. However, it was desirable that each treatment should be fed once in each period. The experimenter concerned found that, for the rectangular lattice design constructed at the end of Section 3, the treatments could be permuted within sheep so that each treatment occurred once in each period: his proposed design is shown in Table 11.

Unfortunately, this design takes no account of the grouping of the 36 experimental units into nine room-periods: each room-period consists of the four observations made in the same test period in the same room. In the notation of Nelder (1965a), the block structure is

$$3 \text{ periods} \times (3 \text{ rooms} \to 4 \text{ sheep}).$$

The stratum projection matrices are given by

$$\begin{aligned}
\mathbf{C}_\mu &= \mathbf{P}_\mu,\\
\mathbf{C}_R &= \mathbf{P}_R - \mathbf{P}_\mu,\\
\mathbf{C}_P &= \mathbf{P}_P - \mathbf{P}_\mu,\\
\mathbf{C}_{RP} &= \mathbf{P}_{RP} - \mathbf{P}_P - \mathbf{P}_R + \mathbf{P}_\mu,\\
\mathbf{C}_S &= \mathbf{P}_S - \mathbf{P}_R,\\
\mathbf{C}_\varepsilon &= \mathbf{P}_\varepsilon - \mathbf{P}_S - \mathbf{P}_{RP} + \mathbf{P}_R,
\end{aligned}$$

where, for example, $\mathbf{P}_{RP}$ is the averaging matrix for room-periods. Although V_μ, V_f, V_s, and V_ε are eigenspaces of $\mathbf{C}_\mu$, $\mathbf{C}_R$, $\mathbf{C}_P$, and $\mathbf{C}_S$, they are *not* eigenspaces of $\mathbf{C}_{RP}$ and $\mathbf{C}_\varepsilon$, because the block design given by the room-periods alone is not in any sense balanced with respect to the treatment decomposition $V_\mu \oplus V_f \oplus V_s \oplus V_\varepsilon$. Thus the design is not generally balanced.

However, it is possible to permute the treatments given to each sheep so that each treatment occurs once in each period and the design is generally balanced. This may be done for $n(n-1)$ treatments in the simple orthogonal block structure

$$(n-1) \text{ periods} \times [(n-1) \text{ rooms} \to n \text{ sheep}]$$

as follows. Ignoring periods, the design is constructed from a set of mutually orthogonal Latin squares $\Lambda_1, \dots, \Lambda_{n-1}$, as in Section 2. A supplementary $(n-1) \times (n-1)$ Latin square Δ is needed, whose letters are the remaining letters of Λ_{n-2}. Let δ_{ap} be the letter in row a and column p of Δ. Then the treatment in the pth period and the ith animal of the ath room is the unique treatment which is in spoke $\mathscr{S}_{ai}$ and in letter δ_{ap} of Λ_{n-2}. In our particular example we may take the supplementary square Δ shown in Table 18: the resulting design is in Table 19.

In the notation of Section 3, V_ε is the main effect of Q_1, where the levels of Q_1 are the remaining letters of Λ_{n-2}. By our construction, Q_1 is completely confounded with room-periods, while all treatment vectors which are orthogonal to Q_1 are also orthogonal to room-periods. Hence the efficiency factors for this extension of the rectangular lattice design are those shown in Table 20.

TABLE 18
Supplementary Latin square

2	3	4
3	4	2
4	2	3

TABLE 19
Generally balanced design for [periods × (rooms → sheep)]

room		1				2				3			
sheep		1	2	3	4	5	6	7	8	9	10	11	12
time period	1	*A*	*D*	*I*	*L*	*G*	*K*	*B*	*F*	*H*	*J*	*C*	*E*
	2	*B*	*F*	*G*	*K*	*J*	*H*	*E*	*C*	*L*	*I*	*D*	*A*
	3	*C*	*E*	*H*	*J*	*D*	*A*	*L*	*I*	*F*	*B*	*K*	*G*

TABLE 20
Efficiency factors of an extended rectangular lattice design

	treatment subspace			
	V_μ	$V_f = Q_2$	$V_s = Q_1Q_2$	$V_\varepsilon = Q_1$
stratum				
mean W_μ	1	0	0	0
rooms W_R	0	0	0	0
periods W_P	0	0	0	0
room-periods W_{PR}	0	0	0	1
sheep W_s	0	$\frac{1}{(n-1)^2}$	$\frac{n}{(n-1)^2}$	0
units W_ε	0	$\frac{n(n-2)}{(n-1)^2}$	$\frac{n^2-3n+1}{(n-1)^2}$	0

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