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1. General remarks. I would like to begin by saying that I enjoyed this paper very much. As with their previous works, both individual and joint, Martin and Yohai have achieved in this paper a nice combination of analytical rigor and practical significance driven by clearly presented intuition. I congratulate the authors on this contribution.

Despite their central role in many areas of robust statistics, the traditional influence curves proposed by Hampel have played a somewhat limited role in the study of robustness properties of statistical signal processing procedures for applications such as communications and control, primarily because of the restriction of their applicability to static models. Other approaches, such as minimax robustness, have proven to be much more useful in this context (see, for example, the recent surveys by Kassam and Poor (1985) and Poor (1986)). However, by allowing for the treatment of dynamic models, the notion of influence functionals as proposed by Martin and Yohai eliminates this principal disadvantage. The introduction of a heuristic tool of this type is thus a major advance from the viewpoint of robust statistical signal processing, and I can foresee a wide range of applications of Martin and Yohai's ideas in this area.

2. System identification. System identification is among the many applications that can be examined in the context of the Martin–Yohai influence functional. For example, consider the simple problem of identifying a first-order time-invariant linear system from measurements of inputs and noisy outputs. This problem corresponds to the model

$$\begin{aligned} s_i &= \theta s_{i-1} + u_i, \qquad i \in \mathbb{Z}, \\ q_i &= s_i + n_i, \qquad i \in \mathbb{Z}, \end{aligned} \tag{1}$$

in which we assume that $\{u_i\}_{i\in\mathbb{Z}}$ and $\{n_i\}_{i\in\mathbb{Z}}$ are independent i.i.d. $\mathcal{N}(0,1)$ sequences and $|\theta| < 1$. The nominal observation process $\{x_i\}_{i\in\mathbb{Z}}$ consists of the inputs and noisy outputs $\left(i.e., x_i = \binom{u_i}{q_i}\right)$, and so we can think of actual observations $y_\iota = \binom{v_\iota}{r_\iota}$ where $\{v_i\}_{i\in\mathbf{Z}}$ and $\{r_\iota\}_{\iota\in\mathbf{Z}}$ are generated by replacement models from $\{u_i\}_{i\in\mathbf{Z}}$ and $\{q_i\}_{i\in\mathbf{Z}}$, respectively.

M-estimates of θ in (1) (see, for example, Poljak and Tsypkin (1980)) are of the form

$$T_n \in \arg\left\{\min_{|t|<1} \sum_{\iota=1}^{n} \rho\left(y_\iota - \sum_{m=1}^{\iota} t^{\iota-m} u_m\right)\right\} \tag{2}$$

for appropriate functions ρ. The estimates of (2) have limit $\tilde{\psi}$ function given by

$$\tilde{\psi}(\mathbf{y}; t) = \psi\left(q_1 - \sum_{m=-\infty}^{1} t^{1-m} v_m\right) \sum_{m=-\infty}^{0} (m-1) t^{-m} v_m, \tag{3}$$

where $\psi = \rho'$.

Within regularity on ψ, the influence functional of (3) for patchy outliers can be evaluated via Martin and Yohai's Theorem 4.2. Upon examination of IF in this case one sees immediately that, for constant outlier-level ζ, the least-squares estimate of θ in (1) is linearly unbounded in ζ for outliers in the output observations and is quadratically unbounded in ζ for outliers in the observations of the input. Also, although the usual robust ψ functions yield bounded influence against output outliers, we see that any nondecreasing ψ is at least linearly unbounded in ζ for input-observation outliers. Thus, from the viewpoint of gross error sensitivity, redescending ψ functions are called for in this model. (Alternatively, $\psi(\xi_1)\xi_2$ could be replaced by a bounded $\eta(\xi_1, \xi_2)$ as in the directly observed AR case discussed in the paper.)

The general trend of the influence of patch length on M estimation in this model is not as obvious as that for the influence of outlier amplitude. For the particular case of least-squares estimation with constant-level outliers, analysis of (3) via Theorem 4.2 shows that patch length is irrelevant for output outliers. However, the influence of input-observation outliers on least-squares is $O(\theta^{-2k}/k)$ where k is the average patch length. Thus, there is clearly a need to consider permutation dependent issues when analyzing robustness in such models.

3. Time-varying models. Although it has nothing particularly to do with the consideration of influence functionals versus influence curves, another useful aspect of Martin and Yohai's formulation is the idea of analyzing estimates via the limiting form $T(\mu)$ regardless of whether or not T_n is actually given by $T(\mu_n)$ for the empirical measure μ_n. This idea allows the analysis of some time-varying models of interest. For example, consider the problem of estimating the amplitude of a signal of known form from noisy observations,

$$x_i = \theta s_\iota + n_i, \qquad i = 1, 2, \ldots, \tag{4}$$

where $\{n_\iota\}_{i=1}^{\infty}$ is i.i.d. and $\{s_i\}_{\iota=1}^{\infty}$ is a known sequence. M-estimates of θ based on corrupted observations $\{y_i\}_{i=1}^{\infty}$ are of the form

$$T_n \in \arg\left\{ \min_{t \in \mathbb{R}} \sum_{i=1}^{n} \rho(y_i - ts_i) \right\}, \tag{5}$$

which can be analyzed via

$$\tilde{\psi}(\mathbf{y}; t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} s_i \psi(y_i - ts_i) \tag{6}$$

(with $\psi = \rho'$), assuming this limit exists in an appropriate sense. Note that, as in the i.i.d. case, $T(\mu)$ (i.e., the solution of $\int \tilde{\psi}(\mathbf{y}; t)\mu(d\mathbf{y}) = 0$) is a function of only the marginal distribution of $\mathbf{y}$ in this situation.

4. Long-term serial dependencies. As a final comment, I mention another type of statistical contamination that would be interesting to examine from the viewpoint of influence functionals. In particular, the influence on time-series procedures of long-term serial dependencies such as those present in electrical systems due to so-called fractional or "$1/f$" noises (see also Graf et al. (1984)) might be studied in this context. For example, one might consider the influence functionals of parameter estimates along a measurement-error-model trajectory $\{\mu_\gamma; 0 \le \gamma < 1\}$ where μ_γ is a Gaussian measure with zero mean and autocovariance $\int y_i y_{i+k} \mu(d\mathbf{y}) = \frac{1}{2}[|k+1|^{\gamma+1} + |k-1|^{\gamma+1} - 2|k|^{\gamma+1}]$. A process described by μ_γ would arise, for example, from the increments of a fractional Brownian motion (Mandelbrot and Van Ness (1968)) with self-similarity parameter $H = (\gamma + 1)/2$. The tail behavior of the spectrum of this process is $O(f^{-\gamma})$, with $\gamma = 0$ yielding white noise. Alternatively, one might consider a mixed error process of the form $(1-\gamma)\varepsilon_i + \gamma w_i$ with $\{\varepsilon_i\}$ white and $\{w_i\}$ the increments of a fixed fractional Brownian motion. Examination of the local behavior of time-series procedures at $\gamma = 0$ in either of these models would give an indication of the tolerance of such procedures to unexpected long-term dependencies.

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