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ESTIMATION PROBLEMS FOR SAMPLES WITH MEASUREMENT ERRORS

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For $x \in \mathbb{R}$ let $N_\alpha(x) := m\alpha$, iff $x \in (\alpha m - \alpha/2, \alpha m + \alpha/2]$. For a sample $X_1, \dots, X_n$ we mainly study the asymptotic properties of the estimators $\overline{N}_\alpha := 1/n\Sigma_{i=1}^n N_\alpha(X_i)$ and $S_\alpha^2 := 1/(n-1)\Sigma_{i=1}^n (N_\alpha(X_i) - \overline{N}_\alpha)^2$ for $\alpha = \alpha_n \to 0$, as $n \to \infty$. For example, if $E(X^2) < \infty$, $E(e^{itX}) = o(|t|^{-k})$, $(|t| \to \infty)$ for some $k \in \mathbb{N}$ and $\alpha_n = O(n^{-1/(2k+2)})$ or $X \sim N(\theta, \sigma^2)$ and $\alpha_n \le 2\pi\sigma(\log n)^{-1/2}$, we prove that $\sqrt{n}(\overline{N}_{\alpha_n} - EX)$ is asymptotically normal. Problems of truncation as well as general maximum likelihood estimation from discrete scale measurements are also considered.

1. Introduction. In many practical estimation problems the statistician is confronted with the following situation: There is given a random sample $X_1, \dots, X_n$, but the variables X_i cannot be observed exactly; they can only be measured by means of some discrete scale. For example most physical data are of this type, and this problem also quite often occurs, if one has to work with secondary statistical material. If the scale has span $\alpha > 0$, the measured variables are $N_\alpha(X_1), \dots, N_\alpha(X_n)$, where for $\alpha(m - \frac{1}{2}) < x \le \alpha(m + \frac{1}{2})$ we set $N_\alpha(x) := m\alpha$ (if x lies in the middle between two scale values, we decide to take the smaller of them).

The classical theory for this type of measurement errors is treated in Kendall and Stuart (1969, pages 78–81) and in Cramér (1974, pages 437–439), essentially leading to "Sheppard's corrections". If one has to estimate an unknown parameter with the aid of the sample $N_\alpha(X_1), \dots, N_\alpha(X_n)$, the method usually recommended in the literature is to choose $N \in \mathbb{N}$, calculate

$$H_m^\alpha := \operatorname{card}\{i \in \{1, \dots, n\} | N_\alpha(X_i) = \alpha m\}, \; m = -N+1, -N+2, \dots, N-1,$$

$$H_N^\alpha := \operatorname{card}\{i \in \{1, \dots, n\} | N_\alpha(X_i) \ge \alpha N\},$$

$$H_{-N}^\alpha := \operatorname{card}\{i \in \{1, \dots, n\} | N_\alpha(X_i) \le -\alpha N\},$$

and then determine the maximum likelihood estimator (mle) with respect to the random quantity $(H_{-N}^\alpha, H_{-N+1}^\alpha, \dots, H_N^\alpha)$. Kulldorf (1961) gives a complete theory of mle estimation for this special case; see also McDonald and Ransom (1979) for alternative procedures. This approach has two main difficulties:

First, a solution of the corresponding mle equation does not necessarily exist with probability 1 [see Kulldorf (1961), page 17, for a set of sufficient conditions for the existence and uniqueness of the mle].

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Secondly, a mle in this context is usually quite difficult to compute. For instance, if one wants to obtain mles for the mean θ and the variance σ^2 of a normal distribution $[X_i \sim N(\theta, \sigma^2)]$, the log-likelihood function is given by

$$\text{const.} + \left(1 - \Phi\left(\left(\left(N\alpha - \frac{\alpha}{2} - \theta\right)\Big/\sigma\right)\right)\left(H_N^\alpha + H_{-N}^\alpha\right)\right.$$
$$+ \sum_{m=-N+1}^{N-1} H_m^\alpha \left[\Phi\left(\left(m\alpha + \frac{\alpha}{2} - \theta\right)\Big/\sigma\right) - \Phi\left(\left(m\alpha - \frac{\alpha}{2} - \theta\right)\Big/\sigma\right)\right]$$

where Φ is the $N(0,1)$ distribution function. Its maximization is only possible by iterative numerical procedures [see Schader and Schmid (1983a, b) for details and numerical applications for the χ^2 test of fit].

The main part of this paper is devoted to a study of the asymptotic behavior of the following two very simple estimators of the mean and the variance of the sample:

$$(1.1) \qquad \overline{N}_\alpha := n^{-1} \sum_{i=1}^{n} N_\alpha(X_i), \qquad S_\alpha^2 := (n-1)^{-1} \sum_{i=1}^{n} \left(N_\alpha(X_i) - \overline{N}_\alpha\right)^2,$$

which seem to be near at hand. "Asymptotic" here means that we let the span α tend to 0 or n tend to infinity or both simultaneously in a certain relation. After deriving quite general moment formulas (exact and asymptotic) it is, e.g., shown that if $\alpha = \alpha_n = O(n^{-1/(2k+2)})$ and $n \to \infty$, $\sqrt{n}(\overline{N}_{\alpha_n} - E(X))$ is asymptotically normal under the assumption that the characteristic function f of X satisfies $f(t) = o(|t|^{-k})$, as $|t| \to \infty$. A similar result holds for $S_{\alpha_n}^2 - (\alpha_n^2/12)$ and for the respective expected values and variances. In Section 3 the normal case is studied in detail. For instance $\overline{N}_{\alpha_n}$ and $S_{\alpha_n}^2 - (\alpha_n^2/12)$ are asymptotically efficient, if $\alpha_n \leq 2\pi\sigma(\log n)^{-1/2}$ or

$$\limsup_{n\to\infty} \alpha_n (\log n)^{1/2} < 2\pi\sigma,$$

respectively. In Section 4 the effect of truncation is considered: Let all measurements of absolute value exceeding some K be neglected in (1.1). We show the order of magnitude in which $K = K_n$ has to tend to infinity for this truncation to be asymptotically unimportant. The final section takes up mle. We compare the mle $\hat{\theta}^n(X_1, \ldots, X_n)$ of a parameter θ with the mle $\hat{\theta}_\alpha^n$ of θ based on $N_\alpha(X_1), \ldots, N_\alpha(X_n)$ by means of their asymptotic variance, and we consider the difference between $\hat{\theta}_\alpha^n$ and the estimator obtained by computing $\hat{\theta}^n(\cdot)$ at $N_\alpha(X_1), \ldots, N_\alpha(X_n)$ ($\overline{N}_\alpha$ and $[(n-1)/n]S_\alpha^2$ are of this type).

Before going into the details we proceed with an example of the type of results obtained below, which possesses an elementary proof. (This derivation as suggested by an Associate Editor.) We assume that $E(X^2) < \infty$ and that the density p of X is differentiable with a continuous and integrable derivative p'

satisfying $p(x) \to 0$, as $|x| \to \infty$. Then if $\alpha_n = O(n^{-1/4})$,

$$n^{1/2}\big(\overline{N}_{\alpha_n} - E(X)\big) \to_D N(0, \operatorname{var}(X)), \tag{1.2}$$

$$n^{1/2}\big(E(\overline{N}_{\alpha_n}) - E(X)\big) \to 0, \tag{1.3}$$

as $n \to \infty$. To see this, let $U_\alpha = U_\alpha(X) = X - N_\alpha(X)$. Then

$$E(U_\alpha) = \sum_{j=-\infty}^{\infty} \int_{-\alpha/2}^{\alpha/2} up(j\alpha + u)\,du = \sum_{j=-\infty}^{\infty} \int_{-\alpha/2}^{\alpha/2} u[p(j\alpha) + up'(j\alpha + \lambda u)]\,du$$

$$= \sum_{j=-\infty}^{\infty} \int_{-\alpha/2}^{\alpha/2} u^2 p'(j\alpha + \lambda u)\,du,$$

where $\lambda = \lambda(u, j, \alpha) \in [-1, 1]$. Thus

$$\frac{\alpha_n^2}{12} \sum_{j=-\infty}^{\infty} \alpha_n \min_{|u| \le \alpha_n/2} p'(j\alpha_n + u) \le E(U_{\alpha_n}) \le \frac{\alpha_n^2}{12} \sum_{j=-\infty}^{\infty} \alpha_n \max_{|u| \le \alpha_n/2} p'(j\alpha_n + u).$$

If $\lim_{n \to \infty} \alpha_n = 0$, both sums in the above expression converge to $\int_{-\infty}^{\infty} p'(u)\,du = 0$ by virtue of the assumptions. Hence

$$E(U_{\alpha_n}) = \frac{\alpha_n^2}{12}\left[\int_{-\infty}^{\infty} p'(u)\,du + o(1)\right] = o(\alpha_n^2), \quad \text{as } n \to \infty.$$

Let $\overline{U}_\alpha := n^{-1}\sum_{i=1}^n U_\alpha(X_i)$. Since $\operatorname{var}(U_\alpha) \le \alpha^2$, we get for all $\varepsilon > 0$

$$P\big(\big|\sqrt{n}\,\overline{U}_{\alpha_n}\big| > \varepsilon\big) \le P\big(\big|\sqrt{n}\,\overline{U}_{\alpha_n} - E(U_{\alpha_n})\big| > \varepsilon - \sqrt{n}\,\big|E(U_{\alpha_n})\big|\big)$$

$$\le \alpha_n^2 / \big[\varepsilon - \sqrt{n}\,\big|E(U_{\alpha_n})\big|\big]^2 \to 0, \quad \text{as } n \to \infty,$$

provided that $\sqrt{n}\,E(U_{\alpha_n}) = o(1)$. If $\alpha_n = O(n^{-1/4})$, this holds, because $\sqrt{n}\,E(U_{\alpha_n}) = o(\sqrt{n}\,\alpha_n^2) = o(1)$. Hence in this case $\sqrt{n}\,\overline{U}_{\alpha_n} = o_P(1)$. (1.2) and (1.3) now follow from

$$\sqrt{n}\,\big(\overline{N}_{\alpha_n} - E(X)\big) = \sqrt{n}\,(\overline{X} - E(X)) - \sqrt{n}\,\overline{U}_{\alpha_n} = \sqrt{n}\,(\overline{X} - E(X)) + o_P(1)$$

and

$$\sqrt{n}\,\big(E\overline{N}_{\alpha_n} - E(X)\big) = -\sqrt{n}\,E\big(\overline{U}_{\alpha_n}\big) = \sqrt{n}\,o\big(\alpha_n^2\big) = o(1), \quad \text{as } n \to \infty.$$

It will follow from Theorem 4 below that if $E(X^2) < \infty$, $p \in C^k(\mathbb{R})$ with integrable $p^{(k)}$, we only need $\alpha_n = O(n^{-1/(2k+2)})$ to ensure (1.2) and (1.3). However, the above elementary proof does not work for $k > 1$, because in the expansion of $E(U_{\alpha_n})$ to higher powers of u, the $o(\alpha_n^2)$ term originating from the Riemann sum approximation of $\int_{-\infty}^{\infty} p'(u)\,du$ persists. So we have to undertake a more complicated analysis.

2. The estimators $\bar{N}_\alpha$ and S_α^2 in the case of polynomially decreasing densities. For $\alpha > 0$ let $I_{m,\alpha} := (m\alpha - \alpha/2, m\alpha + \alpha/2]$ and $N_\alpha(x) := m\alpha$ iff $x \in I_{m,\alpha}$, $U_\alpha(x) := x - N_\alpha(x)$. In what follows we consider independent random variables $X, X_1, X_2, X_3, \dots$ with the common distribution μ. We shall first derive completely general exact formulae for the moments of U_α. For this the following representation of the characteristic function f_α of $U_\alpha(X)$ is needed. Let f be the characteristic function of X.

THEOREM 1. *For all $t \in \mathbb{R}$ for which $\alpha t/2\pi$ is not an integer we have*

$$\begin{aligned} f_\alpha(t) &= i \sin\frac{\alpha t}{2} \sum_{j=-\infty}^{\infty} P\left(X = \left(j + \frac{1}{2}\right)\alpha\right) \\ &\quad + \lim_{n\to\infty} \sum_{j=-n}^{n} (-1)^j f\left(\frac{2\pi j}{\alpha}\right) \frac{\sin(\alpha t/2)}{(\alpha t/2) - j\pi}. \end{aligned} \tag{2.1}$$

PROOF. Let $0 < r < 1$. It is easily seen that for $k = 0, 1, 2, \dots$

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} (-1)^n r^{|n|} f\left(\frac{2\pi n}{\alpha}\right) \frac{d^k}{dt^k}\left[\frac{\sin(\alpha t/2)}{(\alpha t/2) - n\pi}\right] \\ &\qquad = \alpha^{-1} \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-\alpha/2}^{\alpha/2} \left[\int_{-\infty}^{\infty} (iu)^k \exp\left\{itu + \left(2\pi in\frac{x-u}{\alpha}\right)\right\}\mu(dx)\right] du \\ &\qquad = \int_{-\infty}^{\infty}\left[\int_{x-(\alpha/2)}^{x+(\alpha/2)} [i(x-v)]^k e^{it(x-v)} m_{r,\alpha,x}(dv)\right]\mu(dx), \end{aligned} \tag{2.2}$$

where we denote by $m_{r,\alpha,x}$ for the probability measure on $[x - \alpha/2, x + \alpha/2]$ with density function

$$\begin{aligned} h_{r,\alpha,x}(v) &:= \alpha^{-1} \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi inv/\alpha} \\ &= \frac{1 - r^2}{\alpha\left(1 + r^2 - 2r\cos(2\pi v/\alpha)\right)}, \qquad v \in \left[x - \frac{\alpha}{2}, x + \frac{\alpha}{2}\right]. \end{aligned}$$

The first representation shows that $\int h_{r,\alpha,x}(v)\,dv = 1$, while the second one implies $h_{r,\alpha,x} > 0$.

Let ε_y be the unit mass in y. Let $j \in \mathbb{Z}$ be arbitrary. If $(j - \frac{1}{2})\alpha < x < (j + \frac{1}{2})\alpha$, we have $\lim_{r\uparrow 1} h_{r,\alpha,x}(v) = 0$ for all $v \in [x - \alpha/2, x + \alpha/2] \setminus \{j\alpha\}$ and $\lim_{r\uparrow 1} h_{r,\alpha,x}(j\alpha) = \infty$. Thus $m_{r,\alpha,x}$ converges weakly to $\varepsilon_{j\alpha}$, as $r \uparrow 1$. If $x = (j + \frac{1}{2})\alpha$, $h_{r,\alpha,x} = h_{r,\alpha,j\alpha+(\alpha/2)}$ is symmetric around $j\alpha + (\alpha/2)$ and satisfies $\lim_{r\uparrow 1} h_{r,\alpha,j\alpha+(\alpha/2)}(v) = 0$ for all $v \in (j\alpha - \alpha/2, j\alpha + \alpha/2)$. Thus $m_{r,\alpha,j\alpha+(\alpha/2)}$ converges weakly to $\frac{1}{2}(\varepsilon_{j\alpha} + \varepsilon_{(j+1)\alpha})$. Hence for $r \uparrow 1$ the double integral in (2.2) tends to

$$\sum_{j=-\infty}^{\infty}\Bigg[\int_{((j-1/2)\alpha,(j+1/2)\alpha)}[i(x-j\alpha)]^k e^{it(x-j\alpha)}\mu(dx)$$

$$+P\left(X=\left(j+\frac{1}{2}\right)\alpha\right)\frac{1}{2}\left[\left(\frac{i\alpha}{2}\right)^k e^{it\alpha/2}+\left(-\frac{i\alpha}{2}\right)^k e^{-it\alpha/2}\right]\Bigg]$$

$$=\sum_{j=-\infty}^{\infty}\int_{I_{j,\alpha}}[i(x-j\alpha)]^k e^{it(x-j\alpha)}\mu(dx)$$

$$-\left(\frac{i\alpha}{2}\right)^k e^{it\alpha/2}P\left(X=\left(j+\frac{1}{2}\right)\alpha\right)-P\left(X=\left(j+\frac{1}{2}\right)\alpha\right)\frac{d^k}{dt^k}\cos\left(\frac{\alpha t}{2}\right)$$

$$=\frac{d^k}{dt^k}f_\alpha(t)-i\frac{d^k}{dt^k}\left(\sin\left(\frac{\alpha t}{2}\right)\right)\sum_{j=-\infty}^{\infty}P\left(X=\left(j+\frac{1}{2}\right)\alpha\right).$$

Note that the coefficient of $r^{|n|}$ at the left-hand side of (2.2) is of order $O(n^{-k-1})$, as $n\to\infty$, for $k=0,1,2,\dots$ By an application of Littlewood's Tauberian theorem [see, e.g., Wiener (1958), page 104] we can conclude that

$$\frac{d^k}{dt^k}f_\alpha(t)=i\frac{d^k}{dt^k}\sin\left(\frac{\alpha t}{2}\right)\sum_{j=-\infty}^{\infty}P\left(X=\left(j+\frac{1}{2}\right)\alpha\right)$$
$$+\lim_{n\to\infty}\sum_{j=-n}^{n}(-1)^j f\left(\frac{2\pi j}{\alpha}\right)\frac{d^k}{dt^k}\left[\frac{\sin(\alpha t/2)}{(\alpha t/2)-j\pi}\right]. \tag{2.3}$$

(2.1) is the special case $k=0$. □

Corollary. *For an arbitrary random variable X and any $l\in\mathbb{N}$ the following formulae hold*:

$$E\big(U_\alpha(X)^{2l}\big)=\left(\frac{\alpha}{2}\right)^{2l}\Bigg[\frac{1}{2l+1}+(2l)!2\sum_{j=1}^{\infty}\operatorname{Re}f\left(\frac{2\pi j}{\alpha}\right)$$
$$\times\left(\sum_{m=0}^{l-1}\frac{(-1)^{l+j+m+1}}{(2m+1)!(j\pi)^{2(l-m)}}\right)\Bigg] \tag{2.4}$$

$$E\big(U_\alpha(X)^{2l-1}\big)=\left(\frac{\alpha}{2}\right)^{2l-1}\Bigg[\sum_{j=-\infty}^{\infty}P\left(X=\left(j+\frac{1}{2}\right)\alpha\right)+(2l-1)!2$$
$$\times\sum_{j=1}^{\infty}\operatorname{Im}f\left(\frac{2\pi j}{\alpha}\right)\left(\sum_{m=0}^{l-1}\frac{(-1)^{l+j+m}}{(2m+1)!(j\pi)^{2(l-m)-1}}\right)\Bigg]. \tag{2.5}$$

Proof. This follows from the relations

$$(d^k/dt^k)(f_\alpha)(0)=i^k E\big(U_\alpha(X)^k\big)$$

and

$$\frac{d^k}{dt^k}\left\{\frac{\sin(\alpha t/2)}{(\alpha t/2) - j\pi}\right\}\Bigg|_{t=0} = k!\left(\frac{\alpha}{2}\right)^k \sum_{m=0}^{[(k-1)/2]} \frac{(-1)^{m+1}}{(2m+1)!(j\pi)^{k-2m}}, \qquad j \neq 0,$$

$$\frac{d^k}{dt^k}\left\{\frac{\sin(\alpha t/2)}{(\alpha t/2)}\right\}\Bigg|_{t=0} = \begin{cases} \dfrac{(-1)^{k/2}}{k+1}\left(\dfrac{\alpha}{2}\right)^k, & k \text{ even},\\ 0, & k \text{ odd},\end{cases}$$

by some elementary computations. □

REMARK 1. Equation (2.1) generalizes some so-called Poisson formulae from harmonic analysis [see, e.g., Feller (1971), page 632, equation (5.9) and Schempp and Dreseler (1980), pages 140–143].

REMARK 2. If X has a density $p(u)$, then obviously $p_\alpha(u) = \sum_{n=-\infty}^{\infty} p(\alpha n + u)$, $u \in (-\alpha/2, \alpha/2]$, is a density of U_α.Let us additionally assume that $f(t) = O(|t|^{-1})$, as $|t| \to \infty$. This is, e.g., satisfied, if p has an integrable derivative. If p_α is continuous at some point $u \in (-\alpha/2, \alpha/2)$, we further have

$$p_\alpha(u) = \alpha^{-1} \lim_{n\to\infty} \sum_{j=-n}^{n} f(2\pi j/\alpha) e^{-2\pi i j u/\alpha}. \tag{2.6}$$

For, by the above cited Tauberian theorem,

$$\begin{aligned}
&\alpha^{-1} \lim_{n\to\infty} \sum_{j=-n}^{n} f(2\pi j/\alpha) e^{-2\pi i j u/\alpha}\\
&\quad = \alpha^{-1} \lim_{r\to 1} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} r^{|j|} e^{2\pi i j(v-u)/\alpha} p(v)\, dv\\
&\quad = \alpha^{-1} \lim_{r\to 1} \sum_{m=-\infty}^{\infty} \int_{m\alpha-\alpha/2}^{m\alpha+\alpha/2} \sum_{j=-\infty}^{\infty} r^{|j|} e^{2\pi i j v/\alpha} p(v+u)\, dv\\
&\quad = \lim_{r\to 1} \sum_{m=-\infty}^{\infty} \int_{-\alpha/2}^{+\alpha/2} p(v+u+m\alpha) m_{r,\alpha,0}(dv)\\
&\quad = \lim_{r\to 1} \int_{-\alpha/2}^{+\alpha/2} p_\alpha(u+v) m_{r,\alpha,0}(dv) = p_\alpha(u).
\end{aligned}$$

The last equation follows, since $m_{r,\alpha,0} \to \varepsilon_0$ weakly, as $r \to 1$.

The next theorem gives asymptotic formulae for the mixed moments of X and $N_\alpha(X)$ for small α. We need some conditions on the smoothness and the limiting behavior of the density p of X. The proof is similar to the derivation of Sheppard's corrections in Cramér [(1974), chapter 27.9]. As for the assumptions occurring from now on, it should be noted that $p \in C^{2m}(\mathbb{R})$ means that p is *everywhere* $2m$ times differentiable and $p^{(2m)}$ is continuous everywhere. We further remark that the condition $p(x) = O(|x|^{-l-\varepsilon})$, as $|x| \to \infty$, implies that $E(|X|^{l-1}) < \infty$.

THEOREM 2. *Let X have a density $p \in C^{2m}(\mathbb{R})$ such that for some integers $k, j \geq 0$ and some $\varepsilon > 0$ we have $p^{(i)}(x) = O(|x|^{-k-j-1-\varepsilon})$, as $|x| \to \infty$, for $i = 0, 1, \ldots, 2m$. Then, as $\alpha \to 0$,*

$$(2.7)\quad E\big(X^j N_\alpha(X)^k\big) = \sum_{i=0}^{k/2} \binom{k+1}{2i} \frac{1}{k+1} \left(\frac{\alpha}{2}\right)^{k-2i} E(X^{j+2i}) + O(\alpha^{2m}),$$

k even,

$$(2.8)\quad E\big(X^j N_\alpha(X)^k\big) = \sum_{i=0}^{(k-1)/2} \frac{1}{k+1} \binom{k+1}{2i+1} \left(\frac{\alpha}{2}\right)^{k-2i-1} E(X^{j+2i+1}) + O(\alpha^{2m}),$$

k odd.

PROOF. Let $F(y) := y^k \int_{y-\alpha/2}^{y+\alpha/2} u^j p(u)\, du$, $y \in \mathbb{R}$. Then $F \in C^{2m+1}(\mathbb{R})$ and $F^{(2m+1)}(y) = O(|y|^{-1-\varepsilon})$, as $y \to \infty$. Now Euler's summation formula is needed in the following form:

$$(2.9)\qquad \sum_{n=-\infty}^{\infty} F(\alpha n) = \int_{-\infty}^{\infty} F(\alpha u)\, du + \alpha^{2m+1} \int_{-\infty}^{\infty} H_{2m+1}(u) F^{(2m+1)}(\alpha u)\, du,$$

where

$$H_{2m+1}(u) := (-1)^{m+1} \sum_{n=1}^{\infty} \frac{2\sin(n\pi u)}{(2n\pi)^{2m+1}}, \qquad u \in \mathbb{R}$$

[see Cramér (1974), chapter 12.2.]. Using (2.9) and the boundedness of H_{2m+1} we can carry out the following computation:

$$\begin{aligned} E\big(X^j N_\alpha(X)^k\big) &= \sum_{n=-\infty}^{\infty} (\alpha n)^k \int_{\alpha n-\alpha/2}^{\alpha n+\alpha/2} u^j p(u)\, du \\ &= \int_{-\infty}^{\infty} (\alpha x)^k \left(\int_{\alpha x-\alpha/2}^{\alpha x+\alpha/2} u^j p(u)\, du \right) dx \\ &\quad + \alpha^{2m+1} \int_{-\infty}^{\infty} H_{2m+1}(u) F^{(2m+1)}(\alpha u)\, du \\ &= \alpha^k \int_{-\infty}^{\infty} u^j p(u) \left(\int_{u/\alpha-1/2}^{u/\alpha+1/2} x^k\, dx \right) du \\ &\quad + \alpha^{2m+1} O\left(\int_{-\infty}^{\infty} (1+|\alpha u|)^{-1-\varepsilon}\, du \right) \\ &= \frac{\alpha^k}{k+1} \int_{-\infty}^{\infty} u^j p(u) \sum_{l=0}^{k+1} \binom{k+1}{l} \left(\frac{u}{\alpha}\right)^l \\ &\quad \times \left[\left(\frac{1}{2}\right)^{k+1-l} - \left(-\frac{1}{2}\right)^{k+1-l} \right] du + O(\alpha^{2m}) \\ &= \sum_{l=o}^{k+1} \binom{k+1}{l} \left(\frac{1}{2}\right)^{k+1-l} \frac{1+(-1)^{k-l}}{k+1} \alpha^{k-l} E(X^{j+l}) + O(\alpha^{2m}). \end{aligned} \tag{2.10}$$

(2.10) immediately yields (2.7) and (2.8). □

We now turn to a discussion of the estimators $\overline{N}_\alpha$ and S_α^2 introduced in Section 1.

THEOREM 3. (a) *Let X have a density $p \in C^{2m}(\mathbb{R})$ such that $p^{(i)}(x) = O(|x|^{-3-\varepsilon})$, as $|x| \to \infty$, for $i = 0, 1, \dots, 2m$ and some $\varepsilon > 0$. Then the following relations hold for $\alpha \to 0$:*

$$E(\overline{N}_\alpha) = E(X) + o(\alpha^{2m+1}), \tag{2.11}$$

$$\operatorname{var}(\overline{N}_\alpha) = \frac{1}{n}\left(\operatorname{var}(X) + \frac{\alpha^2}{12} + O(\alpha^{2m})\right), \tag{2.12}$$

$$E(S_\alpha^2) = \operatorname{var}(X) + \frac{\alpha^2}{12} + O(\alpha^{2m}). \tag{2.13}$$

(b) *If moreover $p^{(i)}(x) = O(|x|^{-5-\varepsilon})$, as $|x| \to \infty$, for $i = 0, \dots, 2m$ and some $\varepsilon > 0$, we further have*

$$\begin{aligned} \operatorname{var}(S_\alpha^2) &= \frac{1}{n}\left[E\left((X - EX)^4\right) - \operatorname{var}(X)^2 + \frac{\alpha^2}{3}\operatorname{var}(X) + \frac{\alpha^4}{180} + O(\alpha^{2m})\right] \\ &\quad + \frac{1}{n(n-1)}\left[\operatorname{var}(X)^2 + \frac{\alpha^2}{6}\operatorname{var}(X) + \frac{\alpha^4}{144} + O(\alpha^{2m})\right]. \end{aligned} \tag{2.14}$$

PROOF. The existence and integrability of $p^{(2m)}$ implies that $f(t) = o(|t|^{-2m})$, as $|t| \to \infty$ [Feller (1971), page 514]. Thus, setting $l = 1$ in (2.5), we get $E(U_\alpha) = o(\alpha^{2m+1})$, as $\alpha \to 0$. Hence,

$$E(\overline{N}_\alpha) = E(N_\alpha) = E(X) - E(U_\alpha) = E(X) + o(\alpha^{2m+1}).$$

It follows from (2.4) for $l = 1$ that $E(U_\alpha^2) = \alpha^2/12 + o(\alpha^{2m+2})$ and from (2.8) for $j = k = 1$ that $E(XU_\alpha) = E(X^2) - E(XN_\alpha) = O(\alpha^{2m})$. Thus,

$$\begin{aligned} \operatorname{var}(\overline{N}_\alpha) &= \frac{1}{n}\left[E(N_\alpha^2) - (E(N_\alpha))^2\right] \\ &= \frac{1}{n}\left[E(X^2) - 2E(XU_\alpha) + E(U_\alpha^2) - (E(X) + o(\alpha^{2m+1}))^2\right] \\ &= \frac{1}{n}\left[\operatorname{var}(X) + \frac{\alpha^2}{12} + O(\alpha^{2m})\right]. \end{aligned}$$

Further we have

$$E(S_\alpha^2) = \operatorname{var}(N_\alpha) = n \operatorname{var}(\overline{N}_\alpha) = \operatorname{var}(X) + \frac{\alpha^2}{12} + O(\alpha^{2m}).$$

Finally, to prove (2.14) observe that

$$\operatorname{var}(S_\alpha^2) = \frac{\mu_4 - \mu_2^2}{n} + \frac{2}{(n-1)n}\mu_2^2 \tag{2.15}$$

[Kendall and Stuart (1969), page 244, Example 10.13], where $\mu_2^2 := (\mathrm{var}(N_\alpha))^2$, $\mu_4 := E((N_\alpha - E(N_\alpha))^4) = E([X - E(X) - (U_\alpha - E(U_\alpha))]^4)$.

Using Theorem 2 it is easily calculated that

$$E(XU_\alpha^2) = \frac{\alpha^2}{12}E(X) + O(\alpha^{2m}), \qquad E(X^2U_\alpha) = O(\alpha^{2m}),$$

$$E(X^2U_\alpha^2) = \frac{\alpha^2}{12}E(X^2) + O(\alpha^{2m}), \qquad E(X^3U_\alpha) = O(\alpha^{2m}),$$

$$E(XU_\alpha^3) = O(\alpha^{2m}).$$

A lengthy computation using these results will then show (2.14). □

REMARK. It is clear from the above corollary that the tail properties of the characteristic function f of X play an essential role for the asymptotic behavior of $\overline{N}_\alpha$ and S_α^2. The result quoted from Feller (1971) connects the differentiability of p with the tail behavior of f. However, differentiability of p is not necessary for relations of the form $f(t) = O(|t|^{-\beta})$, as the examples of the uniform, the triangular, or the gamma distribution make obvious.

Theorem 3 shows that the variances of the estimators $\overline{N}_\alpha$ and S_α^2 are of order $O(n^{-1})$, but both estimators have a bias that does not converge to 0, as $n \to \infty$. In order to achieve asymptotic unbiasedness of $\overline{N}_\alpha$ and S_α^2 we have to let α and n simultaneously tend to 0 resp. infinity. The following two theorems give the right order of magnitude for refining the measurement scale in dependence of n.

THEOREM 4. *If $E(X^2) < \infty$ and $f(t) = o(|t|^{-k})$, $(|t| \to \infty)$ for some $k \in \mathbb{N}$ [e.g., if $p \in C^k(\mathbb{R})$ and $p^{(k)}$ is integrable] and $\alpha_n = O(n^{-1/(2k+2)})$, as $n \to \infty$, we have*

$$\sqrt{n}\left(\overline{N}_{\alpha_n} - E(X)\right) \to_D N(0, \mathrm{var}(X)), \qquad n \to \infty, \tag{2.16}$$

$$\sqrt{n}\left(E\left(\overline{N}_{\alpha_n}\right) - E(X)\right) \to 0, \qquad n \to \infty. \tag{2.17}$$

If $k = 2m$ and the assumptions of Theorem 3(a) are in force, we further have

$$nE\left(\left(\overline{N}_{\alpha_n} - E(X)\right)^2\right) \to \mathrm{var}(X), \qquad n \to \infty. \tag{2.18}$$

PROOF. Obviously,

$$\sqrt{n}\left(\overline{N}_{\alpha_n} - E(X)\right) = \sqrt{n}\left(\overline{X} - E(X)\right) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n} U_{\alpha_n}(X_i) \tag{2.19}$$

and, for $\varepsilon > 0$,

$$\begin{aligned} P\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} U_{\alpha_n}(X_i)\right| > \varepsilon\right) &\le (n\varepsilon^2)^{-1} E\left(\left[\sum_{i=1}^{n} U_{\alpha_n}(X_i)\right]^2\right) \\ &= (n\varepsilon^2)^{-1}\left[nE\left(U_{\alpha_n}^2\right) + n(n-1)\left(E\left(U_{\alpha_n}\right)\right)^2\right] \\ &= O\left(\alpha_n^2\right) + o\left(n\alpha_n^{2k+2}\right) \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

where for the last equation the corollary of Theorem 1 has to be applied. Thus $n^{-1/2}\sum_{i=1}^{n} U_{\alpha_n}(X_i) \to_P O$ $(n \to \infty)$. Further we conclude from $E(X^2) < \infty$ that $n^{1/2}(\overline{X} - E(X)) \to N(0, \mathrm{var}(X))$. (2.16) now follows from (2.19). Further $E(\overline{N}_{\alpha_n}) - E(X) = E(U_{\alpha_n}) = o(\alpha_n^{k+1}) = o(n^{-1/2})$ yielding (2.17). (2.18) is an immediate consequence of theorem 3. □

For $S_\alpha^2 - \mathrm{var}(X)$ one gets analogous conclusions as for $\overline{N}_\alpha - E(X)$; however, one has to choose $\alpha = \alpha_n = o(n^{-1/4})$ because of the term $\alpha^2/12$ in (2.12). It is therefore preferable to apply Sheppard's corrections.

THEOREM 5. *Let $E(X^4) < \infty$ and $f(t) = o(|t|^{-k})$, as $|t| \to \infty$, for some $k \in \mathbb{N}$. If $\alpha_n = O(n^{-1/(2k+2)})$, as $n \to \infty$, we have*

$$(2.20) \qquad \sqrt{n}\left(S_{\alpha_n}^2 - \frac{\alpha_n^2}{12} - \mathrm{var}(X)\right) \to_D N\Big(0, E\big((X - E(X))^4\big) - \mathrm{var}(X)^2\Big),$$

$$(2.21) \qquad \sqrt{n}\left(E\big(S_{\alpha_n}^2\big) - \frac{\alpha_n^2}{12} - \mathrm{var}(X)\right) \to 0.$$

If $k = 2m$ and the assumptions of Theorem 3(b) *hold, we also have*

$$(2.22) \qquad nE\left(\left(S_{\alpha_n}^2 - \frac{\alpha_n^2}{12} - \mathrm{var}(X)\right)^2\right) \to E\big((X - E(X))^4\big) - \mathrm{var}(X)^2.$$

The proof starts from the identity

$$S_\alpha^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left[X_i - \overline{X} - (U_\alpha(X_i) - \overline{U}_\alpha)\right]^2,$$

where $\overline{U}_\alpha := n^{-1}\sum_{i=1}^{n} U_\alpha(X_i)$. *We omit it, because it is similar to that of Theorem* 7, *Section* 3, *which is carried out in detail.*

Hence if the correction term $-(\alpha^2/12)$ is introduced, the order of magnitude in which the span has to tend to 0 can in most cases be considerably decreased.

3. The normal case. In Section 2 we have mainly considered random variables with densities p with the property $p^{(i)}(x) = O(|x|^{-\beta})$, $(|x| \to \infty)$, $i = 0, 1, \ldots, 2m$, for some β. If p as well as its derivatives are of exponentially small order for large $|x|$, the remainder terms of our approximations will also be exponentially small. We shall not give the details, but restrict ourselves to the important case of the normal distribution. So let $X, X_1, X_2, \ldots$ be independent $N(\theta, \sigma^2)$ variables for some $(\theta, \sigma^2) \in \mathbb{R} \times (0, \infty)$. Then (2.4) and (2.5) take the form

$$(3.1) \qquad E\big(U_\alpha^{2l}\big) = \left(\frac{\alpha}{2}\right)^{2l}\left[\frac{1}{2l+1} - \sum_{j=1}^{\infty} c_{j,2l}\cos\left(\frac{2\pi\theta j}{\alpha}\right)\exp(-2\pi^2\sigma^2 j^2/\alpha^2)\right],$$

$$(3.2) \quad E\big(U_\alpha^{2l-1}\big) = \left(\frac{\alpha}{2}\right)^{2l-1} \sum_{j=1}^{\infty} c_{j,2l-1}\sin(2\pi\theta j/\alpha)\exp(-2\pi^2\sigma^2 j^2/\alpha^2),$$

where

$$c_{jk} := k!2 \sum_{m=0}^{[(k-1)/2]} \frac{(-1)^{j+m+[(k+1)/2]}}{(2m+1)!(j\pi)^{k-2m}}.$$

Here $[(k+1)/2]$ denotes the integer part of $(k+1)/2$. Note that

$$(3.3)\qquad \sum_{j=1}^{\infty} \exp\left(\frac{-2\pi^2\sigma^2 j^2}{\alpha^2}\right) = \exp\left(\frac{-2\pi^2\sigma^2}{\alpha^2}\right)\left(1 + O\left(\exp\left(\frac{-4\pi^2\sigma^2}{\alpha^2}\right)\right)\right),$$

as $\alpha \to 0$. By (3.2) and (3.3) we obtain

$$(3.4)\qquad E(\overline{N}_\alpha) = \theta - E(U_\alpha) = \theta + O\big(\alpha \exp(-2\pi^2\sigma^2/\alpha^2)\big), \quad \text{as } \alpha \to 0.$$

To compute $\operatorname{var}(\overline{N}_\alpha)$, we remark that

$$\operatorname{var}(\overline{N}_\alpha) = \frac{1}{n}\left[E(U_\alpha^2) - \sigma^2 + 2E(N_\alpha\cdot(X-\theta)) - (E(N_\alpha))^2 + \theta^2 - 2\theta E(U_\alpha)\right].$$

We have $(E(N_\alpha))^2 = \theta^2 + O(\alpha \exp(-2\pi^2\sigma^2/\alpha^2))$, $E(U_\alpha^2) = \alpha^2/12 + O(\alpha^2 \exp(-2\pi^2\sigma^2/\alpha^2))$ and, if φ denotes the density of $N(0,1)$,

$$\begin{aligned}
E(N_\alpha(X-\theta)) &= \sum_{n=-\infty}^{\infty} \alpha n \int_{\alpha n-\alpha/2}^{\alpha n+\alpha/2} (x-\theta)\sigma^{-1}\varphi\left(\frac{x-\theta}{\sigma}\right) dx \\
&= \sum_{n=-\infty}^{\infty} \alpha n \sigma \int_{(\alpha n-\alpha/2-\theta)/\sigma}^{(\alpha n+\alpha/2-\theta)/\sigma} y\varphi(y)\,dy \\
&= \alpha\sigma \sum_{n=-\infty}^{\infty} n\left[\varphi\left(\left(\alpha n - \frac{\alpha}{2} - \theta\right)/\sigma\right) - \varphi\left(\left(\alpha n + \frac{\alpha}{2} - \theta\right)/\sigma\right)\right] \\
(3.5)\qquad &= (2\pi)^{-1/2}\alpha\sigma \sum_{n=-\infty}^{\infty} \exp\left\{-\alpha^2\left(n - \frac{1}{2} - \frac{\theta}{\alpha}\right)^2/2\sigma^2\right\} \\
&= \sigma^2 \sum_{n=-\infty}^{\infty} (-1)^n \exp(-2\pi i n\theta/\alpha)\exp(-2\pi^2\sigma^2 n^2/\alpha^2) \\
&= \sigma^2 + 2\sigma^2 \sum_{n=1}^{\infty} (-1)^n \exp(-2\pi^2\sigma^2 n^2/\alpha^2)\cos\left(\frac{2\pi n\theta}{\alpha}\right).
\end{aligned}$$

For the fifth equation we have used the theta function identity

$$\sum_{n=-\infty}^{\infty} \exp\left(-\beta(n-\gamma)^2\right) = (\pi/\beta)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left(-2\pi i n\gamma - (\pi^2 n^2/\beta)\right),$$

where $\beta > 0$, $\gamma \in \mathbb{R}$ [see, e.g., Bellman (1961), page 10]. The asymptotic result for $\operatorname{var}(\overline{N}_\alpha)$ is thus given by

$$(3.6)\qquad \operatorname{var}(\overline{N}_\alpha) = \frac{1}{n}\left[\sigma^2 + \frac{\alpha^2}{12} + O\big(\exp(-2\pi^2\sigma^2/\alpha^2)\big)\right], \quad \text{as } \alpha \to 0.$$

Further,

$$E\left(S_\alpha^2\right) = \sigma^2 + \frac{\alpha^2}{12} + O\left(\exp\left(-2\pi^2\sigma^2/\alpha^2\right)\right), \quad \text{as } \alpha \to 0. \tag{3.7}$$

Now we again turn to the case of a span $\alpha = \alpha_n$ converging to 0, as $n \to \infty$.

THEOREM 6. *If* $\alpha_n \le 2\pi\sigma(\log n)^{-1/2}$,

$$\sqrt{n}\left(\overline{N}_{\alpha_n} - \theta\right) \to_D N(0, \sigma^2), \qquad n \to \infty. \tag{3.8}$$

The sequences of the expected values and of the variances converge to 0 *and* σ^2, *respectively.*

PROOF. One has to proceed as in the proof of Theorem 4 in Section 2, but now using (3.1) and (3.2). The crucial steps are

$$\begin{aligned} P&\left(n^{-1/2}\left|\sum_{i=1}^n U_{\alpha_n}(X_i)\right| > \varepsilon\right) \\ &\le (n\varepsilon^2)^{-1}\left[nE\left(U_{\alpha_n}^2\right) + n(n-1)\left(E\left(U_{\alpha_n}\right)\right)^2\right] \\ &= O\left(\alpha_n^2\right) + O\left(n\alpha_n^2\exp\left(-4\pi^2\sigma^2/\alpha_n^2\right)\right) \\ &= O\left((\log n)^{-1} + O\left((\log n)^{-1}\right), \quad \text{as } n \to \infty \end{aligned}$$

and

$$\begin{aligned} E\left(\sqrt{n}\left(\overline{N}_{\alpha_n} - \theta\right)\right) &= O\left(\sqrt{n}\,\alpha_n\exp\left(-2\pi^2\sigma^2/\alpha_n^2\right)\right) \\ &= O(\log n)^{-1/2}\big), \quad \text{as } n \to \infty. \end{aligned}$$

□

THEOREM 7. *If* $\limsup_{n\to\infty}\alpha_n(\log n)^{1/2} =: \gamma < 2\pi\sigma$,

$$\sqrt{n}\left(S_{\alpha_n}^2 - \frac{\alpha_n^2}{12} - \sigma^2\right) \to_D N(0, 2\sigma^4), \qquad n \to \infty. \tag{3.9}$$

The sequences of the expected values and of the variances converge to 0 *and* $2\sigma^4$, *respectively.*

PROOF. We have

$$\begin{aligned} \sqrt{n}\left(S_{\alpha_n}^2 - \frac{\alpha_n^2}{12} - \sigma^2\right) &= \sqrt{n}\left[\frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2 - \sigma^2\right] \\ &\quad + \sqrt{n}\left[\frac{1}{n-1}\sum_{i=1}^n \left(U_{\alpha_n}(X_i) - \overline{U}_{\alpha_n}\right)^2 - \frac{\alpha_n^2}{12}\right] \\ &\quad - \frac{2\sqrt{n}}{n-1}\sum_{i=1}^n (X_i - \overline{X})\left(U_{\alpha_n}(X_i) - \overline{U}_{\alpha_n}\right). \end{aligned} \tag{3.10}$$

The first term at the right-hand side converges in distribution to $N(0, 2\sigma^4)$, and we shall now show that the other terms tend to 0 in probability. Let us first consider the third sum. Using Chebychev's and Hölder's inequality, for fixed $\varepsilon > 0$, we obtain

$$P\left(\frac{2\sqrt{n}}{n-1}\left|\sum_{i=1}^{n} X_i U_{\alpha_n}(X_i) - n\overline{X}\overline{U}_{\alpha_n}\right| \geq \varepsilon\right)$$

$$\leq P\left(\frac{2\sqrt{n}}{n-1}\left|\sum_{i=1}^{n} X_i U_{\alpha_n}(X_i)\right| \geq \frac{\varepsilon}{2}\right) + P\left(\frac{2\sqrt{n}}{n-1}|n\overline{X}\overline{U}_{\alpha_n}| \geq \frac{\varepsilon}{2}\right)$$

$$= O\left(\frac{1}{n}\right)E\left(\left[\sum_{i=1}^{n} X_i U_{\alpha_n}(X_i)\right]^2\right) + O(n)E\left(\overline{X}^2\overline{U}_{\alpha_n}^2\right)$$

$$= O\left(\frac{1}{n}\right)\left[nE\left(X^2U_{\alpha_n}^2\right) + n(n-1)\left(E\left(XU_{\alpha_n}\right)\right)^2\right]$$

$$+ O\left(\frac{1}{n^3}\right)E\left(\sum_{i,j,k,l=1}^{n} X_i X_j U_{\alpha_n}(X_k) U_{\alpha_n}(X_l)\right)$$

$$= O(1)\left[E(X^4)E\left(U_{\alpha_n}^4\right)\right]^{1/2} + O\left(n\left(E\left(XU_{\alpha_n}\right)\right)^2\right)$$

$$+ O\left(\frac{1}{n^2}\right)E\left(X^2U_{\alpha_n}^2\right) + O\left(\frac{1}{n}\right)\left[E\left(X^2U_{\alpha_n}\right)E\left(U_{\alpha_n}\right) + E\left(XU_{\alpha_n}^2\right)E(X)\right]$$

$$+ O(1)\left[E(X^2)\left(E\left(U_{\alpha_n}\right)\right)^2 + E\left(XU_{\alpha_n}\right)E(X)E\left(U_{\alpha_n}\right)\right.$$

$$\left. + (E(X))^2\left(E\left(U_{\alpha_n}^2\right)\right)\right]$$

$$+ O(n)(E(X))^2\left(E\left(U_{\alpha_n}\right)\right)^2$$

$$= O\left((\log n)^{-1}\right).$$

Here we have applied the relations

$$E\left(U_{\alpha_n}\right) = O\left(\alpha_n \exp\left(-2\pi^2\sigma^2/\alpha_n^2\right)\right) = O\left((\log n)^{-1/2} n^{-(1/2)-\delta}\right),$$

$$E\left(U_{\alpha_n}^2\right) = O\left(\alpha_n^2\right) = O\left((\log n)^{-1}\right),$$

$$E\left(U_{\alpha_n}^4\right) = O\left(\alpha_n^4\right) = O\left((\log n)^{-2}\right),$$

$$E\left(XU_{\alpha_n}\right) = O\left(\exp\left(-2\pi^2\sigma^2/\alpha_n^2\right) = O\left(n^{-(1/2)-\delta}\right)\right.$$

where $\delta = (4\pi^2\sigma^2 - \gamma^2)/2\gamma^2 > 0$.

We now turn to the middle term at the right-hand side of (3.10). The following computation will finish the proof of (3.9):

$$P\left(n^{-1/2}\left|\sum_{i=1}^{n} U_{\alpha_n}(X_i)^2 - n\overline{U}_{\alpha_n}^2 - \frac{n\alpha_n^2}{12}\right| \geq \varepsilon\right)$$

$$\leq P\left(\left|n^{-1}\sum_{i=1}^{n}\left(U_{\alpha_n}(X_i)^2 - E\left(U_{\alpha_n}^2\right)\right)\right| \geq \frac{\varepsilon}{2\sqrt{n}} - \left|E\left(U_{\alpha_n}^2\right) - \frac{\alpha_n^2}{12}\right|\right) + P\left(\overline{U}_{\alpha_n}^2 \geq \varepsilon/2\sqrt{n}\right)$$

$$\leq \left[\frac{\varepsilon\sqrt{n}}{2} - n\left|E\left(U_{\alpha_n}^2\right) - \frac{\alpha_n^2}{12}\right|\right]^{-2} \operatorname{var}\left(\sum_{i=1}^{n} U_{\alpha_n}(X_i)^2\right) + O(n)E\left(\overline{U}_{\alpha_n}^2\right)$$

$$= O(1)E\left(U_{\alpha_n}^4\right) + O(n)\left[n^{-1}E\left(U_{\alpha_n}^2\right) + O(1)\left(E\left(U_{\alpha_n}\right)\right)^2\right]$$

$$= O\left((\log n)^{-2}\right) + O\left((\log n)^{-1}\right) + O\left((\log n)^{-1}n^{-2\delta}\right).$$

The assertions about the expected value and the variance are similarly proved. □

REMARK. If α_n tends sufficiently fast to 0, as $n \to \infty$, Theorems 6 and 7 combined with the Cramér–Rao inequality tell us that $\overline{N}_{\alpha_n}$ and $S_{\alpha_n}^2 - (\alpha_n^2/12)$ are asymptotically normal estimators of θ and σ^2 with asymptotically minimal variance. If σ is known, one can take $\alpha_n = 2\pi\sigma(\log n)^{-1/2}$ for this optimality property to hold for $\overline{N}_{\alpha_n}$. If σ is unknown, one may choose $\alpha_n = o((\log n)^{-1/2})$ for $\overline{N}_{\alpha_n}$ as well as for $S_{\alpha_n}^2$.

4. The effect of truncation. In most applications one does not divide the whole real axis into intervals of equal length; often this partition is only carried out for a "central region," say the interval $[-K, K]$, whereas the two unbounded intervals $(-\infty, K)$ and (K, ∞) are not subdivided. If we neglect the observations not falling into $[-K, K]$ (for which it is not so clear then how to take them into consideration for $\overline{N}_\alpha$ and S_α^2), we arrive at the modified estimators

$$\overline{N}_\alpha(K) := \frac{1}{n} \sum_{i:\,|N_\alpha(X_i)| \leq K} N_\alpha(X_i), \tag{4.1}$$

$$S_\alpha^2(K) := \frac{1}{n-1} \sum_{i:\,|N_\alpha(X_i)| \leq K} \left[N_\alpha(X_i) - \overline{N}_\alpha(K)\right]^2. \tag{4.2}$$

We shall show that $\overline{N}_\alpha$ and $\overline{N}_\alpha(K)$ as well as S_α^2 and $S_\alpha^2(K)$ are asymptotically equivalent, if $n \to \infty$, $K \to \infty$, and (in the second case) $\alpha \to 0$ in a properly connected manner. Thus neglecting the "outliers" asymptotically has no effect for our estimation procedure.

Again we shall first consider the "smooth" situation of Section 2 and then the normal case.

Let the density p be in $C^{2k}(\mathbb{R})$ and $p^{(i)}(u) = O(|u|^{-\beta})$ for some $\beta > 2$, where $i = 0, 1, \dots, 2k$, for some $k \in \mathbb{N}$. Suppose that $K_n \geq \alpha_n$ for all $n \in \mathbb{N}$.

THEOREM 8. *Let*
(a) $\lim_{n\to\infty} n^{-1/2} K_n^{\beta-2} = \infty$, *if* α_n *remains bounded;*
(b) *if* $\alpha_n \to \infty$, *suppose* $\lim_{n\to\infty} \alpha_n^{-2k} n^{-1/2} K_n^{\beta-2} = \infty$.
Then $\overline{N}_{\alpha_n}$ *and* $\overline{N}_{\alpha_n}(K_n)$ *are asymptotically equivalent in the sense that*

$$\sqrt{n}\left(\overline{N}_{\alpha_n} - \overline{N}_{\alpha_n}(K_n)\right) \to_P 0, \qquad n \to \infty. \tag{4.3}$$

PROOF. Again let $F(t) := t\int_{t-(\alpha/2)}^{t+(\alpha/2)} p(u)\,du$. By Euler's summation formula,

$$\begin{aligned}\sum_{n=m}^{N} F(\alpha n) = {}& \int_m^N F(\alpha x)\,dx + \frac{1}{2}(F(\alpha N) - F(\alpha m)) \\ & + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!}\alpha^{2i-1}\left(F^{(2i-1)}(\alpha N) - F^{(2i-1)}(\alpha m)\right) \\ & + \alpha^{2k+1}\int_m^N H_{2k+1}(x) F^{(2k+1)}(\alpha x)\,dx.\end{aligned} \tag{4.4}$$

First let $N \to \infty$ and then let $m = m_n = [K_n/\alpha_n] + 1$. It is easily seen that the last term is $O(\alpha_n^{2k} K_n^{2-\beta})$. The two middle terms at the right-hand side are of order $O(\alpha_n K_n^{1-\beta})$. Further,

$$\begin{aligned}\int_m^\infty F(\alpha x)\,dx &= \alpha\int_{-\infty}^{\infty}\left[\int_{u/\alpha-1/2}^{u/\alpha+1/2} x 1_{(m,\infty)}(x)\,dx\right] p(u)\,du \\ &= \int_{m\alpha+(\alpha/2)}^{\infty} up(u)\,du + \alpha\int_{m\alpha-(\alpha/2)}^{m\alpha+(\alpha/2)} p(u)\frac{1}{2}\left(\left(\frac{u}{\alpha}+\frac{1}{2}\right)^2 - m^2\right)du \\ &\leq \int_{m\alpha-(\alpha/2)}^{\infty} up(u)\,du \\ &= O\left(K_n^{2-\beta}\right).\end{aligned}$$

Next we observe that

$$E\left(|\overline{N}_\alpha - \overline{N}_\alpha(K)|\right) \leq E\left(N_\alpha 1_{\{N_\alpha > K\}}\right) + E\left(N_\alpha 1_{\{N_\alpha < -K\}}\right). \tag{4.5}$$

Now the above calculations show that

$$\begin{aligned}n^{1/2} E\left(N_{\alpha_n} 1_{\{N_{\alpha_n} > K_n\}}\right) &= O\left(n^{1/2}\alpha_n^{2k} K_n^{2-\beta}\right) + O\left(n^{1/2}\alpha_n K_n^{1-\beta}\right) \\ &\qquad + O\left(n^{1/2} K_n^{2-\beta}\right) \\ &= o(1), \quad \text{as } n \to \infty.\end{aligned} \tag{4.6}$$

A similar result holds for $E(N_{\alpha_n} 1_{\{N_{\alpha_n} < -K_n\}})$. (4.3) now is an easy consequence of the Markov inequality. □

Hence if α_n remains bounded, it is sufficient for (4.3) that $K_n/n^{1/2(\beta-2)} \to \infty$. If is even possible that $\overline{N}_{\alpha_n}$ and $\overline{N}_{\alpha_n}(K_n)$ are asymptotically equivalent in the case that α_n tends to infinity: For example, if $\alpha_n = o(n^{1/2(\beta-2)})$, it suffices to choose K_n such that $\alpha_n^{2k/(\beta-2)} n^{1/2(\beta-2)} = o(K_n)$.

REMARK. If $p(u)$ and $up(u)$ are monotone decreasing on (b, ∞) for some $b > 0$, we have, for sufficiently large $m\alpha$,

$$\begin{aligned}
&\int_{m\alpha-(\alpha/2)}^{\infty} up(u)\,du - E\left(N_\alpha 1_{\{N_\alpha > K\}}\right)\\
&\quad = \sum_{n=m}^{\infty}\left[\int_{n\alpha-(\alpha/2)}^{n\alpha+(\alpha/2)} up(u)\,du - n\alpha \int_{n\alpha-(\alpha/2)}^{n\alpha+(\alpha/2)} p(u)\,du\right]\\
&\quad \le \sum_{n=m}^{\infty}\left[\alpha\left(n\alpha - \frac{\alpha}{2}\right)p\left(n\alpha - \frac{\alpha}{2}\right) - n\alpha^2 p\left(n\alpha + \frac{\alpha}{2}\right)\right]\\
&\quad = \alpha^2\left[\sum_{n=m+1}^{\infty} p\left(n\alpha - \frac{\alpha}{2}\right) + mp\left(m\alpha - \frac{\alpha}{2}\right) - \frac{1}{2}\sum_{n=m}^{\infty} p\left(n\alpha - \frac{\alpha}{2}\right)\right]\\
&\quad = \alpha^2\left[\left(m - \frac{1}{2}\right)p\left(m\alpha - \frac{\alpha}{2}\right) + \frac{1}{2}\sum_{n=m+1}^{\infty} p\left(n\alpha - \frac{\alpha}{2}\right)\right],
\end{aligned}$$

and a lower estimate is similarly derived.

Especially let $p(u) = \varphi_{\theta,\sigma}(u) = (2\pi\sigma^2)^{-1/2}\exp\{-(u-\theta)^2/2\sigma^2\}$. Then it follows from our estimate that in this case

$$E\left(N_\alpha 1_{\{N_\alpha > K\}}\right) = O\left((1+\alpha K)e^{-K^2/2\sigma^2}\right). \tag{4.7}$$

Therefore in the normal case a sufficient condition for (4.3) is given by the relation

$$\frac{K_n^2}{2\sigma^2} - \log(1+\alpha_n K_n) - \frac{1}{2}\log n \to \infty, \qquad n \to \infty. \tag{4.8}$$

The right order of magnitude for K_n can then be determined for the various cases. For instance, if α_n is bounded and σ^2 is known, K_n can be chosen to be of order $(\log n)^{1/2}$; if σ^2 is unknown, the order of $1/K_n$ may be taken as $o((\log n)^{-1/2})$.

For a comparison of S_α^2 and $S_\alpha^2(K)$ we suppose that $\alpha_n \to 0$.

THEOREM 9. *Assume that* $p \in C^{2k}(\mathbb{R})$ *and* $p^{(i)}(x) = O(|x|^{-\beta})$, *as* $|x| \to \infty$, *for some* $\beta > 3$, *where* $i = 0, 1, \ldots, 2k$. *Let*

$$\alpha_n = O(n^{-1/(2k+2)}), \qquad n \to \infty, \tag{4.9}$$

$$n^{-1/2}K_n^{\beta-3} \to \infty, \qquad n \to \infty. \tag{4.10}$$

Then

$$\sqrt{n}\left(S_{\alpha_n}^2 - S_{\alpha_n}^2(K_n)\right) \to_P 0, \qquad n \to \infty. \tag{4.11}$$

PROOF. Let $I_n(K)$ be the number of indices $i \in \{1, \dots, n\}$ for which $|N_\alpha(X_i)| \le K$. We subtract

$$S_\alpha^2(K) = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i:\,|N_\alpha(X_i)|\le K} N_\alpha(X_i)^2 - 2\overline{N}_\alpha(K)^2 + \frac{1}{n}\overline{N}_\alpha(K)^2 I_n(K)\right]$$

from

$$S_\alpha^2 = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n} N_\alpha(X_i)^2 - \overline{N}_\alpha^2\right].$$

The result is

$$S_\alpha^2 - S_\alpha^2(K) = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i:\,|N_\alpha(X_i)|> K} N_\alpha(X_i)^2 - (\overline{N}_\alpha + \overline{N}_\alpha(K))(\overline{N}_\alpha - \overline{N}_\alpha(K)) + \overline{N}_\alpha(K)^2((n - I_n(K))/n).\right] \tag{4.12}$$

Theorem 8 and Theorem 4 yield

$$\sqrt{n}\left(\overline{N}_{\alpha_n} - \overline{N}_{\alpha_n}(K_n)\right) \to_P 0,$$

$$\overline{N}_{\alpha_n} + \overline{N}_{\alpha_n}(K_n) \to_P 2E(X),$$

$$\overline{N}_{\alpha_n}(K_n)^2 \to_P (E(X))^2.$$

Note further that $n - I_n(K_n)$ has a binomial distribution with parameters n and $p_n := P(|X| > \alpha_n[K_n/\alpha_n] + (\alpha_n/2)) = O(K_n^{1-\beta})$. Therefore we can conclude that

$$\sqrt{n}\left[1 - (I_n(K_n)/n)\right] \to_P 0,$$

because $\sqrt{n}\,K_n^{1-\beta} \to 0$ by (4.10). Thus it remains to evaluate the remaining sum in (4.12). As in Theorem 8, the proof can be finished by the Markov inequality, if we can show that

$$\sqrt{n}\,E\left(N_{\alpha_n}^2 1_{\{N_{\alpha_n} > K_n\}}\right) \to 0, \qquad n \to \infty. \tag{4.13}$$

This can be done again using (4.4), but with F now defined by

$$F(t) := t^2 \int_{t-(\alpha/2)}^{t+(\alpha/2)} p(u)\,du.$$

It is not difficult to verify that the right-hand side of (4.4) is then of order $O(K_n^{3-\beta}) + O(\alpha_n K_n^{2-\beta}) + O(\alpha_n^{2k} K_n^{3-\beta})$, which altogether is $O(K_n^{3-\beta})$.

Thus if $\sqrt{n}\,K_n^{3-\beta} \to 0$, (4.11) follows. This relation is however implied by (4.10). □

We now turn to the normal case. Without restriction of generality let $\theta = 0$. A close look at the proof of Theorem 9 will show that if $\alpha_n(\log n)^{1/2} \le 2\pi\sigma$, in order to establish (4.11) we have to choose K_n such that (4.8) and the following relations hold:

$$\sqrt{n}\,P(X > K_n) \to 0, \qquad \sqrt{n}\,E\Big(N_{\alpha_n}^2 1_{\{N_{\alpha_n} > K_n\}}\Big) \to 0. \tag{4.14}$$

Now, for some absolute constants M, $\tilde{M}$, $\tilde{\tilde{M}}$

$$\begin{aligned} E\Big(N_\alpha^2 1_{\{N_\alpha > K\}}\Big) &\le \sum_{n=m}^{\infty} (\alpha n)^2 \alpha \varphi_{0,\sigma}\Big(\alpha n - \frac{\alpha}{2}\Big) \\ &\le M\alpha^3 \sum_{n=m-1}^{\infty} n^2 (2\pi\sigma^2)^{-1/2} \exp(-\alpha^2 n^2/2\sigma^2) \\ &\le M\tilde{M}\alpha^3 \exp\big(-\alpha^2(m-1)^2/2\sigma^2\big)\left(m^3 + \left(\frac{2\sigma^2}{\alpha^2}\right)^{3/2}\right) \\ &\le \tilde{\tilde{M}} K^3 \exp(-K^2/2\sigma^2). \end{aligned} \tag{4.15}$$

Here we have used the upper bound

$$\sum_{n=m}^{\infty} n^2 \exp(-\gamma n^2) \le \tilde{M} \exp(-\gamma m^2)(m^3 + \gamma^{-3/2}), \qquad m \in \mathbb{N}, \gamma > 0. \tag{4.16}$$

Now choose K_n such that

$$K_n^2 - 6\sigma^2 \log K_n - \sigma^2 \log n \to \infty, \qquad n \to \infty. \tag{4.17}$$

In this case (4.8) is satisfied and

$$\begin{aligned} \sqrt{n}\,P(X > K_n) &= \sqrt{n}\,(2\pi\sigma^2)^{-1/2} \int_{K_n}^{\infty} \exp(-x^2/2\sigma^2)\,dx \\ &= O\big(\sqrt{n}\,K_n^{-1} \exp(-K_n^2/2\sigma^2)\big) \to 0, \qquad n \to \infty. \end{aligned} \tag{4.18}$$

By (4.15), the second relation in (4.14) is also valid.

Thus in the normal case we have $n^{1/2}(S_{\alpha_n}^2 - S_{\alpha_n}^2(K_n)) \to_P 0$, if $\alpha_n \le 2\sigma(\log n)^{-1/2}$ and K_n satisfies (4.17).

5. Maximum likelihood estimation. Let $X_1, \dots, X_n$ be independent and identically distributed with real density $p_\theta(x)$ for some $\theta \in \Theta$, where $\Theta \subset \mathbb{R}$ is an open interval. Let $H_{m,n}^\alpha$ be the number of $i \in \{1, \dots, n\}$ for which $N_\alpha(X_i) = m\alpha$. In this final chapter we consider the mle $\hat{\theta}_\alpha^n$ of θ based on $(H_{m,n}^\alpha)_{m \in \mathbb{Z}}$. The usual mle of θ, which is based on $X_1, \dots, X_n$, will be denoted by $\hat{\theta}^n$. Under suitable regularity conditions we have $n^{1/2}(\hat{\theta}_\alpha^n - \theta) \to_D N(0, I_\alpha(\theta)^{-1})$ and $n^{1/2}(\hat{\theta}^n - \theta) \to_D N(0, I(\theta)^{-1})$, where $I(\theta)$ and $I_\alpha(\theta)$ are the respective Fisher information numbers: $I(\theta) := E_\theta([(\partial/\partial\theta)\log p_\theta(X_1)]^2)$ and, setting $g_\theta^\alpha(t) := \int_{t-(\alpha/2)}^{t+(\alpha/2)} p_\theta(u)\,du$, $I_\alpha(\theta) := E_\theta([(\partial/\partial\theta)\log g_\theta^\alpha(N_\alpha(X_1))]^2)$.

It is therefore of interest to know in which manner $I_\alpha(\theta)$ differs from $I(\theta)$. Under quite general conditions the relation $I_\alpha(\theta) \to I(\theta)$ $(\alpha \to 0)$ holds [see Bhattacharya (1976), page 1786, for a related result]. Here we shall prove an expansion

$$I_\alpha(\theta) - I(\theta) = a(\theta)\alpha^2 + O(\alpha^4), \quad \text{as } \alpha \to 0, \tag{5.1}$$

which is valid for all $\theta \in \Theta$. $a(\theta)$ is given in (5.2). A lot of technical assumptions are needed to derive (5.1):

ASSUMPTIONS.
(i) $p_\theta \in C^4(\mathbb{R})$, $p_\theta(x) > 0 \ \forall\, x \in \mathbb{R}$.
(ii) $\lim_{|x| \to \infty}(d^i/dx^i)(g_\theta^\alpha(\partial^2/\partial\theta^2)\log g_\theta^\alpha)(x) = 0$, $i = 0, 1, 3$.
(iii) $\limsup_{\alpha \to 0}\int_{-\infty}^{\infty}|(d^5/dx^5)(g_\theta^\alpha(\partial^2/\partial\theta^2)\log g_\theta^\alpha)(x)|\, dx < \infty$.
(iv) The following functions exist and are integrable:

$$\frac{\partial^2}{\partial\theta^2}p_\theta, \quad \frac{\partial^2}{\partial\theta^2}p_\theta'', \quad \left(\frac{\partial}{\partial\theta}p_\theta''\right)^2\Big/p_\theta, \quad \frac{\partial}{\partial\theta}p_\theta\frac{\partial}{\partial\theta}p_\theta''\Big/p_\theta, \quad p_\theta''\left(\frac{\partial}{\partial\theta}p_\theta\right)^2\Big/p_\theta^2,$$

$$p_\theta''\left(\frac{\partial}{\partial\theta}p_\theta''\right)^2\Big/p_\theta^2.$$

(v) Let

$$h_{\theta,i}(t) := \sup_{|t-v|<\varepsilon}\left|\frac{\partial^i}{\partial\theta^i}p_\theta^{(4)}(v)\right|.$$

Then for some $\varepsilon > 0$, $(\theta, x) \to h_{\theta,1}(x)$ and $(\theta, x) \to h_{\theta,2}(x)$ are bounded and the following functions are integrable:

$$h_{\theta,2}, \quad h_{\theta,1}^2/p_\theta, \quad h_{\theta,1}\left(\frac{\partial}{\partial\theta}p_\theta\right)\Big/p_\theta, \quad h_{\theta,1}\left(\frac{\partial}{\partial\theta}p_\theta''\right)\Big/p_\theta,$$

$$h_{\theta,0}\left(\frac{\partial}{\partial\theta}p_\theta\right)^2\Big/p_\theta^2, \quad h_{\theta,0}\left(\frac{\partial}{\partial\theta}p_\theta''\right)^2\Big/p_\theta^2.$$

(vi) For some $K > 0$ we have $[p_\theta''(x)]/p_\theta(x) \geq -K > -\infty$ and $[h_{\theta,0}(x)]/p_\theta(x) \geq -K > -\infty$ for all $x \in \mathbb{R}$.

THEOREM 10. *If Assumptions* (i)–(vi) *hold,*

$$I_\alpha(\theta) - I(\theta) = \frac{\alpha^2}{24}\int_{-\infty}^{\infty}\left(\frac{\partial^2}{\partial\theta^2}p_\theta''(x) - 2\frac{\frac{\partial}{\partial\theta}p_\theta(x)\frac{\partial}{\partial\theta}p_\theta''(x)}{p_\theta(x)} + \frac{p_\theta''(x)}{p_\theta(x)^2}\left[\frac{\partial}{\partial\theta}p_\theta(x)\right]^2\right)dx + O(\alpha^4), \tag{5.2}$$

as $\alpha \to 0$.

PROOF. By Taylor expansion of p_θ around αx we obtain

$$g_\theta^\alpha(\alpha x) = \int_{\alpha x-(\alpha/2)}^{\alpha x+(\alpha/2)} \Bigg[p_\theta(\alpha x) + p_\theta'(\alpha x)(u-\alpha x) + p_\theta''(\alpha x)\frac{(u-\alpha x)^2}{2}$$

(5.3)
$$+p_\theta'''(\alpha x)\frac{(u-\alpha x)^3}{6} + \frac{1}{24}\int_{\alpha x}^{u}(u-v)^4 p_\theta^{(4)}(v)\,dv\Bigg]\,du$$

$$= \alpha p_\theta(\alpha x) + \frac{\alpha^3}{24}p_\theta''(\alpha x) + R_\theta^\alpha(\alpha x),$$

where we have set

$$R_\theta^\alpha(t) := \frac{1}{24}\int_{t-(\alpha/2)}^{t+(\alpha/2)}\left[\int_t^u (u-v)^4 p_\theta^{(4)}(v)\,dv\right]du.$$

By Euler's summation formula and assumptions (ii) and (iii) we get

$$E_\theta\left(\frac{\partial^2}{\partial\theta^2}[\log g_\theta^\alpha(N_\alpha(X_1))]\right)$$

$$= \sum_{n=-\infty}^{\infty} g_\theta^\alpha(\alpha n)\frac{\partial^2}{\partial\theta^2}[\log g_\theta^\alpha(\alpha n)]$$

$$= \int_{-\infty}^{\infty} g_\theta^\alpha(\alpha x)\frac{\partial^2}{\partial\theta^2}[\log g_\theta^\alpha(\alpha x)]\,dx$$

$$+ \alpha^5\int_{-\infty}^{\infty}\frac{d^5}{dx^5}\left[g_\theta^\alpha\frac{\partial^2}{\partial\theta^2}\log g_\theta^\alpha\right](\alpha x)H_5(x)\,dx$$

(5.4)
$$= \int_{-\infty}^{\infty}\left[\left(\frac{\partial^2}{\partial\theta^2}g_\theta^\alpha\right)(\alpha x) - \frac{\left[\left(\frac{\partial}{\partial\theta}g_\theta^\alpha\right)(\alpha x)\right]^2}{g_\theta^\alpha(\alpha x)}\right]dx + O(\alpha^4)$$

$$= \int_{-\infty}^{\infty}\frac{\partial^2}{\partial\theta^2}p_\theta(\alpha x)\,dx + \int_{-\infty}^{\infty}\frac{\alpha^3}{24}\frac{\partial^2}{\partial\theta^2}p_\theta''(\alpha x)\,dx$$

$$+ \int_{-\infty}^{\infty}\frac{\partial^2}{\partial\theta^2}R_\theta^\alpha(\alpha x)\,dx - \int_{-\infty}^{\infty}\frac{\left[\left(\frac{\partial}{\partial\theta}g_\theta^\alpha\right)(\alpha x)\right]^2}{\alpha p_\theta(\alpha x)}\,dx$$

$$- \int_{-\infty}^{\infty}\left[\left(\frac{\partial}{\partial\theta}g_\theta^\alpha\right)(\alpha x)\right]^2\left[\frac{1}{g_\theta^\alpha(\alpha x)} - \frac{1}{\alpha p_\theta(\alpha x)}\right]dx + O(\alpha^4).$$

The first two integrals at the right-hand side of (5.4) are equal to

$$\int_{-\infty}^{\infty}\frac{\partial^2}{\partial\theta^2}p_\theta(x)\,dx + \frac{\alpha^2}{24}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial\theta^2}p_\theta''(x)\,dx.$$

Further we have, by Assumption (v),

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial\theta^2} R_\theta^\alpha(\alpha x)\, dx = O(\alpha^5), \quad \text{as } \alpha \to 0.$$

Note that

$$\left(\frac{\partial}{\partial\theta} g_\theta^\alpha\right)^2 = \alpha^2\left(\frac{\partial}{\partial\theta} p_\theta\right)^2 + \frac{\alpha^6}{24^2}\left(\frac{\partial}{\partial\theta} p_\theta''\right)^2 + \frac{\alpha^4}{12}\frac{\partial}{\partial\theta} p_\theta \frac{\partial}{\partial\theta} p_\theta''$$
$$+ \left(\frac{\partial}{\partial\theta} R_\theta^\alpha\right)^2 + 2\left(\alpha\frac{\partial}{\partial\theta} p_\theta + \frac{\alpha^3}{24}\frac{\partial}{\partial\theta} p_\theta''\right)\frac{\partial}{\partial\theta} R_\theta^\alpha$$

and

$$\frac{1}{\alpha p_\theta} - \frac{1}{g_\theta^\alpha} = \frac{1}{\alpha p_\theta}\left(\frac{\alpha^2}{24}\frac{p_\theta''}{p_\theta} + \frac{R_\theta^\alpha}{\alpha p_\theta}\right)\Bigg/\left(1 + \frac{\alpha^2}{24}\frac{p_\theta''}{p_\theta} + \frac{R_\theta^\alpha}{\alpha p_\theta}\right),$$

the omitted arguments all being αx. By Assumptions (iv) and (v), the fourth integral at the right-hand side of (5.4) can thus be written as

$$-\int_{-\infty}^{\infty} \frac{\left[\frac{\partial}{\partial\theta} p_\theta(x)\right]^2}{p_\theta(x)}\, dx - \frac{\alpha^2}{12}\int_{-\infty}^{\infty} \frac{\left[\frac{\partial}{\partial\theta} p_\theta(x)\right]\left[\frac{\partial}{\partial\theta} p_\theta''(x)\right]}{p_\theta(x)}\, dx + O(\alpha^4).$$

Finally, the Assumptions (iv), (v), and (vi) yield, for the last integral in (5.4), the relation

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta} g_\theta^\alpha(\alpha x)\right]^2\left[\frac{1}{\alpha p_\theta(\alpha x)} - \frac{1}{g_\theta^\alpha(\alpha x)}\right] dx$$
$$= \frac{\alpha^2}{24}\int_{-\infty}^{\infty} \frac{p_\theta''(x)}{p_\theta(x)^2}\left[\frac{\partial}{\partial\theta} p_\theta(x)\right]^2 dx + O(\alpha^4), \quad \text{as } \alpha \to 0.$$

Collecting terms we arrive at (5.2). □

Due to the computational complexity of the mle equation a procedure often used in practice is to take the mle $\hat{\theta}^n$, but based on the measured values $N_\alpha(X_1), \dots, N_\alpha(X_n)$, instead of $\hat{\theta}_\alpha^n$. For instance, if θ is the mean (resp. variance) of a normal population, this leads to the estimator $\overline{N}_\alpha$ (resp. $(n-1)n^{-1}S_\alpha^2$), whereas it takes considerable effort to calculate $\hat{\theta}_\alpha^n$ approximatively.

For $x \in \mathbb{R}^n$ let $\{x\}_\alpha := (N_\alpha(x_1), \dots, N_\alpha(x_n))$, $p_{\theta,n}(x) := \prod_{i=1}^n p_\theta(x_i)$, $g_{\theta,n}^\alpha(x) := \prod_{i=1}^n g_\theta^\alpha(x_i)$.

Let $\hat{\theta}_\alpha^n(\{x\}_\alpha)$ resp. $\hat{\theta}^n(\{x\}_\alpha)$ be a mle of θ from $g_{\theta,n}(\{x\}_\alpha)$ resp. $p_{\theta,n}(\{x\}_\alpha)$, i.e., $\hat{\theta}_\alpha^n(\{x\}_\alpha)$ resp. $\hat{\theta}^n(\{x\}_\alpha)$ is a value of θ maximizing $\theta \to g_{\theta,n}(\{x\}_\alpha)$ resp. $\theta \to p_{\theta,n}(\{x\}_\alpha)$. First we shall give conditions under which $\hat{\theta}_\alpha^n(\{x\}_\alpha) - \hat{\theta}^n(\{x\}_\alpha) \to 0$, as $\alpha \to 0$. Our final theorem will then sharpen this result to the form

$$\hat{\theta}_\alpha^n(\{x\}_\alpha) - \hat{\theta}^n(\{x\}_\alpha) = \alpha^2 h(\{x\}_\alpha) + o(\alpha^2), \quad \text{as } \alpha \to 0,$$

where h is given in (5.9).

LEMMA. *Let $x \in \mathbb{R}^n$ be fixed. Suppose that $(\theta, y) \to p_\theta(y)$ is uniformly continuous on $\Theta \times U$, where U is some neighbourhood of x, and that $\lim_{\theta \to a_i} p_{\theta, n}(x) = 0$, $i = 1,2$, where $\Theta = (a_1, a_2)$. Suppose that the mle $\hat{\theta}^n(x)$ of x from $p_{\theta, n}(x)$ is uniquely determined. Then*

$$\lim_{\alpha \to 0} \hat{\theta}^n_\alpha(\{x\}_\alpha) = \lim_{\alpha \to 0} \hat{\theta}^n(\{x\}_\alpha) = \hat{\theta}^n(x). \tag{5.5}$$

PROOF. If $\hat{\theta}^n(\{x\}_{\alpha_k}) \to \theta_1 \in \Theta$ for some sequence $\alpha_k \to 0$, we get for all $\theta \in \Theta$ by the uniform continuity assumption

$$p_{\theta, n}(x) \leftarrow p_{\theta, n}(\{x\}_{\alpha_k}) \le p_{\hat{\theta}^n(\{x\}_{\alpha_k}), n}(\{x\}_{\alpha_k}) \to p_{\theta_1, n}(x), \qquad \alpha_k \to 0.$$

Since $\hat{\theta}^n(x)$ is unique, it follows that $\hat{\theta}^n(x) = \theta_1$. If $\hat{\theta}^n(\{x\}_{\alpha_k}) \to a_i$, $p_{\hat{\theta}^n(\{x\}_{\alpha_k}), n}(\{x\}_{\alpha_k}) \to 0$, but $p_{\hat{\theta}^n(x), n}(\{x\}_{\alpha_k}) \to p_{\hat{\theta}^n(x), n}(x) > 0$, which contradicts the definition of $\hat{\theta}^n(\{x\}_{\alpha_k})$ for large k. Thus the second equation in (5.5) is proved.

Further for all $\varepsilon > 0$ there is a $\alpha_\varepsilon > 0$ such that for all $\alpha \in (0, \alpha_\varepsilon)$ and for all $\theta \in \Theta$

$$p_{\theta, n}(\{x\}_\alpha) - \varepsilon \le \alpha^{-1} g^\alpha_{\theta, n}(\{x\}_\alpha) \le p_{\theta, n}(\{x\}_\alpha) + \varepsilon. \tag{5.6}$$

Suppose $\hat{\theta}^n_{\alpha_k}(\{x\}_{\alpha_k}) \to \theta_0 \in \Theta$ for some sequence $\alpha_k \to 0$. Then clearly

$$\alpha_k^{-1} g^{\alpha_k}_{\hat{\theta}^n_{\alpha_k}(\{x\}_{\alpha_k}), n}(\{x\}_{\alpha_k}) \to p_{\theta_0, n}(x) \tag{5.7}$$

so that by (5.6)

$$p_{\theta_0, n}(x) \ge p_{\theta, n}(x) - \varepsilon, \qquad \forall\, \theta \in \Theta, \forall\, \varepsilon > 0.$$

The uniqueness of $\hat{\theta}^n(x)$ implies that $\theta_0 = \hat{\theta}^n(x)$. Further note that convergence of $\hat{\theta}^n_{\alpha_k}(\{x\}_{\alpha_k})$ to a_1 or a_2 is impossible, because in this case

$$\lim_{k \to \infty} \alpha_k^{-1} g^{\alpha_k}_{\hat{\theta}^n_{\alpha_k}(\{x\}_{\alpha_k}), n}(\{x\}_{\alpha_k}) = 0,$$

but

$$\lim_{k \to \infty} \alpha_k^{-1} g^{\alpha_k}_{\hat{\theta}^n(x), n}(\{x\}_{\alpha_k}) = p_{\hat{\theta}^n(x), n}(x) > 0$$

yielding a contradiction for large k. The lemma is proved. □

Now fix $x \in \mathbb{R}^n$. Assume that $\hat{\theta}^n(\{x\}_\alpha)$ and $\hat{\theta}^n_\alpha(\{x\}_\alpha)$ are roots of the mle equations

$$\frac{\partial}{\partial \theta} \log p_{\theta, n}(\{x\}_\alpha) = 0, \qquad \frac{\partial}{\partial \theta} \log g^\alpha_{\theta, n}(\{x\}_\alpha) = 0 \tag{5.8}$$

and that $(\partial^2/\partial\theta^2) \log p_{\theta, n}(\{x\}_\alpha)|_{\theta = \hat{\theta}^n(\{x\}_\alpha)} \neq 0$. Also (5.5) has to be satisfied. Further we suppose that there is an open neighbourhood $U = U_1 \times \cdots \times U_n$ of x such that $p_{\theta, n}(z) > 0$ for all $(\theta, z) \in \Theta \times U$, $p_{\theta, n} \in C^4(U)$, and $p^{(i)}(t) \in C^3(\Theta)$ for all $t \in \bigcup_{i=1}^n U_i$ and $i = 0, \ldots, 4$.

THEOREM 11. *Under the above conditions we have for $\alpha \to 0$*

$$\hat{\theta}^n(\{x\}_\alpha) - \hat{\theta}^n_\alpha(\{x\}_\alpha) \tag{5.9}$$
$$= \frac{\alpha^2}{24}\left[\left[\frac{\partial^2}{\partial\theta^2}\log p_{\theta,n}(\{x\}_\alpha)\right]^{-1}\sum_{i=1}^{n}\frac{(\partial/\partial\theta)p''_\theta(N_\alpha(x_i))}{p_\theta(N_\alpha(x_i))}\right]\Bigg|_{\theta=\hat{\theta}^n(\{x\}_\alpha)}$$
$$+o(\alpha^2).$$

PROOF. There are open sets $V = V_1 \times \cdots \times V_n$ and $W \subset \Theta$ such that $x \in V \subset U$ and for all $t \in \bigcup_{i=1}^n V_i$ and $\theta \in W$

$$\frac{\partial}{\partial\theta}\log g_\theta^\alpha(t) = \frac{\partial}{\partial\theta}\log\left(\alpha p_\theta(t) + \frac{\alpha^3}{24}p''_\theta(t) + R_\theta^\alpha(t)\right)$$
$$= \frac{\partial}{\partial\theta}\log p_\theta(t) + \frac{\alpha^2}{24}\frac{(\partial/\partial\theta)p''_\theta(t)}{p_\theta(t)} + O(\alpha^4), \tag{5.10}$$

$$\frac{\partial^2}{\partial\theta^2}\log g_\theta^\alpha(t) = (1 + O(\alpha^2))\frac{\partial^2}{\partial\theta^2}\log p_\theta(t) \tag{5.11}$$

[recall (5.4)], where the both O-terms hold uniformly with respect to (t,θ). We write $\hat{\theta}_\alpha$ for $\hat{\theta}^n_\alpha(\{x\}_\alpha)$ and $\hat{\hat{\theta}}_\alpha$ for $\hat{\theta}^n(\{x\}_\alpha)$. Taylor expansion shows that for some $\tilde{\theta}$ between $\hat{\theta}_\alpha$ and $\hat{\hat{\theta}}_\alpha$

$$\frac{\partial}{\partial\theta}\log g^\alpha_{\theta,n}(\{x\}_\alpha)\Big|_{\theta=\hat{\hat{\theta}}_\alpha} = \left(\hat{\hat{\theta}}_\alpha - \hat{\theta}_\alpha\right)\frac{\partial^2}{\partial\theta^2}\log g^\alpha_{\theta,n}(\{x\}_\alpha)\Big|_{\theta=\hat{\hat{\theta}}_\alpha} \tag{5.12}$$
$$-\left(\hat{\hat{\theta}}_\alpha - \hat{\theta}_\alpha\right)^2\frac{\partial^3}{\partial\theta^3}\log g^\alpha_{\theta,n}(\{x\}_\alpha)\Big|_{\theta=\tilde{\theta}}.$$

Here we have used $(\partial/\partial\theta)\log g^\alpha_{\theta,n}(\{x\}_\alpha)|_{\theta=\hat{\theta}_\alpha} = 0$. Inserting (5.10) and (5.11) into (5.12) and recalling $(\partial/\partial\theta)\log p_{\theta,n}(\{x\}_\alpha)|_{\theta=\hat{\hat{\theta}}_\alpha} = 0$ yields

$$\frac{\alpha^2}{24}\sum_{i=1}^{n}\frac{(\partial/\partial\theta)p''_\theta(N_\alpha(x_i))}{p_\theta(N_\alpha(x_i))}\Bigg|_{\theta=\hat{\hat{\theta}}_\alpha} + O(\alpha^4)$$
$$= \left(\hat{\hat{\theta}}_\alpha - \hat{\theta}_\alpha\right)\left[(1 + O(\alpha^2))\frac{\partial^2}{\partial\theta^2}\log p_{\theta,n}(\{x\}_\alpha)\Big|_{\theta=\hat{\hat{\theta}}_\alpha}\right. \tag{5.13}$$
$$\left.-\left(\hat{\hat{\theta}}_\alpha - \hat{\theta}_\alpha\right)\frac{\partial^3}{\partial\theta^3}\log g^\alpha_{\theta,n}(\{x\}_\alpha)\Big|_{\theta=\tilde{\theta}}\right].$$

The expression in square brackets is clearly equal to $(\partial^2/\partial\theta^2)\log p_{\theta,n}(\{x\}_\alpha)|_{\theta=\hat{\hat{\theta}}_\alpha} + o(1)$, as $\alpha \to 0$. (5.9) is now easily derived. □

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REFERENCES

BELLMAN, R. (1961). *A Brief Introduction to Theta Functions*. Holt, Rinehart and Winston, New York.

BHATTACHARYA, P. K. (1967). Efficient estimation of a shift parameter from grouped data. *Ann. Math. Statist.* **38** 1770–1787.

CRAMÉR, H. (1974). *Mathematical Methods of Statistics*. Princeton Univ. Press, 13th printing.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, 2nd ed. Wiley, New York.

KENDALL, M. G. and STUART, A. (1969). *The Advanced Theory of Statistics* **1**, 3rd ed. Griffin, London.

KULLDORFF, G. (1957). On the conditions for consistency and asymptotic efficiency of maximum likelihood estimates. *Scand. Actuar. J.* **40** 129–144.

KULLDORFF, G. (1961). *Contributions to the Theory of Estimation from Grouped and Partially Grouped Samples*, Almqvist & Wiksell, Stockholm.

MCDONALD, J. B. and RANSOM, M. R. (1979). Alternative parameter estimators based upon grouped data. *Comm. Statist. A—Theory Methods* **A8**(9) 899–917.

SCHADER, M. and SCHMID, F. (1983a). Maximum likelihood estimation of μ and σ from a grouped sample of a normal population. *Discussion Papers in Statistics and Quantitative Economics* **15**. Hochschule der Bundeswehr, Hamburg.

SCHADER, M. and SCHMID, F. (1983b). Rejection probabilities in small samples when testing the hypothesis "X is normally distributed" with a χ^2-test of fit using various estimation methods for μ and σ. To appear in *Proc. 8. Symposium Op. Res., Karlsruhe.*

SCHEMPP, W. and DRESELER, B. (1980). *Einführung in die harmonische Analyse*. Teubner, Stuttgart.

STADJE, W. (1984). Wrapped distributions and measurement errors. *Metrika* **31** 303–317.

WIENER, N. (1958). *The Fourier Integral and Certain of Its Applications*. Dover, New York.

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