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COMBINING INDEPENDENT ONE-SIDED NONCENTRAL t OR NORMAL MEAN TESTS

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The admissibility or inadmissibility of procedures for combining several one-sided tests of significance into one overall test when the individual tests are based on independent normal or noncentral t variables is considered. Minimal complete classes are found, from which the following results (with some exceptions) are obtained. The likelihood ratio tests and Tippett's procedure are admissible in both cases, the inverse logistic and sum of significance levels procedures are inadmissible in both cases, and Fisher's and the inverse normal procedure are admissible in the normal case but inadmissible in the t case.

1. Introduction. The admissibility or inadmissibility of several methods for combining independent tests when the individual test statistics are noncentral χ^2 or F variables was determined in Marden (1982a) and Marden and Perlman (1982). In this paper we consider one-sided testing problems based on independent normal or noncentral t statistics. Oosterhoff (1969) has considered combining nonindependent t tests, and Oosterhoff and van Zwet (1967) have considered the normal case. Birnbaum (1955) states the minimal complete class for the normal case, which we repeat in Section 2. That for the t case is presented in Section 3.

The basic setup has $T_1, \dots, T_p$ independent, where the distribution of T_i depends on θ_i. We test

$$(1.1)\qquad H_0\colon \boldsymbol{\theta} = \mathbf{0} \quad \text{versus} \quad H_A\colon \boldsymbol{\theta} \in \Theta_A \equiv \Theta - \{\mathbf{0}\},$$

where

$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_p) \quad \text{and} \quad \Theta = \{\boldsymbol{\theta} \in \mathbb{R}^p | \theta_i \geq 0 \text{ for all } i\}.$$

One class of procedures is based on the observed significance levels p_i of the individual T_i's, i.e.,

$$(1.2)\qquad p_i = p_i(t_i) \equiv P_{\mathbf{0}}(T_i \geq t_i)$$

when $T_i = t_i$ is observed. $P_{\mathbf{0}}$ represents the null distribution of $\mathbf{T} \equiv (T_1, \dots, T_p)$.

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Procedures proposed include those with the following rejection regions:

$$-2\sum \log p_i > \chi^2_{2p,\alpha} \qquad \text{(Fisher);} \tag{1.3}$$

$$\min_i p_i < 1 - (1-\alpha)^{1/p} \qquad \text{(Tippett);} \tag{1.4}$$

$$-\sum \Phi^{-1}(p_i) > \sqrt{p}\,\Phi^{-1}(1-\alpha) \qquad \text{(inverse normal);} \tag{1.5}$$

$$-\sum \log\left[p_i/(1-p_i)\right] > b_\alpha \qquad \text{(inverse logistic);} \tag{1.6}$$

$$\sum p_i < c_\alpha \qquad \text{(sum of } p_i\text{'s).} \tag{1.7}$$

Here, Φ is the standard normal distribution function.

In the normal situation we have

$$T_i \sim N(\theta_i, 1). \tag{1.8}$$

Other tests which are appropriate in this case include the likelihood ratio test (LRT), which rejects H_0 when

$$\sum (T_i^+)^2 > d_\alpha \quad [z^+ = \max(0, z)], \tag{1.9}$$

and linear combination tests which reject H_0 when

$$\sum \gamma_i T_i > \left(\sum \gamma_i^2\right)^{1/2} \Phi^{-1}(1-\alpha) \tag{1.10}$$

for $\boldsymbol{\gamma} \in \Theta_A$. Test (1.10) is most powerful for alternatives along the ray $\boldsymbol{\theta} = a\boldsymbol{\gamma}$, $a > 0$. When $\gamma_1 = \cdots = \gamma_p$, test (1.10) becomes

$$\sum T_i > s_\alpha \quad \text{(sum test),} \tag{1.11}$$

which is equivalent to the inverse normal procedure (1.5). Oosterhoff (1969) finds the most stringent test for $p = 2$, which rejects H_0 when

$$\sum e^{rT_i} > m_\alpha, \tag{1.12}$$

where the constant r depends on α.

[Define the power envelope of a class $\mathscr{D}$ of tests to be

$$e(\boldsymbol{\theta}, \mathscr{D}) \equiv \sup_{\phi \in \mathscr{D}} E_{\boldsymbol{\theta}}(\phi) \quad \text{for } \boldsymbol{\theta} \in \Theta_A. \tag{1.13}$$

The *maximum regret* of a test $\phi \in \mathscr{D}$ *relative to* $\mathscr{D}$ is

$$\sup_{\boldsymbol{\theta} \in \Theta_A} \left[e(\boldsymbol{\theta}, \mathscr{D}) - E_{\boldsymbol{\theta}}(\phi)\right]. \tag{1.14}$$

The most stringent level α test is that with smallest maximum regret relative to the level α tests.]

In the t case we have

$$T_i \sim t_{\nu_i}(\theta_i), \tag{1.15}$$

i.e., T_i is noncentral t with ν_i degrees of freedom and noncentrality parameter θ_i. If we are given

$$\left\{Z_{ij};\ i = 1, \ldots, p,\ j = 1, \ldots, \nu_i + 1\right\}, \qquad Z_{ij} \sim N\left(\mu_i, \sigma_i^2\right), \tag{1.16}$$

then

$$T_i \equiv \sqrt{\nu_i + 1}\,\bar{Z}_i \Big/ \left(\sum \left(Z_{ij} - \bar{Z}_i \right)^2 / \nu_i \right)^{1/2}, \tag{1.17}$$

where $\bar{Z}_i \equiv \Sigma Z_{ij}/(\nu_i + 1)$, is distributed as in (1.15) for $\theta_i = \sqrt{\nu_i + 1}\,\mu_i/\sigma_i$. If we consider problem (1.1) based on (1.16), then the LRT has rejection region

$$\sum (\nu_i + 1)\log\left(1 + \nu_i (T_i^+)^2\right) > e_\alpha. \tag{1.18}$$

This test is also appropriate for testing (1.1) based on **T**. We will henceforth refer to (1.18) as the LRT, although it is not the LRT for our original problem based on **T**. Unlike in the normal case, there is no most powerful test along a ray $\boldsymbol{\theta} = a\boldsymbol{\gamma}$, $a > 0$. The locally most powerful (as $a \to 0$) test in the direction $\boldsymbol{\gamma}$ rejects H_0 when

$$\sum \gamma_i \frac{\Gamma\left(\dfrac{\nu_i}{2} + 1\right)}{\Gamma\left(\dfrac{\nu_i}{2} + \dfrac{1}{2}\right)} 2^{1/2} X_i > f_\alpha, \tag{1.19}$$

where

$$X_i = \frac{T_i}{\left(\nu_i + T_i^2\right)^{1/2}}. \tag{1.20}$$

The asymptotically most powerful (as $a \to \infty$) test in the direction $\boldsymbol{\gamma}$ has rejection region

$$\sum \gamma_i^2 (X_i^+)^2 > g_\alpha. \tag{1.21}$$

Linear combination tests, as in (1.10), may also have good power along a given ray.

The optimality properties of test (1.10) in the normal case, and tests (1.19) and (1.21) in the t case, are essentially unique, hence those tests are admissible for their respective problems. Table 1 exhibits the status of many of the other tests. These results are proved in Section 2 (for the normal case) and Section 4 (for the t case) using the appropriate minimal complete class.

In Table 2 we summarize the admissibility/inadmissibility results in the four problems (normal and noncentral χ^2, F, and t) for the omnibus tests. For simplicity of presentation we assume $p > 2$, $\alpha < \frac{1}{2}$, and the denominator degrees of freedom in the F and t cases exceed two. In all cases the LRT and Tippett's procedure (1.4) are admissible, and the inverse logistic (1.6) and sum of p_i's (1.7) procedures are inadmissible. Fisher's procedure (1.3) is admissible except in the t case and certain situations in the F case. The inverse normal procedure (1.5) is admissible only in the normal case.

REMARK 1.1. A test can also be evaluated by determining whether it is *parameter consistent*, i.e., whether $\|\boldsymbol{\theta}\| \to \infty$ automatically implies that the power approaches one [see Anderson and Perlman (1979)]. It is easy to see that

TABLE 1
Admissibility (A) or inadmissibility (I) of several tests

procedure	t case		normal case
Fisher's (1.3)	$\nu_i = 2$ for all i	A	A
	$\nu_i \neq 2$ for some i	I	
Tippett's (1.4)		A	A
Inverse normal (1.5)	$p > 2$	I	A
	$p = 2,\ \nu_1 \leq 2,\ \nu_2 \geq 2$ $(\nu_1, \nu_2) \neq (2, 2)$	I	
Inverse logistic (1.6)	$p > 2$	I	I
	$p = 2,\ \nu_1 \neq \nu_2$	I	
Sum of p_i's (1.7)	$\nu_i > 2$ for some i, $\alpha < \frac{1}{2}$	I	I
	$\nu_i = 2$ for all i	A	
LRT (1.9) or (1.8)	$\alpha \leq 1 - 2^{-p}$	A	A
Sum (1.11)	$p = 2,\ \nu_1 = \nu_2,\ \alpha = \frac{1}{2}$	A	A
	otherwise	I	

TABLE 2
Admissibility (A) or inadmissibility (I) of several tests

procedure	normal case	noncentral t case	noncentral χ^2 case	noncentral F case
LRT	A	A	A	A
Fisher's (1.3)	A	I	A	A/I*
Tippett's (1.4)	A	A	A	A
Inverse normal (1.5)	A	I	I	I
Inverse logistic (1.6)	I	I	I	I
Sum of p_i's (1.7)	I	I	I	I

$p > 2$, $\alpha < \frac{1}{2}$, denominator degrees of freedom in t and F cases exceed 2.
*I if some numerator degree of freedom is one, A otherwise.

most of the tests considered in this paper are parameter consistent. Some exceptions follow.

Suppose $c_\alpha < p - 1$ in the sum of p_i's test (1.7). Then as $\theta_1 \to \infty$, with $\theta_2, \dots, \theta_p$ fixed, the power approaches

$$P_{(\theta_2, \dots, \theta_p)}\left(\sum_{i=2} p_i \leq c_\alpha \right) < 1.$$

Since $c_\alpha < p - 1$ if and only if $\alpha < 1 - (p!)^{-1}$, the test is not parameter consistent at the usual levels. Similarly, if in one of the locally most powerful tests (1.19) in the t situation we have

$$\frac{f_\alpha - \gamma_1 \Gamma(\nu_1/2 + 1) 2^{1/2}}{\Gamma(\nu_1/2 + 1/2)} > 0,$$

the power will remain bounded away from one if $\theta_2, \dots, \theta_p$ are bounded.

TABLE 3

Noncentral t case (maximum regrets relative to tests considered in percent), $\alpha = 0.05$, $p = 2$.

procedure	Tippett's	LRT	Fisher's	sum	inverse logistic	inverse normal	sum of p_i's
ν_1, ν_2	(1.4)	(1.18)	(1.3)	(1.11)	(1.6)	(1.5)	(1.7)
1, 1	57	30	21	45	31	42	68
2, 2	39	19	11	17	25	37	68
3, 3	30	14	8	9	21	33	68
5, 5	23	10	6	12	17	28	68
10, 10	18	8	4	15	13	23	68
20, 20	16	7	4	16	11	20	68
50, 50	15	7	3	17	10	19	68
∞, ∞*	14	6	3	18	10	18	68
2, 10	27	14	19	35	30	40	68
5, 50	18	9	9	24	20	30	68
10, 20	17	8	5	18	13	23	68

* Normal case.

Normal case (maximum regrets relative to all level α tests in parentheses)

α	Tippett's	most stringent (1.12)	LRT (1.9)	Fisher's	inverse logistic	sum	sum of p_i's
0.001	29	—	8 (10.5)	7 (14.6)	13	37 (42.2)	96
0.01	20	—	7 (10.9)	5 (14.1)	12	26 (33.4)	86
0.05	14	10 (10.7)	6 (11.0)	3 (13.0)	10	18 (25.5)	68
0.10	11	— (10.8)	6 (10.8)	2 (12.1)	8	13 (21.3)	55

REMARK 1.2. We compared the powers of the tests when $p = 2$. Table 3 contains the maximum regrets (1.14) relative to the tests considered in the table. Fisher's procedure (1.3) is best in this sense when $\nu_1 = \nu_2$. In fact, when $\nu_1 = \nu_2 \geq 10$ and $\alpha = 0.05$, none of the other tests beats Fisher's by more than 4% in power. The LRT (1.9) or (1.8) is almost as good. When ν_1 and ν_2 are disparate, the LRT is somewhat better that Fisher's. The sum of p_i's test (1.7) looks very bad. Since the other tests are parameter consistent, and as $\theta_1 \to \infty$ with $\theta_2 = 0$, the power of the sum of p_i's test approaches $(2\alpha)^{1/2}$, the maximum regret of this test is $1 - (2\alpha)^{1/2}$.

The tests are ordered in Table 3 so that the farther to the left (right) the test is, the relatively more power the test has along the axes (equiangular line). Thus Tippett's procedure (1.4) is generally better that the LRT when exactly one θ_i is positive, while the LRT is better when $\theta_1 \approx \theta_2$, etc. We note that the sum test (1.11) in the normal case appears to dominate the sum of p_i's tests everywhere (by very little when $\theta_1 = \theta_2$ and by a great deal even a moderate distance from the equiangular line).

Table 3 also includes (in parentheses) the maximum regrets (1.14) relative to *all* level α tests for some of the tests in the normal case. These are taken from Oosterhoff [(1969), Table 2.5.1]. The most stringent test (1.12) minimizes this value among all level α tests. A drawback to this criterion is that it is obtained by comparing each test to all other tests, while we are looking at permutation symmetric ($\phi(x_1, x_2) = \phi(x_2, x_1)$) tests. An alternative criterion, to which the maximum regrets relative to the tests considered is an approximation, would be to find the maximum regrets relative to all permutation symmetric level α tests.

2. The normal case. Birnbaum (1955) states that the minimal complete class of tests of (1.1) based on $\mathbf{T}$ as in (1.8) consists of all tests of the form

$$\phi = \begin{cases} 1 & \text{if } \mathbf{t} \notin C, \\ 0 & \text{if } \mathbf{t} \in C, \quad \text{a.e. } [\mu], \end{cases} \tag{2.1}$$

where C is a closed, convex and nonincreasing subset of $\mathbb{R}^p$ and μ is Lebesgue measure. By nonincreasing we mean

$$\mathbf{t} \in C, \qquad \mathbf{s} \le \mathbf{t} \Rightarrow \mathbf{s} \in C \; [\mathbf{s} \le \mathbf{t} \text{ means } s_i \le t_i \text{ for all } i.] \tag{2.2}$$

The proof uses the fact that $\mathbf{T}$ has an exponential family density. See also Oosterhoff [(1969), Section 1.4] and Eaton (1970) for this result.

It is easy to see that tests (1.4), (1.5) ≡ (1.11), (1.9), and (1.12) have acceptance regions of the correct form C, hence they are admissible. Birnbaum (1954) states that Fisher's procedure (1.3) has convex acceptance region. To see this, note that

$$-\frac{\partial}{\partial t} \log p_i(t) = \frac{e^{-t^2/2}}{\int_t^\infty e^{-z^2/2}\, dz} = \left[\int_0^\infty e^{-u(u+2t)/2}\, du\right]^{-1}, \tag{2.3}$$

where we make the substitution $u = z - t$. Since $u > 0$, (2.3) is increasing in t. Thus the statistic in (1.3) is convex in $\mathbf{t}$, and is clearly increasing in $\mathbf{t}$, so that the acceptance region is convex and nonincreasing proving the test admissible.

Arthur Cohen has shown (personal communication) that the inverse logistic procedure (1.6) does not have convex acceptance region unless $p = 2$ and $\alpha = \frac{1}{2}$, in which case the test is equivalent to the sum test (1.11) with $s_{1/2} = 0$. Fix $(t_3^0, \dots, t_p^0)$ and look at the slice $\{(t_1, t_2, t_3^0, \dots, t_p^0) | (t_1, t_2) \in \mathbb{R}^2\}$. Consider the line $t_2 = \varepsilon - t_1$ for fixed ε. Using l'Hospital's Rule we find that

$$\lim_{t_1 \to \pm\infty} -\sum_{i=1}^{2} \log[p_i(t_i)/(1 - p_i(t_i))] = \begin{cases} \infty & \text{if } \varepsilon > 0, \\ 1 & \text{if } \varepsilon = 0, \\ -\infty & \text{if } \varepsilon < 0 \end{cases}. \tag{2.4}$$

Thus for any $\varepsilon > 0$, if t_1 is sufficiently close to either $\pm\infty$, then

$$\left(t_1, \varepsilon - t_1, t_3^0, \dots, t_p^0\right) \notin \text{acceptance region}$$

and

$$\left(t_1, -\varepsilon - t_1, t_3^0, \dots, t_p^0\right) \in \text{acceptance region}.$$

Hence the boundary of the acceptance region in this slice is asymptotic to the line $t_1 + t_2 = 0$ as $t_1 \to \pm\infty$. The only way for this to occur with the acceptance region convex is to have the boundary be the line $t_1 + t_2 = 0$. This cannot happen for all $(t_3^0, \dots, t_p^0)$ unless $p = 2$, hence the only time this test has a convex acceptance region is when $p = 2$ and $b_\alpha = 0$, which gives $\alpha = \frac{1}{2}$.

Finally, take $0 < c_\alpha < p - 1$ in test (1.7) so that $0 < \alpha < 1 - 1/p!$. Choose $(t_3^0, \ldots, t_p^0)$ so that

$$0 < c_\alpha - \sum_{i=3}^{p} p_i(t_i^0) = c_0 < 1. \tag{2.5}$$

The acceptance region in the slice contains $\{(t_i, t_2) | t_1 \le p_1^{-1}(c_0) \text{ or } t_2 \le p_2^{-1}(c_0)\}$, where $p_i^{-1}(c_0) \in (-\infty, \infty)$ since $0 < c_0 < 1$. Thus this acceptance region cannot be convex, showing the test inadmissible.

3. The t case: minimal complete class theorem. We find it convenient to work with $\mathbf{X} \equiv (X_1, \ldots, X_p)$, where X_i is defined in (1.20). The space of $\mathbf{X}$ is

$$\mathscr{X} \equiv \{\mathbf{x} \in \mathbb{R}^p | -1 < x_i < 1 \text{ for all } i\}. \tag{3.1}$$

Several definitions are needed before describing the minimal complete class. Let $f(x_i; \theta_i, \nu_i)$ be the density of X_i when θ_i obtains, and define

$$R(\mathbf{x}; \boldsymbol{\theta}) = \prod_{i=1}^{p} R(x_i; \theta_i, \nu_i), \tag{3.2}$$

where

$$R(z; \tau, \nu) \equiv (1 + \tau^2)^{(\nu+1)/2} \exp(\tau^2/2) f(z; \tau, \nu)/f(z; 0, \nu) \tag{3.3}$$

$$= (1 + \tau^2)^{(\nu+1)/2} c(\nu)^{-1} \int_0^\infty \exp(-a^2/2) \exp(a\tau z) a^\nu \, da \tag{3.4}$$

$$= (1 + \tau^2)^{(\nu+1)/2} \sum_{k=0}^{\infty} [c(\nu + k)/c(\nu)](\tau z)^k/k!, \tag{3.5}$$

and

$$c(\nu) = \Gamma\left(\frac{\nu + 1}{2}\right) 2^{(\nu-1)/2}. \tag{3.6}$$

By Lemma 5.1(a) we can extend the definition of $R(z; \tau, \nu)$ to values of $\tau = \infty$ by taking

$$R(z; \infty, \nu) = \begin{cases} c(\nu)^{-1} \Gamma(\nu + 1) |z|^{-(\nu+1)} & \text{if } z < 0, \\ \infty & \text{if } z \ge 0 \end{cases}. \tag{3.7}$$

We can therefore take $R(\mathbf{x}; \boldsymbol{\theta})$ to be a continuous function of $\boldsymbol{\theta}$ for $\boldsymbol{\theta}$ in

$$\overline{\Theta} \equiv \{\boldsymbol{\theta} | 0 \le \theta_i \le \infty, \ i = 1, \ldots, p\}$$

for each $\mathbf{x}$.

Define $\mathscr{W} \equiv \{\boldsymbol{\omega} \in \mathbb{R}^p | 0 \le \omega_i < 1, \ i = 1, \ldots, p\}$ and the map

$$\boldsymbol{\omega}: \mathscr{X} \to \mathscr{W}; \qquad \omega_i(\mathbf{x}) = (x_i^+)^2 \tag{3.8}$$

[see (1.9)]. Let $\mathscr{C}_\omega$ be the class of closed, convex, and nonincreasing [see (2.2)] subsets of $\mathscr{W}$.

For a set $D \subseteq \mathbb{R}^p$, let

$$J(D) = \{i | \text{there exists } \mathbf{x} \in D \text{ with } x_i > 0\} \tag{3.9}$$

and

$$\Theta(D) = \{\boldsymbol{\theta} \in \overline{\Theta} | \theta_i < \infty \text{ for } i \in J(D)\}. \tag{3.10}$$

Define Φ to be the class of tests ϕ: $\mathscr{X} \to [0, 1]$ of the form

$$\phi = \begin{cases} 1 & \text{if } \mathbf{x} \notin \boldsymbol{\omega}^{-1}(C_\omega), \\ 1 & \text{if } d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b) > c, \\ 0 & \text{otherwise} \quad \text{a.e. } [\mu], \end{cases} \tag{3.11}$$

where $C_\omega \in \mathscr{C}_\omega$, $\boldsymbol{\lambda} \in \Theta$, π_a is a finite measure on $\Theta_a \equiv \{\boldsymbol{\theta} \in \Theta_A | \Sigma\theta_i \leq 1\}$, π_b is a locally finite measure on $\overline{\Theta}_b$, the closure of $\Theta_b \equiv \{\boldsymbol{\theta} \in \Theta_A | \Sigma\theta_i > 1\}$ in $\Theta(C_\omega)$, $|c| < \infty$,

$$\begin{aligned} d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b) = & \sum \lambda_i \frac{c(\nu_i + 1)}{c(\nu_i)} x_i \\ & + \int_{\Theta_a} \frac{R(\mathbf{x}; \boldsymbol{\theta}) - 1}{\Sigma\theta_i} \pi_a(d\boldsymbol{\theta}) + \int_{\overline{\Theta}_b} R(\mathbf{x}; \boldsymbol{\theta}) \pi_b(d\boldsymbol{\theta}), \end{aligned} \tag{3.12}$$

and

$$|d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b)| < \infty \quad \text{for } \mathbf{x} \in \text{interior } \boldsymbol{\omega}^{-1}(C_\omega). \tag{3.13}$$

The main result follows.

THEOREM 3.1. *The class of tests* Φ *is minimal complete for* (1.1) *based on* $\mathbf{T}$ *as in* (1.15).

The set $\boldsymbol{\omega}^{-1}(C_\omega)$ is convex and nonincreasing for $C_\omega \in \mathscr{C}_\omega$. Using (3.4) and (3.7), we see that for each $\boldsymbol{\theta}$, $R(\mathbf{x}; \boldsymbol{\theta})$ is convex and nondecreasing in $\mathbf{x}$, so that d of (3.12) is also convex and nondecreasing. Thus the acceptance region of a test (3.11) is essentially $\boldsymbol{\omega}^{-1}(C_\omega) \cap \{\mathbf{x} | d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b) \leq c\}$, a convex and nonincreasing set. Hence we have the following useful corollary.

COROLLARY 3.2. *A test* ϕ *is admissible for problem* (1.1) *based on* $\mathbf{T}$ *as in* (1.15) *only if it is of the form*

$$\phi = \begin{cases} 1 & \text{if } \mathbf{x} \notin A, \\ 0 & \text{if } \mathbf{x} \in A \quad \text{a.e. } [\mu], \end{cases} \tag{3.14}$$

where A is closed, convex, and nonincreasing.

We turn to the proof of the theorem. We will first present some local and asymptotic properties of $R(\mathbf{x}; \boldsymbol{\theta})$, and then use these properties to show how to prove Theorem 3.1 using the proof of Theorem 2.1 in Marden (1982b). The proof of the asymptotic properties we defer to Section 5.

LOCAL PROPERTIES. Let $l_i(\mathbf{x}) \equiv (\partial/\partial\theta_i) R(\mathbf{x}; \boldsymbol{\theta})|_{\boldsymbol{\theta}=0} = [c(\nu_i + 1)/c(\nu_i)] x_i$ by (3.5). For each $\mathbf{x}$, as $\Sigma\theta_i \to 0$,

$$R(\mathbf{x}; \boldsymbol{\theta}) = 1 + \sum \theta_i l_i(\mathbf{x}) + o\left(\sum \theta_i\right), \tag{3.15}$$

and

$$\sup_{\boldsymbol{\theta} \in \Theta_a} \frac{|R(\mathbf{x}; \boldsymbol{\theta}) - 1|}{\sum \theta_i} \leq k \tag{3.16}$$

for some $K < \infty$ independent of $\mathbf{x}$.

(3.15) follows from (3.5). For (3.16), write

$$R(\mathbf{x}; \boldsymbol{\theta}) - 1 = \sum_{j=1}^{p} \left(R(x_j; \theta_j, \nu_j) - 1 \right) \prod_{i=0}^{j-1} R(x_i; \theta_i, \nu_i), \tag{3.17}$$

where $R(x_0; \theta_0, \nu_0) \equiv 1$. When $\boldsymbol{\theta} \in \Theta_a$ with $\theta_i > 0$ for all i,

$$\frac{|R(\mathbf{x}; \boldsymbol{\theta}) - 1|}{\sum \theta_i} \leq \sum_{j=1}^{p} \frac{|R(x_j; \theta_j, \nu_j) - 1|}{\theta_j} \prod_{i=0}^{j-1} R(x_i; \theta_i, \nu_i), \tag{3.18}$$

where by (3.5), since $\boldsymbol{\theta} \in \Theta_a$ implies that $\theta_j \leq 1$,

$$|R(x_j; \theta_j, \nu_j) - 1|/\theta_j \leq R(1; 1, \nu_j) - 1 < \infty$$

and

$$R(x_i; \theta_i, \nu_i) \leq R(1; 1, \nu_i) < \infty.$$

Thus the left-hand side of (3.18) is bounded by some K for all $\boldsymbol{\theta} \in \Theta_a$ with $\theta_i > 0$ for all i, hence by the same K for all $\boldsymbol{\theta} \in \Theta_a$ proving (3.16).

Asymptotic properties. Let $C \subseteq \mathscr{X}$ be such that for some sequence $\{\pi_n\}$ of proper measures on Θ_A,

(3.19)

$$C = \text{closure } C'; \qquad C' = \left\{ \mathbf{x} \in \mathscr{X} \,|\, \limsup_{n \to \infty} \int R(\mathbf{x}; \boldsymbol{\theta}) \pi_n(d\boldsymbol{\theta}) < \infty \right\}.$$

For any such C, there exists a $C_\omega \in \mathscr{C}_\omega$ with

$$C = \omega^{-1}(C_\omega). \tag{3.20}$$

For any $\mathbf{x} \notin C$ of (3.19), there exists a subsequence $\{m(n)\} \subseteq \{n\}$ (possibly depending on $\mathbf{x}$) for which

$$\lim_{m \to \infty} \int R(\mathbf{x}'; \boldsymbol{\theta}) \pi_m(d\boldsymbol{\theta}) = \infty \quad \text{whenever } \mathbf{x}' \geq \mathbf{x}. \tag{3.21}$$

For any $\mathbf{x} \in$ interior C, there exists an $\mathbf{x}' \in$ interior C such that

$$\lim_{k \to \infty} \sup_{\sigma(\theta) \geq k} \left[R(\mathbf{x}; \boldsymbol{\theta}) / R(\mathbf{x}'; \boldsymbol{\theta}) \right] = 0, \tag{3.22}$$

where

$$\sigma(\boldsymbol{\theta}) = \sigma(\boldsymbol{\theta}, C) \equiv \sum_{i \in J(C)} \theta_i^2 \tag{3.23}$$

[see (3.9)].

Finally, take $\boldsymbol{\xi} \in \Theta_A$ and $t \geq 0$. Letting

$$\alpha(\boldsymbol{\theta}) = \prod_{i=1}^{p} \left(1 + \theta_i^2\right)^{-(\nu_i + 1/2)} \tag{3.24}$$

we have that

$$\lim_{s \to \infty} \alpha(s\boldsymbol{\xi}) \exp\left(\frac{-s^2 t}{2}\right) R(\mathbf{x}; s\boldsymbol{\xi}) = \begin{cases} \infty & \text{if } \sum \xi_i^2 \omega_i(\mathbf{x}) > t, \\ 0 & \text{if } \sum \xi_i^2 \omega_i(\mathbf{x}) \leq t, \end{cases} \tag{3.25}$$

and for some $K < \infty$,

(3.26)

$$\sup_{s > 0} \alpha(s\boldsymbol{\xi}) \exp\left(\frac{-s^2 t}{2}\right) R(\mathbf{x}; s\boldsymbol{\xi}) \leq K \quad \text{for } \mathbf{x} \text{ such that } \sum \xi_i^2 \omega_i(x) \leq t.$$

PROOF OF THEOREM 3.1. Marden (1982b), which will be referred to as 82b in this proof, presents minimal complete class theorems for testing problems such as (1.1). Two cases are considered: Case A, in which $R(\mathbf{x}; \boldsymbol{\theta})$ grows polynomially in $\boldsymbol{\theta}$ as $\boldsymbol{\theta}$ approaches the perimeter of the parameter space for fixed $\mathbf{x}$; and Case B, in which the growth is exponential. Unfortunately, the present problem has elements of both cases. When $z < 0$, $R(z; \tau, v)$ grows polynomially, and when $z > 0$, it grows exponentially (see Lemma 5.1). However, the proof of Theorem 2.1 in 82b can be modified for the present Theorem 3.1 as follows. The proof is in two parts.

PART I. Using the local properties above, the first two paragraphs in Part I of the proof of Theorem 2.1 in 82b can be followed exactly. There, $V = \Theta$ and $\sigma \equiv \Sigma\theta_i$. Define C as in (3.19) with $\pi_n = \pi_{bn}$ as in the third paragraph in 82b. We want to show that

$$\phi = 1 \quad \text{a.e. } [\mu] \quad \text{for } \mathbf{x} \notin C \tag{3.27}$$

and that along some subsequence $\{m(n)\} \subseteq \{n\}$,

$$\int_{\Theta_b} R(\mathbf{x}; \boldsymbol{\theta}) \pi_{bn}(d\boldsymbol{\theta}) \to \int_{\overline{\Theta}_b} R(\mathbf{x}; \boldsymbol{\theta}) \pi_b(d\boldsymbol{\theta}) < \infty \quad \text{for } \mathbf{x} \in \text{interior } C \tag{3.28}$$

for some locally finite measure π_b on $\overline{\Theta}_b$.

From (3.20) and (3.8) it can be seen that C is nonincreasing. Hence

$$C^c = \bigcup_{\mathbf{x} \in C^c} \{\mathbf{x}' \in \mathscr{X} | \mathbf{x}' \geq \mathbf{x}\}. \tag{3.29}$$

As in the fourth paragraph in Part I, (3.21) and (3.29) can be used to show (3.27). The sets B_z in 82b should be replaced by the sets $\{\mathbf{x}' \in \mathscr{X} | \mathbf{x}' \geq \mathbf{x}\}$.

Now consider (3.28). As in paragraph five of Part I, there is a locally finite measure π_b on $\overline{\Theta}_b$ and subsequence $\{m(n)\} \subseteq \{n\}$ such that

$$\pi_{bm} \to \pi_b \text{ vaguely.}$$

Take $\mathbf{x} \in$ interior C and follow the fifth paragraph with the set $\{\boldsymbol{\theta} | 1 \leq \sigma \leq i\}$

replaced by

$$A_i \equiv \{\boldsymbol{\theta} \in \overline{\Theta} | 1 \leq \theta_j \leq \infty \text{ for } j \notin J(C), \sigma(\theta) \leq i\}$$

[see (3.23)] until (3.14) of 82b. Take $\mathbf{x}' \in$ interior C as in (3.22). With $B_i \equiv \overline{\Theta}_b - A_i$,

$$\limsup_{n\to\infty} \int_{B_i} R(\mathbf{x}; \boldsymbol{\theta})\pi_{bn}(d\boldsymbol{\theta}) \leq \sup_{\sigma(\theta)\geq i} [R(\mathbf{x}; \boldsymbol{\theta})/R(\mathbf{x}'; \boldsymbol{\theta})] \cdot \limsup_{n\to\infty} \int R(\mathbf{x}'; \boldsymbol{\theta})\pi_{bn}(d\boldsymbol{\theta}). \tag{3.30}$$

As below (3.15) in 82b, (3.30) and (3.22) prove (3.28).

To complete Part I we use the sixth paragraph in Part I of 82b and the fact that

$$\mu(\{\mathbf{x} | d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b) = c\}) = 0, \tag{3.31}$$

whenever $(\boldsymbol{\lambda}, \pi_a, \pi_b, c) \neq (\mathbf{0}, 0, 0, 0)$. If $(\boldsymbol{\lambda}, \pi_a, \pi_b) \neq (\mathbf{0}, 0, 0)$, then by (3.4) and (3.12), $d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b)$ is strictly increasing in at least one of the x_i's so that (3.31) must hold. If $(\boldsymbol{\lambda}, \pi_a, \pi_b) = (\mathbf{0}, 0, 0)$, and $c \neq 0$, then (3.31) holds trivially. Part I is finished.

PART II. We present a sequence of proper measures $\{\pi_n\}$ on Θ_A, which can be used as the sequence in (3.20) of 82b is used for Part II there. A proof similar to the one in 82b [using (3.16) and (3.26)] will complete the proof of our Theorem 3.1.

We are given ϕ as in (3.11), with its C_ω, $\boldsymbol{\lambda}$, π_a, and π_b. Since C_ω is convex and nonincreasing, there exist countable sets $\{\boldsymbol{\xi}^{(i)}\} \subseteq \Theta_A$ and $\{t^{(i)}\} \subseteq [0, \infty)$ for $i \in I$ such that

$$C_\omega = \bigcap_{i \in I} \left\{\boldsymbol{\omega} \in \mathscr{W} | \sum \left(\xi_j^{(i)}\right)^2 \omega_j(\mathbf{x}) \leq t^{(i)}\right\}. \tag{3.32}$$

Define the function u_n of $\boldsymbol{\theta}$ by

$$u_n(\boldsymbol{\theta}) = \left(\frac{\theta_1 n}{n + \theta_1 + 1}, \ldots, \frac{\theta_p n}{n + \theta_p + 1}\right),$$

and let $\Theta_{bn} = u_n(\overline{\Theta}_b)$. Note that $\Theta_{bn} \subseteq \Theta_A$. Let π_{bn} be the finite measure on Θ_{bn} defined for $B \subseteq \Theta_{bn}$ by

$$\pi_{bn}(B) = \pi_b(u_n^{-1}(B)),$$

so that for any integrable function f,

$$\int_{\Theta_{bn}} f(\theta)\pi_{bn}(d\theta) = \int_{\overline{\Theta}_b} f(u_n(\boldsymbol{\theta}))\pi_b(d\boldsymbol{\theta}).$$

Let $\rho_n(d\boldsymbol{\theta})$ be the finite measure on Θ_A:

$$\rho_n(d\boldsymbol{\theta}) = \sum_{i \in I} 2^{-i}\alpha(n\boldsymbol{\xi}^{(i)})\exp(-n^2 t^{(i)})\delta(d\boldsymbol{\theta}; n\boldsymbol{\xi}^{(i)})$$

where $\delta(d\boldsymbol{\theta}; \boldsymbol{\tau})$ represents the measure placing point mass 1 at $\boldsymbol{\theta} = \boldsymbol{\tau}$. The measure π_n is defined by

$$\pi_n(d\boldsymbol{\theta}) = h(\boldsymbol{\theta})\Big[n\delta(d\boldsymbol{\theta}; \boldsymbol{\lambda}/n) + \left(\sum\theta_i\right)^{-1}\pi_a(d\boldsymbol{\theta})I_{\{1/n<\Sigma\theta_i\le 1\}} + \pi_{bn}(d\boldsymbol{\theta})I_{\Theta_{bn}} + \rho(d\boldsymbol{\theta})\Big],$$

where $h(\boldsymbol{\theta}) = \prod(1+\theta_i^2)^{(\nu_i+1)/2}\exp(\Sigma\theta_i^2/2)$.

4. The *t* case: admissibility or inadmissibility of specific tests. Using Theorem 3.1, we have that any test of the form

$$\phi = \begin{cases} 1 & \text{if } \mathbf{x} \notin \boldsymbol{\omega}^{-1}(C_\omega), \\ 0 & \text{if } \mathbf{x} \in \boldsymbol{\omega}^{-1}(C_\omega), \end{cases} \tag{4.1}$$

where $C_\omega \in \mathscr{C}_\omega$ is admissible. When $\alpha \le 1 - 2^{-p}$, Tippett's procedure (1.4) and the LRT (1.18) are of this form:

$$\text{Tippett:}\quad C_\omega = \{\boldsymbol{\omega} \in \mathscr{W} \mid \omega_i \le k_i,\ i = 1,\dots,p\};$$

$$\text{LRT:}\quad C_\omega = \left\{\boldsymbol{\omega} \in \mathscr{W} \mid -\sum(\nu_i+1)\log(1-\omega_i) \le c\right\}.$$

We consider the other tests individually.

FISHER'S PROCEDURE (1.3). Suppose $\nu_1 \ne 2$. Fix $(x_3^0,\dots,x_p^0)$ and look at the boundary $x_2(x_1)$ of the acceptance region in the slice defined by

$$-2\log p_1(x_1) - 2\log p_2(x_2(x_1)) = c + \sum_{i=3}^{p} 2\log p_i(x_i^0). \tag{4.2}$$

Since as $x_1 \to -1$, $x_2(x_1) \to x_2^0 \in (-1,1)$, we can show that

$$\lim_{x_1\to -1} \frac{dx_2(x_1)}{dx_1} = \begin{cases} 0 & \text{if } \nu_1 > 2, \\ -\infty & \text{if } \nu_1 = 1. \end{cases} \tag{4.3}$$

If test (1.3) is admissible, then by Theorem 3.1 it must be of the form (3.11). We show below that (4.3) cannot hold for a test of that form, hence test (1.3) must be inadmissible for $\nu_1 \ne 2$.

Suppose (3.1) is of the form (3.11). Since the statistic in (1.3) is strictly increasing in each $x_i \in (-1,1)$, the set C_ω has the property that

$$\omega^{-1}(C_\omega) \cap A \supset \{d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b) \le c\} \cap A,$$

where $A = \{\mathbf{x} \mid x_i \le 0 \text{ for some } i\}$. Thus by (4.2) we must have that

$$d\big((x_1, x_2(x_1), x_3^0,\dots,x_p^0); \boldsymbol{\lambda}, \pi_a, \pi_b\big) = c,\ -1 < x_1 < x_1^0 \tag{4.4}$$

for some $x_1^0 \in (-1,0]$. We can extend the definition of R and d to $\mathscr{X}^* = \{\mathbf{x} \in \mathbb{R}^p \mid -1 \le x_i < 1 \text{ for all } i\}$ by continuity since R is increasing in x_i. Also, for any $\mathbf{x} \in \mathscr{X}^*$, it can be shown that

$$\frac{\partial}{\partial x_i} d(\mathbf{x}; \boldsymbol{\lambda}, \pi_a, \pi_b) = \lambda_i \frac{c(\nu_i+1)}{c(\nu_i)} + \int_{\Theta_a} \frac{\partial R(\mathbf{x};\boldsymbol{\theta})/\partial x_i}{\sum\theta_i}\pi_a(d\boldsymbol{\theta}) + \int_{\Theta_b} \frac{\partial R(\mathbf{x};\boldsymbol{\theta})}{\partial x_i}\pi_b(d\boldsymbol{\theta}) < \infty. \tag{4.5}$$

Now $(\partial/\partial x_i)R(\mathbf{x};\boldsymbol{\theta})$ is strictly positive [see (3.4)] unless $\theta_i = 0$, so that the right-hand side of (4.5) is strictly positive unless $\lambda_i = 0$ and π_a and π_b place zero measure on the set where $\theta_i > 0$. The latter possibility implies that d does not depend at all on x_i, a situation which does not apply here. Hence from (4.4),

$$(4.6)\qquad \lim_{x_1\to -1}\frac{dx_2(x_1)}{dx_1} = -\left.\frac{\partial d(\mathbf{x};\boldsymbol{\lambda},\pi_a,\pi_b)/\partial x_1}{\partial d(\mathbf{x};\boldsymbol{\lambda},\pi_a,\pi_b)/\partial x_2}\right|_{\mathbf{x}=(-1,x_2^0,\dots,x_p^0)} \varepsilon(-\infty,0),$$

since each derivative is strictly between 0 and $+\infty$. (4.6) contradicts (4.3) when $\nu_1 \neq 2$.

Next, consider the test which rejects H_0 when

$$(4.7)\qquad \prod(1-x_i)^{-1} > k.$$

We will show that for any set of ν_i's, (4.7) is Bayes, hence admissible. When $\nu_i = 2$, $p_i(x_i) = (1-x_i)/2$, hence (4.7) coincides with (1.3) when $\nu_i = 2$ for all i, proving the latter test admissible.

Define the measure $\rho(d\tau;\nu)$ on $(0,\infty)$ through $\tau = \alpha^{1/2}\beta$ where $\alpha \sim$ gamma$(\nu/2+1;1/2)$ and $\beta \sim$ beta$(1,\nu)$, so that

$$(4.8)\qquad \begin{aligned}&\int_0^\infty \frac{f(z;\tau,\nu)}{f(z;0,\nu)}(1+\tau^2)^{(\nu+1)/2}\exp(\tau^2/2)\rho(d\tau;\nu)\\ &= \int_0^1\int_0^\infty R(z;\alpha^{1/2}\beta,\nu)\frac{\Gamma(\nu+1)}{\Gamma(\nu)}\frac{\Gamma\left(\frac{\nu}{2}+1\right)}{2^{\nu/2+1}}e^{-\alpha/2}\alpha^{\nu/2}(1-\beta)^{\nu-1}\,d\alpha\,d\beta.\end{aligned}$$

Use (3.5) to show that the coefficient of z^k in the final expression of (4.8) is 1. It is helpful to use the duplication formula for the gamma function:

$$\Gamma\left(\frac{\nu+1+k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\nu+1+k}{2}\right) = \Gamma(\nu+1+k)(2\pi)^{1/2}2^{-(\nu+1+k)}.$$

Hence

$$(4.9)\qquad \int_0^\infty R(z;\tau,\nu)\rho(d\tau;\nu) = (1-z)^{-1}.$$

Let $\pi(d\boldsymbol{\theta})$ be the measure defined by

$$\pi(d\boldsymbol{\theta}) = \prod(1+\theta_i^2)^{(\nu_i+1)/2}\exp(\theta_i^2/2)\rho(d\theta_i;\nu_i).$$

Then by (4.8), (4.9), and (3.2),

$$\int\prod\frac{f(x_i;\theta_i,\nu_i)}{f(x_i;0,\nu_i)}\pi(d\boldsymbol{\theta}) = \prod(1-x_i)^{-1},$$

which proves that test (4.8) is Bayes.

INVERSE NORMAL PROCEDURE (1.5). We will use the following lemma here and elsewhere.

LEMMA 4.1. *Suppose $h_1, \dots, h_p$ are continuous, strictly increasing functions from $(-1, 1)$ onto $\mathbb{R}$ such that $h_i(0) = 0$. Consider the test which rejects H_0 when*

$$\sum h(X_i) > c. \tag{4.10}$$

If for some $(x_3^0, \dots, x_p^0)$ there exists $x \in (-1, 1)$ such that

$$h_1(x) + h_2(-x) > c - \sum_{i=3}^{p} h_i(x_i^0), \tag{4.11}$$

then the test is inadmissible. In particular, (4.11) *will hold if $p > 2$.*

PROOF. Let A be the acceptance region of test (4.10), and suppose the test is admissible. By Corollary 3.2, A must be convex. Since

$$\lim_{x \to \pm 1} h_i(x) = \pm\infty,$$

there are points $(x_1, x_2, x_3^0, \dots, x_p^0)$ arbitrarily close to the points $(1, -1, x_3^0, \dots, x_p^0)$ and $(-1, 1, x_3^0, \dots, x_p^0)$ which are in A. Since A is closed and convex, it must be that $(y, -y, x_3^0, \dots, x_p^0) \in A$ for all y. Since this fact is clearly violated by (4.11), the test must be inadmissible.

If $p > 2$ for any x, $(x_3^0, \dots, x_p^0)$ can be chosen large enough so that (4.11) holds.

Lemma 4.1 with $h_i(x) = -\Phi^{-1}(p_i(x))$ shows that test (1.5) is inadmissible when $p > 2$. If $\alpha > \frac{1}{2}$, then $c < 0$, so that by taking $x = 0$ in (4.11) we can again show the test inadmissible.

We need another lemma.

LEMMA 4.2. *Suppose $p = 2$ and h_1 and h_2 are strictly increasing functions with continuous second derivatives. The test which rejects H_0 when*

$$h_1(X_1) + h_2(X_2) > c \tag{4.12}$$

is inadmissible unless

$$\sum_{i=1}^{2} \frac{h_i''(x_i)}{h_i'(x_i)^2} > 0 \quad \textit{whenever} \sum_{i=1}^{2} h_i(x_i) = c. \tag{4.13}$$

PROOF. As in Lemma 5.1 of Marden (1982a), if (4.13) is violated then the acceptance region of (4.12) is not convex, hence by Corollary 3.1 the test is inadmissible.

For test (1.5), (4.13) is

$$-\sum_{i=1}^{2} \phi(\Phi^{-1}(p_i(x_i)))(\nu_i - 2)x_i/d_i(1 - x_i^2)^{\nu_i/2} + c > 0 \tag{4.14}$$

whenever

$$-\sum \Phi^{-1}(p_i(x_i)) = c. \tag{4.15}$$

Here, $\phi(z) = \Phi'(z)$.

Using l'Hospital's Rule,

$$(4.16)\qquad -\frac{\phi(\Phi^{-1}(p_i(x_i)))(\nu_i-2)x_i}{d_i(1-x_i^2)^{\nu_i/2}} \to \begin{cases} +\infty & \text{if } \nu_i<2 \text{ and } x_i\to+1, \\ & \text{or } \nu_i>2 \text{ and } x_i\to-1, \\ -\infty & \text{if } \nu_i<2 \text{ and } x_i\to-1, \\ & \text{or } \nu_i>2 \text{ and } x_i\to+1, \\ 0 & \text{if } \nu_i=2. \end{cases}$$

Since as $x_1 \to 1$, $x_2 \to -1$ along (4.15), (4.16) shows that (4.14) is violated as $x_1 \to 1$ if $\nu_1 \geq 2$, $\nu_2 \leq 2$, and at least one $\nu_i \neq 2$. Hence Lemma 4.2 proves (1.5) inadmissible.

INVERSE LOGISTIC PROCEDURE (1.6). With $h_i(x) = -\log p_i(x)/(1-p_i(x))$, Lemma 4.1 shows test (1.6) is inadmissible if $p > 2$, or $\alpha > \frac{1}{2}$ and $p = 2$. Suppose $p = 2$ and $\nu_1 > \nu_2$. Now

$$h_1(x) + h_2(-x) = \log\frac{1-p_1(x)}{p_1(x)}\frac{1-p_2(-x)}{p_2(-x)}.$$

Again using l'Hospital's Rule,

$$\lim_{x\to1}\frac{1-p_1(x)}{p_1(x)}\frac{1-p_2(-x)}{p_2(-x)} = \lim_{x\to1}\frac{d_2(1-x^2)^{\nu_2/2-1}}{d_1(1-x^2)^{\nu_1/2-1}} = \infty.$$

Thus (4.11) is violated for x close to 1, proving the test inadmissible.

SUM OF p_i'S TEST (1.7). If all ν_i's are 2, then test (1.7) is equivalent to the local test (1.19) Σx_i since $p_i(x_i) = (1-x_i)/2$. Thus it is admissible.

Suppose $\nu_1 > 2$ and $\alpha < 1 - 1/p!$ so that $c_\alpha < p - 1$ in (1.7). Take $x_3^0, \dots, x_p^0$ so that

$$0 < c_0 \equiv c_\alpha - \sum_{i=3} p_i(x_i^0) < 1.$$

Consider the boundary of the acceptance region in the slice

$$p_1(x_1) + p_2(x_2(x_1)) = c_0.$$

As $x_1 \to 1$, $x_2(x_1) \to x_2^0 \in (-1,1)$ since $c_0 < 1$. Now

$$(4.17)\qquad \lim_{x_1\to1}\frac{dx_2(x_1)}{dx_1} = \lim_{x_1\to1}\frac{d_2(1-x_1^2)^{(\nu_1/2)-1}}{d_1(1-x_2^2)^{(\nu_2/2)-1}} = 0$$

since $\nu_1 > 2$. But (4.17) shows that the acceptance region cannot be convex since dx_2/dx_1 would have to be nonpositive and decreasing to zero. Thus the test is inadmissible.

SUM TEST (1.11). Taking $h_i(x_i) = \sqrt{\nu_i}x_i(1 - x_i^2)^{-1/2}$, Lemma 4.1 proves the test (1.11) inadmissible when $p > 2$. Take $p = 2$. If $s_\alpha < 0$, then (4.11) holds with $x = 0$, so that the test is inadmissible. Otherwise, note that (4.13) is

$$(4.18) \qquad \sum \frac{t_i}{\nu_i + t_i^2} > 0 \quad \text{if } t_1 + t_2 = s_\alpha.$$

Set $t_2 = s_\alpha - t_1$. It is straightforward to show that

$$\left(\frac{t_1}{\nu_1 + t_1^2} + \frac{s_\alpha - t_1}{\nu_2 + (s_\alpha - t_1)^2} \right) < 0$$

for sufficiently large t_1 if either $s_\alpha < 0$ or $s_\alpha = 0$ and $\nu_1 > \nu_2$. Hence Lemma 4.2 proves the test inadmissible in these situations.

Finally, if $s_\alpha = 0$ and $\nu_1 = \nu_2$, the test is the level $\frac{1}{2}$ local test (1.19), $x_1 + x_2 > 0$, hence admissible.

5. The *t* case: proof of asymptotic properties. We start with two lemmas.

LEMMA 5.1. (a) *As* $\tau \to \infty$

$$R(z; \tau, \nu) \sim \begin{cases} c(\nu)^{-1}\Gamma(\nu + 1)|z|^{-(\nu+1)} & \text{if } z < 0, \\ \tau^{\nu+1} & \text{if } z = 0, \\ c(\nu)^{-1}(2\pi)^{1/2} z^\nu \tau^{2\nu+1} \exp(\tau^2 z^2/2) & \text{if } z > 0. \end{cases}$$

(b) *Define the p vector* sgn(**x**) *for* $\mathbf{x} \in \mathscr{X}$ *by* $\text{sgn}(\mathbf{x})_i = \text{sgn}(x_i)$, *i.e.*, 1, 0, *or* -1 *as* $x_i > 0$, $= 0$, *or* < 0. *If* $\text{sgn}(\mathbf{x}) = \text{sgn}(\mathbf{y})$, *then*

$$(5.1) \qquad \begin{aligned} 0 < i(\mathbf{x}, \mathbf{y}) &\equiv \inf_{\theta \in \Theta_A} \frac{\exp\left(-\sum \theta_i^2 \omega_i(\mathbf{x})/2\right) R(\mathbf{x}; \boldsymbol{\theta})}{\exp\left(-\sum \theta_i^2 \omega_i(\mathbf{y})/2\right) R(\mathbf{y}; \boldsymbol{\theta})} \\ &\leq \sup_{\theta \in \Theta_A} \frac{\exp\left(-\sum \theta_i^2 \omega_i(\mathbf{x})/2\right) R(\mathbf{x}; \boldsymbol{\theta})}{\exp\left(-\sum \theta_i^2 \omega_i(\mathbf{y})/2\right) R(\mathbf{y}; \boldsymbol{\theta})} \equiv s(\mathbf{x}, \mathbf{y}) < \infty. \end{aligned}$$

PROOF. (a) First take $z < 0$, and make the substitution $b = -a\tau z$ in (3.4) to obtain

$$(5.2) \quad R(z; \tau, \nu) = (1 + \tau^2)^{(\nu+1)/2} \tau^{-(\nu+1)} c(\nu)^{-1} |z|^{-(\nu+1)} \int_0^\infty e^{-b^2/2\tau^2 z^2} e^{-b} b^\nu \, db.$$

The first line of (a) follows fairly quickly from (5.2) by letting $\tau \to \infty$. The second line follows directly from (3.5). Make the transformation $u = a/\tau$ in (3.4) to show that

$$(5.3) \qquad \begin{aligned} &c(\nu)(2\pi)^{-1/2} \tau^{-2\nu-1} \exp(-\tau^2 z^2/2) R(z; \tau, \nu) \\ &= (1 + \tau^{-2})^{(\nu+1)/2} E_\tau\left[U^\nu I_{\{0, \infty\}}(U)\right], \end{aligned}$$

where $U \sim N(z, \tau^{-2})$. Thus the final line in (a) follows by letting $\tau \to \infty$ in (5.3).

(b) If $\operatorname{sgn}(x_i) = \operatorname{sgn}(y_i)$, then by (a)

$$\lim_{\theta_i \to \infty} \frac{\exp(-\theta_i^2 \omega_i(\mathbf{x})/2) R(x_i; \theta_i, \nu_i)}{\exp(-\theta_i^2 \omega_i(\mathbf{y})/2) R(y_i; \theta_i, \nu_i)} = \begin{cases} |y_i/x_i|^{\nu_i+1} & \text{if } \operatorname{sgn}(x_i) = -1, \\ 1 & \text{if } \operatorname{sgn}(x_i) = 0, \\ (x_i/y_i)^{\nu_i} & \text{if } \operatorname{sgn}(x_i) = 1, \end{cases} \tag{5.4}$$

which is finite and positive in any event. Clearly the ratio in (5.1) is 1 at $\boldsymbol{\theta} = \mathbf{0}$, finite and positive for any $\boldsymbol{\theta}$, and continuous in $\boldsymbol{\theta}$, so that by (5.4), (5.1) must hold.

Before presenting the next lemma we need to set some definitions. Define

$$\mathscr{X}^+ \equiv \{\mathbf{x} \in \mathscr{X} | x_i \geq 0 \text{ for all } i\}. \tag{5.5}$$

For any set $D \subseteq \mathscr{X}$ let

$$\mathbb{E}(D) = \{\mathbf{x} \in \mathscr{X} | x_i > 0 \text{ for } i \in J(D),\ x_i = 0 \text{ for } i \notin J(D)\}, \tag{5.6}$$

where $J(D)$ is given in (3.9).

LEMMA 5.2. (a) *Any set C' as in* (3.19) *is convex and nonincreasing* (2.2).

(b) *For C as in* (3.19), *$\mathbf{x} \in C$ if and only if $\mathbf{x}^+ \in C$, where $\mathbf{x}^+ = (x_1^+, \dots, x_p^+)$.*

(c) *If $D \subseteq \mathscr{X}^+$* (5.5) *is convex, then*

$$\text{closure}\,(D \cap \mathbb{E}(D)) = \text{closure}\,(D).$$

(d) *Let C' be as in* (3.19). *There exists $C'_\omega \subseteq \mathscr{W}$ which is convex, nonincreasing, and satisfies $C'_\omega \subseteq$ closure $\mathbb{E}(C')$* (5.6), *such that*

$$C' \cap \mathbb{E}(C') = \boldsymbol{\omega}^{-1}(C'_\omega) \cap \mathbb{E}(C'). \tag{5.7}$$

PROOF. (a) From (3.4) it is clear that $R(x; \boldsymbol{\theta})$ is convex and nondecreasing in $\mathbf{x}$ for each $\boldsymbol{\theta}$, hence $\int R(\mathbf{x}; \boldsymbol{\theta})\pi_n(d\boldsymbol{\theta})$ is for each n. Thus (a) holds.

(b) If $\mathbf{x}^+ \in C$, $\mathbf{x} \in C$ since C is nonincreasing and $\mathbf{x}^+ \geq \mathbf{x}$. Suppose $\mathbf{x} \in C'$. Consider

$$Y(\mathbf{x}) \equiv \{\mathbf{x} \in \mathscr{X} | \operatorname{sgn}(\mathbf{y}) = \operatorname{sgn}(\mathbf{x}) \text{ and } y_i = x_i \text{ if } x_i > 0\}.$$

Since $\mathbf{x}^+ = \mathbf{y}^+$ for $\mathbf{y} \in Y(\mathbf{x})$, $\boldsymbol{\omega}(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{y})$. By Lemma 5.1(b),

$$R(\mathbf{x}; \boldsymbol{\theta}) \geq i(\mathbf{x}, \mathbf{y}) R(\mathbf{y}; \boldsymbol{\theta}) \quad \text{for } \mathbf{y} \in Y(\mathbf{x}).$$

Since $\mathbf{x} \in C'$, by definition (3.19) of C', $Y(\mathbf{x}) \subseteq C'$. Also, $\mathbf{x}^+ \in$ closure $Y(\mathbf{x})$ so that $\mathbf{x}^+ \in C \equiv$ closure C'. It follows that if $\mathbf{x} \in C$, $\mathbf{x}^+ \in C$, since the "plus" function in (1.9) is continuous.

(c) Clearly closure $(D \cap \mathbb{E}(D)) \subseteq$ closure (D). Suppose $\mathbf{x} \in D$. Since D is convex, there exists an $\mathbf{x}^0 \in D$ such that

$$x_i^0 > 0 \quad \text{for } i \in J(D). \tag{5.8}$$

Now

$$D_0 \equiv \{\alpha\mathbf{x} + (1 - \alpha)\mathbf{x}^0 | 0 < \alpha < 1\} \subseteq D \cap \mathbb{E}(D) \tag{5.9}$$

since D is convex and $(\alpha\mathbf{x} + (1 - \alpha)\mathbf{x}^0)_i \geq (1 - \alpha)x_i^0 > 0$ for $i \in J(D)$. Thus by taking the closure of both sides of (5.9) we have $\mathbf{x} \in$ closure $(D \cap \mathbb{E}(D))$, from which follows closure $(D) \subseteq$ closure $(D \cap \mathbb{E}(D))$, proving (c).

(d) Take $\mathbf{y}_0 \in C'$. By Lemma 5.1(b),

$$\limsup_{n\to\infty} \int R(\mathbf{x};\boldsymbol{\theta})\pi_n(d\boldsymbol{\theta}) < \infty \quad \text{if and only if}$$

$$\limsup_{n\to\infty} \int \exp\left(\sum \theta_i^2 \omega_i(\mathbf{x})/2\right) g_0(\boldsymbol{\theta})\pi_n(d\boldsymbol{\theta}) < \infty,$$

where $g_0(\boldsymbol{\theta}) = \exp(-\Sigma\theta_i^2\omega_i(\mathbf{y}_0)/2)R(\mathbf{y}_0;\boldsymbol{\theta})$. Now

$$C'_\omega \equiv \left\{\boldsymbol{\omega} \in \mathscr{W} \,\middle|\, \limsup_{n\to\infty} \int \exp\left(\sum \theta_i^2 \omega_i(\mathbf{x})/2\right) g_0(\boldsymbol{\theta})\pi_n(d\boldsymbol{\theta}) < \infty\right\}$$

is convex and nonincreasing. By definition (3.9), $\mathbf{x} \in C'$ implies that $x_i \leq 0$ for $i \notin J(C')$. Since $x_i \leq 0$ implies that $\omega_i(\mathbf{x}) = 0$ [see (3.8)], $\boldsymbol{\omega} \in C'_\omega$ implies that $\omega_i = 0$ for $i \notin J(C')$. Hence $C'_\omega \subseteq$ closure $\mathbb{E}(C')$. Also, $x_i > 0$ implies that $\omega_i(\mathbf{x}) > 0$, so that (5.7) holds.

The proof of Lemma 5.2 is concluded. We now turn to the asymptotic properties in Section 3. Use Lemma 5.2(d) to obtain C'_ω as described. Note that as a function from $\mathbb{E}(C')$ to $\mathbb{E}(C')$, $\boldsymbol{\omega}$ of (3.8) is a homeomorphism. Thus from (5.7),

$$\text{closure}\,(C') \cap \mathbb{E}(C') = \boldsymbol{\omega}^{-1}(\text{closure}\, C'_\omega) \cap \mathbb{E}(C'). \tag{5.10}$$

Apply Lemma 5.2(c) to (5.10) to obtain

$$\text{closure}\,(C') \cap \mathscr{X}^+ = \boldsymbol{\omega}^{-1}(\text{closure}\, C'_\omega), \tag{5.11}$$

or by taking $C_\omega \equiv$ closure $C'_\omega \in \mathscr{C}_\omega$, obtain from (3.19)

$$C \cap \mathscr{X}^+ = \boldsymbol{\omega}^{-1}(C_\omega) \cap \mathscr{X}^+. \tag{5.12}$$

(5.11) holds since by Lemma 5.2(a), closure $[C' \cap \mathbb{E}(C')]$ is convex, and is contained in $\mathscr{X}^+$ by definition (5.6) of $\mathbb{E}(C')$. Since by Lemma 5.2(b), $\mathbf{x} \in C$ if and only if $\mathbf{x}^+ \in C$, and by (3.8), $\boldsymbol{\omega}(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x}^+)$, (5.12) yields (3.20).

Now consider (3.21). Take C as in (3.19) and $\mathbf{x} \notin C$. By definition of C, there exists a subsequence $\{m(n)\} \subseteq \{n\}$ such that

$$\lim_{m\to\infty} \int R(\mathbf{x};\boldsymbol{\theta})\pi_m(d\boldsymbol{\theta}) = \infty. \tag{5.13}$$

Since $R(z;\tau,\nu)$ is nondecreasing in z, $\mathbf{x}' \geq \mathbf{x}$ implies that $R(\mathbf{x}';\boldsymbol{\theta}) \geq R(\mathbf{x};\boldsymbol{\theta})$. Thus (3.21) follows from (5.13). Next take $\mathbf{x} \in$ interior C. It can be shown that $\mathbf{x}^*$ is also in interior C, where

$$x_i^* = x_i \quad \text{if } i \notin J(C), \qquad x_i^* = x_i^+ \quad \text{if } i \in J(C) \tag{5.14}$$

so that $\mathbf{x}^* \geq \mathbf{x}$. Since $x_i^* \geq 0$ for $i \in J(C)$ and $\mathbf{x}^*$ is in the interior of C, we can find $\mathbf{x}'$ with $x'_i > 0$ for $i \in J(C)$, $x'_i = x_i$, otherwise, $\mathbf{x}' \in$ interior C, and

$$\mathbf{x}' \geq \mathbf{x}. \tag{5.15}$$

Take $\alpha \in (0,1)$ so that $\mathbf{y} \equiv \alpha\mathbf{x}' + (1-\alpha)\mathbf{x} \geq \mathbf{x}$ and $y_i > 0$ for $i \in J(C)$. Thus $\text{sgn}(\mathbf{x}') = \text{sgn}(\mathbf{y})$ and $\omega_i(\mathbf{x}') > \omega_i(\mathbf{y})$ for $i \in J(C)$. Now (3.22) will hold if

$$\lim_{k\to\infty} \sup_{\sigma(\boldsymbol{\theta})\geq k} [R(\mathbf{y};\boldsymbol{\theta})/R(\mathbf{x}';\boldsymbol{\theta})] = 0. \tag{5.16}$$

But by Lemma 5.1(b), the left-hand side of (5.16) is bounded from above by

$$\lim_{k\to\infty} \sup_{\sigma(\boldsymbol{\theta})\geq k} \left\{ s(\mathbf{y},\mathbf{x}')\exp\left[-\sum \theta_i^2(\omega_i(\mathbf{x}') - \omega_i(\mathbf{y}))/2\right]\right.$$

$$\leq \lim_{k\to\infty} s(\mathbf{y},\mathbf{x}')\exp\left[-k \cdot \min_{j\in J(c)} \left\{ \left(\omega_j(\mathbf{x}') - \omega_j(\mathbf{y})\right)\right\}\right]$$

$$= 0.$$

Letting α be as in (3.24), Lemma 5.1. and (3.2) can be used to prove (3.25). Furthermore, we have that

$$\sup_{s>0} \alpha(s\boldsymbol{\xi})\exp(-s^2t/2)R(\mathbf{x}; s\boldsymbol{\xi}) \leq \exp\left(-s^2\left[\sum \xi_i^2\omega_i(\mathbf{x}) - t\right]/2\right)$$

$$\cdot \prod_{i:\, x_i>0} \sup_{t>0} (1+t^2)^{-(\nu_i+1/2)} R(x_i; t, \nu_i)$$

since $R(z; \tau, \nu) \leq 1$ if $z \leq 0$ [see (3.4)]. Lemma 5.1(a) shows that the terms in the final product are finite since the supremands are clearly finite and continuous for $0 \leq t < \infty$. Hence (3.26) holds.

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