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TESTING LINEAR REGRESSION FUNCTION ADEQUACY WITHOUT REPLICATION

BY JAMES W. NEILL AND DALLAS E. JOHNSON

The Upjohn Company and Kansas State University

The well known pure error–lack of fit test, which can be used to assess the adequacy of a linear regression model, is generalized to accommodate the case of nonreplication. The asymptotic null distribution of the proposed test statistic is derived. Also, the proposed test statistic is shown to be asymptotically comparable under general alternatives to the test statistic obtained in the case of replication. Consistency properties associated with pseudo lack of fit and pure error mean squares are given which parallel those obtained in the case of replication. In addition, the test statistic is invariant with respect to location and scale changes made to the regression variables.

1. Introduction. A linear regression model with replication can be represented by

$$Y_{ik} = \sum_{j=1}^{p} \beta_j x_{ij} + \varepsilon_{ik}, \qquad (1)$$

where $i = 1, 2, \ldots, M$; $k = 1, 2, \ldots, n_i$ and $n_i > 1$ for at least one i. The random errors ε_{ik} are assumed to be independent and identically distributed with $E(\varepsilon_{ik}) = 0$ and $E(\varepsilon_{ik}^2) = \sigma^2$ where σ^2 is an unknown real-valued parameter. The x_{ij} are fixed observable real numbers and the β_j constitute a $p \times 1$ vector of unknown parameters defined in $\mathbb{R}^p$. For notational simplicity, only the case for which $n_i = n$, $i = 1, 2, \ldots, M$, will be discussed; the extension to the unbalanced case is straightforward. Thus, the total number of observations is $N = Mn$.

The model given by (1) and some reasons for taking repeated observations at fixed values of the regression variables are discussed by Graybill (1976). One important aspect of this model is that it allows one to test for linear model adequacy against a general alternative.

Let $\sum_{j=1}^{p} \beta_j x_{ij}$, $i = 1, 2, \ldots, M$, be written in matrix form as $\overline{\mathbf{X}}\boldsymbol{\beta}$ where $\overline{\mathbf{X}} = [x_{ij}]$ has size $M \times p$ and rank p. In this paper, $\mathbf{I}$ denotes the identity matrix of appropriate dimension, and $\mathbf{J}$ and $\mathbf{j}$ represent $n \times n$ and $n \times 1$ matrices of ones, respectively. Also, $\overline{\mathbf{X}}^-$ is the Moore–Penrose generalized inverse of the matrix $\overline{\mathbf{X}}$ and the notation $\otimes$ represents the Kronecker matrix product (Graybill, 1983). To test

$$H_0\colon E(\mathbf{Y}) = (\overline{\mathbf{X}} \otimes \mathbf{j})\boldsymbol{\beta}$$

vs.

$$H_a\colon E(\mathbf{Y}) = (\overline{\mathbf{X}} \otimes \mathbf{j})\boldsymbol{\beta} + (\overline{\mathbf{X}}^* \otimes \mathbf{j})\boldsymbol{\beta}^*,$$

first partition the residual sum of squares for the model given by (1) into two

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parts: the so-called lack of fit sum of squares,

$$\text{(2)} \qquad \text{SSLOF} = \mathbf{Y}'[(\mathbf{I} - \overline{\mathbf{X}}\overline{\mathbf{X}}^{-}) \otimes (1/n)\mathbf{J}]\mathbf{Y},$$

and the pure error sum of squares,

$$\text{(3)} \qquad \text{SSPE} = \mathbf{Y}'[\mathbf{I} \otimes (\mathbf{I} - (1/n)\mathbf{J})]\mathbf{Y}.$$

Next observe that

$$\text{(4)} \qquad F = [(N-M)/(M-p)](\text{SSLOF}/\text{SSPE})$$

is distributed according to the noncentral F distribution $F(M-p, N-M, \lambda)$ under H_a and normally distributed errors where

$$\text{(5)} \qquad \lambda = (n/2\sigma^2)(\overline{\mathbf{X}}^*\boldsymbol{\beta}^*)'[\mathbf{I} - \overline{\mathbf{X}}\overline{\mathbf{X}}^{-}](\overline{\mathbf{X}}^*\boldsymbol{\beta}^*).$$

Thus, assuming that the matrix $[\overline{\mathbf{X}}, \overline{\mathbf{X}}^*]$ has full column rank, a strictly unbiased size α test of H_0 vs. H_a is: Reject H_0 if the observed value of F exceeds $F_{\alpha, M-p, N-M}$, where $F_{\alpha, M-p, N-M}$ is the $(1-\alpha)$th quantile of the central F distribution $F(M-p, N-M)$.

This pure error–lack of fit test is general in the sense that $\overline{\mathbf{X}}^*$ need not be specified, except for power calculations. Thus, $\overline{\mathbf{X}}^*$ may even consist of unknown parameters. As a result, the test described above is a test of the adequacy of $\overline{\mathbf{X}}\boldsymbol{\beta}$, the deterministic portion of the model given by (1).

Replication is not always possible or available in many experimental applications. Several procedures exist that may help one to assess the adequacy of a proposed model for the case of nonreplication; Neill and Johnson (1984a) provided a review of such procedures. In particular, the works by Green (1971), Lyons and Proctor (1977), Shillington (1979), Daniel and Wood (1980), Draper and Smith (1981) and Utts (1982) were cited. Each of these test procedures is based on a pseudo pure error estimator of the error variance. However, the proposed estimators are biased under the hypothesis of model adequacy and/or the alternative, and, as a result, the power of the procedures may be adversely affected.

The pure error–lack of fit test is generalized in this paper to accommodate the case of nonreplication. The generalization is based on a pseudo pure error estimator that is consistent for σ^2 regardless of whether or not the specified model is correct. An extension of (1) to the case of nonreplication is defined in Section 2, and a test statistic for model adequacy is developed. Consistency results and asymptotic distributions are derived in Sections 3 and 4, respectively.

2. The nonreplicated case. A linear regression model without replication can be represented by

$$\text{(6)} \qquad Y_{ik} = \sum_{j=1}^{p} \beta_j x_{ijk} + \varepsilon_{ik},$$

where $i = 1, 2, \dots, M$; $k = 1, 2, \dots, n_i$. As in Section 1, only the case for which $n_i = n$, $i = 1, 2, \dots, M$, will be discussed. The models given by (1) and (6) are identical except for the additional subscript k on the regression variables in (6) that allows for nonreplication. Also, x_{ijk} is considered to be of the form $x_{ij} + \delta_{ijk}$ where x_{ij} and δ_{ijk} are fixed observable real numbers. Thus, δ_{ijk} characterizes the perturbation in the jth regressor variable for the kth observation in the ith group. Let $\mathbf{X} = [x_{ijk}]$ denote the $N \times p$ matrix of regression variables for model (6), and note that $\mathbf{X} = (\overline{\mathbf{X}} \otimes \mathbf{j}) + \boldsymbol{\Delta}$ where $\overline{\mathbf{X}} = [x_{ij}]$ and $\boldsymbol{\Delta} = [\delta_{ijk}]$. The matrices $\mathbf{X}$ and $\overline{\mathbf{X}}$ are assumed to have rank equal to p. A test of

$$H_0(\boldsymbol{\Delta}): E(\mathbf{Y}) = [(\overline{\mathbf{X}} \otimes \mathbf{j}) + \boldsymbol{\Delta}]\boldsymbol{\beta}$$

vs.

$$H_a(\boldsymbol{\Delta}, \boldsymbol{\Delta}^*): E(\mathbf{Y}) = [(\overline{\mathbf{X}} \otimes \mathbf{j}) + \boldsymbol{\Delta}]\boldsymbol{\beta} + [(\overline{\mathbf{X}}^* \otimes \mathbf{j}) + \boldsymbol{\Delta}^*]\boldsymbol{\beta}^*$$

will be discussed in the remainder of this section. In the following, $\mathbf{X}^*$ will represent the matrix $(\overline{\mathbf{X}}^* \otimes \mathbf{j}) + \boldsymbol{\Delta}^*$ where $[\mathbf{X}, \mathbf{X}^*]$ has full column rank.

Suppose the model given by (6) is correct and let $\mathbf{Y}^* = \mathbf{Y} - \boldsymbol{\Delta}\boldsymbol{\beta}$. Notice that $\mathbf{Y}^*$, if observable, would conform to the model with replication as given by (1). Model adequacy could then be determined by the usual lack of fit test with $\mathbf{Y}$ replaced by $\mathbf{Y}^*$ in (4). Since $\mathbf{Y}^*$ is not observable, one might replace $\mathbf{Y}^*$ with an observable vector $\hat{\mathbf{Y}}^*$ that, at least asymptotically, is comparable to $\mathbf{Y}^*$. Let $\hat{\mathbf{Y}}^* = \mathbf{Y} - \boldsymbol{\Delta}\hat{\boldsymbol{\beta}}$ where $\hat{\boldsymbol{\beta}}$ is the least squares estimator of $\boldsymbol{\beta}$ under model (6). Then, under conditions to be stated below, $\mathbf{Y}_{ik}^* - \hat{\mathbf{Y}}_{ik}^* \to 0$ with probability one (w.p.1) as $N \to \infty$. Now replace $\mathbf{Y}$ in (4) with $\hat{\mathbf{Y}}^*$ and denote this statistic by $\hat{F}^*$. Model adequacy is then rejected if the observed value of $\hat{F}^*$ exceeds $F_{\alpha, M-p, N-M}$.

In order to obtain asymptotic properties of this proposed test statistic, suppose that the design space is partitioned into M cells. The partition is then refined in such a way that the volume of the largest cell converges to zero as $M \to \infty$ where n is either fixed or $n \to \infty$. To facilitate the proofs of certain results given in the following sections, it will be assumed that the partition sequence is regular. The data points for a common cell would comprise a group of near replicate observations, and asymptotic repeatability is assured since the cell volume converges to zero. With the preceding scheme of asymptotics, it will be shown that the test procedure described above provides an asymptotic size α test of $H_0(\boldsymbol{\Delta})$ vs. $H_a(\boldsymbol{\Delta}, \boldsymbol{\Delta}^*)$. In addition, $\hat{F}^*$ will be shown to be asymptotically comparable, under general alternatives, to the test statistic obtained when replication actually exists.

The test procedure based on $\hat{F}^*$ is naturally implemented by choosing $\overline{\mathbf{X}} = [\bar{x}_{ij.}]$ and $\overline{\mathbf{X}}^* = [\bar{x}_{ij.}^*]$. This choice for $\overline{\mathbf{X}}$ and $\overline{\mathbf{X}}^*$ will be used for the remainder of this paper. In addition, it will be assumed that $\overline{\mathbf{X}}'\overline{\mathbf{X}}/M \to \boldsymbol{\Sigma}_{xx}$ as $N \to \infty$ where $\boldsymbol{\Sigma}_{xx}$ is a positive definite matrix and $\overline{\mathbf{X}}'\overline{\mathbf{X}}^*/M \to \boldsymbol{\Sigma}_{xx^*}$ as $N \to \infty$.

The test based on $\hat{F}^*$ is general in the sense that $\mathbf{X}^*$ need not be specified, except for power calculations. Thus, $\mathbf{X}^*$ may even consist of unknown parameters. As a result, the test based on $\hat{F}^*$ is a test of the adequacy of $\mathbf{X}\boldsymbol{\beta}$, the deterministic portion of the model given by (6). As in the case of replication, $\hat{F}^*$ is invariant with respect to location (for models with an intercept term) and scale changes made to the regression variables.

A simulation study by Neill and Johnson (1984b) has suggested that the test procedure based on $\hat{F}^*$ is useful for small samples as well. The approach consistently provided a test with the desired size, and the power was observed to be comparable to the power obtained in the case of replication. Included in the study were extreme cases in which the sample points were not near replicates but uniformly separated.

3. Consistency results. In this section some consistency results useful in proving the asymptotic distributional properties of $\hat{F}^*$ will be presented. These results parallel those obtained when replication actually exists.

LEMMA 1. (a) *Under* $H_0(\Delta)$, $\hat{\boldsymbol{\beta}} = \mathbf{X}^-\mathbf{Y} \to \boldsymbol{\beta}$ *w.p.*1 *as* $N \to \infty$.
(b) *Under* $H_a(\Delta, \Delta^*)$, $\hat{\boldsymbol{\beta}} \to \boldsymbol{\beta} + \boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xx^*}\boldsymbol{\beta}^*$ *w.p.*1 *as* $N \to \infty$.

PROOF. Since (a) is a special case of (b) with $\mathbf{X}^*\boldsymbol{\beta}^* = \mathbf{0}$, only part (b) will be proved. Under $H_a(\Delta, \Delta^*)$,

$$\hat{\boldsymbol{\beta}} = \mathbf{X}^-\mathbf{Y} = \boldsymbol{\beta} + \mathbf{X}^-\mathbf{X}^*\boldsymbol{\beta}^* + \mathbf{X}^-\boldsymbol{\varepsilon}.$$

By Lemma A1 in the appendix,

$$\mathbf{X}^-\mathbf{X}^*\boldsymbol{\beta}^* \to \boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xx^*}\boldsymbol{\beta}^* \quad \text{as } N \to \infty.$$

It is next shown that $\mathbf{X}^-\boldsymbol{\varepsilon}$ converges strongly to the p dimensional zero vector. From the proof of Lemma A1 it can be observed that

$$\mathbf{X}'\mathbf{X}/N \to \boldsymbol{\Sigma}_{xx} \quad \text{as } N \to \infty,$$

where $\boldsymbol{\Sigma}_{xx}$ is a positive definite matrix. Since the random errors ε_{ik} are assumed to be independent and identically distributed with $E(\varepsilon_{ik}) = 0$ and $E(\varepsilon_{ik}^2) = \sigma^2$, it follows by Lemma 1 of Christopeit and Helmes (1980) that

$$\mathbf{X}^-\boldsymbol{\varepsilon} \to \mathbf{0} \quad \text{w.p.1} \quad \text{as } N \to \infty.$$

Part (b) is a consequence of these observations. □

In the following, let $\mathbf{F}$ and $\mathbf{E}$ denote the matrices for the quadratic forms given by (2) and (3), respectively. Thus, $\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p)$ and $\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^*/(N-M)$ represent the pseudo lack of fit and pure error mean squares whose ratio forms $\hat{F}^*$. In Lemmas 2 and 3 it will be assumed that the random errors have finite fourth moments. Also, the notation $\operatorname{tr}(\cdot)$ will denote the trace function of a square matrix.

LEMMA 2. $\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^*/(N-M) \to_p \sigma^2$ *as* $N \to \infty$ *under* $H_0(\Delta)$ *and* $H_a(\Delta, \Delta^*)$.

PROOF. Since $H_0(\Delta)$ is a special case of $H_a(\Delta, \Delta^*)$ with $\mathbf{X}^*\boldsymbol{\beta}^* = \mathbf{0}$, only the case for $H_a(\Delta, \Delta^*)$ will be proved. Under $H_a(\Delta, \Delta^*)$,

$$\hat{\mathbf{Y}}^* = (\overline{\mathbf{X}} \otimes \mathbf{j})\boldsymbol{\beta} + (\overline{\mathbf{X}}^* \otimes \mathbf{j})\boldsymbol{\beta}^* + \boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) + \boldsymbol{\Delta}^*\boldsymbol{\beta}^* + \boldsymbol{\varepsilon}.$$

Then, since $\mathbf{E}(\overline{\mathbf{X}} \otimes \mathbf{j}) = \mathbf{0}$ and $\mathbf{E}(\overline{\mathbf{X}}^* \otimes \mathbf{j}) = \mathbf{0}$,

$$\begin{aligned}\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^*/(N-M) &= \boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\varepsilon}/(N-M) + (\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))'\mathbf{E}\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})/(N-M) \\ &\quad + (\boldsymbol{\Delta}^*\boldsymbol{\beta}^*)'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M) + 2\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})/(N-M) \\ &\quad + 2\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M) + 2(\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M).\end{aligned}$$

Next note that

$$E[\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\varepsilon}/(N-M)] = [\sigma^2/(N-M)]\mathrm{tr}(\mathbf{E}) + [E(\boldsymbol{\varepsilon})]'\mathbf{E}E(\boldsymbol{\varepsilon})/(N-M) = \sigma^2$$

and

$$\begin{aligned}\mathrm{var}[\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\varepsilon}/(N-M)] &= \left[(\mu_4 - 3\sigma^4)/(N-M)^2\right]\mathbf{e}'\mathbf{e} + \left[2\sigma^4/(N-M)^2\right]\mathrm{tr}(\mathbf{E}^2)\\ &= (\mu_4 - 3\sigma^4)/N + 2\sigma^4/(N-M)\end{aligned}$$

for each N where $\mu_4 = E(\varepsilon_{ik}^4)$ and $\mathbf{e}$ is the $N \times 1$ vector of the diagonal elements of $\mathbf{E}$ (Seber, 1977). Since $\mathrm{var}[\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\varepsilon}/(N-M)] \to 0$ as $N \to \infty$,

$$\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\varepsilon}/(N-M) \to_P \sigma^2 \quad \text{as } N \to \infty.$$

Also,

$$\begin{aligned}&(\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))'\mathbf{E}\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})/(N-M)\\ &\quad = \sum_{i=1}^{M}\sum_{j=1}^{p}\sum_{j'=1}^{p}(\beta_j - \hat{\beta}_j)(\beta_{j'} - \hat{\beta}_{j'})\sum_{k=1}^{n}\delta_{ijk}\delta_{ij'k}/(N-M).\end{aligned}$$

As shown in the proof of Lemma A1, $\sum_{i=1}^{M} a_{iM}/M \to 0$ as $N \to \infty$ where $a_{iM} = \sum_{k=1}^{n}\delta_{ijk}\delta_{ij'k}/n$. Thus, by interchanging summations and using the convergence of $\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}$ derived in Lemma 1,

$$(\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))'\mathbf{E}\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})/(N-M) \to_P 0 \quad \text{as } N \to \infty.$$

The convergence of $(\boldsymbol{\Delta}^*\boldsymbol{\beta}^*)'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M)$ to zero is similar. By the Schwarz inequality and the fact that $\mathbf{E}$ is a projection matrix,

$$\begin{aligned}&|(\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M)|^2\\ &\quad \le [(\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))'\mathbf{E}\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})/(N-M)][(\boldsymbol{\Delta}^*\boldsymbol{\beta}^*)'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M)].\end{aligned}$$

Since the quadratic forms in the preceding inequality converge to zero in probability,

$$(\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M) \to_P 0 \quad \text{as } N \to \infty.$$

The convergence of $\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\Delta}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})/(N-M)$ and $\boldsymbol{\varepsilon}'\mathbf{E}\boldsymbol{\Delta}^*\boldsymbol{\beta}^*/(N-M)$ to zero in probability is similar. Hence,

$$\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^*/(N-M) \to_P \sigma^2 \quad \text{as } N \to \infty \text{ under } H_a(\boldsymbol{\Delta}, \boldsymbol{\Delta}^*).$$

This completes the proof of the lemma. □

Lemma 2 shows that the denominator of $\hat{F}^*$ is consistent for σ^2 regardless of whether or not the specified model is correct. Lemma 3 provides consistency and expectation properties for the numerator of $\hat{F}^*$.

LEMMA 3. (a) *Under* $H_0(\boldsymbol{\Delta})$, $\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p) \to_P \sigma^2$ *as* $N \to \infty$.
(b) *Under* $H_a(\boldsymbol{\Delta}, \boldsymbol{\Delta}^*)$, $E[\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p)] = \sigma^2 + 2\sigma^2\lambda/(M-p)$ *for all* N *where* λ *is given by* (5).

PROOF. Under $H_a(\Delta, \Delta^*)$,

$$\hat{\mathbf{Y}}^* = \mathbf{Y}_0 + \Delta(\beta - \hat{\beta}) + \Delta^*\beta^*,$$

where

$$\mathbf{Y}_0 = (\overline{\mathbf{X}} \otimes \mathbf{j})\beta + (\overline{\mathbf{X}}^* \otimes \mathbf{j})\beta^* + \varepsilon.$$

Then $\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^* = \mathbf{Y}_0'\mathbf{F}\mathbf{Y}_0$ for all N by the fact that $\mathbf{F}$ is a projection matrix and $\mathbf{F}\Delta = \mathbf{0}$ and $\mathbf{F}\Delta^* = \mathbf{0}$. Since

$$E\left[\mathbf{Y}_0'\mathbf{F}\mathbf{Y}_0/(M-p)\right] = \left[\sigma^2/(M-p)\right]\mathrm{tr}(\mathbf{F}) + \left[E(\mathbf{Y}_0)\right]'\mathbf{F}E(\mathbf{Y}_0)/(M-p)$$
$$= \sigma^2 + n(\overline{\mathbf{X}}^*\beta^*)'[\mathbf{I} - \overline{\mathbf{X}}\overline{\mathbf{X}}^-](\overline{\mathbf{X}}^*\beta^*)/(M-p),$$

it follows that

$$E\left[\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p)\right] = \sigma^2 + 2\sigma^2\lambda/(M-p) \quad \text{for all } N \text{ under } H_a(\Delta, \Delta^*),$$

where λ is given by (5). This completes the proof of part (b). To establish part (a), first note that $E[\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p)] = \sigma^2$ under $H_0(\Delta)$ by part (b). In addition, since $\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^* = \mathbf{Y}_0'\mathbf{F}\mathbf{Y}_0$ for all N under $H_0(\Delta)$,

$$\mathrm{var}\left[\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p)\right] = \left[(\mu_4 - 3\sigma^4)/(M-p)^2\right]\mathbf{f}'\mathbf{f} + \left[2\sigma^4/(M-p)^2\right]\mathrm{tr}(\mathbf{F}^2)$$

for each N where $\mu_4 = E(\varepsilon_{ik}^4)$ and $\mathbf{f}$ is the $N \times 1$ vector of the diagonal elements of $\mathbf{F}$ (Seber, 1977). Since $\mathbf{f}'\mathbf{f} \le \mathrm{tr}(\mathbf{F}) = M - p$, $\mathrm{var}[\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p)] \to 0$ as $N \to \infty$ and thus,

$$\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*/(M-p) \to_P \sigma^2 \quad \text{as } N \to \infty \text{ under } H_0(\Delta).$$

This completes the proof of part (a). □

4. Asymptotic distributions. In this section the asymptotic null distribution of $\hat{F}^*$ will be derived. In addition, a result which relates $\hat{F}^*$ under general alternatives to the test statistic obtained when replication actually exists will be proved. Since the proof of the following theorem makes use of Lemmas 2 and 3, it will be assumed that the random errors have finite fourth moments.

THEOREM. (a) *Under* $H_0(\Delta)$, $F_0 - \hat{F}^* \to_P 0$ *as* $N \to \infty$ *where* F_0 *is defined by* (4) *under* $H_0(\Delta = \mathbf{0})$.

(b) *Under* $H_a(\Delta, \Delta^*)$, $\hat{F}^*/F_0 \to_P 1$ *as* $N \to \infty$ *where* F_0 *is defined by* (4) *under* $H_a(\Delta = \mathbf{0}, \Delta^* = \mathbf{0})$.

PROOF. (a) With $\mathbf{Y}_0$ defined by $\mathbf{Y}_0 = (\overline{\mathbf{X}} \otimes \mathbf{j})\beta + \varepsilon$,

$$F_0 - \hat{F}^* = [(N-M)/(M-p)]\left[\mathbf{Y}_0'\mathbf{F}\mathbf{Y}_0 - (\mathbf{Y}_0'\mathbf{E}\mathbf{Y}_0/\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^*)\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^*\right]/\mathbf{Y}_0'\mathbf{E}\mathbf{Y}_0.$$

By Lemma 2 and the fact that $\mathbf{Y}_0'\mathbf{E}\mathbf{Y}_0 = \varepsilon'\mathbf{E}\varepsilon$,

$$\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^*/(N-M) \to_P \sigma^2 \quad \text{as } N \to \infty \text{ under } H_0(\Delta)$$

and

$$\mathbf{Y}_0'\mathbf{E}\mathbf{Y}_0/(N-M) \to_P \sigma^2 \quad \text{as } N \to \infty \text{ under } H_0(\Delta = \mathbf{0}).$$

By Lemma 3,

$$\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^{*}/(M-p) \to_P \sigma^2 \quad \text{as } N \to \infty \text{ under } H_0(\Delta).$$

Part (a) follows from the preceding results and the fact that $\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^{*} = \mathbf{Y}_0'\mathbf{F}\mathbf{Y}_0$ for all N under $H_0(\Delta)$, as noted in the proof of Lemma 3.

(b) With $\mathbf{Y}_0$ defined by $\mathbf{Y}_0 = (\overline{\mathbf{X}} \otimes \mathbf{j})\boldsymbol{\beta} + (\overline{\mathbf{X}}^* \otimes \mathbf{j})\boldsymbol{\beta}^* + \boldsymbol{\varepsilon}$,

$$\hat{F}^*/F_0 = [\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^{*}/\mathbf{Y}_0'\mathbf{F}\mathbf{Y}_0][\mathbf{Y}_0'\mathbf{E}\mathbf{Y}_0/\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^{*}].$$

By Lemma 2, $\hat{\mathbf{Y}}^{*\prime}\mathbf{E}\hat{\mathbf{Y}}^{*}/(N-M)$ and $\mathbf{Y}_0'\mathbf{E}\mathbf{Y}_0/(N-M)$ converge to σ^2 in probability as $N \to \infty$ under $H_a(\Delta, \Delta^*)$ and $H_a(\Delta = \mathbf{0},\ \Delta^* = \mathbf{0})$, respectively. Part (b) follows immediately since $\hat{\mathbf{Y}}^{*\prime}\mathbf{F}\hat{\mathbf{Y}}^{*} = \mathbf{Y}_0'\mathbf{F}\mathbf{Y}_0$ for all N under $H_a(\Delta, \Delta^*)$. □

Since F_0 under $H_0(\Delta = \mathbf{0})$ and normality is distributed according to the central F distribution with $M - p$ and $N - M$ degrees of freedom, Theorem (a) implies the test procedure based on $\hat{F}^*$ is an asymptotic size α test of $H_0(\Delta)$ vs. $H_a(\Delta, \Delta^*)$. By Theorem (b), $\hat{F}^*$ is asymptotically comparable under general alternatives to F_0, the test statistic obtained when replication actually exists.

APPENDIX

A matrix limit result useful for establishing Lemma 1 will now be proved. The notation is given in Section 2.

LEMMA A1. $\mathbf{X}^-\mathbf{X}^* \to \boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xx^*}$ *as* $N \to \infty$.

PROOF. Observe that

$$\begin{aligned}\mathbf{X}^-\mathbf{X}^* &= \{[(\overline{\mathbf{X}} \otimes \mathbf{j}) + \Delta]'[(\overline{\mathbf{X}} \otimes \mathbf{j}) + \Delta]\}^{-1}[(\overline{\mathbf{X}} \otimes \mathbf{j}) + \Delta]'[(\overline{\mathbf{X}}^* \otimes \mathbf{j}) + \Delta^*] \\ &= \{\overline{\mathbf{X}}'\overline{\mathbf{X}}/M + (\overline{\mathbf{X}} \otimes \mathbf{j})'\Delta/N + \Delta'(\overline{\mathbf{X}} \otimes \mathbf{j})/N + \Delta'\Delta/N\}^{-1} \\ &\quad \cdot \{\overline{\mathbf{X}}'\overline{\mathbf{X}}^*/M + (\overline{\mathbf{X}} \otimes \mathbf{j})'\Delta^*/N + \Delta'(\overline{\mathbf{X}}^* \otimes \mathbf{j})/N + \Delta'\Delta^*/N\}.\end{aligned}$$

Since $\delta_{ijk} = x_{ijk} - \bar{x}_{ij}$, $(\overline{\mathbf{X}} \otimes \mathbf{j})'\Delta = [\sum_{i=1}^{M}\bar{x}_{ii'}\delta_{ij'}]_{p\times p} = \mathbf{0}$. Similarly, $(\overline{\mathbf{X}} \otimes \mathbf{j})'\Delta^* = \mathbf{0}$ and $(\overline{\mathbf{X}}^* \otimes \mathbf{j})'\Delta = \mathbf{0}$. Next note that

$$\Delta'\Delta/N = \left[\sum_{i=1}^{M}\sum_{k=1}^{n}\delta_{ii'k}\delta_{ij'k}/Mn\right]_{p\times p}.$$

The term $\sum_{k=1}^{n}\delta_{ii'k}\delta_{ij'k}/n = a_{iM} \to 0$ as $N \to \infty$ by the Schwarz inequality and the manner in which the partition is refined. Since the partition sequence is assumed to be regular, $\{a_{iM}\colon i = 1, 2, \ldots, M\}$ are uniformly convergent to zero. Thus, $\sum_{i=1}^{M}a_{iM}/M \to 0$ as $N \to \infty$ and hence, $\Delta'\Delta/\mathrm{N} \to \mathbf{0}$ as $N \to \infty$. Similarly, $\Delta'\Delta^*/N \to \mathbf{0}$ as $N \to \infty$. Since $\overline{\mathbf{X}}'\overline{\mathbf{X}}/M \to \boldsymbol{\Sigma}_{xx}$ where $\boldsymbol{\Sigma}_{xx}$ is a positive definite matrix and $\overline{\mathbf{X}}'\overline{\mathbf{X}}^*/M \to \boldsymbol{\Sigma}_{xx^*}$ as $N \to \infty$, it follows that $\mathbf{X}^-\mathbf{X}^* \to \boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xx^*}$ as $N \to \infty$. □

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THE UPJOHN COMPANY
CLINICAL BIOSTATISTICS UNIT
KALAMAZOO, MICHIGAN 49001

DEPARTMENT OF STATISTICS
KANSAS STATE UNIVERSITY
MANHATTAN, KANSAS 66506