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RANDOM DISCRIMINANTS[1]

BY I-LI LU AND DONALD RICHARDS

University of Virginia

Let $X_1, X_2, \dots, X_n$ be a random sample from a continuous univariate distribution F, and let $\Delta = \prod_{1 \le i < j \le n}(X_i - X_j)^2$ denote the discriminant, or square of the Vandermonde determinant, constructed from the random sample. The statistic Δ arises in the study of moment matrices and inference for mixture distributions, the spectral theory of random matrices, control theory and statistical physics. In this paper, we study the probability distribution of Δ. When $X_1, \dots, X_n$ is a random sample from a normal, gamma or beta population, we use Selberg's beta integral formula to obtain stochastic representations for the exact distribution of Δ. Further, we obtain stochastic bounds for Δ in the normal and gamma cases. Using the theory of U-statistics, we derive the asymptotic distribution of Δ under certain conditions on F.

1. Introduction. For continuous random variables $X_1, \dots, X_n$, let

$$\Delta \equiv \Delta(X_1, \dots, X_n) = \prod_{1 \le i < j \le n} (X_i - X_j)^2$$

be the *discriminant*, or square of the Vandermonde determinant, constructed from the X_i's. In this paper, we investigate the exact and asymptotic distributions of Δ when $X_1, \dots, X_n$ is a random sample from an absolutely continuous distribution function F.

Random discriminants have a wide variety of applications, arising in hypothesis testing [Andersson, Brøns and Jensen (1983)]; spectral theory of random matrices [Girko (1988)]; control theory [Kalman (1961)]; and statistical physics [Mehta (1990)]. Our primary motivation for studying the distribution of Δ stems from the work of Lindsay (1989a, b). Let X be a real random variable and $M_n = (E(X^{i+j-2}))$ be the *moment matrix* corresponding to X. Then Lindsay (1989a) proves that

$$\det(M_n) = c_n E[\Delta(X_1, \dots, X_n)], \tag{1.1}$$

where c_n is a constant. When X belongs to certain natural exponential families, then Lindsay (1989b) uses the representation (1.1) to show how Δ can be utilized in the construction of estimators for the mixing distribution function in certain finite mixture populations.

Although a variety of methods are available for deriving the exact distributions of random determinants [Muirhead (1982); Girko (1988)], very few techniques appear to be available in the case of the statistic Δ. When the random sample is from a normal, gamma or beta population, we apply a

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famous integral formula of Selberg (1944) to obtain stochastic representations for Δ [cf. Mehta (1990), Chapter 17 or Richards (1989), for a recapitulation of Selberg's proof]. When X is a normal or gamma variable we determine that, up to a constant factor, Δ is equal in distribution to a triangular product of independent gamma variables. When X has a beta distribution and $n \le 4$, we show also that the random variable Δ is equal in distribution to a product of polynomials in independent beta variables.

These stochastic representations for Δ suggest that its exact probability density function is too complicated to be of great practical use. Therefore we use a stochastic analog of Gauss' multiplication formula [Gordon (1989)] to prove that, in the normal and gamma cases, Δ is stochastically bounded above by a power of a product of at most two gamma variables. This stochastic bound provides a simple, easily applicable, numerical lower bound on the distribution function of Δ. We will also use these methods to obtain stochastic upper bounds for Δ, in the normal and gamma cases.

Using the theory of U-statistics [Serfling (1980)], we also derive the asymptotic distribution of Δ under certain assumptions on F. In the normal, gamma or beta case, these assumptions on F may be verified directly with the aid of Selberg's integral. We will show that, after centering and scaling, $\log \Delta(X_1, \dots, X_n)$ is asymptotically standard normal for large n. In general, the rate of convergence is $O(n^{-1/2})$; but in the normal and gamma cases, the rate is $O(n^{-1})$.

The layout of our results is as follows. In Section 2, we review Selberg's integral and some of its limiting cases. In Section 3, we apply the Selberg-type integrals to derive the exact distributions of Δ in the normal, gamma and beta cases. In Section 4, we apply the results of Section 3 to obtain stochastic bounds for Δ in the normal and gamma cases. In Section 5, we obtain the asymptotic distribution of Δ. Finally, in Section 6, we conclude with some remarks on how the results of Sections 2–5 may be extended to other situations. These extensions include the cases when the random variables $X_1, \dots, X_n$ are i.i.d. F-distributed; scale mixtures of the normal, gamma or F distributions; or even when $X_1, \dots, X_n$ follow a Dirichlet distribution.

2. Selberg's integral.

THEOREM 2.1 [Selberg (1944)]. *Suppose that* α, β *and* z *are complex numbers where* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ *and* $\operatorname{Re} z > -\min\{1/n, \operatorname{Re}\alpha/(n-1), \operatorname{Re}\beta/(n-1)\}$. *Then*

$$\int_0^1 \cdots \int_0^1 \Delta(x_1, \dots, x_n)^z \prod_{j=1}^n x_j^{\alpha-1}(1-x_j)^{\beta-1}\,dx_j = \prod_{j=1}^n \frac{\Gamma(1+jz)\Gamma(\alpha+(j-1)z)\Gamma(\beta+(j-1)z)}{\Gamma(1+z)\Gamma(\alpha+\beta+(n+j-2)z)}. \tag{2.1}$$

There have been several applications of Selberg's integral in statistics and probability theory. As examples, we refer to Karlin and Studden (1966), Askey (1980), Andersson, Brøns and Jensen (1983) and Peddada and Richards (1991). Many important integrals are special or limiting cases of Selberg's integral, and the integral (2.1) has led to the solution of important problems in other areas [Andrews (1986)]. We will need to review two limiting cases of (2.1), the first being an integral also treated by Mehta and Dyson [cf. Mehta (1990)].

COROLLARY 2.2 [Askey (1980)]. *If* $\operatorname{Re} z > -1/n$, *then*

$$(2.2)\qquad \begin{aligned}(2\pi)^{-n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(x_1,\dots,x_n)^z\prod_{j=1}^{n}\exp\left(-x_j^2/2\right)dx_j\\ =\prod_{j=1}^{n}\frac{\Gamma(jz+1)}{\Gamma(z+1)}.\end{aligned}$$

PROOF. In (2.1), set $\alpha=\beta$ and make the change of variables $x_j\to\frac{1}{2}(1+(2\alpha)^{-1/2}x_j)$, $j=1,\dots,n$. After collecting terms, we have

$$\begin{aligned}\int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}}\cdots\int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}}\Delta(x_1,\dots,x_n)^z\prod_{j=1}^{n}\left(1-\frac{x_j^2}{2\alpha}\right)^{\alpha-1}dx_j\\ =\left[2^{2(\alpha-1)}\left(2\sqrt{2\alpha}\right)^{(n-1)z+1}\right]^n\prod_{j=1}^{n}\frac{\Gamma(jz+1)\left(\Gamma(\alpha+(j-1)z)\right)^2}{\Gamma(z+1)\Gamma(2\alpha+(n+j-2)z)}.\end{aligned}$$

Letting $\alpha\to\infty$ and applying Stirling's formula for the gamma function [Whittaker and Watson (1927), subsection 13.6], then (2.2) follows by dominated convergence. □

A second limiting case of (2.1), also derived by Askey (1980), corresponds to the gamma distribution in the same way that (2.1) corresponds to the beta distribution. The proof of this result is also similar to the proof of (2.2). In (2.1), replace x_i by x_i/β and then let $\beta\to\infty$. Then we obtain the following result.

COROLLARY 2.3 [Askey (1980)]. *If* $\operatorname{Re}\alpha>0$ *and* $\operatorname{Re} z>-1/n$, *then*

$$(2.3)\qquad \begin{aligned}\int_0^{\infty}\cdots\int_0^{\infty}\Delta(x_1,\dots,x_n)^z\prod_{j=1}^{n}x_j^{\alpha-1}e^{-x_j}\,dx_j\\ =\prod_{j=1}^{n}\frac{\Gamma(\alpha+(j-1)z)\Gamma(jz+1)}{\Gamma(z+1)}.\end{aligned}$$

Statisticians usually encounter (2.1)–(2.3) in the special case $z=1/2$ within the context of distribution theory of real, symmetric random matrices

[Muirhead (1982)]. Similarly, the case $z = 1$ arises frequently in the physics literature in the study of complex Hermitian random matrices [Mehta (1990)].

3. The exact distributions of Δ.

3.1. *The normal case.* Throughout, we denote by $N(\mu, \sigma)$ the normal distribution with mean μ and standard deviation σ.

LEMMA 3.1. *Suppose that* $X_1, \dots, X_n$ *are i.i.d.* $N(\mu, \sigma)$ *variables. Then for any complex number* k *with* $\mathrm{Re}(k) \geq 0$,

$$E(\Delta^k) = \sigma^{n(n-1)k}\left(\prod_{j=1}^{n} j^{jk}\right)\prod_{1\leq i<j\leq n}\frac{\Gamma(k+i/j)}{\Gamma(i/j)}. \tag{3.1}$$

PROOF. Let $Y_i = (X_i - \mu)/\sigma$, $i = 1, \dots, n$. Since the Y_i are $N(0,1)$ random variables, then it follows from (2.2) that

$$E(\Delta^k) = \sigma^{kn(n-1)}\prod_{j=1}^{n}\frac{\Gamma(jk+1)}{\Gamma(k+1)}. \tag{3.2}$$

Applying Gauss' multiplication formula [Carlson (1977)],

$$\Gamma(z) = k^{z-1/2}(2\pi)^{-(k-1)/2}\prod_{j=1}^{k}\Gamma\left(\frac{z+j-1}{k}\right), \tag{3.3}$$

to decompose the gamma products in (3.2) we obtain

$$\begin{aligned} E(\Delta^k) &= \sigma^{n(n-1)k}\left(\prod_{j=1}^{n} j^{jk}\right)\prod_{j=2}^{n}\frac{\Gamma(k+1/j)\cdots\Gamma(k+(j-1)/j)}{(2\pi)^{(j-1)/2}j^{-1/2}} \\ &= \sigma^{n(n-1)k}\left(\prod_{j=1}^{n} j^{jk}\right)\prod_{j=2}^{n}\frac{\Gamma(k+1/j)\cdots\Gamma(k+(j-1)/j)}{\Gamma(1/j)\cdots\Gamma((j-1)/j)}, \end{aligned}$$

which agrees with (3.1). □

Let $X \sim G(\alpha, \beta)$ denote that the random variable X follows a gamma distribution with shape and scale parameters α and β, respectively. Further, we use the notation $=_{\mathrm{st}}$ to denote stochastic equality (or equality in distribution). Then we have the following stochastic representation for Δ.

THEOREM 3.2. *Let* $X_1, \dots, X_n$ *be i.i.d.,* $N(\mu, \sigma)$, *random variables. Then*

$$\Delta(X_1, \dots, X_n) =_{\mathrm{st}} c_n \prod_{1\leq i<j\leq n} X_{ij}, \tag{3.4}$$

where $c_n = \sigma^{n(n-1)}\prod_{j=1}^{n} j^j$, *and* $\{X_{ij}: 1 \leq i < j \leq n\}$ *are mutually independent with* $X_{ij} \sim G(i/j, 1)$.

PROOF. Since $X_{ij} \sim G(i/j, 1)$ then for $\mathrm{Re}(k) \geq 0$,

$$E(X_{ij}^k) = \frac{\Gamma(k + i/j)}{\Gamma(i/j)}.$$

By Lemma 3.1 and the mutual independence of the X_{ij},

$$\begin{aligned} E\left[\left(\frac{\Delta}{c_n}\right)^k\right] &= \prod_{j=2}^{n} \frac{\Gamma(k + 1/j) \cdots \Gamma(k + (j-1)/j)}{\Gamma(1/j) \cdots \Gamma((j-1)/j)} \\ &= \prod_{j=2}^{n} E(X_{1j}^k) E(X_{2j}^k) \cdots E(X_{j-1,j}^k) \\ &= E\left(\prod_{1 \leq i < j \leq n} X_{ij}\right)^k. \end{aligned} \tag{3.5}$$

By Selberg's formula (2.1), it follows that the characteristic function of the random variable $D = \log(\Delta/c_n)$ is

$$E(e^{itD}) = E\left[\left(\frac{\Delta}{c_n}\right)^{it}\right] = \prod_{j=2}^{n} \frac{\Gamma(it + 1/j) \cdots \Gamma(it + (j-1)/j)}{\Gamma(1/j) \cdots \Gamma((j-1)/j)}, \tag{3.6}$$

and (3.6) is valid for all real t. Now define

$$D^* = \log\left(\prod_{1 \leq i < j \leq n} X_{ij}\right).$$

Then it is not difficult to check that D^* has the same characteristic function as D. Therefore $D =_{\mathrm{st}} D^*$; equivalently, $\exp(D) =_{\mathrm{st}} \exp(D^*)$, and this establishes (3.4). □

Note that the moment generating function of Δ exists only at the origin; this follows immediately from (3.1). Further, these moments do not satisfy the Carleman criterion [Shohat and Tamarkin (1943), page 19] so we cannot deduce Theorem 3.2 directly from the solution to the moment problem. However, Selberg's formula allows us to analytically continue the moments of Δ from integer to complex powers so we are able to circumvent these difficulties by first deriving the characteristic function of $\log \Delta$. This explains why we resorted to Selberg's formula.

3.2. *The gamma case.*

LEMMA 3.3. *Let $X_1, \dots, X_n$ be i.i.d. $G(\alpha, \beta)$ variables. Then for $k = 0, 1, 2, \dots,$*

$$\begin{aligned} E(\Delta^k) &= n^{nk} \beta^{n(n-1)k} \left(\prod_{j=1}^{n-1} j^{2jk}\right) \\ &\quad \times \prod_{0 \leq i < j \leq n-1} \frac{\Gamma(k + (\alpha + i)/j)}{\Gamma((\alpha + i)/j)} \prod_{1 \leq i < j \leq n} \frac{\Gamma(k + (i/j))}{\Gamma(i/j)}. \end{aligned} \tag{3.7}$$

PROOF. Proceeding as in the proof of Lemma 3.1, it follows from (2.3) and the multiplication formula (3.3) that

$$\begin{aligned} E(\Delta^k) &= \beta^{kn(n-1)} \prod_{j=1}^{n} \frac{\Gamma(\alpha + (n-j)k)\Gamma(jk+1)}{\Gamma(\alpha)\Gamma(k+1)} \\ &= \beta^{kn(n-1)} \prod_{j=1}^{n} j^j \prod_{j=1}^{n-1} \frac{\Gamma(\alpha + (n-j)k)}{\Gamma(\alpha)} \\ &\quad \times \prod_{j=2}^{n} \frac{\Gamma(k + 1/j) \cdots \Gamma(k + (j-1)/j)}{\Gamma(1/j) \cdots \Gamma((j-1)/j)}. \end{aligned}$$

Then (3.7) is obtained by again applying the multiplication formula to the individual gamma factors. □

THEOREM 3.4. *Let $X_1, \dots, X_n$ be i.i.d., $G(\alpha, \beta)$, variables. Then*

$$\Delta(X_1, \dots, X_n) =_{\text{st}} c_n \prod_{0 \le i < j \le n-1} X_{ij} \prod_{1 \le i < j \le n} Y_{ij}, \tag{3.8}$$

where $c_n = n^n \beta^{n(n-1)} \prod_{j=1}^{n-1} j^{2j}$, and the random variables $\{X_{ij}\}$ and $\{Y_{ij}\}$ are mutually independent with $X_{ij} \sim G((\alpha + i)/j, 1)$ and $Y_{ij} \sim G(i/j, 1)$.

PROOF. The proof is similar to the proof of Theorem 3.2. First, we use Lemma 3.1 and the independence of the X_{ij} and Y_{ij} to check that, for any nonnegative integer k,

$$E\left[(\Delta/c_n)^k\right] = E\left(\prod_{0 \le i < j \le n-1} X_{ij} \prod_{1 \le i < j \le n} Y_{ij}\right)^k \tag{3.9}$$

Then the rest of the proof follows that of Theorem 3.2. □

3.3. *The beta case.* Throughout, we denote the beta distribution with parameters α, β by $B(\alpha, \beta)$. If $X_1, \dots, X_n$ are i.i.d. $B(\alpha, \beta)$ random variables then, by Selberg's integral formula (2.1),

$$\begin{aligned} E(\Delta^k) &= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n \\ &\quad \times \prod_{j=1}^{n} \frac{\Gamma(\alpha + (n-j)k)\Gamma(\beta + (n-j)k)\Gamma(jk+1)}{\Gamma(\alpha + \beta + (2n-j-1)k)\Gamma(k+1)}. \end{aligned} \tag{3.10}$$

For $n = 2, 3, 4$, we now use (3.10) to derive stochastic representations for $\Delta(X_1, \dots, X_n)$.

PROPOSITION 3.5. *If X_1, X_2 are i.i.d. $B(\alpha, \beta)$, then $\Delta(X_1, X_2) =_{st} Y_1 Y_2 Y_3$, where Y_1, Y_2 and Y_3 are mutually independent*; $Y_1 \sim B(\alpha, \beta)$, $Y_2 \sim B(1/2,$

$(\alpha + \beta)/2)$ *and*

$$Y_3 \sim \begin{cases} B(\min(\alpha,\beta), |\alpha - \beta|/2), & \alpha \neq \beta, \\ 1, & \alpha = \beta. \end{cases}$$

PROOF. When $n = 2$ it follows from Selberg's formula (2.1) and the multiplication formula (3.3) that

$$E(\Delta^k) = 2^{1-\alpha-\beta} \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(k+\alpha)\Gamma(k+\beta)\Gamma(k+\frac{1}{2})}{\Gamma(k+\alpha+\beta)\Gamma(k+\frac{1}{2}(\alpha+\beta))\Gamma(k+\frac{1}{2}(\alpha+\beta+1))}.$$

Since the distribution of Δ is symmetric in α and β then, without loss of generality, we may assume that $\alpha \geq \beta$. If $\alpha > \beta$, then by another application of the duplication formula for the gamma function, we obtain

$$\begin{aligned} E(\Delta^k) &= \frac{\Gamma(k+\alpha)\Gamma(\alpha+\beta)}{\Gamma(k+\alpha+\beta)\Gamma(\alpha)} \frac{\Gamma((\alpha+\beta+1)/2)\Gamma(k+\frac{1}{2})}{\Gamma(1/2)\Gamma(k+\frac{1}{2}(\alpha+\beta+1))} \\ &\quad \times \frac{\Gamma((\alpha+\beta)/2)\Gamma(k+\beta)}{\Gamma(\beta)\Gamma(k+\frac{1}{2}(\alpha+\beta))} \\ &= E(Y_1^k)E(Y_2^k)E(Y_3^k), \end{aligned}$$

where Y_1, Y_2 and Y_3 are as specified above. When $\alpha = \beta$, the same procedure leads to the result

$$E(\Delta^k) = \frac{\Gamma(k+\alpha)\Gamma(2\alpha)\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2}+\alpha)}{\Gamma(k+2\alpha)\Gamma(\alpha)\Gamma(k+\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} = E(Y_1^k)E(Y_2^k),$$

then $Y_3 \equiv 1$. Since the density of Δ is compactly supported, then its distribution is uniquely determined by its moments [Shohat and Tamarkin (1943)]. Then the result follows. □

When $n = 3, 4$, the above procedure leads to the following:

PROPOSITION 3.6. *Suppose* X_1, X_2, X_3 *are i.i.d.* $B(\alpha, \beta)$ *with* $\alpha + \beta \geq 3/2$. *Then* $\Delta(X_1, X_2, X_3) =_{st} \prod_{i=1}^{5} Y_i$, *where* $Y_1, \ldots, Y_5$ *are mutually independent,* $Y_1 = W_1(1 - W_1)$ *with* $W_1 \sim B(\alpha, \beta)$; $Y_2 = W_2(1 - W_2)$ *with* $W_2 \sim B(\alpha, \beta)$; $Y_3 \sim B(1/3, (\alpha + \beta)/3)$, $Y_4 \sim B(2/3, (\alpha + \beta)/3)$,

$$Y_5 \sim \begin{cases} B\left(\frac{1}{2}, \frac{\alpha+\beta}{3} - \frac{1}{2}\right), & \alpha + \beta > \frac{3}{2}, \\ 1, & \alpha + \beta = \frac{3}{2}. \end{cases}$$

If X_1, X_2, X_3, X_4 *are i.i.d.* $B(\alpha, \beta)$ *with* $\alpha + \beta \geq 2$, *then* $\Delta(X_1, \dots, X_4) =_{st} \prod_{i=1}^{9} Y_i$, *where* $Y_1, \dots, Y_9$ *are mutually independent*; $Y_1 = W_1(1 - W_1)$, $Y_2 = W_2(1 - W_2)$, $Y_3 = W_3(1 - W_3)$, *with* $W_1, W_2, W_3 \sim B(\alpha, \beta)$; $Y_4 \sim B(1/2, (\alpha + \beta)/6)$, $Y_5 \sim B(1/3, (\alpha + \beta)/6)$, $Y_6 \sim B(2/3, (\alpha + \beta)/6)$, $Y_7 \sim B(3/4, (\alpha + \beta)/6 + 1/12)$, $Y_8 \sim B(1/4, (\alpha + \beta)/6 - 1/4)$,

$$Y_9 \sim \begin{cases} B\left(\dfrac{1}{2}, \dfrac{\alpha + \beta}{6} - \dfrac{1}{3}\right), & \alpha + \beta > 2, \\ 1, & \alpha + \beta = 2. \end{cases}$$

When $n > 4$ it does not appear that Δ can be represented as simply as above, and we were unable to represent Δ as a product of independent variables.

We close this section with remarks on the nonuniqueness of these stochastic representations. In the normal and gamma cases, nonuniqueness follows from the fact that any gamma variable can be decomposed into products of powers of independent gamma variables (cf. Lemma 4.3). In the beta case, nonuniqueness again follows from a well-known decomposition of any beta variable [cf. Rohatgi (1976), page 216] as a product of an arbitrary number of independent beta variables. Even if the number of independent variables is restricted, nonuniqueness is again encountered in the beta case. For example, if $n = 2$ and $\alpha = \beta > 1/4$ then by Proposition 3.5 we know that $\Delta =_{\text{st}} Y_1 Y_2$ with $Y_1 \sim B(\alpha, \alpha)$ independently of $Y_2 \sim B(1/2, \alpha)$. However it can also be checked that $\Delta =_{\text{st}} W_1 W_2$ where $W_1 = W(1 - W)$, $W \sim B(\alpha, \alpha)$, $W_2 \sim B(1/2, 2\alpha - 1/2)$, and W_1 and W_2 are independent.

4. Stochastic bounds for Δ. In this section, we obtain upper and lower stochastic bounds for Δ in the normal and gamma cases. Here, we will use the notation $X \geq_{\text{st}} Y$ to mean that X is *stochastically greater than* Y; that is, $\Pr(X > t) \geq \Pr(Y > t)$ for all t [cf. Marshall and Olkin (1979)]. Our main result is the following.

THEOREM 4.1. *In the normal case,*

$$\left(\prod_{j=1}^{n} j^j\right)\left(\frac{2\sigma^2}{n(n-1)} G(n(n-1)/4, 1)\right)^{n(n-1)/2} \geq_{st} \Delta(X_1, \dots, X_n).$$

In the gamma case,

$$n^n \left(\prod_{j=1}^{n} j^j\right)\left(\frac{4\beta^2}{n^2(n-1)^2} G_1 G_2\right)^{n(n-1)/2} \geq_{st} \Delta(X_1, \dots, X_n),$$

where $G_1 \sim G(n(n-1)(2\alpha + 1)/4, 1)$, $G_2 \sim G(n(n-1)/4, 1)$, *and* G_1 *is independent of* G_2.

Note that in the normal case, the bound is an equality when $n = 2$.

We will need three preliminary results, two being stochastic analogs of classical formulas for the gamma function, and the third being an inequality involving the exponential distribution. All three results appear in the recent paper of Gordon (1989). For the first two results we will provide proofs which are alternative to those given by Gordon (1989). Below, we let $\gamma = -\Gamma'(1) = 0.5772\ldots$ denote Euler's constant.

LEMMA 4.2 [Gordon (1989)]. *Let* $G \sim G(\alpha, 1)$. *Then*

$$\log G =_{st} -\gamma + \sum_{j=0}^{\infty} \left(\frac{1}{j+1} - \frac{Y_j}{\alpha + j} \right)$$

where the $\{Y_j\}$ *are i.i.d. exponential random variables with mean* 1.

PROOF. Let Y denote the series on the right-hand side. To prove that Y is well defined, we need to show that Y is finite, almost surely (a.s.). Since

$$\sum_{j=0}^{\infty} E\left(\frac{1}{j+1} - \frac{Y_j}{\alpha + j} \right) = (\alpha - 1) \sum_{j=0}^{\infty} \frac{1}{(j+1)(\alpha + j)} < \infty$$

and

$$\sum_{j=1}^{\infty} \operatorname{Var}\left(\frac{1}{j+1} - \frac{Y_j}{\alpha + j} \right) = \sum_{j=1}^{\infty} \left(\frac{1}{\alpha + j} \right)^2 \operatorname{Var}(Y_j) = 2 \sum_{j=1}^{\infty} \left(\frac{1}{\alpha + j} \right)^2 < \infty,$$

then, by Ash [(1972), page 73], Y is finite, a.s. Next, by independence of the Y_j's, we obtain the characteristic function of Y as

$$\begin{aligned} E(e^{itY}) &= E\left(e^{-it\gamma} \prod_{j=0}^{\infty} \exp\left(\frac{it}{j+1} - \frac{itY_j}{\alpha + j} \right) \right) \\ &= e^{-it\gamma} \prod_{j=0}^{\infty} \exp\left(\frac{it}{j+1} \right) E(e^{-itY_j/\alpha + j}) \\ &= e^{-it\gamma} \prod_{j=0}^{\infty} \exp\left(\frac{it}{j+1} \right) \left(1 + \frac{it}{\alpha + j} \right)^{-1}. \end{aligned}$$

It is well known [Carlson (1977)] that $\gamma = \sum_{j=1}^{\infty} (j^{-1} - \log(1 + j^{-1}))$ and

$$\Gamma(z) = \frac{1}{z} \prod_{j=1}^{\infty} \left(1 + \frac{1}{j} \right)^z \left(1 + \frac{z}{j} \right)^{-1}, \qquad \operatorname{Re} z > 0.$$

Therefore

$$E(e^{itY}) = \exp\left(-it\sum_{j=1}^{\infty}\left(\frac{1}{j} - \log\left(1 + \frac{1}{j}\right)\right)\right)\prod_{j=0}^{\infty}\exp\left(\frac{it}{j+1}\right)\left(1 + \frac{it}{\alpha + j}\right)^{-1}$$

$$= \prod_{j=1}^{\infty}\left(1 + \frac{1}{j}\right)^{it}\left(\frac{\alpha + j}{\alpha + j + it}\right)\left(\frac{\alpha}{\alpha + it}\right)$$

$$= \frac{\Gamma(\alpha + it)}{\Gamma(\alpha)}.$$

Defining $G = e^Y$, then

$$E(G^{it}) = E(e^{itY}) = \frac{\Gamma(\alpha + it)}{\Gamma(\alpha)},$$

which is the characteristic function of a $G(\alpha, 1)$ random variable. Therefore, $\log G = Y \sim G(\alpha, 1)$. □

Now we present the stochastic version of Gauss' multiplication formula (3.3).

LEMMA 4.3 [Gordon (1989)]. *Suppose $G_0, \dots, G_{r-1}$ are mutually independent gamma variables, with $G_j \sim G(\alpha + j/r, \beta)$. Then $G = r\prod_{j=0}^{r-1} G_j^{1/r} \sim G(r\alpha, \beta)$.*

PROOF. Without loss of generality, assume $\beta = 1$. Then for any nonnegative integer k, it follows by mutual independence of the G_j and the multiplication formula (3.3), that

$$E(G^k) = r^k \prod_{j=0}^{r-1} E\left(G_j^{k/r}\right)$$

$$= \prod_{k=0}^{r-1} \frac{\Gamma(\alpha + (j + k)/r)}{\Gamma(\alpha + j/r)}$$

$$= \frac{\Gamma(r\alpha + k)}{\Gamma(r\alpha)}.$$

This expression is the kth moment of a gamma variable, $G(r\alpha, 1)$. Since the gamma distribution is uniquely determined by its moments, then the result follows. □

LEMMA 4.4 [Gordon (1989)]. *Let Y_1 and Y_2 be i.i.d., exponential random variables. Then for fixed c and $t > 0$,*

$$\Pr\left(\frac{Y_1}{c - \delta} + \frac{Y_2}{c + \delta} > t\right)$$

is monotone increasing in δ for $0 < \delta < c$.

The following result is crucial in the derivation of the stochastic bounds for Δ.

LEMMA 4.5. *Let X_{ij} be independently distributed as $G(i/j, 1)$, $1 \le i < j \le n$. Then*

$$G(n(n-1)/4, 1) \ge_{st} \tfrac{1}{2}n(n-1)\Bigg(\prod_{1 \le i < j \le n} X_{ij}\Bigg)^{2/(n(n-1))}.$$

PROOF. By Lemma 4.2,

$$\log \prod_{1 \le i < j \le n} X_{ij} =_{\mathrm{st}} -\frac{1}{2}n(n-1)\gamma + \sum_{1 \le i < j \le n} \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{Y_{i,j,k}}{k+i/j} \right), \tag{4.1}$$

where the $Y_{i,j,k}$ are i.i.d. $G(1,1)$ variables. For a given j and k, let $\delta = (j-2i)/2j$. Then it follows from Lemma 4.4 that

$$\begin{aligned} \frac{Y_{i,j,k}}{k+i/j} + \frac{Y_{j-i,j,k}}{k+(j-i)/j} &= \frac{Y_{i,j,k}}{k+\frac{1}{2}-\delta} + \frac{Y_{j-i,j,k}}{k+\frac{1}{2}+\delta} \\ &\ge_{\mathrm{st}} \frac{Y_{i,j,k}}{k+\frac{1}{2}} + \frac{Y_{j-i,j,k}}{k+\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\log \prod_{1 \le i < j \le n} X_{ij} \le_{\mathrm{st}} -\frac{1}{2}n(n-1)\gamma + \sum_{1 \le i < j \le n} \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{Y_{i,j,k}}{k+\frac{1}{2}} \right).$$

For each k, let the $\{Y'_{l,k}: 0 \le l \le \frac{1}{2}n(n-1) - 1\}$ denote the set of all $Y_{i,j,k}$ as i varies over $\{1, \dots, n-1\}$. Then

$$\begin{aligned} \log \prod_{1 \le i < j \le n} X_{ij} &\le_{\mathrm{st}} -\frac{1}{2}n(n-1)\gamma + \sum_{1 \le i < j \le n} \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{Y_{i,j,k}}{k+\frac{1}{2}} \right) \\ &\le_{\mathrm{st}} -\frac{1}{2}n(n-1)\gamma \\ &\quad + \sum_{l=0}^{(1/2)n(n-1)-1} \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{Y'_{l,k}}{k+\frac{1}{2}+2l/(n(n-1))} \right) \\ &= \log \prod_{l=0}^{(1/2)n(n-1)-1} G_l, \end{aligned}$$

by Lemma 4.2, with each $G_l \sim G(1/2 + 2l/(n(n-1)), 1)$. Then the result follows from Lemma 4.3. □

LEMMA 4.6. *Let X_{ij} be independently distributed as $G((\alpha + i)/j, 1)$, $0 \leq i < j \leq n - 1$. Then*

$$G((2\alpha + 1)n(n - 1)/4, 1) \geq_{st} \tfrac{1}{2}n(n - 1)\left(\prod_{0 \leq i < j \leq n-1} X_{ij} \right)^{2/n(n-1)}.$$

PROOF. Clearly,

$$\log \prod_{0 \leq i < j \leq n-1} X_{ij} = \log \prod_{1 \leq j \leq n-1} X_{0j} + \log \prod_{1 \leq i < j \leq n-1} X_{ij}.$$

Applying Lemma 4.2 to each term, and then stochastically majorizing both terms as in the proof of Lemma 4.5, we obtain

$$\log \prod_{0 \leq i < j \leq n-1} X_{ij} \leq_{\text{st}} -\frac{1}{2}n(n - 1)\gamma + \sum_{j=1}^{n-1} \sum_{k=0}^{\infty} \left(\frac{1}{k + 1} - \frac{Y_{0,j,k}}{k + \alpha/j} \right)$$

$$+ \prod_{1 \leq i < j \leq n-1} \sum_{k=0}^{\infty} \left(\frac{1}{k + 1} - \frac{Y_{i,j,k}}{k + \alpha/j + \frac{1}{2}} \right),$$

where $\{Y_{i,j,k}\}$ are i.i.d. exponential, mean 1, random variables. In both infinite series, we replace the quantities $k + \alpha/j$ and $k + \alpha/j + 1/2$ by the larger value $k + \alpha + 1/2 + 2j/(n(n - 1))$. Then, again as in the proof of Lemma 4.5, we have

$$\log \prod_{0 \leq i < j \leq n-1} X_{ij} \leq_{\text{st}} -\frac{1}{2}n(n - 1)\gamma$$

$$+ \sum_{l=0}^{(1/2)n(n-1)-1} \sum_{k=0}^{\infty} \left(\frac{1}{k + 1} - \frac{Y'_{l,k}}{k + \alpha + \frac{1}{2} + 2l/(n(n - 1))} \right),$$

where the $\{Y'_{l,k}\}$ are a renaming of the $\{Y_{i,j,k}\}$. Then the conclusion again follows from Lemma 4.3. □

PROOF OF THEOREM 4.1. In the normal case the upper bound follows from the stochastic representation given in Theorem 3.2 and Lemma 4.4. In the gamma case, the bound follows from Theorem 3.4 and Lemma 4.5. □

To end this section, we derive stochastic lower bounds for Δ in the normal and gamma cases. As before, the constant c_n is defined as in Theorem 3.2 (for the normal case) or in Theorem 3.4 (for the gamma case).

PROPOSITION 4.7. *In the normal case,*

$$\Delta(X_1, \dots, X_n) \geq_{st} n^n \sigma^{n(n-1)} \prod_{j=2}^{n} G_j^{j-1},$$

where $G_2, \dots, G_n$ are mutually independent and $G_j \sim G(j^{-1}, 1)$.

In the gamma case,

$$\Delta(X_1, \ldots, X_n) \geq_{st} n^n \beta^{n(n-1)} \prod_{j=2}^{n} G_{1j}^{j-1} G_{2j}^{j-1},$$

where the G_{ij} are mutually independent, $G_{1j} \sim G(j^{-1}, 1)$, and $G_{2j} \sim G(\alpha, 1)$.

PROOF. In the normal case, by Theorem 3.2, $\Delta/c_n =_{\text{st}} \prod_{1 \leq i < j \leq n} X_{ij}$ where the mutually independent $X_{ij} \sim G(i/j, 1)$. If $1 \leq i < j \leq n$, then it is straightforward to check that

$$\frac{i}{j} \geq \frac{1}{j(j-1)} + \frac{i-1}{j-1}.$$

Since the $G(\alpha, 1)$ distributions are stochastically increasing in α (i.e., $G(\alpha, 1) \geq_{\text{st}} G(\beta, 1)$ for $\alpha \geq \beta$), then for any fixed j, $2 \leq j \leq n$,

$$\prod_{i=1}^{j-1} X_{ij} \geq_{\text{st}} \prod_{i=1}^{j-1} G\left(\frac{1}{j(j-1)} + \frac{i-1}{j-1}, 1\right) =_{\text{st}} (j-1)^{-(j-1)} \left[G(j^{-1}, 1)\right]^{j-1},$$

where the stochastic equality follows from Lemma 4.3. Then the stochastic representation for Δ implies that

$$\Delta \geq_{\text{st}} c_n \prod_{j=2}^{n} (j-1)^{-(j-1)} \left[G(j^{-1}, 1)\right]^{j-1} = n^n \sigma^{n(n-1)} \prod_{j=2}^{n} G_j^{j-1}.$$

In the gamma case, the proof is similar. Starting with the stochastic representation in Theorem 3.4,

$$\Delta(X_1, \ldots, X_n) =_{\text{st}} c_n \prod_{0 \leq i < j \leq n-1} X_{ij} \prod_{1 \leq i < j \leq n} Y_{ij},$$

we first obtain a lower bound for $\prod_{1 \leq i < j \leq n} Y_{ij}$ by proceeding exactly as in the normal case. Further, since the X_{ij} are i.i.d. $G((\alpha + i)/j, 1)$ then by Lemma 4.3,

$$\prod_{0 \leq i < j \leq n-1} X_{ij} =_{\text{st}} \prod_{j=2}^{n} (j-1)^{-(j-1)} G_{2j}^{j-1}$$

where the G_{2j} are i.i.d. $G(\alpha, 1)$. Combining these results, we obtain the desired bound. □

Note also that both lower bounds are equalities for $n = 2$.

5. The asymptotic distribution of Δ. Suppose $X_1, \ldots, X_n$ is a random sample from an absolutely continuous univariate distribution F with finite mean μ and variance σ^2. Define

$$U_n = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} \log(X_i - X_j)^2. \tag{5.1}$$

Then U_n is the U-statistic for the parameter $E[\log(X_1 - X_2)^2]$, with the corresponding kernel $h(x, y) = \log(x - y)^2$ [cf. Serfling (1980), Chapter 5]. Denoting $\operatorname{Var}\{E[h(X_1, X_2)|X_1]\}$ by ξ, we have the following result.

THEOREM 5.1. *Let $X_1, \dots, X_n$ be i.i.d., absolutely continuous random variables with finite mean μ and variance σ^2 and let $V_n = \log \Delta(X_1, \dots, X_n)$. Then as $n \to \infty$,*

$$\frac{V_n - E(V_n)}{\sqrt{\operatorname{Var}(V_n)}} \to_d N(0,1),$$

provided that $\operatorname{Var}(h(X_1, X_2)) < \infty$.

PROOF. Since $\operatorname{Var}(h(X_1, X_2)) < \infty$, we have $E(h^2) < \infty$. Also, the assumption of absolute continuity of the X_i implies that $E[h(X_1, X_2)|X_1]$ is nonconstant almost surely; therefore, $\xi > 0$. By the theory of U-statistics [Serfling (1980), page 192],

$$\frac{U_n - E(U_n)}{\sqrt{4\xi/n}} \to_d N(0,1).$$

Now $4\xi \le 2\operatorname{Var}(h(X_1, X_2))$ and $\operatorname{Var}(U_n) = 4\xi n^{-1} + O(n^{-2})$, as $n \to \infty$ [Serfling (1980), page 183]. Therefore, by Slutsky's theorem and the fact that $V_n = \binom{n}{2} U_n$, the result follows immediately. □

COROLLARY 5.2. *Let $X_1, \dots, X_n$ be i.i.d. random variables from a normal, gamma or beta population and let $V_n = \log \Delta(X_1, \dots, X_n)$. Then as $n \to \infty$,*

$$\frac{V_n - E(V_n)}{\sqrt{\operatorname{Var}(V_n)}} \to_d N(0,1). \tag{5.2}$$

PROOF. Since $E[e^{th(X_1, X_2)}] = E[(X_1 - X_2)^{2t}]$ then, by Selberg's integral (2.1), the moment generating function of $h(X_1, X_2)$ exists in a sufficiently small neighborhood of the origin. Therefore $\operatorname{Var}(h(X_1, X_2)) < \infty$, and the result follows from Theorem 5.1. □

REMARK 5.3. (a) If $E(|h|^3) < \infty$, a condition which holds in the normal, gamma or beta cases, it also follows from Serfling [(1980), page 193, Theorem B] that the rate of convergence in (5.2) is $O(n^{-1/2})$. In the normal and gamma cases, we can obtain the asymptotic normality of V_n more directly by using Theorems 3.2 and 3.4 to represent $\log \Delta$ as a sum of independent random variables and applying central limit theory.

(b) In the normal, beta and gamma cases, explicit formulas for the mean and variance of V_n can be calculated using Selberg's integral [Lu (1991)]. For example, in the normal case, setting $W_n = \log(\Delta/\sigma^{n(n-1)})$, we have $E(W_n) = -\gamma n(n-1)/2$ and $\operatorname{Var}(W_n) = \pi^2 n(n-1)(2n+5)/36$. In the gamma case, if we let $W_n = \log(\Delta/\beta^{n(n-1)})$, then

$$E(W_n) = -\frac{1}{2}n(n-1)\left[2\gamma + \frac{1}{\alpha} - \sum_{m=1}^{\infty}\left(\frac{1}{m} - \frac{1}{m+\alpha}\right)\right]$$

and

$$\operatorname{Var}(W_n) = \frac{1}{36}\pi^2 n(n-1)(2n+5) + \frac{1}{6}n(n-1)(2n-1)\sum_{m=0}^{\infty}\left(\frac{1}{m+\alpha}\right)^2.$$

(c) There is also the problem of developing the asymptotic distribution of Δ when $X_1, \dots, X_n$ are i.i.d. with distributions other than those considered in Theorem 5.1. In this situation, asymptotic results can be obtained from the results of Rubin and Vitale (1980); the limiting distributions can be represented as weighted sums of infinite products of Hermite polynomials in standard normal distributions.

To complete this section we comment on the numerical behavior of the asymptotic distribution and the stochastic bounds in the normal and gamma cases. In Figure 1, which pertains to the normal case, the upper and lower bounds and the asymptotic normal distribution are compared with the exact distribution of Δ for various values of n under the assumption that $X_1, \dots, X_n$ are i.i.d. $N(0, 1)$ variables. Define the random variables $R_1 = (V_n - E(V_n))/\sqrt{\operatorname{Var}(V_n)}$, the standardized form of $V_n = \log \Delta$; $R_2 = (\log U - E(V_n))/\sqrt{\operatorname{Var}(V_n)}$, where U denotes the stochastic upper bound given in Theorem 4.1; and $R_3 = (\log L - E(V_n))/\sqrt{\operatorname{Var}(V_n)}$, where L denotes the stochastic lower bound given in Proposition 4.7. Further, denote the distributions of R_i by F_i, $i = 1, 2, 3$. Then Figure 1 provides plots of the simulated values of the differences $F_1(t) - F_2(t)$, $F_1(t) - \Phi(t)$ and $F_1(t) - F_3(t)$ against t, where $\Phi(t)$ denotes the standard normal distribution function. The simulated empirical distributions of the R_i were performed by Monte Carlo methods using the well-known statistical software package SAS. For each plot, 100,000 realizations of each R_i were generated.

From these plots it becomes clear that the upper bound is a better approximation to the exact distribution than the lower bound. This is to be expected, given the relative ease with which the lower bound was derived and the simplicity of the upper bound. Further, both bounds worsen as n increases. The plots also illustrate the rapid convergence to normality of the distribution of R_1.

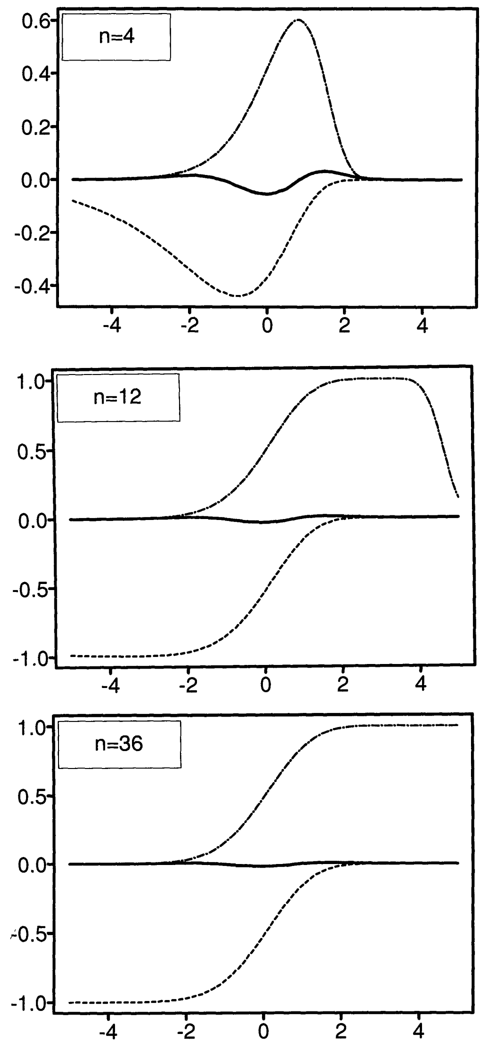

FIG. 1. *Comparisons with the exact distribution of* log Δ *in the normal case.*

In the gamma case we assume, without loss of generality, that $\beta \equiv 1$. With $\alpha = 4$, Figure 2 provides plots of the differences $F_1(t) - F_2(t)$, $F_1(t) - \Phi(t)$ and $F_1(t) - F_3(t)$, where the F_i are defined as before and again 100,000 realizations of each R_i were generated.

It is also noteworthy that the functions plotted in Figure 2 are quite robust to the choice of α. Consider, for example, the function $F_1(t) - \Phi(t)$. With $n = 4$, this function was plotted for $\alpha = 1$ and $\alpha = 10$; it was observed that the maximum variation between the two graphs was less than 0.02. Similar remarks apply to the other two functions plotted in Figure 2.

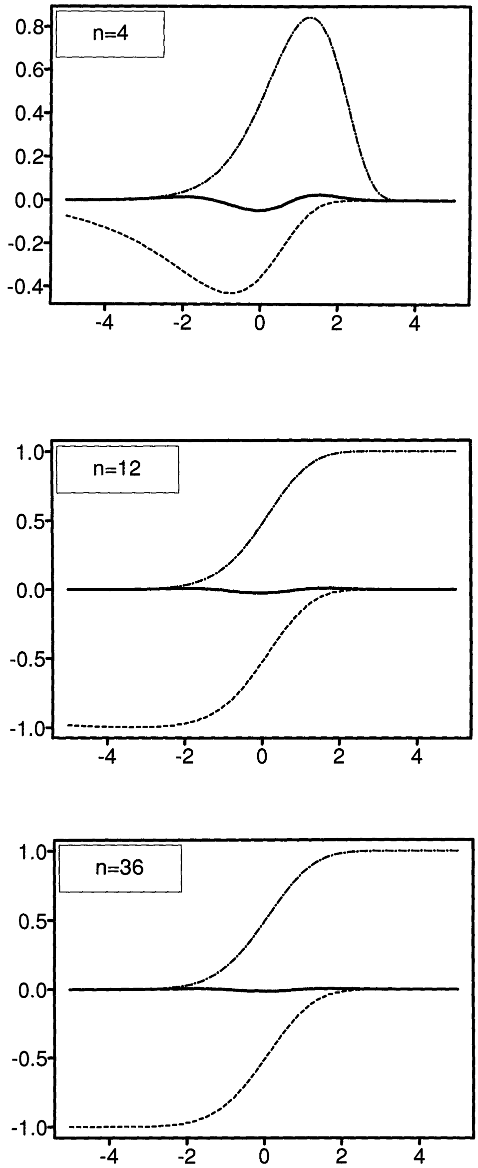

FIG. 2. *Comparisons with the exact distribution of* $\log \Delta$ *in the gamma case,* $\alpha = 4$.

6. Concluding remarks. The basic problem of deducing the exact distribution of $\Delta(X_1, \ldots, X_n)$ can be considered for any random vector $(X_1, \ldots, X_n)$. In general, this is a difficult problem since it requires evaluation of a Selberg-type integral for the distribution of $\Delta(X_1, \ldots, X_n)$. If $X_1, \ldots, X_n$ are i.i.d. F random variables then the corresponding Selberg integral has been obtained by transformations from Selberg's beta integral [Askey (1980)]. If $(X_1, \dots, X_n)$ has a Dirichlet distribution, then the moments of $\Delta(X_1, \dots, X_n)$ can be obtained from a Selberg-type integral of Askey and Richards (1989). Therefore, in these two cases, the distribution of $\Delta(X_1, \dots, X_n)$ can be analyzed as in the normal, gamma and beta cases. More generally, since Δ is homogeneous, we can evaluate the moments of $\Delta(SX_1, \dots, SX_n)$ if the mixing variable S is independent of $(X_1, \dots, X_n)$, and $X_1, \dots, X_n$ have distributions belonging to any of the classes considered above. In principle, these moments can then be analyzed as before.

There is also the problem of deriving stochastic representations for the random variable $\tilde{\Delta} = \Pi_{i<j}(X_i - X_j)$ instead of its square, $\Delta(X_1, \dots, X_n)$. In the case when $(X_1, \dots, X_n)$ is compactly supported, these follow from the results for Δ. In this situation, once it has been proved that $\Delta =_{\text{st}} \Pi X_{ij}$ for mutually independent, nonnegative, X_{ij}, then it is straightforward to check that if U is uniformly distributed on the set $\{1, -1\}$ then $\tilde{\Delta}$ and $U \Pi X_{ij}^{1/2}$ have the same moment sequence; hence, stochastic equality follows from the fact that both variables are compactly supported.

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STATISTICAL RESEARCH DIVISION
BUREAU OF THE CENSUS
WASHINGTON, D.C. 20233

DIVISION OF STATISTICS
HALSEY HALL TRAILERS
UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA 22903