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GOODNESS OF FIT TESTS FOR SPECTRAL DISTRIBUTIONS[1]

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The spectral distribution function of a stationary stochastic process standardized by dividing by the variance of the process is a linear function of the autocorrelations. The integral of the sample standardized spectral density (periodogram) is a similar linear function of the autocorrelations. As the sample size increases, the difference of these two functions multiplied by the square root of the sample size converges weakly to a Gaussian stochastic process with a continuous time parameter. A monotonic transformation of this parameter yields a Brownian bridge plus an independent random term. The distributions of functionals of this process are the limiting distributions of goodness of fit criteria that are used for testing hypotheses about the process autocorrelations. An application is to tests of independence (flat spectrum). The characteristic function of the Cramér–von Mises statistic is obtained; inequalities for the Kolmogorov–Smirnov criterion are given. Confidence regions for unspecified process distributions are found.

1. Introduction. A model used frequently for time series analysis is a stationary stochastic process. If the process is Gaussian, it is completely determined by the mean of the process (a location parameter), the variance of the process (a scale parameter) and the sequence of autocorrelations. The analysis of time series differs from many other statistical analyses because of the possible dependence among observations; that dependence may be characterized by the autocorrelation sequence. For any time series analysis it is essential to make inferences about the autocorrelations.

The Fourier transform of the autocorrelation sequence provides an alternative view of the pattern of dependence. For many purposes it may be more enlightening. In this paper we consider the standardized spectral distribution function as an appropriate description of the pattern of dependence, and we study problems of inference concerning it. "Standardized" means that the spectral distribution is defined in terms of correlations, rather than covariances. The same information is contained in the autocorrelation sequence, the standardized spectral density and the standardized spectral distribution, but the three forms differ in presentation.

The first inference problem treated here is the testing of a null hypothesis that completely specifies the pattern of dependence; for example, the null hypothesis might be that all of the autocorrelations are zero or, equivalently,

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that the spectral density is constant. To test this hypothesis we compare the sample standardized spectral distribution with the process standardized spectral distribution by means of a goodness of fit criterion, such as the Cramér–von Mises criterion or the Kolmogorov–Smirnov criterion. Asymptotic and other approximate distributions are obtained. The mathematics is similar to that of goodness of fit tests of probability distributions, but differs in an essential way. A goodness of fit test usually is consistent against all alternatives, in this case against all correlation structures different from the null hypothesis.

The Kolmogorov–Smirnov criterion can be inverted to give a confidence region for an unspecified standardized spectral distribution. Such a confidence region can be used to infer the increase in the distribution over various intervals of frequency.

Grenander and Rosenblatt (1952, 1957) studied the asymptotic distribution of the difference between the conventional sample spectral distribution function and the conventional process spectral distribution function. They argued that as a process it converges to Brownian motion with a transformed time parameter under the condition that the eighth-order moments of the innovations in the stationary linear process are finite. They proved that the Kolmogorov–Smirnov criterion for the conventional spectral distribution converges in distribution to the supremum of the limit process. This unstandardized spectral distribution, however, is not suited to questions of dependence (i.e., patterns of correlation), and the limiting distributions depend on fourth-order cumulants.

Bartlett (1954, 1966) proposed the sample standardized spectral distribution for testing hypotheses about correlations and asserted that the asymptotic distribution would not depend on fourth-order cumulants, but he did not find any of these distributions. Bartlett treated in more detail an analogue, namely, the integral (or sum) of the sample spectral density (periodogram) divided by the hypothetical process spectral density. This definition leads to the Brownian bridge, and the maximum of the difference between this function of the frequency and the frequency (over $[0, \pi]$) has the asymptotic distribution of the Kolmogorov–Smirnov statistic for goodness of fit of probability distributions. Priestley [(1981), Section 6.2.6] summarized these developments. See also Dzhaparidze and Osidze (1980).

Dahlhaus (1985a) showed that the difference between the sample and process standardized spectral distributions multiplied by the square root of the sample size converges weakly to a Gaussian process under several alternative conditions, always assuming finite eighth-order moments. He obtained the covariance function, but expressed it differently from the form used in this paper. He showed that the supremum of the absolute value of the limiting process does not have the Kolmogorov–Smirnov distribution in general and expressed the probability in terms of a boundary crossing probability involving the Brownian motion process. Dahlhaus (1988) gave a brief formal treatment of the problem with estimated parameters.

The thrust of this paper is to develop the treatment of tests of goodness of fit and confidence regions based on the knowledge of the limiting Gaussian distribution to actual applications. This study includes methods of computing

the goodness of fit statistics, finding their limiting distributions, providing probability inequalities and developing asymptotic confidence regions. As noted above, in general the process with transformed time parameter is different from the Brownian bridge. The limiting distributions are valid under weak conditions, not requiring eighth-order moments or stationarity.

2. The empirical process. Consider a stationary stochastic process $\{y_t\}$, $t = \ldots, -1, 0, 1, \ldots$, with $\mathcal{E}y_t = 0$, autocovariance function

$$\mathcal{E}y_t y_{t+h} = \sigma(h), \qquad h = \ldots, -1, 0, 1, \ldots, \tag{2.1}$$

and autocorrelation function

$$\rho_h = \sigma(h)/\sigma(0), \qquad h = \ldots, -1, 0, 1, \ldots . \tag{2.2}$$

We define the standardized spectral density as

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho_h \cos \lambda h, \qquad -\pi \leq \lambda \leq \pi. \tag{2.3}$$

Note that the coefficients of the trigonometic functions are the autocorrelations, not the autocovariances. The Fourier transform of the standardized spectral density is

$$\rho_g = \int_{-\pi}^{\pi} f(\lambda) \cos \lambda g \, d\lambda, \qquad g = \ldots, -1, 0, 1, \ldots . \tag{2.4}$$

Knowledge of the standardized spectral density is equivalent to knowledge of the autocorrelations. The pattern of correlation can be described equivalently in terms of the autocorrelations or the standardized spectral density.

Since $f(\lambda) = f(-\lambda)$, we define the standardized spectral distribution as

$$F(\lambda) = 2\int_0^{\lambda} f(\nu)\, d\nu = \frac{1}{\pi}\left(\lambda + 2 \sum_{h=1}^{\infty} \rho_h \frac{\sin \lambda h}{h}\right). \tag{2.5}$$

Note that $F(\pi) = 1$; the standardized spectral distribution has the properties (nonnegative increments) of a probability distribution on $[0, \pi]$. In this paper we shall be concerned with inference about the standardized spectral density or distribution.

Inference is based on a sample $y_1, \ldots, y_T$. We define the sample autocovariance sequence

$$c_h = c_{-h} = \frac{1}{T} \sum_{t=1}^{T-h} y_t y_{t+h}, \qquad h = 0, 1, \ldots . \tag{2.6}$$

The sample autocovariance is a biased estimator of the process autocovariance ($h > 0$), but it is asymptotically unbiased. We define the sample autocorrelation sequence

$$r_h = r_{-h} = \frac{c_h}{c_0}, \qquad h = 0, 1, \ldots, \tag{2.7}$$

the standardized sample spectral density (popularly mislabelled as the peri-

odogram)

$$(2.8)\quad I_T(\lambda) = \frac{1}{2\pi c_0}\left|\sum_{t=1}^{T} y_t e^{i\lambda t}\right|^2 = \frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1} r_h \cos\lambda h, \qquad -\pi \le \lambda \le \pi,$$

and the standardized sample spectral distribution function

$$(2.9)\qquad \hat{F}_T(\lambda) = 2\int_0^{\lambda} I_T(\nu)\,d\nu = \frac{1}{\pi}\left(\lambda + 2\sum_{h=1}^{T-1} r_h \frac{\sin\lambda h}{h}\right).$$

We shall study inference based on $\sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)]$, $0 \le \lambda \le \pi$; the limiting distribution will be obtained as $T \to \infty$.

Because patterns of dependence can be described in terms of the autocorrelations, the standardized spectral density and distribution are relevant to questions of dependence, rather than the usual functions defined in terms of autocovariances; the scale parameter of the process is irrelevant. An additional advantage of the standardization is that the asymptotic distributions are valid under much more general conditions than without the standardization, but general conditions will be stated later in another paper [Anderson (1993)]. Another advantage is that $F(\lambda)$ and $\hat{F}_T(\lambda)$ have properties of theoretical and empirical probability distribution functions, respectively.

The asymptotic theory is developed for linear processes

$$(2.10)\qquad y_t = \sum_{s=0}^{\infty} \gamma_s u_{t-s}, \qquad t = \ldots, -1, 0, 1, \ldots,$$

where $\sum_{s=0}^{\infty}\gamma_s^2 < \infty$, $\mathscr{E}u_t = 0$, $\mathscr{E}u_t^2 = \sigma^2$ and $\mathscr{E}u_t u_s = 0$, $t \ne s$. In particular, if the u_t's are independently identically distributed,

$$(2.11)\qquad \sum_{s=0}^{\infty} |\gamma_s| < \infty$$

and

$$(2.12)\qquad \sum_{s=0}^{\infty} \sqrt{s}\,\gamma_s^2 < \infty,$$

then for any integer H

$$(2.13)\qquad \left[\sqrt{T}(r_1 - \rho_1), \ldots, \sqrt{T}(r_H - \rho_H)\right] \to_{\mathrm{d}} N(0, W),$$

where the (g, h) element of W is

$$(2.14)\qquad w_{gh} = \sum_{r=-\infty}^{\infty}\left(\rho_{r+g}\rho_{r+h} + \rho_{r-g}\rho_{r+h} - 2\rho_h\rho_r\rho_{r+g} - 2\rho_g\rho_r\rho_{r+h} + 2\rho_g\rho_h\rho_r^2\right).$$

Note that $\sigma(h) = \sigma^2\sum_{s=0}^{\infty}\gamma_s\gamma_{s+h}$, $h = 0, 1, \ldots$ and $\rho_h = \sum_{s=0}^{\infty}\gamma_s\gamma_{s+h}/\sum_{s=0}^{\infty}\gamma_s^2$, $h = 0, 1, \ldots$. [If y_t is defined by (2.10), $\{\rho_h\}$, $f(\lambda)$ and $F(\lambda)$ are defined even if the process is not stationary.]

The limiting distribution (2.13) was given by Bartlett (1946) under the (implicit) assumption that $\mathscr{E}u_t^4 < \infty$. That the limiting distribution is valid under the assumption of only $\mathscr{E}u_t^2 < \infty$ was shown by Anderson (1959) for autoregressive processes and by Anderson and Walker (1964) for $y_t = \sum_{s=-\infty}^{\infty}\gamma_s u_{t-s}$ and $\sum_{s=-\infty}^{\infty}|s|\gamma_s^2 < \infty$. Hannan and Heyde (1972) relaxed the condition on $\{\gamma_s\}$ to (2.12) when the sum was over $0, 1, 2, \ldots$ and the condition of iid u_t to martingale differences. Anderson (1992) has further relaxed the conditions on the martingale differences.

Consider

$$\begin{aligned}\sqrt{T}\left[\hat{F}_T(\lambda) - F(\lambda)\right] &= \frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}\sqrt{T}(r_h - \rho_h)\\ &\quad - \frac{2}{\pi}\sum_{h=T}^{\infty}\frac{\sin\lambda h}{h}\sqrt{T}\rho_h.\end{aligned} \tag{2.15}$$

We treat $\sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)]$ as a stochastic process on $[0, \pi]$. As $T \to \infty$, this process converges weakly to a Gaussian process with covariance function

$$\begin{aligned}&4\pi\{G[\min(\lambda,\nu)] - G(\lambda)F(\nu) - F(\lambda)G(\nu) + G(\pi)F(\lambda)F(\nu)\}\\ &\quad = 4\pi G(\pi)\left\{\frac{G[\min(\lambda,\nu)]}{G(\pi)} - \frac{G(\lambda)}{G(\pi)}\frac{G(\nu)}{G(\pi)}\right.\\ &\qquad \left. + \left[\frac{G(\lambda)}{G(\pi)} - F(\lambda)\right]\left[\frac{G(\nu)}{G(\pi)} - F(\nu)\right]\right\},\end{aligned} \tag{2.16}$$

where

$$G(\lambda) = 2\int_0^{\lambda} f^2(\nu)\,d\nu. \tag{2.17}$$

The first term in (2.16) was given by Grenander and Rosenblatt (1957). Durlauf (1991) derived the special case of (2.16) when $f(\lambda) = 1/(2\pi)$. Dahlhaus (1985b) gave the first form of the covariance function. See also Shaman (1971).

We can simplify the covariance function of the process by making the monotonic transformation

$$u = \frac{G(\lambda)}{G(\pi)}, \qquad 0 \le \lambda \le \pi, \tag{2.18}$$

to $0 \le u \le 1$. The inverse transformation [defined properly if $f(\lambda) > 0$, $0 \le \lambda \le \pi$] is

$$\lambda = G^{-1}[G(\pi)u], \qquad 0 \le u \le 1. \tag{2.19}$$

The covariance function of the limiting distribution of

$$Y_T(u) = \sqrt{T}\left(\hat{F}_T\{G^{-1}[G(\pi)u]\} - F\{G^{-1}[G(\pi)u]\}\right) \tag{2.20}$$

is

$$4\pi G(\pi)\{\min(u,v) - uv + q(u)q(v)\}, \qquad 0 \le u, v \le 1, \tag{2.21}$$

where

$$q(u) = u - F\{G^{-1}[G(\pi)u]\}. \tag{2.22}$$

Note that $q(0) = q(1) = 0$. It is of interest that $q(u) \equiv 0$, $0 \le u \le 1$, is equivalent to

$$\frac{G(\lambda)}{G(\pi)} - F(\lambda) \equiv 2\int_0^\lambda \left[\frac{f^2(\nu)}{G(\pi)} - f(\nu)\right] d\nu \equiv 0, \qquad 0 \le \lambda \le 1, \tag{2.23}$$

which in turn is equivalent to $f(\nu)[f(\nu) - G(\pi)]/G(\pi) = 0$ a.e. In particular, $q(u) \equiv 0$ for $f(\nu) = 1/(2\pi)$ or $\rho_1 = \rho_2 = \cdots = 0$. Durlauf (1991) has studied tests of lack of correlation.

Let $B(u)$ be the Brownian bridge; that is, $\mathscr{E}B(u) = 0$,

$$\mathscr{E}B(u)B(v) = \min(u, v) - uv, \tag{2.24}$$

$B(u)$ is Gaussian and sample paths are continuous with probability 1. Then

$$\frac{1}{2\sqrt{\pi G(\pi)}} Y_T(u) \to_w B(u) + q(u)X, \tag{2.25}$$

where X has the standard normal distribution $N(0, 1)$, and the covariance matrix of $B(u) + q(u)X$ is

$$k(u, v) = \min(u, v) - uv + q(u)q(v). \tag{2.26}$$

This covariance function is larger than the covariance function of $B(u)$, $\min(u, v) - uv$, in the Loewner sense; that is,

$$\int_0^1\int_0^1 k(u, v)l(u)l(v)\,du\,dv \ge \int_0^1\int_0^1 [\min(u, v) - uv]l(u)l(v)\,du\,dv \tag{2.27}$$

for any $l(\cdot)$ for which the integrals are defined. Thus

$$\Pr\{B(u) + Xq(u) \in \mathscr{C}\} \le \Pr\{B(u) \in \mathscr{C}\} \tag{2.28}$$

for any convex symmetric $\mathscr{C}$ [Anderson (1955)].

3. Test of a specific hypothesis.

3.1. *Test criteria.* Consider testing the null hypothesis

$$H\colon f(\lambda) = f_0(\lambda), \tag{3.1}$$

where $f_0(\lambda)$ is completely specified. Among the criteria available to test this hypothesis are the Cramér–von Mises criterion

$$\frac{1}{4\pi G(\pi)}\int_0^1 Y_T^2(u)\,du = \frac{T}{2\pi G^2(\pi)}\int_0^\pi \left[\hat{F}_T(\lambda) - F_0(\lambda)\right]^2 f_0^2(\lambda)\,d\lambda, \tag{3.2}$$

the Kolmogorov–Smirnov criterion

$$\frac{1}{2\sqrt{\pi G(\pi)}} \sup_{0\le u\le 1} |Y_T(u)| = \sup_{0\le\lambda\le\pi} \frac{\sqrt{T}}{2\sqrt{\pi G(\pi)}} \left|\hat{F}_T(\lambda) - F_0(\lambda)\right| \tag{3.3}$$

and the Anderson–Darling statistic. If the null hypothesis is $f_0(\lambda) = 1/(2\pi)$, that is, complete lack of correlation, the asymptotic tests are exactly those of goodness of fit of probability distributions. See, for example, Shorack and Wellner (1986) for a review of such tests.

To carry out a test procedure, we would like to know the limiting distribution of the criterion under the null hypothesis. This is the distribution of the functional when the limiting distribution of $Y_T(u)$ is Gaussian with covariance function (2.21); under the null hypothesis $q(u)$ is specified. The justification for this assertion is the continuous mapping theorem [Theorem 5.1, Billingsley (1968), for example].

3.2. *The Cramér–von Mises criterion.* We shall show how to obtain the characteristic function of the Cramér–von Mises statistic. Any function $h(u, v)$, $0 \le u, v \le 1$, that is symmetric in u and v, continuous and square integrable (in one and both variables) can be expressed as

$$h(u,v) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} f_j(u) f_j(v), \tag{3.4}$$

where λ_i is an eigenvalue and $f_i(u)$ the corresponding normalized eigenfunction of the integral equation

$$f(u) = \lambda \int_0^1 h(u,v) f(v)\, dv, \tag{3.5}$$

and

$$\int_0^1 f_i(u) f_j(u)\, du = \delta_{ij}, \tag{3.6}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$, $i \neq j$. If $h(0,0) = h(1,1) = 0$, then $f_i(0) = f_i(1) = 0$. If $h(u,v)$ is the covariance function of a stochastic process $Z(u)$, $0 \le u \le 1$, then $h(u,v)$ is positive definite and $\lambda_i > 0$. The series (3.4) converges absolutely and uniformly in the unit square. If $Z(u)$ is Gaussian and $\mathscr{E}Z(u) = 0$, then it has the representation

$$Z(u) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} X_j f_j(u), \tag{3.7}$$

where $X_1, X_2, \ldots$ are independent $N(0,1)$ variables. With probability 1,

$$\int_0^1 B^2(u)\, du = \int_0^1 \sum_{i,j=1}^{\infty} \frac{1}{\sqrt{\lambda_i}} X_i f_i(u) \frac{1}{\sqrt{\lambda_j}} X_j f_j(u)\, du = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} X_j^2. \tag{3.8}$$

The characteristic function of (3.8) is

$$\mathscr{E} \exp\left(it \sum_{j=1}^{\infty} \frac{X_j^2}{\lambda_j} \right) = \prod_{j=1}^{\infty} \left(1 - \frac{2it}{\lambda_j} \right)^{-1/2}. \tag{3.9}$$

The function $D(\lambda) = \prod_{j=1}^{\infty}(1 - \lambda/\lambda_j)$ is known as the Fredholm determinant of the integral equation (3.5).

The Brownian bridge with covariance function $h(u, v) = \min(u, v) - uv$ has the representation (3.7) with $\lambda_j = (\pi j)^2$ and $f_j(u) = \sqrt{2}\sin j\pi u$. The characteristic function (3.9) is $(\sin\sqrt{2it}\,/\sqrt{2it}\,)^{-1/2}$. Anderson and Darling (1952) have given the distribution and a table of it.

We now turn to the case of $q(u) \neq 0$. We define

$$\begin{aligned}\alpha_j &= \int_0^1 q(u) f_j(u)\, du \\ &= \frac{2\sqrt{2}}{G(\pi)}\int_0^{\pi} \sin\left[j\pi\frac{G(\lambda)}{G(\pi)}\right]\left[\frac{G(\lambda)}{G(\pi)} - F(\lambda)\right] f^2(\lambda)\, d\lambda.\end{aligned} \tag{3.10}$$

Then $q(u) = \sum_{j=1}^{\infty}\alpha_j f_j(u)$. The process $B(u) + Xq(u)$ has the representation

$$B(u) + Xq(u) = \sum_{j=1}^{\infty}\left(\frac{X_i}{\sqrt{\lambda_j}} + \alpha_j X\right) f_j(u), \tag{3.11}$$

and the Cramér–von Mises criterion has the representation

$$\begin{aligned}S &= \int_0^1 [B(u) + Xq(u)]^2\, du \\ &= \int_0^1 \left[\sum_{j=1}^{\infty}\left(\frac{X_j}{\sqrt{\lambda_j}} + \alpha_j X\right) f_j(u)\right]^2 du \\ &= \sum_{j=1}^{\infty}\left(\frac{X_j}{\sqrt{\lambda_j}} + \alpha_j X\right)^2 = \sum_{j=1}^{\infty} Y_j^2,\end{aligned} \tag{3.12}$$

where $Y_j = X_j/\sqrt{\lambda_j} + \alpha_j X$. The Y_j's are normally distributed with $\mathscr{E}Y_j = 0$, $\mathscr{E}Y_j^2 = 1/\lambda_j + \alpha_j^2$ and $\mathscr{E}Y_iY_j = \alpha_i\alpha_j$, $i \neq j$. The statistic (3.12) can be approximated by a finite sum

$$S_N = \sum_{j=1}^{N} Y_j^2. \tag{3.13}$$

The difference between (3.12) and (3.13) has expectation

$$\mathscr{E}\sum_{j=N+1}^{\infty} Y_j^2 = \sum_{j=N+1}^{\infty}\left(\frac{1}{\lambda_j} + \alpha_j^2\right), \tag{3.14}$$

which can be made arbitrarily small by taking N sufficiently large. Hence, as $N \to \infty$, the distribution of S_N converges to the distribution of S and the characteristic function of S_N approaches the characteristic function of S.

Let Y_N be the N-vector with Y_j as the jth component. The covariance matrix of Y_N is $\mathscr{E}Y_NY_N' = \Lambda_N + a_Na_N'$, where Λ_N is the diagonal matrix with $1/\lambda_j$ as the jth diagonal element and a_N is the vector with α_j as the jth

component. Then the characteristic function of S_N is

$$\mathcal{E}e^{itY_N'Y_N} = |I_N - 2it(\Lambda_N + a_N a_N')|^{-1/2} = \prod_{j=1}^{N}(1 - 2it\phi_{jN})^{-1/2}, \tag{3.15}$$

where ϕ_{jN} is the jth characteristic root of $\Lambda_N + a_N a_N'$, that is, the jth zero of

$$\begin{aligned} |\Lambda_N + a_N a_N' - \phi I_N| &= \begin{vmatrix} 1 & a_N' \\ -a_N & \Lambda_N - \phi I_N \end{vmatrix} \\ &= |\Lambda_N - \phi I_N|\{1 + a_N'(\Lambda_N - \phi I_N)^{-1} a_N\} \\ &= |\Lambda_N - \phi I_N|\left\{1 + \sum_{j=1}^{N} \frac{\alpha_j^2}{1/\lambda_j - \phi}\right\} \end{aligned} \tag{3.16}$$

for $\phi \neq 1/\lambda_j$, $j = 1, \ldots, N$. We shall write

$$\begin{aligned} D_N^*(\nu) &= |I_N - \nu(\Lambda_N + a_N a_N')| \\ &= \prod_{i=1}^{N}\left(1 - \frac{\nu}{\lambda_i}\right)\left\{1 - \nu^2 \sum_{j=1}^{N} \frac{\alpha_j^2}{\lambda_j - \nu} - \nu \sum_{j=1}^{N} \alpha_j^2\right\}. \end{aligned} \tag{3.17}$$

Since $\sum_{j=1}^{\infty}\lambda_j^{-1} < \infty$ and $\sum_{j=1}^{\infty}\alpha_j^2 = \int_0^1 q^2(u)\,du < \infty$, $D_N^*(\nu)$ converges to

$$D^*(\nu) = \prod_{i=1}^{\infty}\left(1 - \frac{\nu}{\lambda_i}\right)\left\{1 - \nu^2 \sum_{j=1}^{\infty} \frac{\alpha_j^2}{\lambda_j - \nu} - \nu \sum_{j=1}^{\infty} \alpha_j^2\right\} \tag{3.18}$$

as $N \to \infty$. The characteristic function of S is $1/\sqrt{D^*(2it)}$.

The process $B(u) + Xq(u)$ has the covariance function $k(u, v)$ given by (2.26). Then $k(u, v)$ has a representation (3.4) where now the λ_j and $f_j(u)$ are the eigenvalues of the integral equation

$$g(u) = \nu\int_0^1 k(u, v)g(v)\,dv. \tag{3.19}$$

To avoid confusion we denote these eigenvalues and eigenfunctions by ν_j and g_j. The process has a representation (3.7) and S has a representation (3.8) with characteristic function (3.9) with λ_j and $f_j(u)$ replaced by ν_j and $g_j(u)$.

The characteristic function, which must be identical to $1/\sqrt{D^*(2it)}$, can be expressed directly as an integral without calculating the coefficients α_j, as we shall show. If $q(u)$ is twice differentiable, (3.19) can be differentiated twice with respect to u to yield

$$g''(u) + \nu g(u) = \nu C q''(u), \tag{3.20}$$

where $C = \int_0^1 q(u)g(u)\,du$. J. B. Keller has pointed out that the solution of

(3.20) depending on ν is, for $\nu \neq \pi^2 j^2$, $j = 1, 2, \ldots$,

$$(3.21)\quad g(u;\nu) = -\frac{C\nu^{3/2}}{\sin\sqrt{\nu}}\left\{\sin\left[\sqrt{\nu}\,(u-1)\right]\int_0^u \sin(\sqrt{\nu}\,t)q(t)\,dt + \sin(\sqrt{\nu}\,u)\int_u^1 \sin\left[\sqrt{\nu}\,(t-1)\right]q(t)\,dt\right\} + \nu C q(u).$$

When we multiply (3.21) by $q(u)$, integrate from 0 to 1 and divide by C, we obtain

$$(3.22)\quad 0 = 1 - \nu^2\int_0^1\int_0^1 c(u,t;\nu)q(u)q(t)\,du\,dt - \nu\int_0^1 q^2(u)\,du,$$

where

$$(3.23)\quad c(u,t;\nu) = \begin{cases} -\dfrac{1}{\sqrt{\nu}\,\sin\sqrt{\nu}}\sin(\sqrt{\nu}\,u)\sin\left[\sqrt{\nu}\,(t-1)\right], & u \le t, \\ -\dfrac{1}{\sqrt{\nu}\,\sin\sqrt{\nu}}\sin\left[\sqrt{\nu}\,(u-1)\right]\sin(\sqrt{\nu}\,t), & u \ge t. \end{cases}$$

The eigenvalues of $k(u,v)$ are the values of ν $(\neq \pi^2 j^2)$ that satisfy (3.22). The function $c(u,t;\nu)$ is the *resolvent* or *resolving kernel* of the kernel $\min(u,t) - ut$; that is, it satisfies

$$(3.24)\quad c(u,v;\nu) = \min(u,v) - uv + \nu\int_0^1 c(u,t;\nu)[\min(t,v) - tv]\,dt.$$

See Goursat (1964), for example.

The resolvent has the representation

$$(3.25)\quad c(s,t;\nu) = \sum_{j=1}^{\infty}\frac{1}{\lambda_j - \nu}f_j(s)\,f_j(t),$$

and

$$(3.26)\quad \int_0^1\int_0^1 c(s,t;\nu)q(s)q(t)\,ds\,dt = \sum_{j=1}^{\infty}\frac{\alpha_j^2}{\lambda_j - \nu}.$$

Thus

$$(3.27)\quad D^*(\nu) = \frac{\sin\sqrt{\nu}}{\sqrt{\nu}}\left\{1 - \nu^2\int_0^1\int_0^1 c(u,t;\nu)q(u)q(t)\,du\,dt - \nu\int_0^1 q^2(u)\,du\right\}.$$

This is the Fredholm determinant of the integral equation (3.19).

When the explicit form (3.27) of the Fredholm determinant is intractable or cannot be inverted, it can be approximated by the characteristic function of S_N given by (3.15). The values $\phi_{1N}, \ldots, \phi_{NN}$ are the zeros of (3.16). Bunch, Nielsen and Sorenson (1978) have given an algorithm for finding these zeros.

The cumulants of S are given by

$$\kappa_j = 2^{j-1}(j-1)! \sum_{i=1}^{\infty} \left(\frac{1}{\nu_i}\right)^j. \tag{3.28}$$

They can also be calculated from the kernel (2.26). Let $k_1(s,t) = k(s,t)$ and $k_{j+1}(s,t) = \int_0^1 k_j(s,u)k(u,t)\,du$. Then

$$\kappa_j = 2^{j-1}(j-1)! \int_0^1 k_j(s,s)\,ds. \tag{3.29}$$

See Anderson and Darling (1952). The expression (3.28) is obtained from the expansion of the logarithm of the characteristic function of S in powers of it; the coefficient of $(it)^j/j!$ is the jth cumulant. Equivalently, it is the coefficient of $\nu^j/j!$ in the expansion of $-\frac{1}{2}\log D^*(\nu)$, given by (3.18) or (3.27). The first two cumulants are

$$\kappa_1 = \mathscr{E}S = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} + \sum_{j=1}^{\infty} \alpha_j^2 = \frac{1}{6} + \int_0^1 q^2(u)\,du, \tag{3.30}$$

$$\begin{aligned}\kappa_2 &= \operatorname{Var} S \\ &= \sum_{j=1}^{\infty} \frac{2}{\lambda_j^2} + 4\sum_{j=1}^{\infty} \frac{\alpha_j^2}{\lambda_j} + 2\left(\sum_{j=1}^{\infty} \alpha_j^2\right)^2 \\ &= \frac{2}{45} + 2\int_0^1\int_0^1 [\min(u,v) - uv]q(u)q(v)\,du\,dv \\ &\quad + 2\left(\int_0^1 q^2(u)\,du\right)^2.\end{aligned} \tag{3.31}$$

3.3. *Calculation of the Cramér–von Mises criterion.* The Cramér–von Mises criterion can be calculated or approximated by the integral (3.2). However, usually in a time series analysis the autocorrelations r_j will be available and it will be easier to calculate the criterion from them.

If we omit $(2\sqrt{T}/\pi)\sum_{h=T}^{\infty}\rho_h \sin\lambda h/h$, the Cramér–von Mises criterion (3.2) can be written as $T/[2\pi G^2(\pi)]$ times

$$\begin{aligned}&\int_0^{\pi}\left[\frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}(r_h - \rho_h)\right]^2\left[\frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\rho_r e^{i\lambda r}\right]^2 d\lambda \\ &\quad = \frac{1}{2\pi^4}\sum_{g,h=1}^{T-1}\frac{(r_g-\rho_g)(r_h-\rho_h)}{gh} \\ &\qquad \times \sum_{r,s=-\infty}^{\infty}\rho_r\rho_s\int_{-\pi}^{\pi}\sin\lambda g\sin\lambda h\, e^{i\lambda(r-s)}\,d\lambda.\end{aligned} \tag{3.32}$$

Straightforward integration shows that (3.32) is

$$\frac{T}{8\pi^4 G^2(\pi)} \sum_{r=-\infty}^{\infty} \left[\sum_{g=1}^{T-1} \frac{(r_g - \rho_g)(\rho_{r+g} - \rho_{r-g})}{g} \right]^2. \tag{3.33}$$

In the special case of $f_0(\lambda) = 1/(2\pi)$ (i.e., $\rho_1 = \rho_2 = \cdots = 0$), the Cramér–von Mises criterion (except for the part of the sum not depending on the sample) is

$$\frac{T}{\pi^2} \sum_{g=1}^{T-1} \frac{r_g^2}{g^2}. \tag{3.34}$$

In this case any finite set of $\sqrt{T}\, r_g$ has a limiting normal distribution in which the variables are independent standard normal variables. On this basis the limiting distribution of (3.34) is consistent with the limiting distribution of the Cramér–von Mises statistic as indicated in subsection 3.2. [The 5% significance point is 0.46136 and the 1% point is 0.74346; see Anderson and Darling (1952).] It may be of interest to compare (3.34) with the Box–Pierce statistic $T\sum_{g=1}^{K} r_g^2$ for some fixed $K < T$.

4. The Kolmogorov–Smirnov criterion. To test H_0: $f(\lambda) = f_0(\lambda)$ on a large-sample basis, we want to find a constant c such that

$$\Pr\left\{ \frac{1}{2\sqrt{\pi G(\pi)}} \sup_{0 \le u \le 1} |Y_T(u)| \le c \right\} \to 1 - \alpha \tag{4.1}$$

for a specified α, $0 < \alpha < 1$, as $T \to \infty$. We want to evaluate

$$\Pr\left\{ \sup_{0 \le u \le 1} |B(u) + q_0(u)X| \le c \right\}. \tag{4.2}$$

First we derive some inequalities that permit comparison of (4.2) with simple distributions. Let $d = \sup_{0 \le u \le 1} |q_0(u)|$. Then

$$\sup_{0 \le u \le 1} |B(u) + q_0(u)X| \le \sup_{0 \le u \le 1} |B(u)| + |X|d. \tag{4.3}$$

Thus

$$\begin{aligned} \Pr\left\{ \sup_{0 \le u \le 1} |B(u)| + |X|d \le c \right\} &\le \Pr\left\{ \sup_{0 \le u \le 1} |B(u) + Xq_0(u)| \le c \right\} \\ &\le \Pr\left\{ \sup_{0 \le u \le 1} |B(u)| \le c \right\}. \end{aligned} \tag{4.4}$$

The right-hand inequality follows from (2.28). The last probability is the Kolmogorov–Smirnov distribution tabulated by Smirnov (1948):

$$\Pr\left\{ \sup_{0 \le u \le 1} |B(u)| \le c \right\} = 1 + 2 \sum_{j=1}^{\infty} (-1)^j e^{-2j^2c^2}. \tag{4.5}$$

Then the first probability in (4.4) is the convolution of (4.5) and the distribution of $d|X|$; this is (for $w > 0$)

$$\Pr\left\{\sup_{0\le u\le 1}|B(u)| + d|X| \le w\right\}$$

$$= 2\Phi\left(\frac{w}{d}\right) - 1$$

$$+ 4\sum_{j=1}^{\infty}(-1)^j \frac{\exp\left[-2j^2w^2/(1+4d^2j^2)\right]}{\sqrt{1+4d^2j^2}} \tag{4.6}$$

$$\times\left[\Phi\left(\frac{w}{d\sqrt{1+4d^2j^2}}\right) - \Phi\left(\frac{-4\,dj^2\,w}{\sqrt{1+4d^2j^2}}\right)\right].$$

If d is small, (4.5) and (4.6) are approximations to (4.2). The difference between the right-hand side and left-hand sides of (4.4) is an upper bound to either error of approximation. Note that the series (4.6) converges rapidly. [A table of values of (4.6) was given in Anderson (1991).]

If (4.2) is $1-\alpha$ and α is small, then α is approximately 2 times

$$\Pr\left\{\sup_{0\le u\le 1}\left[B(u) + Xq_0(u)\right] \ge c\right\}. \tag{4.7}$$

Suppose $0 \le q_0(u) \le d$, $0 \le u \le 1$. Then for $X > 0$

$$\sup_{0\le u\le 1} B(u) \le \sup_{0\le u\le 1}\left[B(u) + Xq_0(u)\right] \le \sup_{0\le u\le 1} B(u) + Xd, \tag{4.8}$$

and for $X < 0$ the inequalities in (4.8) are reversed. Then

$$\tfrac{1}{2}\Pr\left\{\sup_{0\le u\le 1} B(u) \ge c\right\} + \tfrac{1}{2}\Pr\left\{\sup_{0\le u\le 1} B(u) - |X|d \ge c\right\}$$

$$\le \Pr\left\{\sup_{0\le u\le 1}\left[B(u) + Xq_0(u)\right] \ge c\right\} \tag{4.9}$$

$$\le \tfrac{1}{2}\Pr\left\{\sup_{0\le u\le 1} B(u) + |X|d \ge c\right\} + \tfrac{1}{2}\Pr\left\{\sup_{0\le u\le 1} B(u) \ge c\right\}.$$

Since $\Pr\{\sup_{0\le u\le 1} B(u) \le y\} = 1 - e^{-2y^2}$, $y \ge 0$, we have for $w \ge 0$

$$\Pr\left\{\sup_{0\le u\le 1} B(u) + d|X| \le w\right\}$$

$$= 2\Phi\left(\frac{w}{d}\right) - 1 - \frac{2}{\sqrt{4d^2+1}}\exp\left(\frac{-2w^2}{4d^2+1}\right) \tag{4.10}$$

$$\times\left[\Phi\left(\frac{w}{d\sqrt{4d^2+1}}\right) - \Phi\left(\frac{-4\,dw}{\sqrt{4d^2+1}}\right)\right];$$

Similarly,

$$\Pr\Big\{\sup_{0\le u\le 1} B(u) - d|X| \le w\Big\}$$

$$(4.11)\qquad = \begin{cases} 2\Phi\left(\dfrac{w}{d}\right) - \dfrac{2}{\sqrt{4d^2+1}}\exp\left(\dfrac{-2w^2}{4d^2+1}\right)\Phi\left(\dfrac{w}{d\sqrt{4d^2+1}}\right), & w \le 0, \\ 1 - \dfrac{2}{\sqrt{4d^2+1}}\exp\left(\dfrac{-2w^2}{4d^2+1}\right)\Phi\left(\dfrac{-4\,dw}{\sqrt{4d^2+1}}\right), & w \ge 0. \end{cases}$$

The right-hand side or the left-hand side of (4.9) is an approximation to the probability (4.7).

The extremes of $q_0(u)$ can be found by setting to 0 the derivative

$$(4.12)\qquad \frac{d}{d\lambda}\left[\frac{G(\lambda)}{G(\pi)} - F_0(\lambda)\right] = 2f_0(\lambda)\left[\frac{f_0(\lambda)}{G_0(\pi)} - 1\right],$$

that is, at $f_0(\lambda) = 0$ or

$$(4.13)\qquad f_0(\lambda) = G_0(\pi) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\rho_h^2.$$

We now consider the supremum of $|q_0(u)|$ over $0 \le u \le 1$ and all $f_0(\cdot)$; this is the supremum of $|[G(\lambda)/G(\pi)] - F(\lambda)|$. Since $G(\lambda)/G(\pi)$ and $F(\lambda)$ are monotonically nondecreasing in $[0, \pi]$ with $G(0)/G(\pi) = F(0) = 0$ and $G(\pi)/G(\pi) = F(\pi) = 1$, $\sup_{0\le u\le 1,\ f(\cdot)}|q(u)| \le 1$. We shall now find an $f(\cdot)$ such that the upper bound of 1 is approached. Consider

$$(4.14)\qquad f(\lambda) = \begin{cases} c, & 0 \le \lambda \le \nu, \\ d, & \nu < \lambda \le \pi, \end{cases}$$

for some c, d and ν, $0 < \nu < \pi$ and $0 < c \le 1/(2\pi)$. Then $2\int_0^\pi f(\lambda)\,d\lambda = 1$ implies $d = (1 - 2\nu c)/[2(\pi - \nu)]$. Then as $\nu \to \pi$,

$$(4.15)\qquad \frac{G(\nu)}{G(\pi)} - F(\nu) \to -2\pi c.$$

Hence as $c \to 1/(2\pi)$, $\sup_{0\le u\le 1}|q(u)| \to 1$.

Durbin (1985) has studied the first passage density of a continuous Gaussian process to a general boundary. The probability (4.7) is then the integral of this density from 0 to 1. An approximation to this probability is

$$(4.16)\qquad \frac{c}{\sqrt{2\pi}}\int_0^1 \frac{1 - t + q'(t)q(t)}{k(t,t)}\exp\left(\frac{-c^2}{2k(t,t)}\right)dt,$$

where $k(t,t) = t - t^2 + q^2(t)$. In practice (4.16) could be evaluated by numerical integration. Durbin also gives an exact expression as well as two other approximations to the first passage density. As an example he applies his formulas to $\Pr\{B(u) \ge c\}$ for several values of c. For c yielding an exact

probability of 0.1 this approximation has an error of only 0.0021; for larger values of c (smaller values of α) the error is proportionally smaller.

5. Confidence region for the spectral distribution function. An asymptotic confidence region with confidence coefficient $1-\alpha$ for an unknown spectral distribution consists of all monotonic functions $F(\cdot)$ $[F(0)=0,\ F(1)=1]$ such that

$$\sqrt{T}\left|\hat{F}_T(\lambda)-F(\lambda)\right| \le c \quad \forall\, \lambda \in [0,\pi], \tag{5.1}$$

where c is chosen so that (4.2) is $1-\alpha$ and $q_0(u)$ refers to the unknown distribution. Since c depends on $q_0(u)$, we need a consistent estimator of $q_0(u)$.

From the fact that $\sqrt{T}[\hat{F}_T(\lambda)-F(\lambda)]$ has a limiting normal distribution it follows that $\hat{F}_T(\lambda) \to_{\mathrm{p}} F(\lambda)$. Let $\tilde{f}(\lambda)$ be a uniformly consistent estimator of $f(\lambda)$ in the sense that

$$\sup_{0\le\lambda\le\pi}\left|\tilde{f}_T(\lambda)-f(\lambda)\right| \to_{\mathrm{p}} 0. \tag{5.2}$$

For such an estimator the class of admissible $f(\lambda)$ must be restricted. Define

$$\bar{G}_T(\lambda)=2\int_0^{\lambda}\tilde{f}_T^2(\nu)\,d\nu, \qquad \bar{q}_T(u)=u-\hat{F}_T\left\{\bar{G}_T^{-1}\left[\bar{G}_T(\pi)u\right]\right\}. \tag{5.3}$$

Then $\bar{q}_T(u)$ is a consistent estimator of $q_0(u)$. If c_T is the value of c for which (4.2) holds with $q_0(u)$ replaced by $\bar{q}_T(u)$, then $c_T \to_{\mathrm{p}} c$.

The inequality (5.1) can be written

$$\hat{F}_T(\lambda)-\frac{c}{\sqrt{T}} \le F(\lambda) \le \hat{F}_T(\lambda)+\frac{c}{\sqrt{T}} \quad \forall\, \lambda \in [0,\pi]. \tag{5.4}$$

Another problem of interest is testing that the spectral densities of two independent processes are the same; the null hypothesis is $f_1(\lambda)=f_2(\lambda)$. Suppose $\hat{F}_{T_1}(\lambda)$ and $\hat{F}_{T_2}(\lambda)$ are the corresponding two empirical standardized spectral distributions. Under the null hypothesis

$$\sqrt{\frac{T_1T_2}{T_1+T_2}}\left[\hat{F}_{T_1}(\lambda)-\hat{F}_{T_2}(\lambda)\right] \tag{5.5}$$

converges weakly to the Gaussian process with covariance function (2.16), where $F(\lambda)$ and $G(\lambda)$ refer to the common spectral distribution and the integral of the common spectral density squared, respectively. The Kolmogorov–Smirnov criterion $\sup_{0\le\lambda\le\pi}|\hat{F}_{T_1}(\lambda)-\hat{F}_{T_2}\lambda)|$ can be used to test this null hypothesis. If $\tilde{f}_{T_1}(\lambda)$ and $\tilde{f}_{T_2}(\lambda)$ are uniformly consistent estimators of the common spectral density, then

$$\frac{T_1}{T_1+T_2}\tilde{f}_{T_1}(\lambda)+\frac{T_2}{T_1+T_2}\tilde{f}_{T_2}(\lambda) \tag{5.6}$$

can be used to estimate $G(\lambda)$ and $q(u)$.

6. Examples.

6.1. *Moving average of order 1.* Let $y_t = u_t + \alpha u_{t-1}$, where the u_t's are uncorrelated with mean $\mathscr{E}u_t = 0$ and variance $\mathscr{E}u_t^2 = \sigma^2$. Then $\rho_1 = \rho = \alpha/(1 + \alpha^2)$, $-\frac{1}{2} \le \rho \le \frac{1}{2}$, $f(\lambda) = (1 + 2\rho\cos\lambda)/(2\pi)$, $F(\lambda) = (\lambda + 2\rho\sin\lambda)/\pi$, $0 \le \lambda \le \pi$,

$$G(\lambda) = \frac{1}{2\pi^2}\left[(1 + 2\rho^2)\lambda + 4\rho\sin\lambda\right) + \rho^2\sin 2\lambda\Big], \tag{6.1}$$

and $G(\pi) = (1 + 2\rho^2)/(2\pi)$. The difference

$$\frac{G(\lambda)}{G(\pi)} - F(\lambda) = \frac{2\rho\sin\lambda(1 - 2\rho^2 + \rho\cos\lambda)}{\pi(1 + 2\rho^2)}, \qquad 0 \le \lambda \le \pi, \tag{6.2}$$

is nonnegative for $0 \le \rho \le \frac{1}{2}$ and nonpositive for $-\frac{1}{2} \le \rho \le 0$. The maximum of $|q(u)|$ occurs at $f(\lambda) = G(\pi)$, that is, at $\lambda_0 = \cos^{-1}\rho$. Then

$$\frac{G(\lambda_0)}{G(\pi)} - F(\lambda_0) = \frac{2\cos\lambda_0\sin^3\lambda_0}{\pi(1 + 2\cos^2\lambda_0)}. \tag{6.3}$$

The maximum of $|q(u)|$ with respect to $\rho = \cos\lambda_0$ is at a zero of

$$\frac{d}{d\lambda_0}\left[\frac{G(\lambda_0)}{G(\pi)} - F(\lambda_0)\right] = -\frac{2}{\pi}\frac{\sin^2\lambda_0(1 - 6\cos^2\lambda_0 - 4\cos^4\lambda_0)}{(1 + 2\cos^2\lambda_0)^2}. \tag{6.4}$$

The zeros of (6.4) are $\lambda_0 = 0$, π and $\cos^2\lambda_0 = (-3 - \sqrt{13})/4 = 0.15138$. Substitution of 0.15138 shows that the extremum of (6.3) is 0.1483 (which is considerably less than 1).

6.2. *Autoregression of order 1.* Consider the process $y_t = \rho y_{t-1} + u_t$, where the u_t's are uncorrelated with mean $\mathscr{E}u_t = 0$ and variance $\mathscr{E}u_t^2 = \sigma^2$. Then the standardized spectral density is

$$f(\lambda) = \frac{1 - \rho^2}{2\pi(1 + \rho^2 - 2\rho\cos\lambda)}. \tag{6.5}$$

Furthermore, the standardized spectral distribution is

$$F(\lambda) = \frac{2}{\pi}\tan^{-1}\left[\frac{1 + \rho}{1 - \rho}\tan\frac{\lambda}{2}\right], \qquad 0 \le \lambda \le \pi, \tag{6.6}$$

$$G(\lambda) = \frac{2\rho\sin\lambda}{\pi(1 - \rho^2)}f(\lambda) + \frac{1 + \rho^2}{2\pi(1 - \rho^2)}F(\lambda), \qquad 0 \le \lambda \le \pi. \tag{6.7}$$

The difference

$$\frac{G(\lambda)}{G(\pi)} - F(\lambda) = \frac{4\rho}{1 + \rho^2}\sin\lambda\, f(\lambda) \tag{6.8}$$

is positive for $\rho > 0$ and negative for $\rho < 0$. The maximum of (6.8), occurring

at $\cos \lambda_0 = 2\rho/(1 + \rho^2)$, is

$$\frac{G(\lambda_0)}{G(\pi)} - F(\lambda_0) = \frac{1}{\pi} \cos \lambda_0. \tag{6.9}$$

The difference (6.9) approaches $1/\pi = 0.3184$ as $\lambda_0 \to 0$ (i.e., as $\rho \to 1$) and approaches $-1/\pi$ as $\lambda_0 \to \pi$ (i.e., as $\rho \to -1$). The supremum of $|q(u)|$ is 0.3184, less than the maximum of 1 over processes but greater than the maximum of 0.1486 over moving average processes of order 1.

6.3. *Another example.* Suppose $f(\lambda) = (\alpha + 1)|\lambda|^\alpha/(2\pi^{\alpha+1})$, $\alpha \geq 0$. Then $F(\lambda) = \lambda^{\alpha+1}/\pi^{\alpha+1}$, $0 \leq \lambda \leq \pi$,

$$G(\lambda) = \frac{(\alpha + 1)^2}{2(2\alpha + 1)\pi^{2\alpha+2}} \lambda^{2\alpha+1}, \qquad 0 \leq \lambda \leq \pi. \tag{6.10}$$

The transformation $u = \lambda^{2\alpha+1}/\pi^{2\alpha+1}$ yields

$$q(u) = u - u^{(\alpha+1)/(2\alpha+1)} = u^{(\alpha+1)/(2\alpha+1)}(u^{\alpha/(2\alpha+1)} - 1), \tag{6.11}$$

which is negative for $0 < u < 1$. The maximum of $|q(u)|$ in $[0, 1]$ is at

$$u^{\alpha/(2\alpha+1)} = \frac{\alpha + 1}{2\alpha + 1},$$

and the maximum is

$$\left(\frac{\alpha + 1}{2\alpha + 1}\right)^{(\alpha+1)/\alpha} - \left(\frac{\alpha + 1}{2\alpha + 1}\right)^{(2\alpha+1)/\alpha} = \frac{t^t}{(t + 1)^{t+1}}, \tag{6.12}$$

where $t = 1 + 1/\alpha$ (> 1). This is obviously a decreasing function of t. Its supremum for $t > 1$ is 0.25.

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