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ON THE CONSTRUCTION OF ASYMMETRICAL ORTHOGONAL ARRAYS

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General techniques for the construction of asymmetrical orthogonal arrays of strength 2 are presented. These are then applied to special cases to obtain new families of such arrays. Among these are saturated main-effect plans based on s^m runs with factors at s^{v_i} levels, $i = 0, 1, \dots, r$, where $m \geq v_r$, $v_0 = 1$, v_{i-1} divides v_i, $i = 1, 2, \dots, r$, and s is a prime power.

1. Introduction. The current emphasis on quality control and product improvement have rejuvenated research in the area of factorial design. Practical considerations have spurred research in various new or newly emphasized directions. Among these is that of the use and construction of asymmetrical factorial designs, exemplified by research of Cheng (1989), Pu (1989), Wu (1989), Wang and Wu (1991) and Wu, Zhang and Wang (1992). We will present and combine techniques for the construction of arguably the most appealing asymmetrical factorial designs, namely asymmetrical orthogonal arrays. Although not all of the techniques are new, their power for the construction of asymmetrical orthogonal arrays has not been used to the fullest extent, as our results will demonstrate.

Among the most useful factorial designs are those that allow for unbiased estimation of all main effects. When this is accomplished orthogonally by using the best linear unbiased estimators under the assumption of a first order orthogonal polynomial model [see, e.g., Raktoe, Hedayat and Federer (1981)], such a design is called an orthogonal main-effect plan. Orthogonal arrays of strength 2 [Rao (1947, 1973)] are examples of orthogonal main-effect plans with the added advantage that comparable main-effect contrasts for different factors are estimated with the same efficiency.

An extensive study of orthogonal main-effect plans was conducted by Addelman and Kempthorne (1961a). Their report contains various useful ideas for the construction of such plans, some of which are made more accessible and are further extended in Addelman and Kempthorne (1961b) and

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Addelman (1962a, b). The catalogue of orthogonal main-effect plans in Addelman and Kempthorne (1961a) contains also various asymmetrical orthogonal arrays. A more recent catalogue appears in Dey (1985), which may also be used to trace additional references. Some references pertaining to the construction of asymmetrical orthogonal arrays from orthogonal F rectangles can be found in Federer and Mandeli (1986).

In Section 2 we will present the tools for our constructions accompanied by some pertinent references. Applications of these tools to special cases will be discussed in Section 3.

2. Methods of construction. Some of the concepts and definitions that are introduced in this section are not new to the literature. Since they are, however, fundamental for an overall picture of available methodology, we will include concise formulations of these concepts and definitions in addition to a citation of pertinent references.

Asymmetrical orthogonal arrays were formally introduced by Rao (1973), although examples of such arrays appear already in earlier literature, such as Addelman and Kempthorne (1961a). An asymmetrical orthogonal array of strength 2, with N runs and with k factors of which k_i are at s_i levels, $i = 1, \dots, t$, $t \geq 2$, $\Sigma_{i=1}^{t} k_i = k$, $s_i \neq s_{i'}$ if $i \neq i'$, is a $k \times N$ array in which k_i rows are based on symbols from a set S_i of cardinality s_i such that for any $2 \times N$ subarray, say with one row based on S_i and the other on $S_{i'}$, every element from $S_i \times S_{i'}$ appears equally often as a column. We denote such an array by $\mathrm{OA}(N, \Pi_{i=1}^{t} k_i, \Pi_{i=1}^{t} s_i, 2)$; the notation $\Pi_{i=1}^{t}$ is here only used as a symbolic notation. Alternatively, if $t = 2$, for example, we will also write $k_1 \times k_2$ and $s_1 \times s_2$ instead of $\Pi_{i=1}^{2} k_i$ and $\Pi_{i=1}^{2} s_i$, respectively. Clearly, N must be a multiple of $s_i s_{i'}$, $i \neq i'$. If $k_i \geq 2$, N must also be a multiple of s_i^2. The preceding definition could also include the case $t = 1$, but then the array is usually called a (symmetrical) orthogonal array of strength 2 and denoted by $\mathrm{OA}(N, k_1, s_1, 2)$. These arrays were introduced by Rao (1947), although the adjective "orthogonal" seems to have been added by Bush (1950). For simplicity we will no longer demand that $t \geq 2$ and that $s_i \neq s_{i'}$ if $i \neq i'$. The adjectives symmetrical and asymmetrical will only be used when the distinction is thought to be important. Also, the use of orthogonal arrays as fractional factorial designs is sufficiently documented that it requires no further explanation.

A useful concept for the construction of symmetrical orthogonal arrays is that of difference schemes. The first who used it with this objective were Bose and Bush (1952). Using the notation for additive groups, a difference scheme is an $r \times c$ array with entries from a finite Abelian group G of cardinality g such that the vector difference of any two rows of the array, say

$$r_i - r_{i'},$$

contains every element of G exactly c/g times. We will denote such an array by $D(r, c, g)$, although this notation suppresses the relevance of the group G. In most of our examples G will correspond to the additive group associated with a Galois field GF(s). If a $D(r, c, g)$ exists, it can always be constructed so that one of its rows contains only the zero element of G. This property is important for the next section, and we will therefore only use difference schemes that have this property.

By the Kronecker sum $\oplus$ of two matrices A and $B = (b_{ij})$ based on the same additive group G, we will mean

$$A \oplus B = (b_{ij} + A),$$

where $b_{ij} + A$ stands for the matrix obtained by adding b_{ij} to each entry of A and constitutes the ijth block of entries for $A \oplus B$. By E we will mean a $1 \times g$ vector consisting of the elements of a group G, where if such is relevant and not clear from the context it will be specified which group is meant. It is then well known [see Bose and Bush (1952)] that if D is a difference scheme $D(r, c, g)$, then $E \oplus D$ is an orthogonal array OA$(cg, r, g, 2)$.

Two families of difference schemes that are particularly useful in Section 3 are those of the form $D(p^u, p^u, p^v)$ and $D(2p^v, 2p^v, p^v)$. These difference schemes are known to exist whenever $u \geq v$ and p is a prime. A construction for the first family is already given by Bose and Bush (1952) and uses the Galois fields GF(p^u) and GF(p^v). Masuyama (1969) presents the difference schemes of the second family. A slightly simpler form for this family is given by Xu (1979). Although the latter author presents his results only for $v = 1$, they can easily be extended to any positive integer v. This was also observed by Xiang (1983).

The construction of difference schemes has received a considerable amount of attention in the literature, in part through the construction of generalized Hadamard matrices. A reproductive construction of difference schemes that will be useful for our purpose is due to Shrikhande (1964). It simply states that if D_1 and D_2 are difference schemes $D(r_1, c_1, g)$ and $D(r_2, c_2, g)$, respectively, based on the same group G, then

$$D = D_1 \oplus D_2$$

is a difference scheme $D(r_1 r_2, c_1 c_2, g)$ based on G. In conjunction with the difference schemes $D(p^u, p^u, p^v)$ and $D(2p^v, 2p^v, p^v)$ based on the additive group associated with GF(p^v), this result allows us to conclude the existence of difference schemes $D(2^{\mu} p^{\lambda u + \mu v}, 2^{\mu} p^{\lambda u + \mu v}, p^v)$ based on the additive group associated with GF(p^v), where $u \geq v \geq 1$, $\lambda, \mu \geq 0$, $\lambda + \mu \geq 1$ and p is a prime number.

Another essential concept for the construction of orthogonal arrays is that of resolvability. Its importance for the construction of symmetrical orthogonal arrays was already realized by Bose and Bush (1952), but for asymmetrical orthogonal arrays it seems until now a somewhat dormant concept, a conclusion that is supported by the limited use of the concept in Dey (1985). One paper that exhibits some use of the concept of resolvability for construction of asymmetrical orthogonal arrays is Chacko and Dey (1981). We will say that an OA$(N, \prod_{i=1}^{t} k_i, \prod_{i=1}^{t} s_i, 2)$ is $\prod_{i=1}^{t} \beta_i$-resolvable if its columns can be partitioned into $N/(\beta_1 s_1)$ groups of $\beta_1 s_1$ columns each so that for any factor with

s_i levels each possible level occurs β_i times within the columns of each group. This definition requires clearly that $\beta_i s_i$ does not depend on $i \in \{1, \dots, t\}$ and that it divides N.

We can now present the following result. A proof is omitted for brevity, but would follow a reasoning analogous to that in the proof of Theorem 4 in Bose and Bush (1952).

THEOREM 2.1. *The existence of a*

$$\prod_{i=1}^{u} \beta_i\text{-}resolvable\ \mathrm{OA}\left(N, \prod_{i=1}^{u} k_i, \prod_{i=1}^{u} s_i, 2\right)$$

and a

$$\prod_{i=u+1}^{v} \beta_i\text{-}resolvable\ \mathrm{OA}\left(N/(\beta_1 s_1), \prod_{i=u+1}^{v} k_i, \prod_{i=u+1}^{v} s_i, 2\right)$$

imply the existence of a

$$\left(\prod_{i=1}^{u} \beta_i \beta_v s_v \times \prod_{i=u+1}^{v} \beta_i \beta_1 s_1\right)\text{-}resolvable\ \mathrm{OA}\left(N, \prod_{i=1}^{v} k_i, \prod_{i=1}^{v} s_i, 2\right).$$

Addelman and Kempthorne (1961a) describe two methods that convert a given orthogonal array to a new orthogonal array. To one of these methods we will refer as collapsing the levels of a factor, to the other as the method of replacement.

If a factor in an orthogonal array has s_1 levels and s_2 divides s_1, then it can be replaced by a factor with s_2 levels. This is done by partitioning the s_1 symbols into s_2 groups of size s_1/s_2 and by replacing the symbols in the same group by a common symbol. The resulting array is still an orthogonal array. This is the procedure referred to as collapsing the levels of a factor.

Occasionally, depending on the values of s_1 and s_2, a factor with s_1 levels can be replaced by more than one factor at s_2 levels. If there exists an OA($s_1, k, s_2, 2$) then the factor with s_1 levels can be replaced by k factors with s_2 levels. If the columns of the OA($s_1, k, s_2, 2$) are labeled by the s_1 symbols that are used to denote the levels of the factor with s_1 levels, then we can replace each symbol i for that factor with the $k \times 1$ column labeled as i. If the factor with s_1 levels is part of an orthogonal array of strength 2, then the newly constructed array by the described procedure is still an orthogonal array of strength 2. This procedure is referred to as the method of replacement.

3. Use of the methods of construction. We will now demonstrate for some special cases how the methods of Section 2 can be used with some additional techniques to obtain certain orthogonal arrays. For the first case, which will be considered in more detail, we will construct saturated orthogonal arrays of strength 2 with s^m runs and factors at $s^{v_r}, s^{v_{r-1}}, \dots, s^{v_0}$ levels, where s is a prime power, $m \geq v_r$, v_i/v_{i-1} is an integer at least equal to 2, $i = 1, 2, \dots, r$, and $v_0 = 1$. In order to present this construction, we will first introduce some notation. Find nonnegative integers $c_0, c_1, \dots, c_r$ such that

$$m = \sum_{i=0}^{r} c_i v_i, \qquad 0 \le \sum_{i=0}^{j-1} c_i v_i < v_j, \qquad j = 1,2,\dots,r.$$

It is clear that such $c_0, c_1, \dots, c_r$ exist and that they are uniquely determined by these requirements. Define nonnegative integers $b_0, b_1, \dots, b_r$ by

$$b_0 = 0,$$
$$b_i = c_{i-1}v_{i-1} + b_{i-1}, \qquad i = 1,2,\dots,r.$$

Next, define integers $m_0, m_1, \dots, m_{r+1}$ by

$$m_{r+1} = m,$$
$$m_i = v_i + b_i, \qquad i = 0,1,\dots,r.$$

We also define

$$d_r = 0,$$
$$d_i = v_{i+1}/v_i, \qquad i = 0,1,\dots,r-1$$

and

$$k_i = (s^{m_{i+1}} - s^{m_i})/(s^{v_i} - 1), \qquad i = 0,1,\dots,r.$$

The existence of the following orthogonal arrays was already observed by Bose and Bush (1952):

(i) An $s^{(c_i+d_i-2)v_i}$-resolvable OA$(s^{m_{i+1}}, k_i, s^{v_i}, 2)$, $i = 0,1,\dots,r$.
(ii) An $s^{(c_i+d_i-1)v_i}$-resolvable OA$(s^{m_{i+1}}, k_i + 1, s^{v_i}, 2)$, $i = 0,1,\dots,r$.

Since $s^{m_{i+1}}/s^{(c_i+d_i-1)v_i} = s^{m_i}$, we obtain upon repeated applications of Theorem 2.1 the existence of the orthogonal array OA$(s^m, (k_0+1) \times \prod_{i=1}^r k_i, s \times \prod_{i=1}^r s^{v_i}, 2)$. By observing that $(k_0+1)(s-1) + \sum_{i=1}^r k_i(s^{v_i}-1) = s^m - 1$, we see that this is a saturated main-effect plan. However, we will show that we can do slightly better if we use the methods of Section 2 instead of the method by Bose and Bush (1952) to construct the preceding symmetrical orthogonal arrays.

As a first step towards this improvement, observe that $m_{i+1} = (c_i + d_i) \cdot v_i + b_i$, $i = 0,1,\dots,r$. Now let D_{1i} be a difference scheme $D(s^{v_i}, s^{v_i}, s^{v_i})$ and let D_{2i} be a difference scheme $D(s^{v_i+b_i}, s^{v_i+b_i}, s^{v_i})$, $i = 0,1,\dots,r$. The existence of these difference schemes based on the additive group associated with GF(s^{v_i}) was observed in Section 2. With E_i as the $1 \times s^{v_i}$ vector with the elements of GF(s^{v_i}) and with 0_i as the $1 \times s^{v_i}$ vector with all entries equal to the zero element of GF(s^{v_i}), it is then easily seen that the following array is an $s^{(c_i+d_i-2)v_i}$-resolvable OA$(s^{m_{i+1}}, k_i, s^{v_i}, 2)$:

$$\begin{array}{l} E_i \oplus \overbrace{D_{1i} \oplus D_{1i} \oplus \cdots \oplus D_{1i}}^{c_i+d_i-2 \text{ times}} \oplus D_{2i}, \\ 0_i \oplus E_i \oplus D_{1i} \oplus \cdots \oplus D_{1i} \oplus D_{2i}, \\ \quad\vdots \\ 0_i \oplus 0_i \oplus 0_i \oplus \cdots \oplus E_i \oplus D_{2i}. \end{array}$$

An $s^{(c_i+d_i-1)v_i}$-resolvable $\mathrm{OA}(s^{m_{i+1}}, k_i+1, s^{v_i}, 2)$ can be obtained by adding one factor represented by

$$0_i \oplus 0_i \oplus 0_i \oplus \cdots \oplus 0_i \oplus E_i \oplus \bar{0}_i,$$

where $\bar{0}_i$ is a $1 \times s^{b_i}$ vector of zeros. Notice that contrary to the construction by Bose and Bush (1952), this construction requires only computations in two finite fields, namely $\mathrm{GF}(s^{v_i})$ and $\mathrm{GF}(s^{v_i+b_i})$, and only in one field if $b_i = 0$.

The claimed improvement for the construction of asymmetrical orthogonal arrays that are obtained from these symmetrical orthogonal arrays by repeated application of Theorem 2.1 is due to the fact that the current representation of the symmetrical orthogonal arrays allows us at times to combine some lower level factors and replace them by one higher level factor without distorting the orthogonality of the array. More precisely, if $c_i \geq 1$ then we can replace $(s^{v_{i+1}}-1)/(s^{v_i}-1)$ factors in the $\mathrm{OA}(s^{m_{i+1}}, k_i, s^{v_i}, 2)$ by one factor at $s^{v_{i+1}}$ levels without affecting the orthogonality of the array. To see this, note that the following array is a subarray of the $\mathrm{OA}(s^{m_{i+1}}, k_i, s^{v_i}, 2)$ as previously constructed:

$$\begin{array}{l}
E_i \oplus \overbrace{D_{1i} \oplus D_{1i} \oplus \cdots \oplus D_{1i}}^{d_i - 1 \text{ times}} \oplus \overbrace{0_i \oplus \cdots \oplus 0_i}^{c_i - 1 \text{ times}} \oplus 0_i^*,\\
0_i \oplus E_i \oplus D_{1i} \oplus \cdots \oplus D_{1i} \oplus 0_i \oplus \cdots \oplus 0_i \oplus 0_i^*,\\
\quad\vdots\\
0_i \oplus 0_i \oplus 0_i \oplus \cdots \oplus E_i \oplus 0_i \oplus \cdots \oplus 0_i \oplus 0_i^*,
\end{array}$$

where 0_i^* is the $1 \times s^{v_i+b_i}$ vector of zeros. That this is a subarray follows since, as argued in Section 2, both D_{1i} and D_{2i} contain a row of zeros. Also, this subarray consists of $s^{(d_i-1)v_i} + s^{(d_i-2)v_i} + \cdots + s^{v_i} + 1 = (s^{v_{i+1}}-1)/(s^{v_i}-1)$ factors and has only $s^{v_{i+1}}$ different columns. By identifying each of these columns with an element of $\mathrm{GF}(s^{v_{i+1}})$ we can replace these $(s^{v_{i+1}}-1)/(s^{v_i}-1)$ factors by one factor with $s^{v_{i+1}}$ levels, which is of the form $E_{i+1} \oplus \bar{0}_{i+1}$.

We can also replace lower level factors by one higher level factor if $c_i = b_i = 0$. In that case $b_0 = b_1 = \cdots = b_i = b_{i+1} = 0$. If $b_1 = 0$, then $k_0 + 1 = (s^{m_1} - s^{m_0})/(s^{v_0}-1) + 1 = (s^{v_1} - s^{v_0})/(s^{v_0}-1) + 1 = (s^{v_1}-1)/(s^{v_0}-1)$, and the entire $\mathrm{OA}(s^{m_1}, k_0+1, s^{v_0}, 2)$ that is used in the construction of the $\mathrm{OA}(s^m, (k_0+1) \times \prod_{j=1}^r k_j, s^{v_0} \times \prod_{j=1}^r s^{v_j}, 2)$ can be replaced by one factor at s^{v_1} levels in a similar way as before. This leads to an $\mathrm{OA}(s^m, (k_1+1) \times \prod_{j=2}^r k_j, s^{v_1} \times \prod_{j=2}^r s^{v_j}, 2)$. This array contains as a subarray replications of an $\mathrm{OA}(s^{m_2}, k_1+1, s^{v_1}, 2)$ as previously constructed. If $b_2 = 0$, then $k_1 + 1 = (s^{m_2} - s^{m_1})/(s^{v_1}-1) + 1 = (s^{v_2} - s^{v_1})/(s^{v_1}-1) + 1 = (s^{v_2}-1)/(s^{v_1}-1)$. Thus the $\mathrm{OA}(s^{m_2}, k_1+1, s^{v_1}, 2)$ can in its entirety be replaced by one factor at s^{v_2} levels. This process can be continued until we finally reach an $\mathrm{OA}(s^m, (k_{i+1}+1) \times \prod_{j=i+2}^r k_j, s^{v_{i+1}} \times \prod_{j=i+2}^r s^{v_j}, 2)$.

The previous paragraphs are summarized in the following result.

THEOREM 3.1. *Let* $1 = v_0 < v_1 < \cdots < v_r \leq m$ *and* v_{i-1} *divides* v_i, $i = 1, 2, \ldots, r$. *Let* b_i, c_i *and* k_i *be defined as previously and let* s *be a prime power. Then there exists an* $\mathrm{OA}(s^m, \prod_{i=0}^{r} k_i^*, \prod_{i=0}^{r} s^{v_i}, 2)$, *where*

$$k_0^* = k_0 - (s^{v_1} - s^{v_0})/(s^{v_0} - 1)$$

$$k_1^* = \begin{cases} k_1 + 1, & \text{if } c_1 = 0 \text{ and } b_1 > 0, \\ k_1 - (s^{v_2} - s^{v_1})/(s^{v_1} - 1), & \text{otherwise}, \end{cases}$$

$$k_i^* = \begin{cases} k_i + 1, & \text{if } c_{i-1} > 0 \text{ or } b_{i-1} = 0 \text{ and } c_i = 0,\ b_i > 0, \\ k_i, & \text{if } c_{i-1} = 0,\ b_{i-1} > 0 \text{ and } c_i = 0, \\ k_i - (s^{v_{i+1}} - s^{v_i})/(s^{v_i} - 1), & \text{if } c_{i-1} > 0 \text{ or } b_{i-1} = 0 \text{ and } c_i > 0 \text{ or } b_i = 0, \\ k_i - (s^{v_{i+1}} - s^{v_i})/(s^{v_i} - 1) - 1, & \text{if } c_{i-1} = 0,\ b_{i-1} > 0 \text{ and } c_i > 0, \end{cases}$$

where $i = 2, 3, \ldots, r - 1$,

$$k_r^* = \begin{cases} k_r + 1, & \text{if } c_{r-1} > 0 \text{ or } b_{r-1} = 0, \\ k_r, & \text{if } c_{r-1} = 0 \text{ and } b_{r-1} > 0. \end{cases}$$

Observe that the orthogonal array in this theorem is a saturated main-effect plan. This can, for example, be concluded from the fact that the original $\mathrm{OA}(s^m, (k_0 + 1) \times \prod_{i=1}^{r} k_i, s \times \prod_{i=1}^{r} s^{v_i}, 2)$ is a saturated main-effect plan and the manner in which the orthogonal array of Theorem 3.1 is constructed from this array.

We will now illustrate the result of Theorem 3.1 and the construction of the corresponding orthogonal array by two examples.

EXAMPLE 3.1. Let $s = 2$, $m = 9$, $r = 2$, $v_0 = 1$, $v_1 = 2$ and $v_2 = 4$. The values for c_i, b_i, m_i, k_i and k_i^*, $i = 0, 1, 2$, are then as follows:

i	c_i	b_i	m_i	k_i	k_i^*
0	1	0	1	6	4
1	0	1	3	8	9
2	2	1	5	32	32

Thus, we can obtain a saturated main-effect plan $\mathrm{OA}(2^9, 4 \times 9 \times 32, 2 \times 4 \times 16, 2)$. This plan can be represented as follows:

$$\begin{array}{ll} E_2 \oplus D_{22} & \text{32 factors at 16 levels;} \\ \left.\begin{array}{l} 0_2 \oplus E_1 \oplus D_{21} \\ 0_2 \oplus 0_1 \oplus E_1 \oplus \overline{0}_1 \end{array}\right\} & 8 + 1 = 9 \text{ factors at 4 levels;} \\ \left.\begin{array}{l} 0_2 \oplus 0_1 \oplus E_0 \oplus D_{10} \oplus \overline{D}_{10} \\ 0_2 \oplus 0_1 \oplus 0_0 \oplus E_0 \oplus \overline{D}_{10} \\ 0_2 \oplus 0_1 \oplus 0_0 \oplus 0_0 \oplus E_0 \end{array}\right\} & 2 + 1 + 1 = 4 \text{ factors at 2 levels.} \end{array}$$

Here, $\overline{D}_{10}$ is the array obtained from D_{10} by deleting the row of zeros. In this example, our methods do not allow us to combine four-level factors to make an additional 16-level factor because $c_1 = 0$ and $b_1 > 0$.

EXAMPLE 3.2. Let $r = 1$. Then $k_0^* = k_0 - (s^{v_1} - s)/(s - 1)$ and $k_1^* = k_1 + 1$. A saturated main-effects plan $\mathrm{OA}(s^m, k_0^* \times k_1^*, s \times s^{v_1}, 2)$ can thus be obtained for any $m \geq v_1 \geq 2$ and prime power s. Wu (1989) treats the special case $s = 2$, $v_1 = 2$ by a more complicated technique. Wu, Zhang and Wang (1992) extend this technique to generalize the result of Wu (1989) to the result in this example. See also our Example 3.4.

To assume a priori that interaction effects are negligible is usually not recommendable. In view of that and the fact that not all possible factors may be needed in a particular experiment, it is desirable to have some knowledge about the alias structure of the design. There are different approaches that can be used. For one, we could seek a desirable alias structure by an appropriate choice of the difference schemes D_{1i} and D_{2i}. This approach is followed in Pu (1989). As an alternative, the array from Theorem 3.1 can be converted to an $\mathrm{OA}(s^m, \sum_{i=0}^r k_i^*(s^{v_i} - 1)/(s - 1), s, 2)$ by the method of replacement as discussed in Section 2. The alias structure of the asymmetrical orthogonal array can now be studied through that of this symmetrical orthogonal array. This is in the spirit of Wu (1989).

The method of replacement allows also an immediate generalization of Theorem 3.1. Since an $\mathrm{OA}(s^{v_{i+1}}, (s^{v_{i+1}} - 1)/(s^{v_i} - 1), s^{v_i}, 2)$ exists for $i = 0, 1, \ldots, r - 1$, any factor at $s^{v_{i+1}}$ levels can be replaced by $(s^{v_{i+1}} - 1)/(s^{v_i} - 1)$ factors at s^{v_i} levels. This leads to the following generalization of Theorem 3.1.

THEOREM 3.2. *Let $s, m, v_0, \ldots, v_r, k_0^*, \ldots, k_r^*$ be as in Theorem* 3.1. *An* $\mathrm{OA}(s^m, \prod_{i=0}^r l_i, \prod_{i=0}^r s^{v_i}, 2)$ *exists if the following inequalities hold*:

$$\sum_{i=j}^{r} l_i(s^{v_i} - 1) \leq \sum_{i=j}^{r} k_i^*(s^{v_i} - 1), \qquad j = 0, 1, \ldots, r.$$

EXAMPLE 3.3. With the parameters as in Example 3.1, we can conclude that an $\mathrm{OA}(2^9, l_0 \times l_1 \times l_2, 2 \times 4 \times 16, 2)$ exists if

$$l_2 \leq 32,$$
$$3l_1 + 15l_2 \leq 507$$

and

$$l_0 + 3l_1 + 15l_2 \leq 511.$$

EXAMPLE 3.4. With the parameters as in Example 3.2, we can conclude that an $\mathrm{OA}(s^m, l_0 \times l_1, s \times s^{v_1}, 2)$ exists if

$$l_1 \leq (s^m - s^{m_1})/(s^{v_1} - 1) + 1,$$
$$l_0(s - 1) + l_1(s^{v_1} - 1) \leq s^m - 1.$$

The special case of $s = v_1 = 2$ corresponds to the result in Wu (1989), and is generalized to the result in this example by Wu, Zhang and Wang (1992).

Wu, Zhang and Wang (1992) discuss a fairly complicated method to construct orthogonal arrays of the form $OA(s^m, \prod_{i=0}^{r} l_i, \prod_{i=0}^{r} s^{v_i}, 2)$ with s a prime power and $v_0 = 1$, but no further restrictions on the v_i's. Clearly then, not all their arrays can be obtained from the result in Theorem 3.2. However, many arrays constructed by Wu, Zhang and Wang can also be constructed by the methods in our paper with, typically, an important role for Theorem 2.1. We illustrate this by an example.

EXAMPLE 3.5. We will show how to obtain an $OA(2^6, l_0 \times l_1 \times 1, 2 \times 4 \times 8, 2)$, where $l_1 \leq 16$ and $l_0 + 3l_1 \leq 56$. Clearly, the existence of this array does not immediately follow from Theorem 3.2. However, from a difference scheme $D(16, 16, 4)$ we obtain a one-resolvable $OA(2^6, 16, 4, 2)$, while an $OA(2^4, 8 \times 1, 2 \times 8, 2)$ can be obtained as in Theorem 3.1. Using Theorem 2.1, these arrays can be combined to an $OA(2^6, 8 \times 16 \times 1, 2 \times 4 \times 8, 2)$. If a value of $l_1 < 16$ is desired, each redundant four-level factor can be replaced by three two-level factors by the method of replacement.

The chief advantages of our methods are simplicity and ease of extension to other families of orthogonal arrays, as illustrated in the remainder of the paper. For a discussion on advantages of the method by Wu, Zhang and Wang the reader is referred to Wu (1989).

The techniques that were applied for the special case discussed in the preceding paragraphs of this section can also be used for other cases. To illustrate this, we will now discuss a second family of designs. This time, consider a design with $2s^m$ runs instead of s^m runs. Again, let $m \geq v_r > v_{r-1} > \cdots > v_0 = 1$, v_{i-1} divides v_i, $i = 1, 2, \ldots, r$, and s is a prime power. Further, let b_i, c_i, d_i and m_i be defined as for the previous case. From Section 2 we know that a difference scheme $D(2s^{m_i}, 2s^{m_i}, s^{m_i})$ based on the additive group associated with $GF(s^{m_i})$ exists. Analogous to an argument by Bose and Bush (1952), this implies the existence of a difference scheme $D(2s^{m_i}, 2s^{m_i}, s^{v_i})$ based on the additive group associated with $GF(s^{v_i})$. To this latter difference scheme we will refer as D_{3i}, $i = 0, 1, \ldots, r$. The array represented by

$$\begin{array}{l} E_i \oplus \overbrace{D_{1i} \oplus D_{1i} \oplus \cdots \oplus D_{1i}}^{c_i + d_i - 2 \text{ times}} \oplus D_{3i}, \\ 0_i \oplus E_i \oplus D_{1i} \oplus \cdots \oplus D_{1i} \oplus D_{3i}, \\ \vdots \\ 0_i \oplus 0_i \oplus 0_i \oplus \cdots \oplus E_i \oplus D_{3i} \end{array}$$

is then a $s^{(c_i+d_i-2)v_i}$-resolvable $OA(2s^{m_{i+1}}, 2(s^{m_{i+1}} - s^{m_i})/(s^{v_i} - 1), s^{v_i}, 2)$. One factor, represented by

$$0_i \oplus 0_i \oplus 0_i \oplus \cdots \oplus 0_i \oplus E_i \oplus \hat{0}_i,$$

can be added to obtain a $s^{(c_i+d_i-1)v_i}$-resolvable OA($2s^{m_i+1}, 2(s^{m_i+1} - s^{m_i}) + 1, s^{v_i}, 2$), where $\hat{0}_i$ denotes a $1 \times 2s^{b_i}$ vector of zeros. Thus, if we define

$$f_i = 2(s^{m_i+1} - s^{m_i})/(s^{v_i} - 1), \qquad i = 0, 1, \dots, r,$$

then we obtain from repeated applications of Theorem 2.1 an OA($2s^m$, $(f_0 + 1) \times \prod_{i=1}^r f_i, s \times \prod_{i=1}^r s^{v_i}, 2$). Although this family of main-effect plans does not contain any saturated designs, similar considerations as for the previous case to combine lower level factors to one higher level factor can be entertained for this case. This leads to the following analogue of Theorem 3.2.

THEOREM 3.3. *Let $s, m, v_0, \dots, v_r, b_0, \dots, b_r, c_0, \dots, c_r, m_0, \dots, m_r$ be as in the previous paragraphs. An* OA($2s^m, \prod_{i=0}^r l_i, \prod_{i=0}^r s^{v_i}, 2$) *exists if the following inequalities hold*:

$$\sum_{i=j}^{r} l_i(s^{v_i} - 1) \le \sum_{i=j}^{r} f_i^*(s^{v_i} - 1), \qquad j = 0, 1, \dots, r,$$

where $f_0^*, f_1^*, \dots, f_r^*$ are defined by

$$f_0^* = \begin{cases} f_0 + 1, & \text{if } c_0 = 0, \\ f_0 - (s^{v_1} - s^{v_0})/(s^{v_0} - 1), & \text{if } c_0 > 0, \end{cases}$$

$$f_i^* = \begin{cases} f_i + 1, & \text{if } c_{i-1} > 0,\ c_i = 0, \\ f_i, & \text{if } c_{i-1} = 0,\ c_i = 0, \\ f_i - (s^{v_{i+1}} - s^{v_i})/(s^{v_i} - 1), & \text{if } c_{i-1} > 0,\ c_i > 0, \\ f_i - (s^{v_{i+1}} - s^{v_i})/(s^{v_i} - 1) - 1, & \text{if } c_{i-1} = 0,\ c_i > 0, \end{cases}$$

where $i = 1, 2, \dots, r - 1$.

$$f_r^* = \begin{cases} f_r + 1, & \text{if } c_{r-1} > 0, \\ f_r, & \text{if } c_{r-1} = 0. \end{cases}$$

Thus if $c_i > 0$, some factors in our OA($2s^{m_i+1}, f_i, s^{v_i}, 2$) can be combined in the same spirit as for Theorem 3.1 to form one factor at $s^{v_{i+1}}$ levels. This analogy fails, however, if $c_i = b_i = 0$.

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