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SPECIAL CAPACITIES, THE HUNT–STEIN THEOREM AND TRANSFORMATION GROUPS

BY ZBIGNIEW SZKUTNIK

University of Mining and Metallurgy, Cracow

The problem of maximin testing between families of probability measures generated by special capacities and transformation groups is solved by showing that the Hunt–Stein theorem is applicable to such, generally nondominated, families of probability measures and that the problem reduces to maximin testing between neighbourhoods of distributions of maximal invariant, defined by special capacities closely related to the initial ones. An application to testing approximate binormality is also sketched.

1. Introduction. Consider a measure space $(\mathscr{X}, \mathscr{B}, m)$ and suppose that m is positive and σ-finite. Let $\mathscr{Q}_0$ and $\mathscr{Q}_1$ denote disjoint families of probability measures absolutely continuous with respect to (w.r.t.) m and let G be a transformation group acting on $\mathscr{X}$, satisfying regularity conditions stated explicitly, for example, by Bondar and Milnes (1981) and preserving the testing problem $(\mathscr{Q}_0, \mathscr{Q}_1)$. The classical Hunt–Stein (HS) theorem can be formulated as follows: If G is an amenable group, almost G-invariant tests form a maximin complete class, that is, in order to find a maximin test for $(\mathscr{Q}_0, \mathscr{Q}_1)$, one can restrict oneself to the class of almost G-invariant tests. σ-finiteness of the dominating measure m is essential, since it implies the weak compactness of the set of critical functions and this property is, in turn, crucial for the proofs of the HS theorem [Lehmann (1959), Bondar and Milnes (1981)].

In an alternative approach originating from Kiefer (1957) [see also Wesler (1959) and Kiefer (1966), page 263] and resulting in the so-called generalized HS theorem (for general decision problems), the compactness of the corresponding set of decision rules results from a construction of a suitable locally convex topology on the class of decision functions. As a result, the theorem applies to nondominated problems as well. An application of the generalized HS theorem to the testing problem considered in this paper seems not, however, to be straightforward. Hence, we shall work with the classical testing version of the HS theorem and show that it can be applied to a class of nondominated testing problems defined later in this section.

Denote by P_0 and P_1 two probability measures on $(\mathscr{X}, \mathscr{B})$ absolutely continuous w.r.t. m and let f_0 and f_1 be concave functions $[0,1] \to [0,1]$ satisfying $f_i(1) = 1$, $i = 0, 1$. Following Bednarski (1981) and Buja (1980) we define special capacities $\nu_i = \nu_{f_i P_i}$ as set functions ν_i: $\mathscr{B} \to [0,1]$ such that $\nu_i(\varnothing) = 0$ and $\nu_i(A) = f_i[P_i(A)]$ for nonempty $A \in \mathscr{B}$, $i = 0, 1$.

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Bednarski (1981) and Buja (1980, 1986) [see also Birge (1977, 1980)] consider maximin testing between neighbourhoods of P_0 and P_1 defined by special capacities in the following way:

$$\mathscr{A}_i = \{P: P(A) \le \nu_i(A): A \in \mathscr{B}\}, \qquad i = 0, 1.$$

We shall study a more general testing problem $(\mathscr{P}_0, \mathscr{P}_1)$, where $\mathscr{P}_i = \{Pg\colon P \in \mathscr{A}_i, g \in G\}$, $i = 0, 1$. $\mathscr{P}_i$ can be considered as a neighbourhood of the whole parametric family of distributions $\{P_i g\colon g \in G\}$. Assume that $\mathscr{P}_0 \cap \mathscr{P}_1 = \varnothing$. The testing problem $(\mathscr{P}_0, \mathscr{P}_1)$ remains, clearly, invariant under the action of G but, in general, the HS theorem cannot be applied since $\mathscr{P}_0$ and $\mathscr{P}_1$ are not dominated by any σ-finite measure unless $f_i(0) = 0$, which means that ν_i are 2-alternating Choquet capacities provided f_i are continuous in $[0, 1]$ [Bednarski (1981)].

Denote by $\mathscr{M}_m$ the set of all probability measures absolutely continuous w.r.t. m and take $\mathscr{Q}_i = \mathscr{P}_i \cap \mathscr{M}_m$. In regular cases the HS theorem is applicable to the testing problem $(\mathscr{Q}_0, \mathscr{Q}_1)$. It will be shown that the HS theorem applies to $(\mathscr{P}_0, \mathscr{P}_1)$ as well. This will be achieved by an application of the HS theorem to $(\mathscr{Q}_0, \mathscr{Q}_1)$ and proving the existence of an invariant maximin test for this problem which is also maximin for $(\mathscr{P}_0, \mathscr{P}_1)$.

An application to testing approximate binormality will briefly be described in the last section.

2. Assumptions and some preliminary results. In what follows, we shall always assume that:

(A1). $(\mathscr{X}, \mathscr{B})$ is a locally compact Polish space and $\mathscr{B}$ is its Borel σ-algebra.

(A2). G is a σ-compact, locally compact, Hausdorff amenable group acting properly on the left of $\mathscr{X}$.

(A3). $\mathscr{B}$ is G-invariant, that is $gB = \{gx\colon x \in B\} \in \mathscr{B}$ for all $B \in \mathscr{B}$.

(A4). m is a positive and σ-finite measure on $\mathscr{B}$ such that m dominates each measure gm defined for $g \in G$ and $B \in \mathscr{B}$ by $gm(B) = m(gB)$.

(A5). m is relatively invariant under G.

Proper action in (A2) means that the action is continuous and that the inverse image of a compact set under the mapping $(g, x) \to (gx, x)$ is compact. $\mathscr{X}/G$ will denote the space of orbits of G which is given the quotient topology and $\Pi\colon \mathscr{X} \to \mathscr{X}/G$ will be the canonical projection. Properness of the action implies [see Andersson (1982)] that $\mathscr{X}/G$ is Hausdorff and locally compact so that the notion of Radon measures on $\mathscr{X}/G$ can be applied. Moreover, $\mathscr{X}/G$ is σ-compact and paracompact.

Michael (1959) proved that if $h\colon X \to Y$ is an open mapping of a metrizable space X onto a paracompact space Y and, for each $y \in Y$, the set $h^{-1}(y)$ is complete, then Y is metrizable. It can be shown that Π is open. $\mathscr{X}/G$ is Hausdorff which implies that $\{y\}$ is closed. Hence, $\Pi^{-1}(y)$ is complete as a closed subset of the topologically complete space $\mathscr{X}$ and, using Michael's result, $\mathscr{X}/G$ is metrizable.

$\mathscr{X}/G$ is separable since it is metrizable and second countable which, in turn, is implied by its local compactness and the fact that it is a continuous image (under Π) of a second countable space $\mathscr{X}$. Finally, since $\mathscr{X}/G$ is metrizable and locally compact, it is topologically complete.

Hence, $\mathscr{X}/G$ is Polish and the theory developed by Bednarski (1981) and Buja (1980, 1986) applies to measures on $\mathscr{X}/G$.

The last part of (A4) can be formulated in the form: For all $g \in G$ and $B \in \mathscr{B}$, $m(B) = 0$ if and only if $m(gB) = 0$. This, in turn, leads to the following equality, which will be useful later:

$$\mathscr{Q}_i = \{Pg\colon P \in \mathscr{A}_i, g \in G\} \cap \mathscr{M}_m = \{Pg\colon P \in \mathscr{A}_i \cap \mathscr{M}_m, g \in G\}, \quad i = 0,1. \tag{1}$$

(A2), in particular amenability of G, is required in order that the classical HS theorem be applicable.

By $\beta_\psi(P) = E_P\psi(x)$ we shall denote the power function of the test ψ. Suppose that φ is a maximin test at its significance level for $(\mathscr{Q}_0, \mathscr{Q}_1)$. This means that for each test ψ we have the following implication:

$$\begin{aligned}&\text{If } \sup\{\beta_\psi(P)\colon P \in \mathscr{Q}_0\} \le \sup\{\beta_\varphi(P)\colon P \in \mathscr{Q}_0\},\\ &\text{then } \inf\{\beta_\psi(P)\colon P \in \mathscr{Q}_1\} \le \inf\{\beta_\varphi(P)\colon P \in \mathscr{Q}_1\}.\end{aligned} \tag{2}$$

LEMMA. *Let ϕ be a size α maximin test for $(\mathscr{Q}_0, \mathscr{Q}_1)$. If*

$$\begin{aligned}\sup\{\beta_\varphi(P)\colon P \in \mathscr{Q}_0\} &= \sup\{\beta_\varphi(P)\colon P \in \mathscr{P}_0\},\\ \inf\{\beta_\varphi(P)\colon P \in \mathscr{Q}_1\} &= \inf\{\beta_\varphi(P)\colon P \in \mathscr{P}_1\},\end{aligned} \tag{3}$$

then ϕ is a size α maximin test for $(\mathscr{P}_0, \mathscr{P}_1)$.

PROOF. (2) and (3) imply that, for each test ψ,

$$\begin{aligned}&\text{if } \sup\{\beta_\psi(P)\colon P \in \mathscr{Q}_0\} \le \sup\{\beta_\varphi(P)\colon P \in \mathscr{P}_0\},\\ &\text{then } \inf\{\beta_\psi(P)\colon P \in \mathscr{Q}_1\} \le \inf\{\beta_\varphi(P)\colon P \in \mathscr{P}_1\}.\end{aligned} \tag{4}$$

If $\sup\{\beta_\psi(P)\colon P \in \mathscr{P}_0\} \le \sup\{\beta_\varphi(P)\colon P \in \mathscr{P}_0\}$, then also $\sup\{\beta_\psi(P)\colon P \in \mathscr{Q}_0\} \le \sup\{\beta_\varphi(P)\colon P \in \mathscr{P}_0\}$ and (4) implies that $\inf\{\beta_\psi(P)\colon P \in \mathscr{Q}_1\} \le \inf\{\beta_\varphi(P)\colon P \in \mathscr{P}_1\}$. Hence, $\inf\{\beta_\psi(P)\colon P \in \mathscr{P}_1\} \le \inf\{\beta_\varphi(P)\colon P \in \mathscr{P}_1\}$, which completes the proof. □

REMARK. This lemma remains true for all pairs of testing problems $(\mathscr{Q}_0, \mathscr{Q}_1)$ and $(\mathscr{P}_0, \mathscr{P}_1)$ such that $\mathscr{Q}_i \subset \mathscr{P}_i$, $i = 0, 1$, not necessarily having the structure considered in this paper.

3. Main result. Throughout this section we shall not repeat continually that $i = 0, 1$. Whenever i appears it can take value 0 and 1.

Consider first the problem of constructing maximin tests for $(\mathscr{Q}_0, \mathscr{Q}_1)$. Let $T\colon (\mathscr{X}, \mathscr{B}) \to (\mathscr{Y}, \mathscr{B}_Y)$ be a maximal invariant under G. Application of the HS theorem reduces the problem to testing between families of distributions of the maximal invariant, that is, $\mathscr{Q}_i^T = \{PT^{-1}: P \in \mathscr{A}_i \cap \mathscr{M}_m\}$. Consider a related problem of testing between $\mathscr{A}_i^T = \{PT^{-1}: P \in \mathscr{A}_i\}$ and define special capacities $\nu_i^T(B) = f_i[P_i T^{-1}(B)]$, $B \in \mathscr{B}_Y$. Let further

$$\mathscr{D}_i = \{Q: Q(A) \le \nu_i^T(A): A \in \mathscr{B}_Y\}.$$

It is an interesting fact, which can generally be useful in the theory of robust minimax testing, that the equality $\mathscr{A}_i^T = \mathscr{D}_i$ holds, under mild assumptions, in a more general setting when T is a surjective mapping (not necessarily an invariant) and ν_i is a pseudocapacity defined by Buja (1986). The inclusion $\mathscr{A}_i^T \subset \mathscr{D}_i$ is obvious. The reverse inclusion is equivalent to the existence of an extension of any probability measure, say P, defined on the sub σ-algebra $\mathscr{B}_0 = T^{-1}(\mathscr{B}_Y) \subset \mathscr{B}$ and satisfying $P(B) \le \nu_i(B)$ for each $B \in \mathscr{B}_0$, to the σ-algebra $\mathscr{B}$ in such a way that $P \le \nu_i$ for all Borel subsets of $\mathscr{X}$. Existence and properties of such extensions, without any boundedness by capacity condition are discussed by Plachky and Rüschendorf (1984). The general equality $\mathscr{A}_i^T = \mathscr{D}_i$ has recently been proved by Buja (1989).

What we really need in the present context are, however, not the equalities $\mathscr{A}_i^T = \mathscr{D}_i$, but rather the existence of a Huber–Strassen (1973) least favourable pair (Q_0^*, Q_1^*) for the testing problem $(\mathscr{D}_0, \mathscr{D}_1)$ such that $Q_i^* \in Q_i^T$. If one finds such a pair, then the Neyman–Pearson tests for (Q_0^*, Q_1^*) will be maximin not only for $(\mathscr{D}_0, \mathscr{D}_1)$ but for $(\mathscr{Q}_0^T, \mathscr{Q}_1^T)$ as well (lemma applies). Further, according to the HS theorem, they will be maximin for $(\mathscr{Q}_0, \mathscr{Q}_1)$ and, finally, after another application of the lemma and making explicit use of the test invariance, they will be maximin for $(\mathscr{P}_0, \mathscr{P}_1)$.

In what follows we shall use the canonical projection $\Pi: \mathscr{X} \to \mathscr{X}/G$ as a maximal invariant. For a comprehensive treatment of the theory used in this section see, for example, Andersson (1982).

As noted in Section 2, the space $\mathscr{X}/G$ is sufficiently regular to admit the application of Radon measures and of the theory of Bednarski (1981) and Buja (1986). Let β be a right Haar measure on G and Δ_G the modular function of G. Assume that m is relatively invariant under G with multiplier Δ_G^{-1}. This assumption is not restrictive, since any relatively invariant measure can easily be modified to have this property. Then, there exists on $\mathscr{X}/G$ the so-called quotient measure m/β dominating the distributions $\Pi(P_0)$ and $\Pi(P_1)$ of the maximal invariant:

$$\Pi(P_i) = q_i m/\beta, \quad \text{where } q_i[\Pi(x)] = \int_G p_i(gx)\, d\beta(g) \quad \text{and} \quad p_i = dP_i/dm.$$

THEOREM. *Assume* (A1)–(A5) *hold true. If* $\Pi(P_0)$ *and* $\Pi(P_1)$ *are mutually absolutely continuous and there exists* $0 < x_0 < 1$ *such that* $f_0(x_0) = f_1(x_0) = 1$, *then for the testing problem* $(\mathscr{Q}_0, \mathscr{Q}_1)$ *there exists a Huber–Strassen least favourable pair* $(Q_0^*, Q_1^*) \in \mathscr{Q}_0 \times \mathscr{Q}_1$ *such that* $Q_i^* \in \mathscr{Q}_i^T$, $i = 0, 1$. *The Neyman–Pearson tests for* (Q_0^*, Q_1^*) *are maximin not only for* $(\mathscr{Q}_0, \mathscr{Q}_1)$ *but also for* $(\mathscr{P}_0, \mathscr{P}_1)$.

PROOF. It is seen from the proof of Theorem 4.1 of Bednarski (1981) that there exists a least favourable pair (Q_0^*, Q_1^*) such that $Q_i^* \ll \Pi(P_i)$. Hence, $Q_i^* \ll m/\beta$ and $Q_i^*(B) \le f_i[\Pi(P_i)(B)]$ for each measurable $B \in \mathscr{X}/G$.

Let $q_i^* = dQ_i^*/dm/\beta$. Define probability measures P_i^* on $(\mathscr{X}, \mathscr{B})$ such that

$$dP_i^*/dm = p_i^*(x) = q_i^*[\Pi(x)]p_i(x)/q_i[\Pi(x)]. \tag{5}$$

Formulae (15) in Andersson (1982) easily imply that $\int_{\mathscr{X}} p_i^*(x)\,dm(x) = 1$ so that P_i^* are properly defined probability measures. It is also immediately clear that $Q_i^* = \Pi(P_i^*)$. Further, one has to show that $P_i^* \le \nu_i$, that is, $P_i^*(A) = \int_{\mathscr{X}} I_A(x)p_i^*(x)\,dm(x) \le f_i[\int_{\mathscr{X}} I_A(x)p_i(x)\,dm(x)] = f_i[P_i(A)]$ for all $A \in \mathscr{B}$, where I_A is the indicator function of A. Using formulae (15) from Andersson (1982) one can write

$$P_i(A) = \int_{\mathscr{X}/G} q_i(u)h_i^A(z)/q_i(u)\,dm/\beta(u),$$

$$P_i^*(A) = \int_{\mathscr{X}/G} q_i^*(u)h_i^A(z)/q_i(u)\,dm/\beta(u),$$

where $\Pi(z) = u$ and $h_i^A(z) = \int_G I_A(gz)p_i(gz)\,d\beta(g)$.

Denote $h_i^A(z)/q_i(u)$ by $k_i^A(u)$. Obviously, $0 \le k_i^A(u) \le 1$ for each $u \in \mathscr{X}/G$. It must be shown that, if for each measurable $B \in \mathscr{X}/G$,

$$\int_B q_i^*(u)\,dm/\beta(u) \le f_i\left[\int_B q_i(u)\,dm/\beta(u)\right], \tag{6}$$

then

$$\int_{\mathscr{X}/G} q_i^*(u)k_i^A(u)\,dm/\beta(u) \le f_i\left[\int_{\mathscr{X}/G} q_i(u)k_i^A(u)\,dm/\beta(u)\right]. \tag{7}$$

Assume, for a moment, that $k_i^A(u) = \sum_{k=1}^n a_k I_{A_k}(u)$ is a simple function and A_k are disjoint. Then (7) takes the form

$$\sum_{k=1}^n a_k Q_i^*(A_k) \le f_i\left[\sum_{k=1}^n P_i\Pi^{-1}(A_k)\right]. \tag{8}$$

Denote $Q_i^*(A_k)$ by e_k and $P_i\Pi^{-1}(A_k)$ by d_k, $k = 1,\dots,n$. Then (6) implies

$$\begin{aligned} e_k &\le f_i(d_k), \qquad k = 1,\dots,n,\\ e_k + e_l &\le f_i(d_k + d_l), \qquad k,l = 1,\dots,n \text{ and } k \ne l,\\ &\vdots\\ e_1 + e_2 + \cdots + e_n &\le f_i(d_1 + d_2 + \cdots + d_n). \end{aligned} \tag{9}$$

Inequality (8) is equivalent to

$$H(a_1,\dots,a_n) = \sum_{k=1}^n a_k e_k - f_i\left[\sum_{k=1}^n a_k d_k\right] \le 0$$

which must hold for all $a_1,\dots,a_n$ such that $0 \le a_k \le 1$ for all k [since $0 \le k_i^A(u) \le 1$], $\sum e_k a_k \le 1$ and $\sum d_k a_k \le 1$ [since the left-hand sides of these inequalities are equal to $P_i^*(A)$ and $P_i(A)$, respectively]. Concavity of f implies that $H(a_1, \dots, a_n)$ is a convex functional and (9) means that H takes on negative values in all vertices of the n-dimensional cube $[0,1]^n$. Hence, H takes on negative values in all points of the cube, which proves (8).

If $k_i^A(u)$ is not simple, it can be approximated by a nondecreasing sequence of simple functions, say $h_j(u)$, as a measurable function. Then, for each j, one has the inequality

$$\int_{\mathscr{X}/G} q_i^*(u) h_j(u)\, dm/\beta(u) \le f_i\left[\int_{\mathscr{X}/G} q_i(u) h_j(u)\, dm/\beta(u)\right].$$

Passing to the limit ($j \to \infty$) and using the Lebesgue monotone convergence theorem we get (7).

This completes the proof of the first part of the thesis. It remains to prove that the Neyman–Pearson tests for (Q_0^*, Q_1^*) are maximin for $(\mathscr{P}_0, \mathscr{P}_1)$. For a suitably chosen version of the density dQ_1^*/dQ_0^*, the Neyman–Pearson tests for (Q_0^*, Q_1^*) are G-invariant and maximin for $(\mathscr{Q}_0, \mathscr{Q}_1)$. Let φ be such a test. To complete the proof it suffices to show that equalities (3) hold true and to apply the lemma from Section 2. We have $\sup\{\beta_\varphi(P)\colon P \in \mathscr{Q}_0\} = \sup\{\beta_\varphi(Pg)\colon P \in \mathscr{A}_0 \cap \mathscr{M}_m, g \in G\} = \sup\{\beta_\varphi(P)\colon P \in \mathscr{A}_0 \cap \mathscr{M}_m\} = \sup\{\beta_\varphi(P)\colon P \in \mathscr{A}_0\}$, where we used formula (1), the invariance of φ and the fact that (Q_0^*, Q_1^*) is a Huber–Strassen least favourable pair for $(\mathscr{Q}_0, \mathscr{Q}_1)$ and $Q_0^* \in \mathscr{Q}_0^T \subset \mathscr{A}_0^T \subset \mathscr{Q}_0$. (As mentioned previously we even have $\mathscr{A}_0^T = \mathscr{Q}_0$.) The second of equalities (3) can be proved in a similar way. This completes the proof.

Two remarks are in order here: (i) In view of our result, in order to find maximin tests for $(\mathscr{P}_0, \mathscr{P}_1)$, one needs only to solve the testing problem $(\mathscr{Q}_0, \mathscr{Q}_1)$. This can be done through an application of Theorem 5.1 of Bednarski (1981) preceded by finding $d\Pi(P_1)/d\Pi(P_0)$ which, in turn, is given by general formulae of the Stein type [see, for example, Andersson (1982) and Wijsman (1985)]. (ii) In the particular case of generalized contamination neighbourhoods considered, for example, by Huber (1968) and Rieder (1977), the assumption of mutual absolute continuity of $\Pi(P_0)$ and $\Pi(P_1)$ and of $f_i(x_0) = 1$ for some $x_0 < 1$ is not necessary. The existence of Q_0^* and Q_1^* absolutely continuous w.r.t. m/β is implied, in this case, by absolute continuity of $\Pi(P_0)$ and $\Pi(P_1)$ w.r.t. m/β [cf. Hafner (1982)]. Pure ε-contamination case is here a special case.

4. Maximin testing approximate binormality. Application of statistical procedures to data is often preceded by testing goodness-of-fit. However, the question to be answered is not whether the distribution of data is exactly the same as that assumed in the model. One would rather like to know whether the difference between the true distribution and the model is small enough to preclude serious errors in the inference based on the model which is only approximately correct. Using powerful and consistent tests leads to rejection of the null hypothesis if the sample size is large enough and does not give a satisfactory answer to the question considered.

So, it would be of interest to provide tools for testing approximate goodness-of-fit. This will be our aim in this section in the context of testing binormality, the assumption commonly made in the analysis of bivariate data.

Denote by $X = [X_1, \dots, X_n]$ a $(2, n)$ random matrix. The columns X_i of X are interpreted as independent observations. Let $\mathscr{N}_2$ be the standard gaussian measure on R^2 and let F be another, nongaussian distribution on R^2. Take $G = UT(2) \times R^2$: $R^2 \to R^2$ with the action $gY = CY + b$, where $UT(2)$ is the group of upper triangular matrices with positive diagonals $Y \in R^2$, $C \in UT(2)$, $b \in R^2$. Then $\{(\mathscr{N}_2 g)^{\otimes n}, g \in G\}$ and $\{(Fg)^{\otimes n}, g \in G\}$ constitute the null and alternative hypotheses in a G-invariant problem of testing exact binormality against this particular alternative. Maximin solutions of such problems with F being bivariate uniform and bivariate exponential have been given by Szkutnik (1988).

In the usual approach to problems of existence and construction of robust tests one assumes independence and often also identical distributions of observations [cf. Huber (1965), Huber and Strassen (1973)]. Departures from a postulated model are described as departures from the distribution of single observations and, in such cases, the least favourable pair of distributions for product measures is a product of least favourable pairs for single observations. We cannot follow this idea if we take $\mathscr{P}_0 = \{(Pg)^{\otimes n}: P \le \nu_{f_0 \mathscr{N}_2}, g \in G\}$ and $\mathscr{P}_1 = \{(Pg)^{\otimes n}: P \le \nu_{f_1 F}, g \in G\}$ since, for a single observation, the only size α test is the trivial one. Also the problems of applicability of the HS theorem and of construction of maximin tests for $(\mathscr{Q}_0, \mathscr{Q}_1)$ with $\mathscr{Q}_i = \mathscr{P}_i \cap \mathscr{M}_m$ and $\mathscr{P}_i$ defined as above, seem to be nontrivial and remain open.

We adopt another approach by taking $P_0 = \mathscr{N}_2^{\otimes n}$, $P_1 = F^{\otimes n}$, $\mathscr{P}_i = \{Pg: P \le \nu_{f_i P_i}, g \in G\}$ with G acting on $R^{\otimes 2n}$ according to $gX = CX + b\mathbf{1}_n^T$, where $\mathbf{1}_n^T = (1, \dots, 1) \in R^n$. This and the results developed in the preceding sections lead to testing between neighbourhoods of the distributions of a maximal invariant.

Take $f_i(x) = [(1 - \varepsilon_i)x + \delta_i] \wedge 1$ with $0 \le \varepsilon_i \le \delta_i \le 1$, which leads to generalized contamination neighbourhoods and let F be the uniform distribution over $[0, 1]^2$ or bivariate exponential distribution with the probability density function $\psi(\xi_1, \xi_2) = \exp\{-\xi_1 - \xi_2\}$ for $\xi_1, \xi_2 \ge 0$. Maximin tests for $(\mathscr{Q}_0, \mathscr{Q}_1)$ can be constructed with a method described in detail by Hafner (1982). They are based on the truncated ratio of densities of central distributions which has been found for bivariate uniform and bivariate exponential alternatives by Szkutnik (1988). According to our theorem these tests are maximin for $(\mathscr{P}_0, \mathscr{P}_1)$ as well.

We omit the details here and show only some exemplary results obtained on the basis of a numerical study. Let us fix $\varepsilon_0 = \delta_0 = 0.05$ and take $\varepsilon_1 = \delta_1 = 0.25$ in the first and $\varepsilon_1 = \delta_1 = 0.50$ in the second case. Note that this corresponds to assuming pure ε-contamination neighbourhoods of the hypotheses.

For $n = 10$ and bivariate exponential center of the alternative we get nonrandomized tests for $0.08 \le \alpha \le 0.20$ in the first and for $0.06 \le \alpha \le 0.12$ in the second case. Corresponding minimum power varies increasingly in the intervals $[0.54, 0.72]$ and $[0.36, 0.44]$, respectively. For α outside the indicated intervals randomization is necessary. This corresponds to segments of straight lines in graphs of the risk curves [cf. Hafner (1982)]. For comparison, the minimum power of maximin tests between central families, corresponding to $\alpha = 0.06$, 0.12 and 0.20 is 0.86, 0.93 and 0.96, respectively [cf. Szkutnik (1988)].

For $n = 25$ and bivariate uniform center of the alternative we get nonrandomized tests for $0.06 \leq \alpha \leq 0.18$ in the first and $0.06 \leq \alpha \leq 0.14$ in the second case with corresponding minimum power in the intervals [0.53, 0.73] and [0.36, 0.47], respectively, while the minimum power of maximin tests between central families, corresponding to $\alpha = 0.06$, 0.14 and 0.18 is 0.87, 0.98 and 0.99.

Three important facts should be mentioned explicitly. First, the only thing necessary for maximin testing between $\mathscr{P}_0$ and $\mathscr{P}_1$ is a maximin test for testing between transformation families of central distributions and its risk function. Second, in spite of the rather large contamination level and small sample sizes in the above examples, the minimum power is quite reasonable and the tests obtained in this paper can be used for testing binormality against two different types of alternatives. Third, since we take as a hypothesis a neighbourhood of the product measure, the tests are robust against violating the i.i.d. assumptions.

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