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PERMUTATIONAL EXTREME VALUES OF AUTOCORRELATION COEFFICIENTS AND A PITMAN TEST AGAINST SERIAL DEPENDENCE

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Normal approximations, as provided by permutational central limit theorems, conditionally can be arbitrarily bad. Such approximations therefore are poorly suited to the construction of critical values for Pitman (permutation) tests. A classical remedy consists in substituting a beta approximation (over the appropriate conditional interval range) for the normal one. Whereas deriving permutational extreme values for usual, nonserial statistics is generally straightforward, the corresponding problem for serial statistics (e.g., autocorrelation coefficients), however, appears somewhat more difficult. This problem, which is shown to reduce to a particular case of the well-known *travelling salesman problem*, is explicitly solved here for the autocorrelation coefficient of order one, allowing for a simple computation of permutational critical values for Pitman tests against serial dependence. The case of higher order autocorrelations is, however, of a different nature and requires another approach.

1. Introduction.

1.1. *Pitman tests and permutational central limit theorems.* Whereas invariance principles provide the theoretical grounds for the well-developed theory of rank tests, more classical unbiasedness and similarity arguments quite naturally lead to the relatively less familiar class of permutation tests—conveniently referred to as *Pitman tests*.

Pitman tests typically arise whenever the data, or some function thereof, reduce, under the null hypothesis to be tested, to a white noise series (though exchangeability here would be sufficient) with partly or completely unspecified distribution function. The corresponding order statistic then is sufficient complete (under the null hypothesis); conditioning (as suggested by classical Neyman-structure considerations) upon this sufficient statistic yields permutational distributions and Pitman tests.

Pitman tests based on the permutational distributions of classical, normal-theory test statistics can be shown asymptotically as powerful as the latter [see, e.g., Hoeffding (1952)], with, however, the important additional property of remaining valid and unbiased under completely arbitrary distributional

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assumptions. Such tests accordingly have been considered in a variety of situations: one- and two-sample location problems (permutational t-tests; see Fisher (1935), Pitman (1937a), Efron (1969), Cressie (1980) and Edelman (1986, 1990), to name only a few], bivariate correlation problems [Pitman (1937b)], analysis of variance [Pitman (1937c)], regression [Dufour and Hallin (1991a)]. A complete coverage of the subject can be found in Edgington (1980).

Pitman tests against serial dependence also have been considered, mainly in Wald and Wolfowitz (1943), Ghosh (1954) and David and Fix (1966). These three papers, however, essentially concentrate on establishing permutational central limit theorems for a class of statistics including serial correlation coefficients. Now such permutational limit results are of limited practical interest if a strictly conditional point of view is to be adopted. Unlike traditional (unconditional) central limit theorems, permutational (i.e., conditional) convergence results indeed usually cannot be interpreted as conditional approximation results, since convergence is far from uniform, which makes the quality of the approximation heavily dependent on the actual observed values (an important exception, due to distribution-freeness, is the case of rank-based statistics in the absence of ties). As mentioned by David and Fix (1966), the permutational distribution of autocorrelation coefficients can be extremely skew (for any series length n), and normal approximations clearly cannot account for any degree of skewness. Moreover, the critical values derived from permutational central limit theorems usually differ very little from the unconditional ones—thus annihilating most of the theoretical advantages of permutational procedures over their unconditional counterparts.

A classical remedy against the poor fit of a normal approximation to the distribution of a skew, interval-valued statistic, consists in considering a (four-parameter) beta approximation over the adequate interval, adjusting for the first two moments. This device has been successfully employed in approximating permutational distributions in the nonserial case [Pitman (1937a, b, c), as well as in approximating the unconditional distributions of correlation and autocorrelation coefficients [Dixon (1944), Durbin and Watson (1971), Anderson (1971) pages 338–347, Bartels (1982)].

1.2. *Serial correlation and the travelling salesman problem.* If a beta approximation is to be constructed for the permutational distribution of some test statistic, the first step is to identify the exact permutational range of the statistic under study. Obtaining this range is fairly trivial in the case of nonserial statistics, but turns out to be considerably less obvious for serial ones. Consider indeed a general linear, first-order serial statistic of the form

$$S(\mathbf{X}^{(n)}) = \sum_{t=2}^{n} J\left(X_t^{(n)}, X_{t-1}^{(n)}\right), \tag{1.1}$$

where $\mathbf{X}^{(n)} = (X_1^{(n)}, \ldots, X_n^{(n)})$ denotes an observed series of length n and $J(x, y)$ is some function (a *score* function) from $\mathbb{R}^2$ onto $\mathbb{R}$. Define the complete directed graph on the set of nodes $V = \{0, 1, \ldots, n\}$, and attach a

weight $J(X_j, X_i)$ at arc (i, j), with $J(0, X_i) = 0 = J(X_j, 0)$. Then the minimal (maximal) permutational value of $S(\mathbf{X}^{(n)})$ corresponds to a *minimum (maximum) weighted Hamiltonian cycle* through V (a Hamiltonian cycle through V is a cycle which passes exactly once through each node). The problem of determining such optimal Hamiltonian cycles is best known as the *travelling salesman problem*, itself a special case of the linear ordering problem [see Lawler, Lenstra and Rinnooy Kan (1985)]. These problems are of maximal complexity, and no general explicit solution, no fast algorithmic procedure can be expected to exist.

Now, in most cases of practical importance, tests against serial dependence are based on statistic (1.1) with symmetric multiplicative score function of the form $J(x, y) = H(x)H(y)$ (H may also depend on the order statistic $\mathbf{X}_{(\cdot)}$, e.g., through standardizing constants). This is the case for usual autocorrelation coefficients (whatever definition is adopted), Spearman–Wald–Wolfowitz autocorrelations [Wald and Wolfowitz (1943), Bartels (1982)], and also for van der Waerden autocorrelations [David and Fix (1966), Hallin and Puri (1988)]. Taking advantage of this particular structure of the weight function, we derive (Proposition 2.1 and 2.2) an explicit solution to the corresponding travelling salesman problem. This result is used, in Section 3, to derive critical values for Pitman tests based on usual sample autocorrelations.

The mathematical techniques used throughout the paper constitute a rare instance of a statistical application of graph-theoretical methods, which are rather nonstandard in the area.

2. Optimal Hamiltonian cycles. Denote by $\mathbf{x}_{(\cdot)} = (x_{(1)}, \ldots, x_{(n)})$, where $x_{(1)} \leq \cdots \leq x_{(n)}$, an ordered n-tuple of real numbers. Consider the complete, undirected graph on the set of nodes $V = \{1, \ldots, n\}$ and associate a weight $x_{(i)}x_{(j)}$ with edge (i, j). For the sake of clarity, denote by $x_{(i)}$ and $(x_{(i)}, x_{(j)})$ node i and edge (i, j), respectively.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a permutation of $\mathbf{x}_{(\cdot)}$. Define the objective function

$$T(\mathbf{x}) = \sum_{t=1}^{n} x_t x_{t-1} \tag{2.1}$$

with the convention $x_0 = x_n$. Clearly, $\mathbf{x}$ characterizes a Hamiltonian cycle through V. A permutation $\mathbf{x}^*$, or the corresponding cycle is called *minimal* (respectively, *maximal*) if $T(\mathbf{x}^*) = \min T(\mathbf{x})$ [respectively, $T(\mathbf{x}^*) = \max T(\mathbf{x})$], where the minimum (respectively, maximum) is taken over the $n!$ possible permutations of $\mathbf{x}_{(\cdot)}$. The problem of obtaining minimal or maximal permutations (cycles) is a special case of the so-called symmetric travelling salesman problem; due to the particular structure of the objective function (2.1), this problem here admits an explicit solution, which is described in Propositions 2.1 and 2.2 below.

PROPOSITION 2.1. *A minimal permutation of* $\mathbf{x}_{(\cdot)}$ *is*

$$\mathbf{x}_{\min} = (x_{(1)}, x_{(n)}, x_{(2)}, x_{(n-2)}, x_{(4)}, \ldots, x_{(5)}, x_{(n-3)}, x_{(3)}, x_{(n-1)}); \tag{2.2}$$

the corresponding value of the objective function is

$$T_{\min} = x_{(1)}x_{(n)} + \sum_{i=1}^{[(n-1)/2]} (x_{(i)}x_{(n-i)} + x_{(i+1)}x_{(n+1-i)}) + x_{(n/2)}x_{(n/2+1)}I(n \text{ even}), \tag{2.3}$$

where $[m]$ *denotes, as usual, the largest integer smaller than or equal to* m *and* $I(n \text{ even})$ *is one or zero according as* n *is even or odd.*

PROPOSITION 2.2. *A maximal permutation of* $\mathbf{x}_{(\cdot)}$ *is*

$$\mathbf{x}_{\max} = (x_{(n)}, x_{(n-1)}, x_{(n-3)}, x_{(n-5)}, \ldots, x_{(5)}, x_{(3)}, x_{(1)}, x_{(2)}, x_{(4)}, x_{(6)}, \ldots, x_{(n-6)}, x_{(n-4)}, x_{(n-2)}); \tag{2.4}$$

the corresponding value of the objective function is

$$T_{\max} = x_{(1)}x_{(2)} + \sum_{i=1}^{n-2} x_{(i)}x_{(i+2)} + x_{(n-1)}x_{(n)}. \tag{2.5}$$

The proof of Proposition 2.1 relies on two lemmas.

LEMMA 2.1. *Let* $y \in \mathbb{R}$. *Then* $\mathbf{x}^*$ *is a minimal (maximal) permutation of* $\mathbf{x}_{(\cdot)}$ *if and only if* $\mathbf{x}^* + y = (x_1^* + y, \ldots, x_n^* + y)$ *is a minimal (maximal) permutation of* $\mathbf{x}_{(\cdot)} + y = (x_{(1)} + y, \ldots, x_{(n)} + y)$.

LEMMA 2.2. *Let* $0 \le a \le b \le c \le d$. *Then* $ad + bc \le ac + bd \le ab + cd$.

PROOF. Putting $\mathbf{x} + y = (x_1 + y, \ldots, x_n + y)$, we have

$$T(\mathbf{x} + y) = T(\mathbf{x}) + 2y \sum_{i=1}^{n} x_{(i)} + ny^2.$$

The proof of Lemma 2.2 is elementary and is left to the reader. □

PROOF OF PROPOSITION 2.1. For simplicity, we use the same notation $\mathbf{x}$ to denote a permutation $(x_1, \ldots, x_n)$ of $\mathbf{x}_{(\cdot)}$ and the corresponding Hamiltonian cycle $\{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$. Obviously, the $2n$ cyclical permutations of $\mathbf{x}$ and $(x_n, \ldots, x_1)$ all yield the same Hamiltonian cycle: accordingly, we may assume without any loss of generality that $x_1 = x_{(1)}$.

The proof below relies on a branch-and-bound idea. At each step k, a family $\mathscr{E}^{(k-1)}$ of Hamiltonian cycles is subdivided into two nonempty subfamilies $\mathscr{E}^{(k)}$ and $\mathscr{E}^{(k-1)} \setminus \mathscr{E}^{(k)}$; $\mathscr{E}^{(k)}$ then is shown to contain at least one minimal cycle. Starting with the set $\mathscr{E}^{(0)}$ of all Hamiltonian cycles, the process ends up with $\mathscr{E}^{(n)} = \{\mathbf{x}_{\min}\}$.

Letting $k = 4l + m$, $m = 0, 1, 2$ or 3, define $\mathscr{C}^{(k)}$ as the set of all cycles in $\mathscr{C}^{(k-1)}$ containing edge $(x_{(i)}, x_{(j)})$, with

$$(i,j) = \begin{cases} (1,n), & k = 1, \\ (2l, n-2l), & k > 1,\ m = 1, \\ (n-2l-1, 2l+1), & k > 1,\ m = 2, \\ (n-2l, 2l+2), & k > 1,\ m = 3, \\ (2l+1, n-2l+1), & k > 1,\ m = 0. \end{cases}$$

Accordingly, the cycles in $\mathscr{C}^{(k)}$ are of the form

$$\begin{aligned} &(x_{(1)}, x_{(n)}, x_3, \ldots, x_{n-1}, x_n), && k = 1, \\ &(x_{(1)}, x_{(n)}, x_3, \ldots, x_{n-1}, x_{(n-1)}), && k = 2, \\ &(x_{(1)}, x_{(n)}, x_{(2)}, x_4, \ldots, x_{n-1}, x_{(n-1)}), && k = 3, \\ &(x_{(1)}, x_{(n)}, x_{(2)}, x_4, \ldots, x_{n-2}, x_{(3)}, x_{(n-1)}), && k = 4, \\ &(x_{(1)}, x_{(n)}, x_{(2)}, x_{(n-2)}, x_5, \ldots, x_{n-2}, x_{(3)}, x_{(n-1)}), && k = 5, \\ &\qquad\vdots && \vdots \\ &(x_{(1)}, x_{(n)}, x_{(2)}, x_{(n-2)}, x_{(4)}, \ldots, x_{(n-3)}, x_{(3)}, x_{(n-1)}), && k = n. \end{aligned}$$

Consider $\mathbf{x}' \in \mathscr{C}^{(k-1)} \setminus \mathscr{C}^{(k)}$, with $k = 4l + 2$, say. Then $\mathbf{x}'$ is of the form

$$\mathbf{x}' = (x_{(1)}, x_{(n)}, x_{(2)}, \ldots, x_{(2l)}, x_{(n-2l)}, x_{2l+1}, x_{2l+2}, \ldots, x_i, \ldots, x_{n-2l}, x_{(2l+1)}, \ldots, x_{(3)}, x_{(n-1)}),$$

with $x_i = x_{(n-2l-1)}$. Define $\mathbf{x}''$ as

$$\mathbf{x}'' = (x_{(1)}, x_{(n)}, \ldots, x_{(n-2l)}, x_{2l+1}, x_{2l+2}, \ldots, x_{i-1}, x_{n-2l}, x_{n-2l-1}, \ldots, x_{i+1}, x_{(n-2l-1)}, x_{(2l+1)}, \ldots, x_{(3)}, x_{(n-1)}).$$

Obviously, as can be seen from Figure 1, $\mathbf{x}'' \in \mathscr{C}^{(k)}$ and

$$T(\mathbf{x}'') = T(\mathbf{x}') + x_{(n-2l-1)}x_{(2l+1)} + x_{i-1}x_{n-2l} - x_{(n-2l-1)}x_{i-1} - x_{(2l+1)}x_{n-2l}.$$

Since $x_{(2l+1)} \le x_{n-2l} \le x_{(n-2l-1)}$ and $x_{(2l+1)} \le x_{i-1} \le x_{(n-2l-1)}$, it follows from Lemma 2.2 that $T(\mathbf{x}'') \le T(\mathbf{x}')$; $\mathscr{C}^{(k)}$ consequently contains at least one minimal cycle. A similar reasoning holds for $k = 4l + m$, $m = 3$, 1 or 0.

As for the minimal value of T, it is easy to check that $T(\mathbf{x}_{\min})$ actually reduces to (2.3). □

PROOF OF PROPOSITION 2.2. On account of Lemma 2.1, we may assume, without any loss of generality, $x_{(1)} = 0$, $x_{(i)} \ge 0$. In order to fix the notation, assume again $n = 4l + 2$. The proof still holds, with minor changes, for $n = 4l + m$, $m = 0$, 1 or 3. Denote by $\mathbf{x}$ the vector resulting from $\mathbf{x}_{\max}$ by alternating the signs as follows:

$$\tilde{\mathbf{x}} = (-x_{(n)}, x_{(n-1)}, -x_{(n-3)}, \ldots, x_{(5)}, -x_{(3)}, x_{(1)} = 0, -x_{(2)}, x_{(4)}, \ldots, -x_{(n-4)}, x_{(n-2)}). \tag{2.6}$$

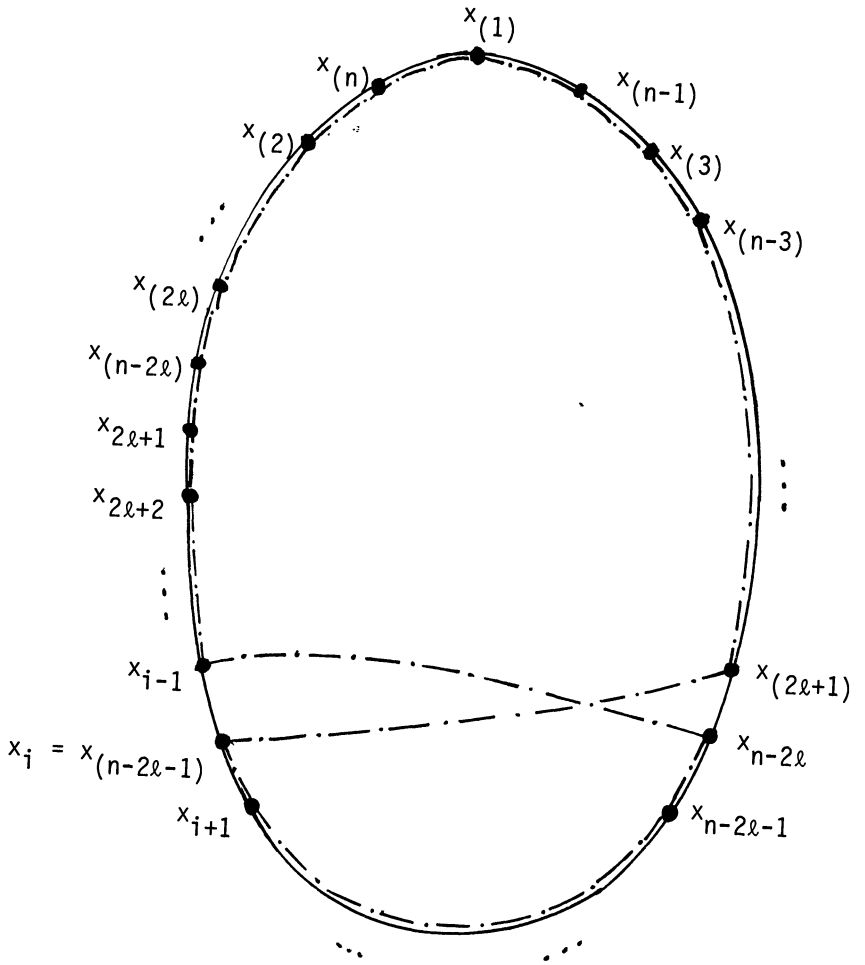

FIG. 1. *Hamiltonian cycles* $\mathbf{x}'$ (*bold*) *and* $\mathbf{x}''$ (*dotted*).

Clearly, $T(\tilde{\mathbf{x}}) = -T(\mathbf{x}_{\max})$ and a sufficient condition for $\mathbf{x}_{\max}$ being a maximal permutation of $\mathbf{x}_{(\cdot)}$ is thus $\tilde{\mathbf{x}}$ being a minimal permutation of $\tilde{\mathbf{x}}_{(\cdot)}$, where $\tilde{\mathbf{x}}_{(\cdot)}$ is the ordered vector resulting from $\tilde{\mathbf{x}}$, namely

$$\tilde{\mathbf{x}}_{(\cdot)} = (-x_{(n)}, -x_{(n-3)}, -x_{(n-4)}, \ldots, -x_{(7)}, -x_{(6)}, -x_{(3)}, -x_{(2)}, x_{(1)} = 0, x_{(4)}, x_{(5)}, x_{(8)}, x_{(9)}, \ldots, x_{(n-5)}, x_{(n-2)}, x_{(n-1)})$$

$$= \left(\tilde{x}_{(1)}, \tilde{x}_{(2)}, \tilde{x}_{(3)}, \ldots, \tilde{x}_{((n/2)-1)}, \tilde{x}_{(n/2)}, \tilde{x}_{((n/2)+1)} = 0, \ldots, \tilde{x}_{(n-1)}, \tilde{x}_{(n)}\right).$$

As can be checked easily from (2.3), (2.6) is precisely the minimal permutation of $\tilde{\mathbf{x}}_{(\cdot)}$ described in Proposition 2.1.

As for (2.5), we leave it to the reader to check that $T_{\max}$ is nothing else than $T(\mathbf{x}_{\max})$. □

3. Pitman tests against serial dependence.

3.1. *First-order autocorrelation coefficient.* Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an observed series of length n. Define

$$Z_t = \begin{cases} (X_t - \bar{X}) \Big/ \left[\sum_{t=1}^{n} (X_t - \bar{X})^2 \right]^{1/2}, & t = 1, \ldots, n, \\ 0, & t = n+1, \end{cases} \tag{3.1}$$

where $\bar{X} = n^{-1} \sum_{t=1}^{n} X_t$. Then the first-order autocorrelation is

$$r_1 = \sum_{t=2}^{n} (X_t - \bar{X})(X_{t-1} - \bar{X}) \Big/ \sum_{t=1}^{n} (X_t - \bar{X})^2 = \sum_{t=1}^{n+1} Z_t Z_{t-1}, \tag{3.2}$$

with the convention $Z_0 = Z_{n+1}$. Alternative definitions of r_1 can be considered. The results below remain valid, provided (3.1) is modified in an obvious fashion. Putting $\mathbf{Z}_{(\cdot)} = (Z_{(1)}, Z_{(2)}, \ldots, Z_{(n+1)})$, with $Z_{(1)} \le Z_{(2)} \le \cdots \le Z_{(n+1)}$, we have the following results.

PROPOSITION 3.1. *The permutational minimal and maximal values of r_1 are*

$$r_{\min} = Z_{(1)} Z_{(n+1)} + \sum_{i=1}^{[n/2]} \left(Z_{(i)} Z_{(n+1-i)} + Z_{(i+1)} Z_{(n+2-i)} \right) + Z_{((n+1)/2)} Z_{((n+3)/2)} I \qquad (n \text{ odd}) \tag{3.3}$$

and

$$r_{\max} = Z_{(1)} Z_{(2)} + \sum_{i=1}^{n-1} Z_{(i)} Z_{(i+2)} + Z_{(n)} Z_{(n+1)}, \tag{3.4}$$

respectively. The permutational mean and variance of r_1 are

$$E(r_1) = -1/n \tag{3.5}$$

and

$$\operatorname{Var}(r_1) = \left[n^2 (n-1) \right]^{-1} \left[n^2 - (n-1) - n(n+1) b_2 \right], \tag{3.6}$$

where $b_2 = \sum_t Z_t^4$ denotes Pearson's usual kurtosis coefficient (computed either from the original series $\mathbf{X}$ or from the transformed one $\mathbf{Z}$).

PROOF. The first part of the proposition follows from Propositions 2.1 and 2.2 by noting that, provided that the $n + 1$ values of Z_t are substituted for the x_t's, the noncircular autocorrelation coefficient (3.2) coincides (with $n + 1$ instead of n) with the circular objective function (2.1); the extremal values (3.3) and (3.4) then result from rewriting (2.3) and (2.5).

As for the permutational moments (3.5) and (3.6), they can be obtained from simple combinatorial arguments; see David and Fix (1966) or Hallin and Mélard (1988) for details. They also are provided by Dufour and Roy (1985, 1986), where they are used in the derivation of an upper bound for the unconditional variance of r_1.

All the elements that are needed for performing Pitman tests based on r_1 are now at hand. According to the available tables, one may either wish to compute p-values or critical points. □

The p-values can be obtained either from tables of the incomplete beta function ratio $I_x(p, q)$ [Pearson (1968); Osborn and Madey (1968)], or from approximations thereof [see Johnson and Kotz (1970), Chapter 24, Section 6.1]. The p-value for an observed value r_1 of the first-order autocorrelation coefficient is then $I_x(\tilde{p}, \tilde{q})$, with

$$x = (r_1 - r_{\min})/(r_{\max} - r_{\min}),$$

$$\tilde{p} = (r_{\min} + n^{-1})(r_{\max} - r_{\min})^{-1}K, \tag{3.7}$$

$$\tilde{q} = -(r_{\max} + n^{-1})(r_{\max} - r_{\min})^{-1}K, \tag{3.8}$$

$$K = \frac{n^2(n-1)(r_{\min} + n^{-1})(r_{\max} + n^{-1})}{n^2 - (n-1) - n(n+1)b_2} + 1 \tag{3.9}$$

[$\tilde{p}$ and $\tilde{q}$ are obtained from formulas (19) and (20) in the above-mentioned chapter of Johnson and Kotz (1970) on replacing the mean μ'_1 and variance μ_2 with (3.5) and (3.6), respectively].

Critical values can be computed from the tables of percentage points given in Pearson and Hartley (1966). Denoting by $x^{\pi}_{p,q}$ the πth quantile of the standard beta distribution with parameters (p, q) [i.e., the solution of $I_x(p, q) = \pi$], one has for r_1 the corresponding percentage point

$$r_{\min} + (r_{\max} - r_{\min})x^{\pi}_{\tilde{p},\tilde{q}}, \tag{3.10}$$

where $\tilde{p}$ and $\tilde{q}$ are as in (3.7) and (3.8), respectively.

3.2. *Higher order coefficients.* Unfortunately, no straightforward extension of Proposition 3.1 to the case of higher order coefficients seems to be possible. The reason for this is as follows. The graph-theoretical version of the problem of order $k > 1$ involves a team of k salesmen. Letting $n = kl + m$, with $l = [n/k]$, divide the set of n nodes into $k - m$ subsets of cardinal l and m subsets of cardinal $l + 1$ in such a way that the cumulated weights of the k minimal (maximal) Hamiltonian cycles through the k subgraphs thus obtained, be minimal (maximal). Now, unlike the optimal Hamiltonian cycles in the k subgraphs [which, according to (2.2) and (2.4), exclusively depend on the *ranks* of the x_i's], the optimal subdivision itself depends on the actual values of the x_i's.

To see this, consider the case $n = 10$, $k = 2$. Let $\varepsilon \in (0.1)$ and

$$\mathbf{x}^{\varepsilon}_{(\cdot)} = (2, 3, 6 - 2\varepsilon, 6 - \varepsilon, 6, 7, 8, 9, 10, 11).$$

The value of ε has no influence upon the ordering of the $x^{\varepsilon}_{(i)}$'s. Whatever this value, $2 < 3 < 6 - 2\varepsilon < 6 - \varepsilon < 6 < 7 < \cdots < 11$. Still, for $1/2 < \varepsilon < 1$, the minimal value of $\sum_{t=3}^{n} x^{\varepsilon}_t x^{\varepsilon}_{t-2}$ is

$$r^{\varepsilon}_{2;\min} = r_{1;\min}(2, 3, 8, 10, 11) + r_{1;\min}(6 - 2\varepsilon, 6 - \varepsilon, 6, 7, 9) = 260 - 43\varepsilon,$$

and is reached for

$$\begin{aligned}\mathbf{x}^{*} &= (11, 9, 2, 6 - 2\varepsilon, 8, 6, 3, 6 - \varepsilon, 10, 7)\\ &= \left(x^{\varepsilon}_{(10)}, x^{\varepsilon}_{(8)}, x^{\varepsilon}_{(1)}, x^{\varepsilon}_{(3)}, x^{\varepsilon}_{(7)}, x^{\varepsilon}_{(5)}, x^{\varepsilon}_{(2)}, x^{\varepsilon}_{(4)}, x^{\varepsilon}_{(9)}, x^{\varepsilon}_{(6)}\right);\end{aligned}$$

whereas for $0 < \varepsilon < 1/2$, the minimal value is

$$r^{\varepsilon}_{2;\min} = r_{1;\min}(2, 3, 9, 10, 11) + r_{1;\min}(6 - 2\varepsilon, 6 - \varepsilon, 6, 7, 8) = 259 - 41\varepsilon,$$

which is reached for

$$\begin{aligned}\mathbf{x}^{**} &= (11, 8, 2, 6 - 2\varepsilon, 9, 6, 3, 6 - \varepsilon, 10, 7)\\ &= \left(x^{\varepsilon}_{(10)}, x^{\varepsilon}_{(7)}, x^{\varepsilon}_{(1)}, x^{\varepsilon}_{(3)}, x^{\varepsilon}_{(8)}, x^{\varepsilon}_{(5)}, x^{\varepsilon}_{(2)}, x^{\varepsilon}_{(4)}, x^{\varepsilon}_{(9)}, x^{\varepsilon}_{(6)}\right).\end{aligned}$$

Different techniques thus should be considered if (exact or approximate) permutational critical values were to be obtained for higher order autocorrelation coefficients. One idea consists in deriving bounds for permutational distribution functions, and is developed in [Dufour and Hallin (1991a, 1991b, 1992)], where various bounds of the exponential, Chebyshev, Berry-Esséen and Eaton types are provided.

4. A numerical illustration. The accuracy of the proposed beta approximation and its expected superiority over normal approximation [based on classical, nonconditional results, such as in Ljung and Box (1978) or Moran (1948), or based on permutational central-limit theorems such as in Wald and Wolfowitz (1943) or Ghosh (1954)] should be investigated through Monte Carlo methods. The very idea of a systematic Monte Carlo study, however, is somewhat contradictory with the conditional nature of permutation tests. Separate Monte Carlo studies indeed should be conducted for every possible value of the conditioning order statistic $\mathbf{X}_{(\cdot)}$ [since the concept of an underlying distribution for $\mathbf{X}_{(\cdot)}$ in some sense has to be ruled out] except in a few specific cases, such as that of rank tests, where all conditional (permutational) distributions coincide with the unconditional one.

The present section is therefore restricted to a brief numerical illustration, for a very short series length ($n = 8$; so that exact permutational distributions still can be obtained through enumeration techniques). Example 1 considers the permutational distribution of r_1 for an observed order statistic $\mathbf{X}_{(\cdot)} = (1, 2, \ldots, 8)$. This distribution thus coincides with that of the Spearman–Wald–Wolfowitz rank autocorrelation coefficient studied by Bartels (1982) and Hallin and Mélard (1988). Except for the shortness of the series, these are

TABLE 1

Exact (a) *and approximate* [(b), ..., (e)] *permutational quantiles of* r_1 *for* $\mathbf{X}_{(\cdot)} = (1, 2, \ldots, 8)$. *The closest approximation is identified by an asterisk. The permutational mean, variance and skewness of* r_1 *are* -0.125, 0.092 *and* 0.072, *respectively. The exact permutational range is* $[r_{\min} = -0.923;\ r_{\max} = 0.720]$.

Example 1	**0.001**	**0.005**	**0.010**	**0.025**	**0.050**	**0.100**	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**
(a) Exact	−0.863	−0.804	−0.756	−0.685	−0.613	−0.530	0.280	0.387	0.470	0.554	0.601	0.673
(b) Beta [−1, 1]	−0.905	−0.834	−0.789	−0.707	−0.621	−0.507	0.507	0.621	0.707	0.789	0.834	0.905
(c) Ljung–Box	−0.914	−0.762	−0.688	−0.580	−0.487	−0.379	0.379	0.487	0.580	0.688	0.762	0.914
(d) Normal	−1.061	−0.906	−0.830	−0.719	−0.623	−0.513	0.263*	0.373*	0.469*	0.580	0.656	0.811
(e) Beta $[r_{\min}; r_{\max}]$	−0.844*	−0.787*	−0.749*	−0.682*	−0.612	−0.517*	0.315	0.409	0.479	0.547*	0.584*	0.642*

TABLE 2

Exact (a) *and approximate* [(b), ..., (e)] *permutational quantiles of* r_1 *for* $\mathbf{X}_{(\cdot)} = (1, 3, 4, 6, 8, 12, 20, 27)$. *The closest approximation is identified by an asterisk. The permutational mean, variance and skewness of* r_1 *are* -0.125, 0.078 *and* 0.348, *respectively. The exact permutational range is* $[r_{\min} = -0.755; r_{\max} = 0.618]$.

Example 2	**0.001**	**0.005**	**0.010**	**0.025**	**0.050**	**0.100**	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**
(a) Exact	−0.720	−0.671	−0.643	−0.591	−0.538	−0.468	0.284	0.380	0.451	0.517	0.547	0.588
(b) Beta [−1, 1]	−0.905	−0.834	−0.789	−0.707	−0.621	−0.507	0.507	0.621	0.707	0.789	0.834	0.905
(c) Ljung–Box	−0.914	−0.762	−0.688	−0.580*	−0.487	−0.379	0.379	0.487	0.580	0.688	0.762	0.914
(d) Normal	−0.990	−0.846	−0.776	−0.674	−0.586	−0.484*	0.234	0.336	0.424*	0.526*	0.596	0.740
(e) Beta $[r_{\min}; r_{\max}]$	−0.690*	−0.641*	−0.610*	−0.554	−0.495*	−0.417	0.279*	0.358*	0.416	0.473	0.504*	0.552*

relatively favorable conditions (skewness value of 0.07) for a reasonable normal approximation. To obtain a skewer, more challenging situation, $\mathbf{X}_{(\cdot)}$ was modified, in Example 2, into $(1, 3, 4, 6, 8, 12, 20, 27)$.

Tables 1 and 2 provide, for each of these two examples successively, the α- and $(1 - \alpha)$-permutational quantiles ($\alpha = 0.001, 0.005, 0.010, 0.025, 0.050$ and 0.100) of r_1 obtained from (a) the exact permutational distribution, (b) the beta approximation over $[-1, 1]$ obtained by adjusting for the permutational mean and variance (3.5) and (3.6) [this approximation is the one proposed—for much larger values of n—by Bartels (1982) and Hallin and Mélard (1988)], (c) the Ljung–Box (1978) (unconditional) normal approximation, namely, $(n^2 + 2n)^{1/2} r_1/(n - 1)^{1/2} \approx N(0, 1)$, (d) the (conditional) normal approximation, based on the exact permutational mean and variance (3.5) and (3.6) and (e) the beta approximation over $[r_{\min}, r_{\max}]$ suggested in Section 3.

Inspection of Tables 1 and 2 reveals that the $[r_{\min}, r_{\max}]$ beta approximation (e) always performs fairly well (recall that the series length is only $n = 8$); it generally provides the best approximate quantiles, mostly in the tails—though the conditional normal approximation (d) is slightly better on a narrow interval. The classical Ljung–Box approximation clearly is not reliable here—though no one presumably would recommend using it for such a short series length.

Assume, for example, that $\mathbf{x} = (8, 6, 1, 4, 3, 12, 20, 27)$ has been observed, yielding a first-order autocorrelation coefficient of 0.5489. Referring to row (e) in Table 2, we reject the white-noise hypothesis at a (two-sided) $\alpha = 1\%$ probability level. None of the other approximate quantiles [rows (b), (c), (d)] would lead to this rejection—though, as indicated by the exact 0.995 quantile value of 0.547 showed in row (a), rejection here is the correct decision.

These two simple examples suggest that the beta approximation proposed in Section 3 provides fairly good approximate permutational quantiles, even for very short series. Other approximating methods, based on asymptotic expansions and inequalities on tail areas are the subject of an ongoing study.

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