

NOTE ON THE TAIL BEHAVIOR OF GENERAL WEIGHTED EMPIRICAL PROCESSES

BY MARTIEN C. A. VAN ZUIJLEN

Katholieke Universiteit Nijmegen

Under minimal conditions precise bounds are obtained for the expectation of the supremum of the weighted empirical process over the interval $(0, 1/(n(\log n)^{d-1}))$, where d is the dimension of the underlying random vectors. The allowed growth of the weight function is optimal in the iid case. The results will have broad applications in the theory of all kinds of nonstandard weighted empirical processes, such as empirical processes based on uniform spacings or U -statistics, where it is often not so easy to show directly (as in the iid case) that the considered suprema converge to 0 in probability.

1. Introduction. Without assuming independence or equal distributions of the underlying d -dimensional sample elements, we will obtain in this article precise bounds for the expectation of the absolute value of the supremum of the weighted empirical process over the interval $(0, 1/(n(\log n)^{d-1}))$. On the remaining part of the unit interval the process can often be controlled by suitable exponential probability inequalities. Our bounds force the statistics to vanish in probability. Even in the one-dimensional iid case the results seem to be new. Because of the mild conditions and the optimality of the considered weighting, the results will have broad applications in the theory of all kinds of nonstandard weighted empirical processes, such as empirical processes based on uniform spacings or U -statistics. In these nonstandard situations it is often not so easy to show directly (as in the iid case) that these suprema converge to 0 in probability. The conditions we impose on the growth of the weight function are implied by the necessary and sufficient growth conditions for the weak convergence of the weighted iid empirical processes [see Einmahl, Ruymgaart and Wellner (1988)]. Our proof is surprisingly simple, uses only elementary analysis and is inspired by Lemma 2 in Silverman (1983).

Similar results for the right tail of the process are immediate by the same technique and therefore omitted.

2. The result. For a given function $k: \mathbb{N} \rightarrow \mathbb{N}$, $0 < \theta < 1$, $d \in \mathbb{N}$ and $n \in \mathbb{N}$, let

$$(1) \quad U_{n,k,\theta} = \sup_{0 < F_{k(n)}^*(t) \leq \theta} \sqrt{n} w(F_{k(n)}^*(t)) \left| \hat{F}_{k(n)}(t) - F_{k(n)}^*(t) \right|,$$

Received January 1990.

AMS 1980 subject classifications. Primary 62E20, 62G99; secondary 62H10.

Key words and phrases. Weighted empirical processes, tail behavior.

where $\hat{F}_{k(n)}$ is the empirical distribution function based on $k(n)$ d -dimensional random vectors $X_1, X_2, \dots, X_{k(n)}$, $w(\cdot)$ is a weight function, $t = (t_1, t_2, \dots, t_d)$ and where $F_{k(n)}^*$ is the average of the respective df's $F_1, F_2, \dots, F_{k(n)}$, which are assumed to be continuous. We will assume without loss of generality that $F_{k(n)}^*$ is concentrated on $I^d := (0, 1)^d$ and has uniform $(0, 1)$ marginal distributions. Finally we denote $|t| = t_1 t_2 \cdots t_d$.

Let us now introduce the following class of weight functions and a condition that plays a role in the theorem:

$$(2) \quad W = \{w: (0, 1/2) \rightarrow R \mid w(\cdot) \text{ is continuous, nonnegative, nonincreasing and } (\cdot)w(\cdot) \text{ is nondecreasing}\}.$$

CONDITION A. $F_{k(n)}^*$ has a continuous density $f_{k(n)}^*$ w.r.t. Lebesgue measure, such that

$$(3) \quad 0 < m = \inf_{t \in I^d} f_{k(n)}^*(t) \leq \sup_{t \in I^d} f_{k(n)}^*(t) = M < \infty.$$

THEOREM. Let $w \in W$ and $d \in \mathbb{N}$. Moreover suppose that condition A is fulfilled and that

$$(4) \quad w(t) = o\left(t^{-1/2}(\log(1/t))^{-(d-1)/2}\right), \quad t \rightarrow 0.$$

Then we have for every function $k: \mathbb{N} \rightarrow \mathbb{N}$:

$$(5) \quad \lim_{n \rightarrow \infty} E(U_{n, k, 1/(n(\log n)^{d-1})}) = 0.$$

REMARK 1. The above theorem reduces to a result in the iid case if we take $k(n) = n$ and independent underlying random vectors with uniform distributions on I^d .

REMARK 2. For some applications, for instance in the theory of empirical U -statistic processes, it is indeed essential to allow ‘‘sample sizes’’ $k(n)$ instead of n in the theorem. See Ruymgaart and van Zuijlen (1990).

REMARK 3. The necessary and sufficient growth condition for the weak convergence of the weighted iid empirical processes is given by

$$(6) \quad \int_0^{1/2} \frac{1}{t} \exp\left(\frac{-\lambda}{tw^2(t)}\right) dt < \infty \quad \text{for all } \lambda > 0,$$

in case $d = 1$ and by (4) and in case $d > 1$ [see Einmahl, Ruymgaart and Wellner (1988)]. It is well known that, because of the monotonicity of w , (6) implies (4) in case $d = 1$.

PROOF. We have for each $0 < \theta < 1/2$:

$$\begin{aligned}
 & E \left(\sup_{0 < F_{k(n)}^*(t) \leq \theta} \sqrt{n} w(F_{k(n)}^*(t)) \left| \hat{F}_{k(n)}(t) - F_{k(n)}^*(t) \right| \right) \\
 & \leq E \left(\sup_{0 < F_{k(n)}^*(t) \leq \theta} \sqrt{n} w(F_{k(n)}^*(t)) \int_0^t d\hat{F}_{k(n)}(s) \right) \\
 & \quad + \sup_{0 < F_{k(n)}^*(t) \leq \theta} \sqrt{n} w(F_{k(n)}^*(t)) F_{k(n)}^*(t) \\
 & \leq E \left(\sup_{0 < F_{k(n)}^*(t) \leq \theta} \sqrt{n} \int_0^t w(F_{k(n)}^*(s)) d\hat{F}_{k(n)}(s) \right) + \sqrt{n} w(\theta) \theta \\
 & \leq E \left(\sqrt{n} \int_{0 < F_{k(n)}^*(s) \leq \theta} w(F_{k(n)}^*(s)) d\hat{F}_{k(n)}(s) \right) + \sqrt{n} \int_0^\theta w(y) dy \\
 & = \sqrt{n} \int_{0 < F_{k(n)}^*(s) \leq \theta} w(F_{k(n)}^*(s)) dF_{k(n)}^*(s) + \sqrt{n} \int_0^\theta w(y) dy \\
 & \leq M\sqrt{n} \int_{0 < m|s| \leq \theta} w(m|s|) ds + \sqrt{n} \int_0^\theta w(y) dy.
 \end{aligned}$$

For convenience we now take $M = m = 1$. The complementary case requires only minor modifications. By the change of variables $y = t_1 = |s|$, $t_2 = s_2, \dots, t_d = s_d$, we obtain the further upper bound

$$\begin{aligned}
 & \leq 2\sqrt{n} \int_0^\theta w(y) (\log(1/y))^{d-1} dy \\
 (7) \quad & = 2\sqrt{n} \theta \int_0^1 w(\theta y) (\log(1/(\theta y)))^{d-1} dy.
 \end{aligned}$$

For each $0 < \theta < 1/e$ we have for some $C = C(d) > 0$:

$$\begin{aligned}
 & \int_0^1 (\theta y)^{-1/2} (\log(1/(\theta y)))^{(d-1)/2} dy \\
 & = \theta^{-1/2} (\log(1/\theta))^{(d-1)/2} \int_0^1 y^{-1/2} \left\{ \frac{(\log(1/(\theta y)))}{(\log(1/\theta))} \right\}^{(d-1)/2} dy \\
 & \leq \theta^{-1/2} (\log(1/\theta))^{(d-1)/2} \int_0^1 y^{-1/2} (1 + \log(1/y))^{(d-1)/2} dy \\
 & \leq C\theta^{-1/2} (\log(1/\theta))^{(d-1)/2},
 \end{aligned}$$

and hence

$$(8) \quad \theta \int_0^1 w(\theta y) (\log(1/(\theta y)))^{d-1} dy = o(\theta^{1/2} (\log(1/\theta))^{(d-1)/2}) \quad \text{as } \theta \rightarrow 0.$$

Taking $\theta = 1/(n(\log n)^{d-1})$, we obtain by combination of (7) and (8) for large n :

$$\begin{aligned} E(U_{n, k, 1/(n(\log n)^{d-1})}) & \\ & \leq 2C\sqrt{n} \theta \int_0^1 w(\theta y) (\log(1/(\theta y)))^{d-1} dy \\ & = o((n\theta)^{1/2} (\log(1/\theta))^{(d-1)/2}) \\ & = o((\log n)^{-(d-1)/2} (\log(n(\log n)^{d-1}))^{(d-1)/2}) \\ & \leq o((\log n)^{-(d-1)/2} (d \log n)^{(d-1)/2}) \\ & = o(1), \end{aligned}$$

which completes the proof of the theorem. \square

Acknowledgment. The author would like to thank J. H. J. Einmahl for a useful comment on an earlier version of this article.

REFERENCES

- EINMAHL, J. H. J., RUYMGAART, F. H. and WELLNER, J. A. (1988). A characterization of weak convergence of weighted multivariate empirical processes. *Acta Sci. Math. (Szeged)* **52** 191–205.
- RUYMGAART, F. H. and VAN ZUIJLEN, M. C. A. (1990). Empirical U -statistics processes. *J. Statist. Plann. Inference*. To appear.
- SILVERMAN, B. W. (1983). Convergence of a class of empirical distribution functions of dependent random variables. *Ann. Probab.* **11** 745–751.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
TOERNOOVELD
6525 ED NIJMEGEN
THE NETHERLANDS