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MINIMAXITY OF THE EMPIRICAL DISTRIBUTION FUNCTION IN INVARIANT ESTIMATION

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Consider the problem of continuous invariant estimation of a distribution function with the weighted Cramér–von Mises loss. The minimaxity of the empirical distribution function, which is also the best invariant estimator, is proved for any sample size. This solves a long-standing conjecture.

1. Introduction. This paper presents results on the minimaxity of the empirical distribution function, which is also the best invariant estimator of a distribution function, for the finite sample size invariant decision problem, involving the weighted Cramér–von Mises loss function. The formulation, introduced by Aggarwal (1955), is as follows.

Let $X_1, \dots, X_n$ be a sample of size n from an unknown continuous distribution function F, which we assume, without loss of generality, to have support on (0, 1). Let $Y_0, \dots, Y_{n+1}$ be the order statistics of $0, X_1, \dots, X_n, 1$, and write

$$\vec{Y} = (Y_1, \dots, Y_n). \tag{1.1}$$

The action space is given by

$$A = \{a(t)\colon a(t) \text{ is a nondecreasing function from } (0,1) \text{ into } [0,1]\}; \tag{1.2}$$

the parameter space is given by

$$\Theta = \{F\colon F \text{ is a continuous distribution function with support in } (0,1)\}; \tag{1.3}$$

and the loss function is

$$L(F, a) = \int (F(t) - a(t))^2 h(F(t))\, dF(t), \tag{1.4}$$

where

$$h(t) = t^{-1}(1-t)^{-1}. \tag{1.5}$$

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The decision problem of estimating F is invariant under monotone transformations. The invariant estimators have the form

$$d(\vec{Y},t) = \sum_{j=0}^{n} u_j 1(Y_j \le t < Y_{j+1}), \tag{1.6}$$

where the u_j's are constants and $1(E)$ is the indicator function of a set E. It can be shown that $d(\vec{Y}, t)$ is of constant risk. So the best invariant estimator, denoted by d_0, exists and has coefficients j/n, $j = 0, 1, \dots, n$. That is, the best invariant estimator is the empirical distribution function (EDF) $\hat{F}(t)$ [see Aggarwal (1955)]. Also, it is asymptotically minimax [see Dvoretzky, Kiefer and Wolfowitz (1956)] and is admissible if and only if the sample size n is 1 or 2 [see Yu (1989a, b, d)]. This also implies that $\hat{F}(t)$ is minimax if $n = 1$ or 2.

Much study has been devoted to the theoretical properties of the best invariant estimator under the above set up with a general $h(t)$ for the loss function (1.4). The admissibility of the best invariant estimator was an interesting open question [see, for example, Cohen and Kuo (1985)]. As is well known, admissibility is a stronger result than minimaxity. When $h(t) = 1$, Brown (1988) proved that the best invariant estimator is inadmissible for all sample sizes $n \ge 1$. When $h(t) = t^{\alpha}(1-t)^{\beta}$, $\alpha, \beta \ge -1$, Yu (1988 and 1989a) extended Brown's result and proved the inadmissibility of the best invariant estimator in the case $\alpha, \beta \in (-1, 0]$ for $n \ge 1$. Also, Yu (1989a) proved the inadmissibility of the best invariant estimator in the case $n \ge 2$, $\alpha = -1$ and $\beta = 0$ or $\alpha = 0$ and $\beta = -1$. When $n = 1$, Yu (1989b) showed that the best invariant estimator is admissible if (1) $\alpha = -1$ and $\beta \ge -1$ or $\alpha \ge -1$ and $\beta = -1$ or (2) either α or $\beta > 0$.

Whether or not the empirical distribution function is minimax for $n \ge 3$ has been an outstanding open question [see, for example, Ferguson (1967), page 197]. Yu (1989c) gave a proof of the minimaxity of the best invariant estimators for $n = 1$ assuming a general $h(t)$ in the loss function (1.4). In this paper, we prove that $\hat{F}(t)$ is minimax for $n \ge 3$ and within the class of estimators $d(\vec{x}, t)$ satisfying the following condition:

$$d(\vec{x}, t) \text{ is nonincreasing in } x_i,\ i = 1, \dots, n, \qquad \text{where } \vec{x} = (x_1, \dots, x_n). \tag{1.7}$$

The minimaxity of $\hat{F}$ actually holds without the previous condition. For ease in understanding and for the sake of space, we present the proof that $\hat{F}(t)$ is minimax for $n = 3$ and under condition (1.7) in detail in Section 2 and outline the approach to generalize the proof to $n \ge 3$ and without the condition (1.7) in Section 4. For details of the proof that $\hat{F}(t)$ is minimax for $n \ge 3$ and under the condition (1.7), see Yu and Chow (1988). For details of the proof that $\hat{F}(t)$ is minimax without the condition (1.7), see Yu (1988b).

A parallel problem is to consider the Kolmogorov–Smirnov loss function $L(F, a) = \sup_t\{|F(t) - a(t)|\}$, which is also invariant under the above monotone transformations. Friedman, Gelman and Phadia (1988) obtained the best invariant estimator d_0 for sample sizes $n \ge 1$ and proved its uniqueness. Two interesting open problems are whether d_0 is minimax or admissible.

In Section 2, the lemmas and theorems needed to prove the minimaxity result of $\hat{F}$ within the class of estimators satisfying (1.7) for $n = 3$ are stated. Then the proof of Theorem 3 is given. The main idea of our approach to prove the minimaxity result is as follows.

Given an estimator d and $\varepsilon > 0$, by the preliminary lemmas and theorems in Section 2, one can find an invariant estimator d_1 and a continuous distribution function F such that $|R(F,d) - R(F,d_1)| < 2\varepsilon$. So $2\varepsilon + R(F,d) \geq R(F,d_1) \geq R(F,d_0)$. Then $\inf_d \sup_F R(F,d) = R(F,d_0)$. Hence d_0 is minimax.

In Section 3, the proofs of preliminary theorems and the construction of d_1 and F previously mentioned are given. In Section 4, an outline of the approach to establish the minimaxity result of $\hat{F}$ for $n \geq 3$ and without the condition (1.7) is given.

2. Minimaxity results under condition (1.7). For convenience, we write $d = d(t) = d(\vec{x}, t) = d(\vec{Y}, t)$. Without loss of generality, we assume that all estimators we consider are functions of the order statistic $\vec{Y}$ [see (1.1)], since they form an essentially complete class.

Given a distribution function $F(t)$, let dF denote the measure induced by F, that is, $dF\{(a,b)\} = F(b) - F(a)$; let $(dF)^k$ denote the product measure $dF \times \cdots \times dF$ with k factors, $k = 2, 3, \dots$. Given a one-dimensional measurable set B, let B^k denote the product set $B \times \cdots \times B$ with k factors. We denote Lebesgue measure by m. By a.e. m, we mean almost everywhere w.r.t. Lebesgue measure. Note that according to our notation, given a measurable set B in R^n, $m^n\{\vec{Y} \in B\} \neq m^n\{(X_1, \dots, X_n) \in B\}$. For example, when $n = 3$, $m^3\{(Y_1, Y_2, Y_3)\colon Y_1 < \frac{1}{2} < Y_2\} = 3!\, m^3\{(X_1, X_2, X_3)\colon X_1 < \frac{1}{2} < X_2\}$, where m^k is the product measure $m \times \cdots \times m$ of k factors. We shall see that restricting consideration to the following class of estimators suffices.

$$(2.1)\quad V = \left\{d\colon d(\vec{Y}, t) = 0 \text{ for } t < Y_1 \text{ and } d(\vec{Y}, t) = 1 \text{ for } t > Y_n \text{ a.e. } m^{n+1}\right\}.$$

Yu (1989d) proved the following lemma related to the set V.

LEMMA 1 [Yu (1989d)]. *Suppose that $n \geq 3$. Under the loss function* (1.4) *and* (1.5), *if an estimator $d \notin V$, then there is an $F \in \Theta$ such that $R(F,d) = +\infty$.*

In order to prove the minimaxity result in Theorem 3, we want to show that for any estimator d satisfying (1.7), there is an $F \in \Theta$ such that

$$(2.2)\qquad R(F,d) \geq R(F,d_0).$$

Note that (2.2) holds for all $d \notin V$ since d_0 is of constant risk and Lemma 1 shows that there is an $F \in \Theta$ such that $R(F,d) = +\infty$. From now on, estimators we consider are limited to the class V.

We first state the following theorem whose proof is in Section 3.

THEOREM 1. *Suppose that* $n = 3$, $[a, b] \subset (0, 1)$, $d \in V$ *and* d *satisfies* (1.7). *For any integers* N, $k > 0$, *there are intervals* $I_0, \dots, I_k$ *and real numbers* $u, v \in [0, 1]$ *such that* $u \le v$ *and*

(i) $I_i = [a_i, b_i]$, $i = 0, \dots, k$, *and* $a = a_0 < b_{j-1} < a_j < b_j < b$, $j = 1, \dots, k$,

(ii) $|d(\vec{Y}, t) - u| < (2/N)$ *if* $Y_1 \in \bigcup_{m=0}^{j-1} I_m$, $t < Y_2$ *and* $t, Y_2, Y_3 \in \bigcup_{m=j}^{k} I_m$, $j = 1, \dots, k$,

(iii) $|d(\vec{Y}, t) - v| < (2/N)$ *if* $Y_1, Y_2 \in \bigcup_{m=0}^{j-1} I_m$, $t < Y_3$ *and* $t, Y_3 \in \bigcup_{m=j}^{k} I_m$, $j = 1, \dots, k$.

Note that $d(\vec{Y}, t) = 0$ for $t < Y_1$ and $d(\vec{Y}, t) = 1$ for $t > Y_3$, since $d \in V$. Furthermore, in statement (ii), $t \in (Y_1, Y_2)$ and in statement (iii), $t \in (Y_2, Y_3)$. So Theorem 1 establishes the fact that on a subset of $(\bigcup_{j=0}^{k} I_j)^{3+1}$, d is very close to an invariant estimator d_1, where

$$d_1 = \begin{cases} 0, & \text{if } t < Y_1, \\ u, & \text{if } Y_1 \le t < Y_2, \\ v, & \text{if } Y_2 \le t < Y_3, \\ 1, & \text{if } Y_3 \le t. \end{cases}$$

By properly choosing k and I_j's (see Section 3), we construct a uniform distribution function F on $\bigcup_{j=0}^{k} I_j$, i.e.,

$$F(t) = \int_{-\infty}^{t} 1\left(x \in \bigcup_{j=0}^{k} I_j\right) \Big/ m\left(\bigcup_{j=0}^{k} I_j\right) dx,$$

which is the F needed in Theorem 2.

THEOREM 2. *Suppose that* $n = 3$ *and* $d \in V$ *is an estimator satisfying* (1.7). *For any* $\delta > 0$ *and* $\eta > 0$, *there exist a continuous distribution function* F *and an invariant estimator* d_1 *of form* (1.6) *such that* $d_1 \in V$ *and*

$$(dF)^4\left(\left\{(\vec{Y}, t) \colon \left|d(\vec{Y}, t) - d_1(\vec{Y}, t)\right| \ge \delta\right\}\right) < \eta. \tag{2.3}$$

The proof of Theorem 2 is in Section 3. Theorem 2 leads to the proof of minimaxity.

THEOREM 3. *For sample size* $n = 3$ *and under the loss function* (1.4) *with* $h(t) = t^{-1}(1 - t)^{-1}$, d_0 *is minimax within the family of estimators satisfying* (1.7).

Before we give the proof, we need the following lemma.

LEMMA 2 [Yu (1989d)]. *Suppose* $n \ge 2$. *For any* $\varepsilon > 0$, *there exists an* $\eta > 0$ *such that for all* $F \in \Theta$ *and* $B \subset R^{n+1}$ *satisfying* $(dF)^{n+1}(B) < \eta$, *we* *have*

$$E\int_{Y_1}^{Y_n} 1(B)h(F(t))\,dF(t) < \varepsilon. \tag{2.4}$$

PROOF OF THEOREM 3. By Lemma 1, it suffices to consider $d \in V$. Suppose that $d \in V$ and d satisfies (1.7). By Theorem 2, there exist an $F \in \Theta$ and an estimator $d_1 \in V$ of form (1.6) and thus of constant risk such that (2.3) holds. To prove the minimaxity of d_0, it suffices to show

$$|R(F,d) - R(F,d_1)| < 2\varepsilon. \tag{2.5}$$

Thus, $2\varepsilon + R(F,d) \geq R(F,d_1) \geq R(F,d_0)$, since d_0 is the best invariant estimator. Note that ε and d are arbitrary, provided that d satisfies (1.7). So

$$\inf\Big\{\sup_{F\in\Theta} R(F,d): d \text{ satisfies } (1.7)\Big\} = R(F,d_0).$$

We now prove (2.5). For any $\varepsilon > 0$, given η as in Lemma 2, let $\delta = \varepsilon/6$ and let

$$B = \Big\{(\vec{Y},t): \big|d(\vec{Y},t) - d_1(\vec{Y},t)\big| \geq \delta\Big\}, \tag{2.6}$$

then $(dF)^4(B) < \eta$ [by (2.3)] and by Lemma 2, $E\int_{Y_1}^{Y_3} 1(B)h(F(t))\,dF(t) < \varepsilon$.

$$\begin{aligned}
&|R(F,d) - R(F,d_1)| \quad (\text{note } d, d_1 \in V)\\
&\quad= \bigg|E\int_{Y_1}^{Y_3}\big[(F-d)^2 - (F-d_1)^2\big]\\
&\qquad\qquad \times\big[1(|d-d_1| \geq \delta) + 1(|d-d_1| < \delta)\big]h(F(t))\,dF(t)\bigg|\\
&\leq E\int_{Y_1}^{Y_3}\Big[1(|d-d_1| \geq \delta)\\
&\qquad + \big|(F-d)^2 - (F-d_1)^2\big|1(|d-d_1| < \delta)\Big]h(F(t))\,dF(t)\\
&\leq E\int_{Y_1}^{Y_3} 1(|d-d_1| \geq \delta)h(F(t))\,dF(t)\\
&\quad + E\int_{Y_1}^{Y_3} 2|d-d_1|1(|d-d_1| \leq \delta)h(F(t))\,dF(t)\\
&\leq E\int_{Y_1}^{Y_3} 1(B)h(F(t))\,dF(t) + 2c\delta\\
&\qquad\qquad \left[B \text{ is as in } (2.6),\ c = E\int_{Y_1}^{Y_3} h(F(t))\,dF(t)\right]\\
&< \varepsilon + 2c\delta = 2\varepsilon \quad (\text{since } \delta = \varepsilon/6,\ c = 3).
\end{aligned}$$

This completes the proof. □

3. Proofs of Theorems 1 and 2. In this section we give the proofs of Theorems 1 and 2 when $n = 3$. Since the proofs are very long, we proceed via a series of lemmas and remarks. We outline the main logic of the proof as follows.

$$(3.1)\quad \text{Let } d = d(\vec{Y}, t) \text{ be an arbitrary estimator in } V \text{ satisfying } (1.7).$$

Hereafter in this section, we assume that d is as in (3.1). We will prove that there exist an estimator d_1 as defined in (3.15) and a distribution function F as defined in (3.20) such that (2.3) holds. Theorem 1 is the key part of the whole proof. It shows that there is a measurable subset of $(\bigcup_{j=0}^{k} I_j)^{3+1}$ on which d is very close to an invariant estimator d_1. d_1 and F are defined after Theorem 1 is established. Theorem 1 is proved by an induction argument. Lemma 3 is the justification of the first step in the induction.

We first define a set $B_{N,\vec{i}} \subset [0.1, 0.9]$ [or any closed interval in (0, 1)], which plays an important role in the following development. Given $\varepsilon > 0$, let

$$(3.2)\qquad N > \frac{2c}{\varepsilon}, \quad \text{where} \quad c = E\int_{Y_1}^{Y_3} h(F(t))\, dF(t) = 3.$$

For any $x \in [0.1, 0.9]$, define

$$(3.3)\qquad \begin{aligned} c_j(x) = \lim_{\delta \to 0^+} \inf\Big\{h\colon m^3\Big\{Y_1, Y_2, Y_3 \in N(x,\delta)\colon \\ d(\vec{Y}, t) > h, Y_{j-1} < t < Y_j\Big\} = 0\Big\}, \end{aligned}$$

$$j = 2, 3, \quad \text{where } N(x,\delta) \text{ is the neighborhood of } x.$$

Define $\vec{i} = (i_2, i_3)$, where the i_j's are positive integers and

$$(3.4)\quad B_{N,\vec{i}} = \left\{x \in [0.1, 0.9]\colon c_j(x) \in \left(\frac{i_j - 1}{N}, \frac{i_j + 1}{N}\right), j = 2, 3\right\}.$$

REMARK 3.1. Here is the explanation of $c_j(x)$ in (3.3).

(i) Let $h(\delta) = \inf\{h\colon m^3\{Y_1, Y_2, Y_3 \in N(x, \delta)\colon d(\vec{Y}, t) > h, Y_{j-1} < t < Y_j\} = 0\}$, then $h(\delta) \downarrow c_j(x)$, as $\delta \downarrow 0$.

(ii) $c_j(x)$ is the essential supremum of $d(\vec{Y}, t)$ [denoted as ess sup $d(\vec{Y}, t)$] in the neighborhood of $(x, x, x, x) \in R^{3+1}$ provided $Y_{j-1} < t < Y_j$, i.e., for any $\delta > 0$, except on a zero-measure set, $d(\vec{Y}, t) \le h(\delta)$ for $(\vec{Y}, t) \in (N(x, \delta))^4 \cap \{Y_{j-1} < t < Y_j\}$. Furthermore, for any $h < h(\delta)$, $m^3\{Y_1, Y_2, Y_3 \in N(x, \delta)\colon d(\vec{Y}, t) > h, Y_{j-1} < t < Y_j\} > 0$.

(iii) If $c_j(x) \in ((i_j - 1)/N, (i_j + 1)/N)$, then there exist real numbers δ and $h(\delta)$ such that $c_j(x) \le h(\delta) < (i_j + 1)/N$ [since $h(\delta) \downarrow c_j(x)$ as $\delta \downarrow 0$ by (i)].

REMARK 3.2. Note that

$$\bigcup_{0 \le i_2,\, i_3 \le N} B_{N,\vec{i}} = [0.1, 0.9].$$

By the Baire category theorem [see Royden (1968)] and without loss of generality, one can assume that there is an interval $[a, b] \subset [0.1, 0.9]$ and some $B_{N,\vec{i}}$ such that

(i) $B_{N,\vec{i}}$ is dense in $[a, b]$,

(ii) $d(\vec{Y}, t) < (i_j + 1)/N$ if $Y_{j-1} < t < Y_j$, $j = 2, 3$, $Y_1, Y_3 \in [a, b]$.

[Otherwise, take some $x \in B_{N,\vec{i}} \cap (a, b)$, then by (ii) and (iii) in Remark 3.1, there exists a $\delta > 0$ such that $d(\vec{Y}, t) \le h(\delta) < (i_j + 1)/N$ if $Y_{j-1} < t < Y_j$, $j = 2, 3$, $Y_1, Y_3 \in N(x, \delta)$ [see (i) in Remark 3.1] and $N(x, \delta) \subset [a, b]$ for a small δ. Then $B_{N,\vec{i}}$ is dense in $N(x, \delta)$ and this $N(x, \delta)$ can be taken to be the new (a, b).]

NOTE. In expression (ii), by $Y_1, Y_3 \in [a, b]$, we mean Y_1, Y_2, Y_3, $t \in [a, b]$, since $Y_1 < Y_2 < Y_3$ and $Y_{j-1} < t < Y_j$. A similar implication applies hereafter for convenience.

From now on, we assume that $B_{N,\vec{i}}$, $[a, b]$ and $\vec{i} = (i_2, i_3)$ are specified as in Remark 3.2. Let

$$u = i_2/N \quad \text{and} \quad v = i_3/N. \tag{3.5}$$

[(u, v) and $(i_2/N, i_3/N)$ will be used interchangeably hereafter for convenience.]

LEMMA 3. *Given N as in (3.2), for any $x \in (a, b) \cap B_{N,\vec{i}}$ and for any $\eta > 0$, there are intervals I_1 and I_2 satisfying*:

(i) $I_i = [a_i, b_i]$, $i = 1, 2$, $a = a_1 < b_1 < a_2 < b_2 < b$ *and* $[b_1, b_2] \subset N(x, \eta)$,

(ii) $|d(\vec{Y}, t) - u| < 2/N$ *if* $a_1 \le Y_1 \le b_1$ *and* $a_2 \le t \le Y_2 \le Y_3 \le b_2$,

(iii) $|d(\vec{Y}, t) - v| < 2/N$ *if* $a_1 \le Y_1 < Y_2 \le b_1$ *and* $a_2 \le t \le Y_3 \le b_2$.

PROOF. Since $B_{N,\vec{i}}$ is dense in $[a, b]$, $(a, b) \cap B_{N,\vec{i}}$ is not empty. Taking an $x \in (a, b) \cap B_{N,\vec{i}}$, we have

$$c_2(x) \in (u - 1/N, u + 1/N) \quad [\text{see (3.4) and (3.5)}]. \tag{3.6}$$

For any η satisfying: $\eta > 0$, $a < x - \eta$ and $x + \eta < b$, it follows from (iii) in Remark 3.1 that there exist real numbers $\delta \in (0, \eta)$ and $h(\delta)$ such that

$$c_2(x) \le h(\delta) < u + 1/N. \tag{3.7}$$

By (ii) in Remark 3.1, $h(\delta) = \operatorname{ess\,sup} d(\vec{Y}, t)$ in $(N(x, \delta))^4 \cap [Y_1 < t < Y_2 < Y_3]$ which has positive measure, thus, there exists $(b_1, y_2, y_3, s_1) \in \{(\vec{Y}, t) \in (N(x, \delta))^4\colon Y_1 < t < Y_2 < Y_3\}$ (note that $b_1 < s_1 < y_2 < y_3$) such that

$$h(\delta) - 1/N < d(b_1, y_2, y_3, s_1) \le h(\delta). \tag{3.8}$$

Take $a_1 = a$, then for all $(\vec{Y}, t)$ satisfying $a_1 \le Y_1 \le b_1 < s_1 \le t \le Y_2 \le Y_3 \le y_2$, we have

$$\begin{aligned} u - 2/N &\le c_2(x) - 1/N \quad [\text{by } (3.6)] \\ &\le h(\delta) - 1/N \quad [\text{by } (3.7)] \\ &\le d(b_1, y_2, y_3, s_1) \quad [\text{by } (3.8)] \\ &\le d(Y_1, Y_2, Y_3, t) = d(\vec{Y}, t) \quad [\text{by monotonicity of } d \text{ in } Y_i\text{'s and } t] \\ &< (i_2 + 1)/N = u + 1/N \\ &\qquad [\text{by (ii) in Remark 3.2, since } (a_1, y_3) \subset [a, b]]. \end{aligned}$$

Thus there exist real numbers a_1, b_1, s_1 and y_2 such that

$$(3.9) \qquad \begin{aligned} &u - 2/N < d(\vec{Y}, t) < u + 1/N \\ &\qquad \text{if} \quad Y_1 \in [a_1, b_1],\ s_1 \le t < Y_2 < Y_3 \le y_2, \end{aligned}$$

which would imply (ii) in the lemma.

Now we try to establish an expression similar to (3.9) and related to v. Since $(s_1, y_2) \subset (a, b)$ by (i) in Remark 3.2, there exists an $x_0 \in B_{N, \vec{i}} \cap (s_1, y_2)$. So we have

$$(3.10) \qquad c_3(x_0) \in (v - 1/N, v + 1/N) \quad [\text{see } (3.3) \text{ and } (3.4)].$$

For any η satisfying $\eta > 0$, $s_1 < x_0 - \eta$ and $x_0 + \eta < y_2$, it follows from (iii) in Remark 3.1 that there exist real numbers $\delta_2 \in (0, \eta)$ and $h(\delta_2)$ such that

$$(3.11) \qquad c_3(x_0) \le h(\delta_2) < v + 1/N.$$

By (ii) in Remark 3.1, $h(\delta_2) = \operatorname{ess\,sup} d(\vec{Y}, t)$ in $(N(x_0, \delta_2))^4 \cap [Y_1 < Y_2 < t < Y_3]$ which has positive measure; therefore, there exists an $(x_1, x_2, b_2, a_2) \in \{(\vec{Y}, t) \in (N(x_0, \delta_2))^4\colon Y_1 < Y_2 < t < Y_3\}$ (note that $s_1 < x_1 < x_2 < a_2 < b_2 < y_2$) such that

$$(3.12) \qquad h(\delta_2) - 1/N < d(x_1, x_2, b_2, a_2) \le h(\delta_2).$$

Then, by an argument similar to that in deriving (3.9), it follows from (3.10)–(3.12) that

$$(3.13) \qquad \begin{aligned} &v - 2/N < d(\vec{Y}, t) < v + 1/N \\ &\qquad \text{if} \quad a_1 \le Y_1 < Y_2 \le b_1,\ a_2 \le t < Y_3 \le b_2. \end{aligned}$$

Note that $(a_2, b_2) \subset (s_1, y_2)$, so by (3.9), we have

$$(3.14) \qquad \begin{aligned} &u - 2/N < d(\vec{Y}, t) < u + 1/N \\ &\qquad \text{if} \quad Y_1 \in [a_1, b_1],\ a_2 \le t < Y_2 < Y_3 \le b_2. \end{aligned}$$

Thus (ii) and (iii) in the lemma follow from (3.13) and (3.14) and (i) in the lemma holds too. This completes the proof of Lemma 3. □

Lemma 3 is the special case of Theorem 1 (where $k = 1$).

PROOF OF THEOREM 1. Given $k \geq 1$, we first construct intervals $[a_j, b_j]$, $j = 0, \dots, k + 1$, and then show these intervals satisfy (i), (ii) and (iii) in Theorem 1.

For $j = 0$, let $B_{N,\vec{i}}$, $[a, b]$ and $\vec{i}$ be the same as in Remark 3.2 and let u and v be as in (3.5). Thus $B_{N,\vec{i}}$ is dense in $[a, b]$ and (ii) in Remark 3.2 is true. By Lemma 3 and the previous assumptions, there exists an $x \in B_{N,\vec{i}} \cap (a, b)$ and there exist real numbers $\eta > 0$ and a_0, b_0, a_1 and h_1 satisfying

$$\text{(T1)} \qquad a = a_0 < b_0 < a_1 < h_1 \quad \text{and} \quad b_0, h_1 \in (a, b) \cap N(x, \eta),$$

$$\text{(T2)} \qquad |d - u| < 2/N \quad \text{if} \quad a_0 \leq Y_1 \leq b_0, a_1 \leq t < Y_2 < Y_3 \leq h_1,$$

$$\text{(T3)} \qquad |d - v| < 2/N \quad \text{if} \quad a_0 \leq Y_1 \leq Y_2 \leq b_0, a_1 \leq t < Y_3 \leq h_1.$$

Note also $[a_1, h_1] \subset (a, b)$.

For $1 \leq j \leq k$, by the induction assumption, we have $[a_j, h_j] \subset (a, b)$ (in particular, from the last paragraph, we have $[a_1, h_1] \subset (a, b)$), i.e., $B_{N,\vec{i}}$ is dense in $[a_j, h_j]$. So by Lemma 3, there exist an $x \in (a_j, h_j) \cap B_{N,\vec{i}}$, $\eta > 0$, b_j, a_{j+1} and h_{j+1} satisfying

$$\text{(T1}') \qquad \begin{gathered} a_j < b_j < a_{j+1} < h_{j+1} \quad \text{and} \\ b_j, h_{j+1} \in (a_j, h_j) \cap N(x, \eta)(\subset (a, b)), \end{gathered}$$

$$\text{(T2}') \quad |d - u| < 2/N, \quad \text{if} \quad a_0 \leq Y_1 \leq b_j, a_{j+1} \leq t < Y_2 < Y_3 \leq h_{j+1},$$

$$\text{(T3}') \quad |d - v| < 2/N \quad \text{if} \quad a_0 \leq Y_1 \leq Y_2 \leq b_j, a_{j+1} \leq t < Y_3 \leq h_{j+1}.$$

Let $I_0 = [a_0, b_0], \dots, I_k = [a_k, b_k]$. By our construction procedure,

$$I_j \subset (a_j, h_j) \subset \cdots \subset (a_1, h_1) \subset (a, b), \qquad j = 1, \dots, k.$$

This means $\bigcup_{m=j}^{k} I_m \subset (a_j, h_j)$. It follows from (T2′) and (T3′) that

$$\text{(T4)} \quad |d - u| < 2/N \quad \text{if} \quad a_0 \leq Y_1 \leq b_j, t < Y_2 \quad \text{and} \quad t, Y_2, Y_3 \in \bigcup_{m=j}^{k} I_m;$$

$$\text{(T5)} \quad |d - v| < 2/N \quad \text{if} \quad a_0 \leq Y_1 \leq Y_2 \leq b_j, t < Y_3 \quad \text{and} \quad t, Y_3 \in \bigcup_{m=j}^{k} I_m.$$

Then (T4) and (T5) imply (ii) and (iii) in Theorem 1, respectively. It is obvious that (i) in Theorem 1 holds. □

Now we are ready to define an invariant estimator d_1 and a continuous distribution function F needed in Theorem 2. Given d as (3.1), let

$$(3.15)\quad d_1 = \begin{cases} 0, & \text{if } t < Y_1, \\ u, & \text{if } Y_1 \le t < Y_2, \\ v, & \text{if } Y_2 \le t < Y_3, \\ 1, & \text{if } Y_3 \le t, \end{cases} \quad \text{where } u \text{ and } v \text{ are as in (3.5).}$$

Note that d_1 is of form (1.6) and has constant risk, hence it satisfies

$$(3.16)\qquad R(F, d_1) \ge R(F, d_0) \quad \text{for any } F \in \Theta.$$

Given $\eta > 0$, let r, s and integer k satisfy:

$$(3.17)\qquad r + s = 1, 0 < r < s \quad \text{and} \quad r/s < \eta/11,$$

$$(3.18)\qquad \begin{aligned} &k > \max\{(1/3)\ln(\eta/4)/\ln s, (1/4)\ln \eta/\ln s\}, \\ &\qquad (\text{i.e., } \max\{s^{4k}/4, s^{3k}\} < \eta/4). \end{aligned}$$

By Theorem 1, given r, s and k as in (3.17) and (3.18), there are disjoint intervals $I_0, \dots, I_k$ such that (i), (ii) and (iii) in Theorem 1 hold. By taking subintervals of I_j's, without loss of generality, we can assume that

$$(3.19)\qquad m(I_0) : m(I_1) : \cdots : m(I_k) = r : rs : \cdots : rs^{k-1} : s^k$$

(note $\sum_{i=0}^{k-1} rs^i + s^k = 1$). Otherwise, since $m(I_j) > 0$ for all j, there are subintervals $I_j^* \subset I_j$ such that

$$m(I_0^*) : m(I_1^*) : \cdots : m(I_k^*) = r : rs : \cdots : rs^{k-1} : s^k.$$

On these I_j^*'s, (i), (ii) and (iii) of Theorem 1 still hold.

Define a uniform distribution function F on $\bigcup_{j=0}^k I_j$ by

$$(3.20)\qquad F(t) = \int_{-\infty}^t 1\left(x \in \bigcup_{j=0}^k I_j\right) \Bigg/ m\left(\bigcup_{j=0}^k I_j\right) dx.$$

REMARK 3.3. Note that $F \in \Theta$ and F has support only on $\bigcup_{j=0}^k I_j$, where $I_j = [a_j, b_j]$ and $b_{j-1} \le a_j$, $j = 1, \dots, k$. Also note that

$$(3.21)\qquad \begin{aligned} F(b_{j-1}) = F(a_j) &= 1 - s^j, \qquad j = 1, \dots, k, \\ dF(I_0) : dF(I_1) : \cdots : dF(I_k) &= r : rs : \cdots : rs^{k-1} : s^k. \end{aligned}$$

PROOF OF THEOREM 2. Given $d \in V$, for any $\eta > 0$ and $N > \varepsilon/(2c)$ as in (3.2), there exist d_1 as in (3.15), s, r, k as in (3.17) and (3.18), I_i's as in Theorem 1 and (3.19) and F as in (3.20). Since $d, d_1 \in V$, it follows immediately that $d = d_1$ if $t < Y_1$ or $t > Y_3$. In order to prove Theorem 2, it suffices to show for $\delta = 2/N$ that

$$(3.22)\qquad (dF)^4\left\{(\vec{Y}, t) : Y_1 < t < Y_2, \left|d(\vec{Y}, t) - u\right| \ge \delta\right\} < \eta/2,$$

$$(3.23)\qquad (dF)^4\left\{(\vec{Y}, t) : Y_2 < t < Y_3, \left|d(\vec{Y}, t) - v\right| \ge \delta\right\} < \eta/2.$$

We verify (3.22) in part (A) and (3.23) in part (B), respectively.

(A) Since the support of F is $\bigcup_{j=0}^{k} I_j$, we only need to check the behavior of $d = d(Y_1, Y_2, Y_3, t)$ on $(\bigcup_{j=0}^{k} I_j)^4 \cap \{Y_1 < t < Y_2\}$. Define

$$H_i = \left\{ \left(\bigcup_{j=i}^{k} I_j \right)^4 \Big\backslash \left(\bigcup_{j=i+1}^{k} I_j \right)^4 \right\} \cap \{Y_1 < t < Y_2\}, \quad i = 0, \dots, k-1, \tag{3.24}$$

$$H_k = (I_k)^4 \cap \{Y_1 < t < Y_2\}.$$

Then the H_j's are disjoint and

$$\bigcup_{j=0}^{k} H_j = \left(\bigcup_{j=0}^{k} I_j \right)^4 \cap \{Y_1 < t < Y_2\}. \tag{3.25}$$

By (ii) in Theorem 1, $\{(\vec{Y}, t) \in H_i: Y_1 \in I_i, t, Y_2, Y_3 \in \bigcup_{j=i+1}^{k} I_j\} \subset \{(\vec{Y}, t) \in H_i: |d(\vec{Y}, t) - u| < \delta\}$, $(\delta = 2/N)$. Thus

$$\begin{aligned} &(dF)^4\left(\left\{(\vec{Y}, t) \in H_i: \left|d(\vec{Y}, t) - u\right| < \delta\right\}\right) \\ &\quad > (dF)^4\left(\left\{(\vec{Y}, t) \in H_i: Y_1 \in I_i, t, Y_2, Y_3 \in \bigcup_{j=i+1}^{k} I_j\right\}\right) \\ &\quad = 3! \int_0^{rs^i} dx_1 \int_0^{s^{i+1}} dt \int_t^{s^{i+1}} dx_2 \int_{x_2}^{s^{i+1}} dx_3 \qquad \text{[see (3.21)]} \\ &\quad = rs^{4i+3}, \end{aligned} \tag{3.26}$$

$$(dF)^4\left(\left\{(\vec{Y}, t) \in \left\{ \left(\bigcup_{j=i}^{k} I_j \right)^4 \cap \{Y_1 < t < Y_2\} \right\}\right\}\right) = \frac{s^{4i}}{4}, \quad i = 0, \dots, k. \tag{3.27}$$

The following partition is helpful for deriving (3.28). For $i = 0, \dots, k-1$,

$$\begin{aligned} H_i = &\left\{(\vec{Y}, t) \in H_i: Y_1 \in I_i \text{ and } t, Y_2, Y_3 \in \bigcup_{j=i+1}^{k} I_j\right\} \qquad \left[\text{measure} = \frac{4}{4} rs^{4i+3}\right] \\ &\cup \left\{(\vec{Y}, t) \in H_i: Y_1, t \in I_i \text{ and } Y_2, Y_3 \in \bigcup_{j=i+1}^{k} I_j\right\} \\ &\qquad \left[\text{measure} = \frac{6}{4} r^2 s^{4i+2}\right] \\ &\cup \left\{(\vec{Y}, t) \in H_i: Y_1, t, Y_2 \in I_i \text{ and } Y_3 \in \bigcup_{j=i+1}^{k} I_j\right\} \left[\text{measure} = \frac{4}{4} r^3 s^{4i+1}\right] \\ &\cup \left\{(\vec{Y}, t) \in H_i: Y_1, t, Y_2, Y_3 \in I_i\right\} \qquad \left[\text{measure} = \frac{1}{4} r^4 s^{4i}\right]. \end{aligned}$$

By (3.24) through (3.27) and the previous partition, we have

$$\frac{(dF)^4\Big(\big\{(\vec{Y},t) \in H_i\colon \big|d(\vec{Y},t) - u\big| \geq \delta\big\}\Big)}{(dF)^4\Big(\big\{(\vec{Y},t) \in H_i\colon \big|d(\vec{Y},t) - u\big| < \delta\big\}\Big)}$$

$$(3.28)\qquad < \frac{(dF)^4(H_i) - rs^{4i+3}}{rs^{4i+3}} = \frac{6r^2s^2 + 4r^3s + r^4}{4rs^3} \quad \text{(see the partition)}$$

$$< \frac{11r}{4s} < \frac{\eta}{4} \quad [\text{by } (3.17)], \qquad i = 0, \dots, k-1.$$

Thus by (3.24) through (3.28) and (3.18), we have (3.22), i.e.,

$$\begin{aligned}
&(dF)^4\Big(\big\{(\vec{Y},t)\colon Y_1 < t < Y_2, \big|d(\vec{Y},t) - d_1\big| \geq \delta\big\}\Big)\\
&\quad = \sum_{i=0}^{k} (dF)^4\Big(\big\{(\vec{Y},t) \in H_i\colon \big|d(\vec{Y},t) - u\big| \geq \delta\big\}\Big) \qquad [\text{by } (3.25)]\\
&\quad \leq \frac{\sum_{i=0}^{k-1}(dF)^4\Big(\big\{(\vec{Y},t) \in H_i\colon \big|d(\vec{Y},t) - u\big| \geq \delta\big\}\Big)}{\sum_{i=0}^{k-1}(dF)^4\Big(\big\{(\vec{Y},t) \in H_i\colon \big|d(\vec{Y},t) - u\big| < \delta\big\}\Big)} + (dF)^4(H_k)\\
&\quad < \frac{\eta}{2},
\end{aligned}$$

where the last inequality holds due to (3.18), (3.27), (3.28) and the following fact:

$$(3.29)\qquad \text{If } a_i, b_i > 0 \text{ and } a_i/b_i < \eta/4,\ i \leq k, \text{ then } \sum_i a_i \Big/ \sum_i b_i < \eta/4.$$

(B) The idea in the proof of (3.23) is the same as that of (3.22). Define

$$D_i = \left[\left\{t, Y_2, Y_3 \in \bigcup_{j=i}^{k} I_j\right\} \Big\backslash \left\{t, Y_2, Y_3 \in \bigcup_{j=i+1}^{k} I_j\right\}\right] \cap \{Y_2 < t < Y_3\}, \qquad 0 \leq i < k,$$

$$D_k = \{t, Y_2, Y_3 \in I_k, Y_2 < t < Y_3\}.$$

Note that $D_0, \dots, D_k$ are disjoint and

$$(3.30)\qquad \bigcup_{j=0}^{k} D_j = \left(\bigcup_{j=0}^{k} I_j\right)^4 \cap \{Y_2 < t < Y_3\}.$$

Similarly to (3.28) in the proof of part (A), we claim that for $i = 0, \dots, k-1$,

$$(3.31)\qquad \frac{(dF)^4\Big(\big\{(\vec{Y},t) \in D_i\colon \big|d(\vec{Y},t) - v\big| \geq \delta\big\}\Big)}{(dF)^4\Big(\big\{(\vec{Y},t) \in D_i\colon \big|d(\vec{Y},t) - v\big| < \delta\big\}\Big)} < \frac{\eta}{4}, \qquad (dF)^4(D_k) < \frac{\eta}{4}.$$

The reason is as follows. Note that for $i = k$,

$$(dF)^4(D_k) = (dF)^4\left(\left\{(\vec{Y},t) \in D_k: Y_1 \in \bigcup_{j=0}^{k-1} I_j \text{ or } Y_1 \in I_k\right\}\right) \qquad \text{[by (3.21)]}$$

$$= 3!\left[\int_0^{1-s^k} dy_1 \int_{1-s^k}^1 \int_{y_2}^1 \int_{y_2}^{y_3} dt\, dy_3\, dy_2 + \int_{1-s^k}^1 dy_1 \int_{y_1}^1 \int_{y_2}^1 \int_{y_2}^{y_3} dt\, dy_3\, dy_2\right]$$

$$\le 3!\left[\int_0^{1-s^k} dy_1 \int_{1-s^k}^1 \int_{y_2}^1 \int_{y_2}^{y_3} dt\, dy_3\, dy_2 + \int_{1-s^k}^1 dy_1 \int_{1-s^k}^1 \int_{y_2}^1 \int_{y_2}^{y_3} dt\, dy_3\, dy_2\right]$$

$$= s^{3k} < \eta/4 \quad \text{[by (3.18)]}.$$

Thus (3.31) holds for $i = k$. For $i < k$,

$$(dF)^4\left(\left\{(\vec{Y},t) \in D_i: \left|d(\vec{Y},t) - v\right| < \delta\right\}\right)$$

$$> (dF)^4\left(\left\{(\vec{Y},t) \in D_i: Y_2 \in I_i,\, t,\, Y_3 \in \bigcup_{j>i}^{k} I_j\right\}\right) \qquad \text{[by (iii) in Theorem 1]}$$

$$(3.32)\qquad = (dF)^4\left(\left\{(\vec{Y},t) \in D_i: Y_1 \in \bigcup_{j=0}^{i-1} I_j,\, Y_2 \in I_i,\, t,\, Y_3 \in \bigcup_{j>i}^{k} I_j\right\}\right) + (dF)^4\left(\left\{(\vec{Y},t) \in D_i: Y_1, Y_2 \in I_i,\, t,\, Y_3 \in \bigcup_{j>i}^{k} I_j\right\}\right)$$

$$= 3!\left[\left(\sum_{m=0}^{i-1} rs^m\right) rs^{3i+2}/2 + r^2 s^{4i+2}/4\right] \quad \text{[by (3.21)]},$$

$$i = 1, \ldots, k-1.$$

$$(dF)^4\left(\left\{(\vec{Y},t) \in D_0: \left|d(\vec{Y},t) - v\right| < \delta\right\}\right)$$

$$(3.33)\qquad \ge (dF)^4\left(\left\{(\vec{Y},t) \in D_0: Y_1, Y_2 \in I_0,\, t,\, Y_3 \in \bigcup_{j>0}^{k} I_j\right\}\right) \qquad \text{[by (iii) in Theorem 1]}$$

$$\ge 3!\, r^2 s^2/4.$$

[Compare to the second term in the end of (3.32) for $i = 0$].

As in part (A), D_i ($i = 1, \dots, k-1$) can be expressed as a union of subsets:

$$D_i = \left[\left\{(\vec{Y}, t) \in D_i \colon Y_2 \in I_i,\, t, Y_3 \in \bigcup_{j>i}^{k} I_j\right\} \cup \left\{(\vec{Y}, t) \in D_i \colon Y_2, t \in I_i,\, Y_3 \in \bigcup_{j>i}^{k} I_j\right\} \cup \left\{(\vec{Y}, t) \in D_i \colon Y_2, t, Y_3 \in I_i\right\}\right] \cap \left[\left\{Y_1 \in \bigcup_{j=0}^{i-1} I_j\right\} \cup \{Y_1 \in I_i\}\right]$$

[essentially 6 (not 3) disjoint subsets]. By this partition and (3.32), we have

$$\begin{aligned} &(dF)^4\left(\left\{(\vec{Y}, t) \in D_i \colon \left|d(\vec{Y}, t) - v\right| \ge \delta\right\}\right) \\ (3.34) \quad &\le (dF)^4\left(\left\{(\vec{Y}, t) \in D_i \colon t, Y_2 \in I_i,\, Y_3 \in \bigcup_{j>i}^{k} I_j;\text{ or } Y_2, Y_3 \in I_i\right\}\right) \\ &= 6\left[(r + \cdots + rs^{i-1})r^2 s^{3i+1}/2 + r^3 s^{4i+1}/6 \right. \qquad \text{[by (3.21)]} \\ &\quad \left. + (r + \cdots + rs^{i-1})r^3 s^{3i}/6 + (rs^i)^4/24\right], \qquad i = 1, \dots, k-1. \end{aligned}$$

Furthermore, D_0 has a similar partition as follows:

$$D_0 = \left\{(\vec{Y}, t) \in D_0 \colon Y_1, Y_2 \in I_0,\, t, Y_3 \in \bigcup_{j>0}^{k} I_j\right\} \cup \left\{(\vec{Y}, t) \in D_0 \colon Y_1, Y_2, t \in I_0,\, Y_3 \in \bigcup_{j>0}^{k} I_j\right\} \cup \left\{(\vec{Y}, t) \in D_0 \colon Y_1, Y_2, t, Y_3 \in I_0\right\}$$

(essentially 3 disjoint subsets). So by the partition and (3.33), we have

$$(3.35) \quad (dF)^4\left(\left\{(\vec{Y}, t) \in D_0 \colon \left|d(\vec{Y}, t) - v\right| \ge \delta\right\}\right) < 6\left[r^3 s/6 + r^4/24\right].$$

The right-hand sides of (3.33) and (3.35) satisfy

$$(3.36) \qquad \frac{r^3 s + r^4/4}{6 r^2 s^2/4} < \frac{5r}{6s} < \frac{\eta}{4} \quad \text{[by (3.17)]}.$$

Note

$$\frac{(r + \cdots + rs^{i-1})\left[r^2 s^{3i+1}/2 + r^3 s^{3i}/6\right]}{(r + \cdots + rs^{i-1}) r s^{3i+2}/2} < \frac{4r}{3s} < \frac{\eta}{4} \qquad \text{[by (3.17)]}.$$

By the above inequality, (3.36) and (3.29), the right-hand sides of (3.32) and (3.34) satisfy

$$(3.37)\qquad \frac{6\left[\dfrac{(r+\cdots+rs^{i-1})r^2s^{3i+1}}{2}+\dfrac{r^3s^{4i+1}}{6}+\dfrac{(r+\cdots+rs^{i-1})r^3s^{3i}}{6}+\dfrac{(rs^i)^4}{24}\right]}{3!\left[\left(\sum_{m=0}^{i-1}rs^m\right)\dfrac{rs^{3i+2}}{2}+\dfrac{r^2s^{4i+2}}{4}\right]} < \frac{\eta}{4}.$$

By (3.33), (3.35) and (3.36), it is easy to verify (3.31) for $i = 0$. By (3.32), (3.34) and (3.37), it is easy to verify (3.31) for $i = 1, \dots, k-1$. Then

$$\begin{aligned}
&(dF)^4\left(\left\{(\vec{Y},t): Y_2 < t < Y_3, \left|d(\vec{Y},t) - d_1\right| \geq \delta\right\}\right)\\
&\leq \sum_{i=0}^{k-1} (dF)^4\left(\left\{(\vec{Y},t) \in D_i: \left|d(\vec{Y},t) - v\right| \geq \delta\right\}\right) + (dF)^4(D_k)\\
&\hspace{22em}[\text{by } (3.30)]\\
&< \frac{\sum_{i=0}^{k-1}(dF)^4\left(\left\{(\vec{Y},t) \in D_i: \left|d(\vec{Y},t) - v\right| \geq \delta\right\}\right)}{\sum_{i=0}^{k-1}(dF)^4\left(\left\{(\vec{Y},t) \in D_i: \left|d(\vec{Y},t) - v\right| < \delta\right\}\right)} + (dF)^4(D_k)\\
&< \frac{\eta}{2} \quad [\text{by } (3.31) \text{ and } (3.29)],
\end{aligned}$$

which is (3.23) and this completes the proofs of part (B) and Theorem 2. □

4. Minimax result for $n \geq 3$ and without condition (1.7). In this section, we state the minimax result for $n \geq 3$ and without condition (1.7). For the sake of space, we only give some comments on the proof of these results. We assume that in this section the setup of the problem is the same as (1.1)–(1.5).

THEOREM 4. *Suppose that $d = d(\vec{Y}, t)$ is a nonrandomized estimator with finite risk and is a (measurable) function of the order statistic $\vec{Y}$. For any $\varepsilon, \delta > 0$, there exist a uniform distribution function $F(t)$ on a positive Lebesgue-measure subset I and an invariant estimator d_1 [of form (1.6)] such that*

$$(dF)^{n+1}\left(\left\{(Y_1,\dots,Y_n,t): \left|d(\vec{Y},t) - d_1(\vec{Y},t)\right| \geq \varepsilon\right\}\right) \leq \delta,$$

where n (≥ 1) is the sample size.

THEOREM 5. *Under the assumptions (1.1)–(1.5) in Section 1, the best invariant estimator $d_0 = \hat{F}(t)$ is minimax for sample size $n \geq 1$.*

In the following we first give some comments on the proof for $n > 3$ and under condition (1.7). Then we give some comments on the proof for $n > 3$ and without condition (1.7).

We first note that, in Section 2, Lemmas 1 and 2 are true for $n \geq 2$ and the proof in Theorem 3 can go through by slightly modifying coefficients and notation (e.g., Y_3 is replaced by Y_n).

We can similarly modify the arguments in Section 3. Of course, all the notation [e.g., (3.2), (3.3) and (3.4)] has to be revised for general n. For example, the general form of Theorem 1 for $n \geq 2$ is Theorem 1*.

THEOREM 1*. *Suppose that the sample size $n \geq 2$ and $[a, b] \subset (0, 1)$, $d \in V$ and d satisfies* (1.7). *For any integers $N, k > 0$, there are integers $0 \leq i_2 \leq \cdots \leq i_n \leq N$ and intervals $I_0, \dots, I_k$ such that*

(i) $I_i = [a_i, b_i]$, $i = 0, \dots, k$, *and* $a = a_0 < b_{j-1} < a_j < b_j < b$, $j = 1, \dots, k$,

(ii) $|d(\vec{Y}, t) - i_q/N| < 2/N$ *if* $Y_1, \dots, Y_{q-1} \in \bigcup_{m=0}^{j-1} I_m$, $t < Y_q$, t, $Y_q, \dots, Y_n \in \bigcup_{m=j}^{k} I_m$, *where* $q = 2, \dots, n$ *and* $j = 1, \dots, k$.

Now we give some comments on how to eliminate condition (1.7), which is a monotonicity assumption on the estimators considered. Under this assumption, any estimator d is continuous almost everywhere. Under only the measurability assumption [i.e., without condition (1.7)], an estimator d is approximately continuous a.e. [see Munroe (1953), pages 291–292], i.e., $d(\vec{x}, t)$ is approximately continuous at $(\vec{x}_0, t_0)$ if for any $\varepsilon, \delta > 0$, there exists a neighborhood $N(r)$ of $(\vec{x}_0, t_0)$ with radius r such that

$$\frac{m^{n+1}(\{(\vec{x}, t) \in N(r): |d(\vec{x}, t) - d(\vec{x}_0, t_0)| > \varepsilon\})}{m^{n+1}(\{(\vec{x}, t) \in N(r)\})} \leq \delta.$$

In the previous sections, the minimaxity of d_0 within the class of estimators satisfying (1.7) is proved by using the fact that d is continuous a.e. m^{n+1}. Hence the minimaxity of d_0 among estimators which are approximately continuous a.e. can be proved similarly. For details of proofs without condition (1.7), see Yu (1988b).

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