The Annals of Statistics
1991, Vol. 19, No. 2, 899–917

TESTING FOR SPHERICAL SYMMETRY OF A MULTIVARIATE DISTRIBUTION

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Rotationally invariant tests based on test statistics of the von Mises type are proposed under the hypothesis of spherical symmetry of a multivariate distribution. The tests are distribution-free when the hypothesis of spherical symmetry is true. The asymptotic distributions of the test statistics are derived under the null hypothesis and under any fixed alternative. A simple criterion for consistency is given. The results are illustrated by numerous examples of test statistics which give rise to tests being consistent against all alternatives.

1. Introduction. There is a considerable body of literature relating to the problem of testing for symmetry of a one-dimensional distribution. However, only a few of the tests proposed for this problem are consistent against any fixed alternative. For example, the analogues of the Kolmogorov–Smirnov test and the Cramér–von Mises test given by Butler (1969) and by Rothman and Woodroofe (1972), respectively, enjoy this property. For the hypothesis of spherical symmetry of a multivariate distribution, the present work aims to develop some broadly useful tests that are consistent against general alternatives. It is assumed that the center of symmetry is known; in Euclidean $(p+1)$-dimensional space, it is taken to be the origin of the coordinate system chosen. A $(p+1)\times 1$ random vector X, then, is said to have a spherically symmetric distribution if X and HX have the same distribution for all $(p+1)\times(p+1)$ orthogonal matrices H. The basis for our tests is the well-known fact that if the distribution of the Euclidean length $|X| = (X'X)^{1/2}$ of X is continuous, the distribution of X is spherically symmetric if and only if $|X|$ and $|X|^{-1}X$ are independent and $|X|^{-1}X$ is uniformly distributed on S_p, the unit hypersphere in $(p+1)$-dimensional space [see Kariya and Eaton (1977)]. Therefore, based on a random sample of $(p+1)\times 1$ random vectors $X_1,\dots,X_n$ with continuous distribution function $F(x) = P(|X_i| \le x)$, testing for spherical symmetry is equivalent to testing simultaneously for independence of $|X_i|$ and $Z_i = |X_i|^{-1}X_i$, and for uniformity of the distribution of the Z_i on S_p. In fact, for the special case $p = 1$, Smith (1977) suggests testing for circular symmetry by using the test statistic

$$U_n = n\int_0^\infty \int_0^{2\pi} Y_n(t,\theta)^2 \, dF_n(t,\theta),$$

Received March 1989; revised April 1990.
AMS 1980 *subject classifications.* Primary 62E20, 62H15; secondary 33A50, 33A65.
Key words and phrases. Invariant tests of spherical symmetry, Gegenbauer polynomials.

where

$$Y_n(t,\theta) = F_n(t,\theta) - \frac{1}{2\pi}F_n(t)\theta - \frac{1}{2\pi}\int_0^{2\pi}\left[F_n(t,\theta) - \frac{1}{2\pi}F_n(t)\theta\right]d\theta, \tag{1.1}$$

$$t \geq 0,\ 0 \leq \theta < 2\pi,$$

$F_n(t,\theta)$ is the empirical distribution function of the polar coordinates $(|X_i|, \theta_i)$ of the X_i, the θ_i being the angles with respect to an arbitrary polar axis, and $F_n(t)$ is the empirical distribution function of the $|X_i|$. The third term on the right-hand side of (1.1) is added to get a test which is invariant under rotations. The limiting null distribution of U_n is seen to be that of a weighted infinite sum of independent χ^2 variables. The test based on U_n is shown to be consistent against all alternatives. In his paper, Smith writes that it is not clear how the present method could be generalized to get a test for spherical symmetry in higher dimensions which is invariant under rotations. However, integrating $Y_n(t,\theta)^2$ with respect to $(1/2\pi)\,d\theta\,dF_n(t)$ instead of $dF_n(t,\theta)$, we get the test statistic

$$T_n = \frac{n}{2\pi}\int_0^\infty\left(\int_0^{2\pi}Y_n(t,\theta)^2\,d\theta\right)dF_n(t),$$

which can be written as

$$T_n = \frac{1}{n}\sum_{i,j=1}^{n} h\left(Z_i'Z_j\right)\min\left(1 - \frac{R_{ni}-1}{n}, 1 - \frac{R_{nj}-1}{n}\right), \tag{1.2}$$

where, for $\nu = 1,\ldots,n$, $R_{n\nu}$ is the rank of $|X_\nu|$ in the sample $|X_1|,\ldots,|X_n|$, and the function $h(t)$, $-1 \leq t \leq 1$, is given by

$$h(t) = \frac{1}{12} - \frac{1}{4\pi}\arccos t + \frac{1}{8\pi^2}(\arccos t)^2, \qquad |t| \leq 1.$$

This function has a representation of the form

$$h(t) = \sum_{q=1}^{\infty}\alpha_q C_q^0(t), \qquad |t| \leq 1,$$

where $C_q^0(t)$ is the Chebyshev polynomial of the first kind of degree q and the coefficients α_q are seen to be

$$\alpha_q = \frac{1}{2\pi^2 q^2}, \qquad q = 1, 2, \ldots.$$

Using the basic trigonometric identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, it is seen that

$$\frac{C_q^0(z_1'z_2)}{2} = \int C_q^0(z_1'z)C_q^0(z'z_2)\,d\omega(z), \qquad z_1, z_2 \in S_1, \tag{1.3}$$

where ω denotes the uniform distribution on S_1. Thus, denoting by μ_n the empirical distribution of the sample $(Z_1, |X_1|), \dots, (Z_n, |X_n|)$, we get

$$(1.4)\quad T_n = n \sum_{q=1}^{\infty} 2\alpha_q \int \left[\int C_q^0(w'z) I_{[0,x]}(y)\, d\mu_n(w,y) \right]^2 d\omega \otimes F_n(z,x).$$

The statistic T_n has the same limiting null distribution as the statistic U_n proposed by Smith. Moreover, for any fixed distribution of the $(Z_i, |X_i|)$, $(1/n)T_n$ tends to

$$\xi = \sum_{q=1}^{\infty} 2\alpha_q \int \left[E\left(C_q^0(Z_1'z) I(|X_1| \le x) \right) \right]^2 d\omega \otimes F(z,x)$$

$$= E\left(h(Z_1'Z_2) \min(1 - F(|X_1|), 1 - F(|X_2|)) \right)$$

in probability. Since all the α_q are positive, we have $\xi = 0$ if and only if

$$(1.5)\quad E\left(C_q^0(Z_1'z) I(|X_1| \le x) \right) = 0 \quad \text{for all } q = 1, 2, \dots,\ z \in S_1,\ x \ge 0,$$

the latter being equivalent to

$$E(\cos(q\theta_1) I(|X_1| \le x)) = 0 = E(\sin(q\theta_1) I(|X_1| \le x))$$

for all $q = 1, 2, \dots,$ $x \ge 0$, where θ_1 is the angle of X_1 with respect to the polar axis. However, this is true iff for F-almost all $x \ge 0$ the Fourier coefficients of order $q \ge 1$ of the conditional distribution of θ_1 given $|X_1| = x$ vanish. Thus, $\xi = 0$ if and only if θ_1 is uniformly distributed on $[0, 2\pi)$ and is independent of $|X_1|$, that is, $\xi = 0$ iff the distribution of the X_i is circularly symmetric. It follows that the test obtained by rejecting the hypothesis of circular symmetry for large values of T_n so as to get a test of given size $\alpha \in (0,1)$ is consistent against any fixed alternative.

Now, any other statistic T_n of the form (1.2) which is based on a function $h(t)$ admitting for $|t| \le 1$ an expansion into a series of Chebyshev polynomials $C_q^0(t)$ with positive coefficients α_q seems to be a suitable test statistic for treating the hypothesis of circular symmetry. Changing from the case $p = 1$ to the general case $p \ge 2$, we have, replacing the Chebyshev polynomials $C_q^0(t)$ by the Gegenbauer polynomials $C_q^\lambda(t)$ of order $\lambda = (p-1)/2$, the corresponding assertions to (1.3) and (1.5). To be precise, let ω be the uniform distribution on S_p, and let $E_q \subset L_2(\omega)$ be an orthonormal basis for the real surface (spherical) harmonics of degree q, $q \ge 0$. Then the following hold:

1. E_q has $\nu(p,q) = \binom{p+q-2}{p-1} + \binom{p+q-1}{p-1}$.
2. The vector space spanned by the union of all the E_q, $q \ge 0$, is dense in the space of continuous functions on S_p with respect to the supremum norm on S_p.
3. Surface harmonics of different degree are orthogonal with respect to ω.

4. The addition formula holds, that is,

$$\sum_{\varphi \in E_q} \varphi(z)\varphi(\omega) = d(p,q) C_q^\lambda(z'\omega), \qquad z, \omega \in S_p, q \geq 1,$$

where $d(p,q) = 2$ or $d(p,q) = \nu(p,q)/C_q^\lambda(1) = (1 + q/\lambda)$ according as $p = 1$ or $p \geq 2$.

Also $E_0 = \{1\}$. [See Stein and Weiss (1971), pages 140–148, and Erdélyi, Magnus, Oberhettinger and Tricomi (1953), pages 232–248.] As a consequence, we get (1.3) and (1.5) again and obtain that, for $p \geq 2$,

$$\frac{C_q^\lambda(z_1'z_2)}{1 + q/\lambda} = \int C_q^\lambda(z_1'z) C_q^\lambda(z'z_2)\, d\omega(z), \qquad z_1, z_2 \in S_p, \tag{1.3'}$$

and

$$E\Big(C_q^\lambda(Z_1'z) I(|X_1| \leq x)\Big) = 0 \quad \text{for all } q = 1, 2, \dots, \; z \in S_p, \; x \geq 0, \tag{1.5'}$$

if and only if the distribution of X_1 is spherically symmetric.

Now, it is obvious to treat the hypothesis of spherical symmetry with a test statistic T_n of the form (1.2), where the function $h(t)$, $|t| \leq 1$, admits an expansion into a series of Gegenbauer polynomials with nonnegative coefficients,

$$h(t) = \sum_{q=1}^{\infty} \alpha_q C_q^\lambda(t), \qquad |t| \leq 1. \tag{1.6}$$

Note that, since $|C_q^\lambda(t)| \leq C_q^\lambda(1)$ for $|t| \leq 1$, the series converges uniformly for $|t| \leq 1$. Moreover, $|h(t)| \leq h(1)$ for $|t| \leq 1$, and h is continuous on $[-1, +1]$. Note also that (up to additive constants) by a theorem of Schoenberg (1942) the class of functions $h(t)$ admitting an expansion of the form (1.6) coincides with the class of continuous functions that are positive definite with respect to S_p. A function $h(t)$, $-1 \leq t \leq +1$, is said to be positive definite with respect to S_p if

$$\sum_{j,k=1}^{n} c_j c_k h(z_j'z_k) \geq 0, \qquad n = 1, 2, \dots,$$

for all real c_j and points $z_j \in S_p$.

As in the two-dimensional case, the statistic T_n can be written as

$$T_n = n \sum_{q=1}^{\infty} \alpha_q^* \int \left[\int C_q^\lambda(w'z) I_{[0,x]}(y)\, d\mu_n(w,y) \right]^2 d\omega \otimes F_n(z,x), \tag{1.4'}$$

where $\alpha_q^* = (1 + q/\lambda)\alpha_q$ and μ_n and F_n are the empirical distributions of the $(Z_i, |X_i|)$ and $|X_i|$, respectively.

We remark that test statistics of the form $(1/n)\sum_{i,j=1}^n h(Z_i'Z_j)$ with h as in (1.6) were proposed by Giné (1975) to treat the hypothesis of uniformity for a random sample $Z_1, \ldots, Z_n$ on S_p. Thus, in a sense, we are dealing with adaptations of Gine's uniformity tests. Besides the motivation given by Giné, there is an intuitive approach to these tests. Because a uniformity test should be invariant with respect to orthogonal transformations of the data, let d be a metric on S_p satisfying $d(Hz, Hz') = d(z, z')$, $z, z' \in S_p$, for all orthogonal $(p+1) \times (p+1)$ matrices H, and put $V_n = (1/n)\Sigma_{i,j=1}^n d(Z_i, Z_j)^2$. Then, intuitively, V_n tends to be larger if the hypothesis of uniformity is true, that is, this hypothesis should be rejected for small values of V_n. Now, Bochner (1941) proved that the metric space (S_p, d) can be isometrically embedded in the Hilbert space l_2 iff the metric d is of the form $d(z_1, z_2) = (h(1) - h(z_1'z_2))^{1/2}$, where h is as in (1.6) with positive coefficients α_q. But then, up to an additive constant, $V_n = -(1/n)\Sigma_{i,j=1}^n h(Z_i'Z_j)$.

Obviously, T_n is rotationally invariant. Moreover, by the continuity assumption on the distribution of $|X_i|$, it is distribution-free when the hypothesis of spherical symmetry is true. The asymptotic distribution of T_n in null as well as nonnull cases is derived in Section 2. Consistency properties of the test obtained by rejecting the hypothesis of spherical symmetry for large values of T_n are also discussed in Section 2. For the proofs, only standard theorems on V-statistics are used. Examples of functions $h(t)$ admitting expansions into series of Chebyshev polynomials or Gegenbauer polynomials with positive coefficients and, therefore, providing tests that are consistent against general alternatives are given in Section 3. Approximations to the limiting null distributions of the test statistics proposed in Section 3 are suggested in Section 4. An impression on the accuracy of these approximations is offered by some numerical computations and simulations. The paper concludes with some remarks concerning possible extensions and applications.

2. Asymptotic properties of the tests and test statistics. Recall that F is the continuous distribution function of the $|X_i|$. Introducing the independent random variables $U_i = F(|X_i|)$, $i = 1, \ldots, n$, being uniformly distributed on the interval $[0, 1]$, and denoting by F_n^* the empirical distribution of the $U_1, \ldots, U_n$, we may write the statistic T_n as

$$T_n = \int \left[\frac{1}{n} \sum_{i,j=1}^{n} h\left(Z_i'Z_j\right) I\left(\max(U_i, U_j) \le u\right) \right] dF_n^*(u).$$

Let us define another statistic,

$$\begin{aligned} T_n' &= \int_0^1 \left[\frac{1}{n} \sum_{i,j=1}^{n} h\left(Z_i'Z_j\right) I\left(\max(U_i, U_j) \le u\right) \right] du \\ &= \frac{1}{n} \sum_{i,j=1}^{n} h\left(Z_i'Z_j\right) \min(1 - U_i, 1 - U_j), \end{aligned}$$

being a V-statistic of degree 2.

THEOREM 2.1. *On the null hypothesis of spherical symmetry, T_n and T_n' are asymptotically distributed as*

$$\sum_{q=1}^{\infty}\sum_{r=1}^{\infty}\tilde{\alpha}_q\left[\left(r-\tfrac{1}{2}\right)\pi\right]^{-2}W_{q,r},$$

where the $W_{q,r}$ are independent random variables distributed as $\chi^2_{\nu(p,q)}$ with

$$\nu(p,q)=\binom{p+q-2}{p-1}+\binom{p+q-1}{p-1},$$

and $\tilde{\alpha}_q=\alpha_q/2$ or $\tilde{\alpha}_q=\alpha_q/(1+q/\lambda)$ and $\lambda=(p-1)/2$ according as $p=1$ or $p\geq 2$.

PROOF. Define the kernel $k((z_1,v_1),(z_2,v_2))$ on $(S_p\times[0,1])\times(S_p\times[0,1])$ by

$$k((z_1,v_1),(z_2,v_2))=h(z_1'z_2)\min(1-v_1,1-v_2).$$

Let $\varphi_q^m(z)$, $m=1,\dots,\nu(p,q)$, be $\nu(p,q)$ linearly independent real surface harmonics of degree q, and let the φ_q^m be orthonormal on S_p with respect to ω, i.e.,

$$\int\left[\varphi_q^m(z)\right]^2\omega(dz)=1,\qquad m=1,\dots,\nu(p,q),$$

$$\int\varphi_q^l(z)\varphi_q^m(z)\omega(dz)=0,\qquad l,m=1,\dots,\nu(p,q),\ l\neq m.$$

Define $\psi_{q,r}^m(z,v)=\sqrt{2}\,\varphi_q^m(z)\cos((r-\frac{1}{2})\pi v)$, $z\in S_p$, $0\leq v\leq 1$. Then

$$\int\int\psi_{q,r}^m(z_1,v_1)k((z_1,v_1),(z_2,v_2))\omega(dz_1)\,dv_1=\lambda_{q,r}^m\psi_{q,r}^m(z_2,v_2),$$
$$z_2\in S_p,\ 0\leq v_2\leq 1,$$

where $\lambda_{q,r}^m=\tilde{\alpha}_q[(r-\frac{1}{2})\pi]^{-2}$. In view of $E(k((Z_1,U_1),(z_2,v_2)))=0$ for all $z_2\in S_p$, $v_2\in[0,1]$, and

$$\sum_{q=1}^{\infty}\sum_{r=1}^{\infty}\sum_{m=1}^{\nu(p,q)}\lambda_{q,r}^m=\left[\sum_{r=1}^{\infty}\left[\left(r-\tfrac{1}{2}\right)\pi\right]^{-2}\right]\left[\sum_{q=1}^{\infty}\alpha_qC_q^{\lambda}(1)\right]<\infty,$$

we can apply Theorem 2.3 of Gregory (1977) to obtain the result for the V-statistic T_n'. When the hypothesis of spherical symmetry is true, we have $E(h(Z_1'z))=0$ for all $z\in S_p$. Therefore, $E(h(Z_1'Z_2))=0$ and $E(h(Z_1'Z_2)h(Z_2'Z_3))=0$. By noting that

$$E\left(\left|U_1-\frac{R_{n1}-1}{n}\right|^2\right)=O\left(\frac{1}{n}\right)$$

and that

$$E\left(\left|\max(U_1, U_2) - \max\left(\frac{R_{n1} - 1}{n}, \frac{R_{n2} - 1}{n}\right)\right|^2\right) = O\left(\frac{1}{n}\right),$$

it is easily seen that $E(|T_n - T_n'|^2) = O(1/n)$. Hence, T_n has the same limiting distribution as T_n'. □

In the work of Baringhaus (1988) the limiting distribution of T_n is obtained also for suitable sequences of contiguous alternatives. It turns out to be the same as that of the infinite sum of random variables stated in Theorem 2.1, the $W_{q,r}$ being noncentral $\chi^2_{\nu(p,q)}$ variables now. For a sequence of contiguous alternatives chosen, one may try to find that function $h(t)$, positive definite with respect to S_p, giving maximal asymptotic power. To derive results in this direction, one may consult the paper of Neuhaus (1976) dealing with this subject in a more general framework.

The asymptotic distribution of T_n for any fixed nonnull distribution is stated in Theorems 2.2 and 2.3.

THEOREM 2.2. *For a fixed alternative, put*

$$\xi = E\big(h(Z_1'Z_2)\min(1 - U_1, 1 - U_2)\big),$$

$$\sigma^2 = \operatorname{Var}(h_1(Z_1, U_1)), \qquad \tau^2 = \operatorname{Var}(g(U_1))$$

and

$$\zeta = \operatorname{Cov}(h_1(Z_1, U_1), g(U_1)),$$

where $h_1(z, u) = E(h(z'Z_2)\min(1 - u, 1 - U_2))$, $z \in S_p$, $0 \le u \le 1$, *and* $g(u) = E(h(Z_1'Z_2)I_{[0,u]}(\max(U_1, U_2)))$, $0 \le u \le 1$. *Assume that* σ^2 *is positive. Then, as* $n \to \infty$, *the limiting distribution of* $\sqrt{n}((1/n)T_n - \xi)$ *is normal with mean* 0 *and variance* $4\sigma^2 + 4\zeta + \tau^2$.

PROOF. Define the statistic T_n' as before. Then from standard results on U-statistics and V-statistics [see, e.g., Hoeffding (1948)] it follows that

$$\sqrt{n}\left(\frac{1}{n}T_n' - \xi\right) = 2\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} h_1(Z_i, U_i) - \xi\right) + o_p(1).$$

Now let τ_n be the empirical distribution of the n^2 random variables $(h(Z_i'Z_j), \min(1 - U_i, 1 - U_j))$, and let τ^* be the distribution of the random variable $(h(Z_1'Z_2), \min(1 - U_1, 1 - U_2))$. Putting

$$W_n(u) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} I_{[0,u]}(1 - U_i) - u\right), \qquad 0 \le u \le 1,$$

the random variable

$$D_n = \sqrt{n}\left(\frac{1}{n}T_n - \frac{1}{n}\sum_{i=1}^{n} g(U_i)\right) - \sqrt{n}\left(\frac{1}{n}T_n' - \xi\right)$$

can be written as $D_n = \int t W_n(u)(\tau_n - \tau^*)(d(t,u))$. Since $W_n(u)$ converges in distribution to the Brownian bridge it follows that D_n tends to zero in probability. Hence,

$$\sqrt{n}\left(\frac{1}{n}T_n - \xi\right) = \sqrt{n}\left(\frac{1}{n}T_n' - \xi\right) + \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(U_i) - \xi\right) + o_p(1)$$
$$= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(2h_1(Z_i, U_i) + g(U_i)\right) - 3\xi\right) + o_p(1),$$

which yields the desired result. □

Before handling the degenerate case $\sigma^2 = 0$, we state the following result.

LEMMA. *The following assertions are equivalent*:

(a) $\sigma^2 = 0$.
(b) $\xi = 0$.
(c) *For almost all* $v \in [0,1]$, *the conditional expectation* $E(h(Z_1'z)|U_1 = v)$ *is zero for all* $z \in S_p$.

PROOF. If $\sigma^2 = 0$, then $h_1(z,u)$ is constant on $S_p \times [0,1]$ with probability 1. Let $u \to 1$ to obtain that the constant is 0. Hence, (a) implies (b). From the representation

$$E\left(h\left(Z_1'Z_2\right)\min(1 - U_1, 1 - U_2)\right)$$
$$= \sum_{q=1}^{\infty} \alpha_q^* \int\int \left[E\left(C_q^\lambda\left(Z_1'z\right)I_{[0,v]}(U_1)\right)\right]^2 \omega(dz)\, dv,$$

we deduce that if $E(h(Z_1'Z_2)\min(1 - U_1, 1 - U_2)) = 0$, then for any q with $\alpha_q \neq 0$,

$$0 = E\left(C_q^\lambda\left(Z_1'z\right)I_{[0,v]}(U_1)\right)$$
$$= E\left(I_{[0,v]}(U_1)E\left(C_q^\lambda\left(Z_1'z\right)\middle|U_1\right)\right) \quad \text{for all } z \in S_p, v \in [0,1].$$

Equivalently, for q with $\alpha_q \neq 0$, we can say that for almost all $v \in [0,1]$.

$$0 = E\left(C_q^\lambda\left(Z_1'z\right)\middle|U_1 = v\right) \quad \text{for all } z \in S_p.$$

Thus, (b) implies (c). Obviously, (c) implies (a). □

THEOREM 2.3. *Let Q be the fixed alternative distribution of* $(Z_i, |X_i|)$. *Assume that* $\sigma^2 = 0$. *Let* $k((z_1, v_1), (z_2, v_2))$ *be the kernel defined in Theorem*

2.1. *Let* $\{\lambda_q,\ q \geq 0\}$ *denote the finite or infinite collection of eigenvalues of* k *corresponding to orthonormal eigenfunctions* $\{\varphi_q,\ q \geq 0\}$, i.e., *for all* q *and* j,

$$\int k((z_1, v_1), (z_2, v_2))\varphi(z_1, v_1)Q(d(z_1, v_1)) = \lambda_q \varphi_q(z_2, v_2) \quad a.e.\ (Q),$$

$\int \varphi_q \varphi_j\, dQ = 0$ *if* $q \neq j$, *and* $\int \varphi_q^2\, dQ = 1$. *Then* T_n *is asymptotically distributed as* $\sum \lambda_q W_q^2$, *where* $W_1, W_2, \dots$ *are i.i.d. standard normal variables.*

Proof. Since the function $h(t)$ is continuous and positive definite with respect to S_p, all eigenvalues λ_k are nonnegative and $\sum \lambda_k < \infty$. Hence, by Theorem 2.3 of Gregory (1977), the statistic T_n' defined previously is asymptotically distributed as $\sum \lambda_q W_q^2$. Using part (c) of the lemma, it can be verified that $E(|T_n - T_n'|^2) = O(1/n)$. □

Let us discuss now the consistency properties of the test φ_n rejecting the hypothesis of spherical symmetry for large values of T_n. For any n let φ_n be of fixed size α, where $0 < \alpha < 1$ is given.

Theorem 2.4. *Let* Q *be the fixed alternative distribution of the* X_i. *The sequence* $\{\varphi_n\}$ *of tests* φ_n *is consistent against* Q *if and only if* $\xi > 0$.

Proof. Since $n^{-1}T_n$ tends to ξ in probability, we have consistency if $\xi > 0$. If $\xi = 0$, then $\sigma^2 = 0$ and T_n is asymptotically distributed as $\sum \lambda_q W_q^2$. It follows at once that for any positive x the limit of $P(T_n \leq x)$ is positive, whence $\{\varphi_n\}$ is not consistent. □

Obviously, if some of the coefficients α_q in the $C_q^\lambda(t)$-expansion of $h(t)$ vanish, there can be quoted easily an alternative distribution Q such that $\{\varphi_n\}$ is not consistent against Q. However, if all the coefficients α_q are positive, then $\xi > 0$ for any fixed alternative distribution Q. Thus, we can state the following result.

Corollary. *If the test statistic* T_n *is based on a function* $h(t)$ *which admits an expansion of the form* (1.6) *with positive coefficients* α_q, $q = 1, 2, \dots$, *then the sequence* $\{\varphi_n\}$ *of tests* φ_n *is consistent against any fixed alternative.*

3. Examples. For applications, it is important to present functions $h(t)$ of the form (1.6) with values that are easy to calculate for all values of t. Moreover, the distribution of the test statistic based on such a function $h(t)$ should converge rapidly to its limiting null distribution. Additionally, to treat the limiting null distribution numerically, the coefficients α_q in the expansion (1.6) should be easy to calculate. Regarding also the results stated in Sections 4 and 5, we assert that the functions $h(t)$ presented in this section comply with these demands. However, there may be other functions that are even better in this respect.

We treat the examples for the cases $p = 1$ and $p \geq 2$ separately. For $p \geq 2$, first we adapt the most interesting test statistic proposed by Giné (1975) and Prentice (1978) for testing of uniformity on S_p. As was shown by Giné (1975) and Prentice (1978), the function

$$h_1(t) = 1 - 2\pi^{-1} \arccos t, \qquad -1 \leq t \leq +1, \tag{3.1}$$

has an expansion into a series of Gegenbauer polynomials of odd degree, and the function

$$h_2(t) = \frac{1}{2} - p\left[\frac{\Gamma(\lambda + \frac{1}{2})}{2\Gamma(\lambda + 1)}\right]^2 (1 - t^2)^{1/2}, \qquad -1 \leq t \leq 1, \tag{3.2}$$

has an expansion into a series of Gegenbauer polynomials of even degree, the coefficients involved being positive for both the functions. Then any linear combination $ch_1(t) + dh_2(t)$ of $h_1(t)$ and $h_2(t)$ with positive weights c and d give rise to a test for spherical symmetry which is consistent against all alternatives. For practical purposes, it is convenient to put $c = 1$ and $d = 2(\pi p)^{-1}[2\Gamma(\lambda + 1)/(\Gamma(\lambda + \frac{1}{2})]^2$, yielding the function

$$h(t) = c_p - 2\pi^{-1}\left[\arccos t + (1 - t^2)^{1/2}\right], \qquad -1 \leq t \leq 1, \tag{3.3}$$

where $c_p = 1 + 4(\pi p)^{-1}[\Gamma(\lambda + 1)/\Gamma(\lambda + \frac{1}{2})]^2$. It has a representation of the form (1.6), where

$$\alpha_{2q} = \left(1 + \frac{2q}{\lambda}\right)\frac{2q - 1}{\pi^2(2q + p)}\left[\frac{\Gamma(\lambda + 1)\Gamma(q - \frac{1}{2})}{\Gamma(q + \lambda + \frac{1}{2})}\right]^2, \qquad q = 1, 2, \dots \tag{3.4}$$

and

$$\alpha_{2q-1} = \left(1 + \frac{2q - 1}{\lambda}\right)\pi^{-2}\left[\frac{\Gamma(\lambda + 1)\Gamma(q - \frac{1}{2})}{\Gamma(q + \lambda + \frac{1}{2})}\right]^2, \qquad q = 1, 2, \dots . \tag{3.5}$$

The test statistic T_n based on $h(t)$ can be written as

$$\begin{aligned} T_n = {} & \frac{(2n + 1)(n + 1)c_p}{6n} \\ & - \frac{4}{\pi n} \sum_{1 \leq i < j \leq n} \left[\widehat{Z_i Z_j} + \sin \widehat{Z_i Z_j}\right] \min\left(1 - \frac{R_{ni} - 1}{n}, 1 - \frac{R_{nj} - 1}{n}\right), \end{aligned} \tag{3.6}$$

where $\widehat{Z_i Z_j} = \arccos Z_i' Z_j$ is the spherical distance (great circle distance) between the points Z_i and Z_j on S_p. Each of the functions $h_i(t)$, $i = 1, 2$, itself provides a suitable test for testing the hypothesis of spherical symmetry. The test based on $h_1(t)$ is an adapted version of Ajne's test for testing of uniformity on S_p [see Mardia (1972)]. The test statistic based on $h_2(t)$ is an adapted version of Giné's statistic for testing of uniformity on H_p, the sphere S_p with antipodes identified. However, the test based on the single function $h_i(t)$ is not consistent against all alternatives. To get a further test enjoying this property we note that $h_1(t) = \sum_{q=1}^{\infty}\alpha_{2q-1}C_{2q-1}^{\lambda}(t)$, where α_{2q-1} is given in (3.5), and

$$h_1(t)^2 = \frac{2\psi'(\lambda+1)}{\pi^2} + \sum_{q=1}^{\infty}\left(1+\frac{2q}{\lambda}\right)2\left[\frac{(q-1)!\Gamma(\lambda+1)}{\pi\Gamma(\lambda+q+1)}\right]^2 C_{2q}^{\lambda}(t), \qquad |t| \le 1, \tag{3.7}$$

where $\psi'(z)$ denotes the first derivative of the digamma function. The representation (3.7) is easily derived from the power series of $(\pi/2 - \arccos t)^2$, $|t| \le 1$, by equating the polynomials $(2t)^{2q}$ as

$$(2t)^{2q} = \frac{(2q)!\Gamma(\lambda)}{\Gamma(\lambda+q+1)}\sum_{k=0}^{q}\frac{2k+\lambda}{(q+\lambda+1)_k(q-k)!}C_{2k}^{\lambda}(t), \qquad |t| \le 1.$$

Any weighted sum of $h_1(t)$ and $h_1(t)^2 - 2\psi'(\lambda+1)/\pi^2$ can serve as a kernel function which provides a test being consistent against all alternatives. We remark that

$$\frac{2\psi'(\lambda+1)}{\pi^2} = \frac{8}{\pi^2}\sum_{q=0}^{\infty}\frac{1}{(2q+p+1)^2}.$$

Giving weights $\frac{1}{16}$ to $h_1(t)$ and $\frac{1}{32}$ to $h_1(t)^2 - 2\psi'(\lambda+1)/\pi^2$, and putting

$$d_p = \frac{3}{32} - \frac{1}{(2\pi)^2}\sum_{q=0}^{\infty}\frac{1}{(2q+p+1)^2},$$

we get the kernel function

$$h(t) = d_p - \frac{\arccos t}{4\pi} + \frac{(\arccos t)^2}{8\pi^2}, \qquad -1 \le t \le 1. \tag{3.8}$$

The test statistic based on this function can be written as

$$T_n = \frac{(2n+1)(n+1)d_p}{6n} - \frac{1}{2\pi n}\sum_{1\le i<j\le n}\left[\widehat{Z_iZ_j} - (2\pi)^{-1}\widehat{Z_iZ_j}^2\right]\min\left(1-\frac{R_{ni}-1}{n}, 1-\frac{R_{nj}-1}{n}\right). \tag{3.9}$$

To obtain other test statistics, we start from the generating function for Gegenbauer polynomials,

$$(1-2tw+w^2)^{-\lambda} = \sum_{q=0}^{\infty}C_q^{\lambda}(t)w^q, \qquad |w| < 1, |t| \le 1.$$

Then, for any fixed real number w, $0 < w < 1$, the function

$$h(t) = (1-2tw+w^2)^{-\lambda} - 1, \qquad |t| \le 1, \tag{3.10}$$

has an expansion of the form (1.6) with positive coefficients $\alpha_q = w^q$. Further examples of this kind are easily obtained by starting from other generating functions for Gegenbauer polynomials.

For the case $p = 1$ the examples posed for $p \geq 2$ can be adopted. In fact, a function $h(t)$ which is positive definite with respect to S_p is also positive definite with respect to $S_{p'}$ when $p' < p$ is a positive integer. If $h(t)$ has the expansion $h(t) = \sum_{q=0}^{\infty} \alpha_q C_q^{\lambda}(t)$, it follows from Gegenbauer's addition formula,

$$C_q^{\lambda}(t) = \sum_{k=0}^{[q/2]} \alpha_{k,q} C_{q-2k}^{\lambda'}(t),$$

with $\lambda' = (p' - 1)/2$ and

$$\alpha_{k,q} = \frac{\Gamma(\lambda')(q - 2k + \lambda')\Gamma(k + \lambda - \lambda')\Gamma(q - k + \lambda)}{\Gamma(\lambda)\Gamma(\lambda - \lambda')k!\Gamma(q - k + \lambda' + 1)},$$

that the expansion of $h(t)$ into a series of Gegenbauer polynomials $C_q^{\lambda'}(t)$ is

$$(3.11) \qquad h(t) = \sum_{q=0}^{\infty} \alpha_q \sum_{k=0}^{[q/2]} \alpha_{k,q} C_{q-2k}^{\lambda'}(t), \qquad -1 \leq t \leq +1.$$

If $p' = 1$ (i.e., $\lambda' = 0$), $\alpha_{k,q} C_{q-2k}^{\lambda'}(t)$ should read

$$2\left\{\frac{\Gamma(k + \lambda)\Gamma(q - k + \lambda)}{\Gamma(\lambda)^2 k!(q - k)!}\right\} C_{q-2k}^{0}(t), \qquad k < \frac{q}{2},$$

$$\left\{\frac{\Gamma(\lambda + q/2)}{\Gamma(\lambda)(q/2)!}\right\}^2, \qquad k = \frac{q}{2}.$$

Note that any coefficient of $C_q^{\lambda'}(t)$, $q = 1, 2, \ldots$, in the $C_q^{\lambda'}(t)$-expansion of $h(t)$ is positive if any coefficient of $C_q^{\lambda}(t)$, $q = 1, 2, \ldots$, in the $C_q^{\lambda}(t)$-expansion of $h(t)$ is positive. One may use formula (3.11) to derive the $C_q^0(t)$-expansions of the functions $h(t)$ stated in (3.3), (3.8) and (3.10). However, in view of $C_q^0(t) = \cos(q \arccos t)$, these expansions can also be obtained from the Fourier series expansions of $h(\cos\theta)$, $0 \leq \theta \leq \pi$. Applying the latter method to the functions

$$(3.12) \qquad h(t) = 1 + \frac{4}{\pi^2} - \frac{2}{\pi}\left[\arccos t + (1 - t^2)^{1/2}\right],$$

$$(3.13) \qquad h(t) = \frac{1}{12} - \frac{\arccos t}{4\pi} + \frac{(\arccos t)^2}{8\pi^2},$$

the coefficients α_q in the $C_q^0(t)$-expansion are seen to be

$$(3.12') \quad \alpha_{2q} = 8\left[(2q - 1)(2q + 1)\pi^2\right]^{-1}, \qquad \alpha_{2q-1} = 8\left[(2q - 1)\pi\right]^{-2},$$

$$(3.13') \qquad \alpha_q = 2^{-1}(\pi q)^{-2},$$

respectively. See Ryshik and Gradstein [(1957), formulas 1.444, 6, and 7] and Hansen [(1975), formula (17.2.8)]. Formula (3.11) may be used to derive the

$C_q^0(t)$-expansion of

$$h(t) = (1 - 2tw + w^2)^{-\lambda} - {}_2F_1(\lambda, \lambda; 1; w^2), \qquad \lambda > 0, 0 < w < 1.$$

An easy computation gives

$$h(t) = 2 \sum_{q=1}^{\infty} w^q \, {}_2F_1(\lambda, \lambda + q; q + 1; w^2) \frac{(\lambda)_q}{q!} C_q^0(t), \qquad -1 \le t \le +1.$$

For the special case $\lambda = 1$, we have

$$h(t) = \frac{2w(t - w)}{1 - 2tw + w^2}, \qquad -1 \le t \le +1, 0 < w < 1, \tag{3.14}$$

which has the expansion

$$h(t) = 2 \sum_{q=1}^{\infty} w^q C_q^0(t), \qquad -1 \le t \le 1, 0 < w < 1. \tag{3.15}$$

Note that this function is one of the generating functions for the Chebyshev polynomials of the first kind.

We point out that the test statistic based on the function given in (3.13) is the modified version of the statistic proposed by Smith, already obtained in Section 1. The considerations given previously suggest that T_n from (3.9) is its counterpart in the case $p \ge 2$.

4. Approximations to sampling distributions. First, we look at the bivariate case and consider the test statistic based on the function

$$h(t) = \frac{t - \frac{1}{4}}{\frac{17}{8} - t}, \qquad -1 \le t \le +1, \tag{4.1}$$

obtained from (3.14) by putting $w = \frac{1}{4}$. In view of

$$\cosh y = \prod_{r=1}^{\infty} \left(1 + \frac{y^2}{\pi^2 (r - \frac{1}{2})^2} \right)$$

and

$$\prod_{q=1}^{\infty} \cosh(2^{-q} y) = \frac{\sinh y}{y},$$

the Laplace transform of the asymptotic null distribution of the test statistic T_n based on $h(t)$ given in (4.1) is $\sqrt{2s}/\sinh\sqrt{2s}$, $s \ge 0$. It equals

$$\prod_{q=1}^{\infty} \left(1 + \frac{2s}{\pi^2 q^2} \right)^{-1}, \qquad s \ge 0,$$

which is known to be the Laplace transform of $W_1 + W_2$, where the W_i, $i = 1, 2$, are independent random variables and the distribution of W_i is the same as the limiting null distribution of the Cramér–von Mises statistic. It is also well known that the distribution of $W_1 + W_2$ is the same as that of $((2/\pi)K)^2$, where the distribution of K is the limiting null distribution of the Kolmogorov–Smirnov statistic.

Let us now have a look at the test statistics based on the functions defined in (3.12) and (3.13). A first-order approximation to upper tail probabilities of the asymptotic null distributions is suggested by some Tauberian theorems for Laplace transforms. A rigorous proof of a theorem which is in this direction was given by Zolotarev (1961). The approximations stated in our paper are based on Zolotarev's work. First, let W be a random variable with Laplace transform

$$\prod_{r=1}^{\infty}\prod_{q=1}^{\infty}\left(1+\left[\frac{\alpha_q}{\left(r-\frac{1}{2}\right)^2\pi^2}\right]s\right)^{-1}, \qquad s \geq 0,$$

where the α_q are positive real numbers satisfying $\sum_{\alpha=1}^{\infty}\alpha_q < \infty$. Let α_1 be the unique largest α_q. Then $P(W > x)$ may be approximated by

$$(4.2)\quad \prod_{r=1}^{\infty}\prod_{q=2}^{\infty}\left(1-\frac{\alpha_q}{\alpha_1(2r-1)^2}\right)^{-1}\prod_{r=2}^{\infty}\left(1-\frac{1}{(2r-1)^2}\right)^{-1}\exp\left(-\frac{\pi^2 x}{4\alpha_1}\right).$$

In view of $\prod_{r=2}^{\infty}(1-(2r-1)^{-2}) = \pi/4$ and $\prod_{r=1}^{\infty}(1-y^2/(2r-1)^2) = \cos(\pi y/2)$, (4.2) reduces to

$$(4.3)\quad \frac{4}{\pi}\left[\prod_{q=2}^{\infty}\cos\left(\frac{\pi}{2}\left(\frac{\alpha_q}{\alpha_1}\right)^{1/2}\right)\right]^{-1}\exp\left(-\frac{\pi^2 x}{4\alpha_1}\right).$$

To give first-order approximations to upper tail probabilities of the asymptotic null distributions of the test statistics proposed in the case $p \geq 2$, let W be a random variable with Laplace transform

$$\prod_{r=1}^{\infty}\prod_{q=1}^{\infty}\left(1+2\left[\frac{\alpha_q}{(1+q/\lambda)\left(r-\frac{1}{2}\right)^2\pi^2}\right]s\right)^{-\nu(p,q)/2}, \qquad s \geq 0,$$

where the α_q are positive real numbers satisfying $\sum_{q=1}^{\infty}\alpha_q C_q^{\lambda}(1) < \infty$. Let α_1 be the unique largest α_q. Then $P(W > x)$ may be approximated by

$$(4.4)\quad \begin{aligned}&\left(\frac{4}{\pi}\right)^{(p+1)/2}\prod_{q=2}^{\infty}\left\{\cos\left(\frac{\pi}{2}\left[\frac{\alpha_q(1+1/\lambda)}{\alpha_1(1+q/\lambda)}\right]^{-1/2}\right)\right\}^{-\nu(p,q)/2}\\ &\times P\left(V > \left(1+\frac{1}{\lambda}\right)\frac{\pi^2 x}{4\alpha_1}\right)\end{aligned}$$

where the random variable V is distributed as χ^2_{p+1}.

Better approximations to upper tail probabilities may be obtained by using a refined version of Zolotarev's theorem stated in a paper of Hoeffding (1964)., There are various methods of expansion and inversion which may give still better approximations. The reader is referred to the work of Blum, Kiefer and Rosenblatt (1961) and the references given therein.

5. Simulations and numerical results. In the case $p = 1$, i.e., the two-dimensional case, let T_{n1}^1, T_{n2}^1 and T_{n3}^1 be the test statistics based on the functions

$$h_1^1(t) = \frac{1}{12} - \frac{\arccos t}{4\pi} + \frac{(\arccos t)^2}{8\pi^2},$$

$$h_2^1(t) = 1 + \frac{4}{\pi^2} - \frac{2}{\pi}\left[\arccos t + (1 - t^2)^{1/2}\right],$$

and

$$h_3^1(t) = \frac{t - \frac{1}{4}}{\frac{17}{8} - t},$$

respectively. Correspondingly, let T_{n1}^2, T_{n2}^2 and T_{n3}^2 be the test statistics based on the functions

$$h_1^2(t) = \frac{1}{16} + \frac{1}{4\pi^2} - \frac{\arccos t}{4\pi} + \frac{(\arccos t)^2}{8\pi^2},$$

$$h_2^2(t) = \frac{3}{2} - \frac{2}{\pi}\left[\arccos t + (1 - t^2)^{1/2}\right],$$

and

$$h_3^2(t) = \left(\frac{2}{\frac{17}{8} - t}\right)^{1/2} - 1,$$

obtained from (3.3), (3.8) and (3.10), respectively, by specializing there to $p = 2$, i.e., the three-dimensional case, putting $w = \frac{1}{4}$ in (3.10).

The rows of Table 1 marked "∞" give numerical values for some upper quantiles of the limiting null distributions of these test statistics. For T_{n3}^1, these are values of $((2/\pi)K_\alpha)^2$, where K_α is the $(1 - \alpha)$-quantile of the limiting null distribution of the Kolmogorov–Smirnov statistic. For the other statistics, the table gives approximate values offered by the approximations suggested in (4.3) and (4.4). Thereby, the various infinite products involved were found by numerical methods to six decimal places. Additionally, for sample sizes of $n = 20$ and $n = 50$, Table 1 presents some empirical upper quantiles of these test statistics obtained by a simulation study with 10,000 replications. From this, one may conclude that the (approximate) upper quantiles of the limiting null distributions provide approximations to the exact quantiles of the test statistics that are satisfactory for sample sizes of $n \geq 20$. However, due to the fast convergence to the limiting distribution, Monte Carlo methods provide an attractive way to calculating critical values.

TABLE 1
Empirical quantiles of the test statistics

	n	0.750	0.900	0.950	0.975	0.990
T_{n1}^1	20	0.0536	0.0729	0.0867	0.1007	0.1183
	50	0.0516	0.0704	0.0852	0.0991	0.1153
	∞	0.0508	0.0696	0.0838	0.0980	0.1169
T_{n2}^1	20	0.8867	1.2058	1.4306	1.6394	1.9416
	50	0.8797	1.1785	1.4000	1.6460	1.9892
	∞	0.8597	1.1607	1.3884	1.6161	1.9171
T_{n3}^1	20	0.4449	0.6321	0.7689	0.9028	1.0710
	50	0.4311	0.6259	0.7759	0.9105	1.0940
	∞	0.4210	0.6090	0.7469	0.8876	1.0756
T_{n1}^2	20	0.0536	0.0672	0.0770	0.0861	0.0990
	50	0.0524	0.0661	0.0760	0.0844	0.0956
	∞	0.0555	0.0682	0.0777	0.0871	0.0994
T_{n2}^2	20	0.9178	1.1467	1.3055	1.4357	1.6383
	50	0.8911	1.1208	1.2762	1.4370	1.6656
	∞	0.9509	1.1531	1.3044	1.4545	1.6517
T_{n3}^2	20	0.2178	0.2905	0.3447	0.3943	0.4564
	50	0.2084	0.2785	0.3332	0.3829	0.4496
	∞	0.2185	0.2879	0.3395	0.3901	0.4567

6. Remarks.

REMARK 1. Due to Theorem 2.1 and the remark following the proof of Theorem 2.1, the large sample theory of our tests is of the general form described in Beran (1975b). Therefore, Beran's conclusions concerning the asymptotic power of the tests apply. To be precise, putting

$$T_{n,q,r} = n \sum_{m=1}^{\nu(p,q)} \left[\frac{1}{n} \sum_{i=1}^{n} \psi_{q,r}^{m}\left(Z_i, \frac{R_{ni}-1}{n} \right) \right]^2,$$

where the $\psi_{q,r}^m$ are defined in the proof of Theorem 2.1, we get using Mercer's theorem that T_n can be represented in the form

$$T_n = \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} \tilde{\alpha}_q \left[(r - \tfrac{1}{2})\pi \right]^{-2} T_{n,q,r}.$$

The limit distributions of the $T_{n,q,r}$ under the null hypothesis and under the given sequence of contiguous alternatives are the central and noncentral χ^2 distributions occurring in the corresponding limit distributions of T_n with the weights $\tilde{\alpha}_q[(r - \frac{1}{2})\pi]^{-2}$. Let T_{n,q_0,r_0}, say, be the component with the (unique) largest weight. Then, following Beran, for sufficiently small significance levels α the asymptotic power of the test equals or exceeds that of the level α test based on T_{n,q_0,r_0}, depending on the noncentrality parameters occurring in the limit distributions of the $T_{n,q,r}$, $(q,r) \neq (q_0, r_0)$. If for a sequence of alternatives the limit distribution of T_{n,q_0,r_0} is central χ^2 while for some T_{n,q_1,r_1} it is noncentral χ^2, the power of the test based on T_n is a small fraction of that based on T_{n,q_1,r_1}. The approximations to the asymptotic power proposed by Beran (1975a) can also be adopted. Additionally, Beran's results concerning the local asymptotic efficiency applies, supporting the assertions on the asymptotic power given previously.

REMARK 2. The (local) asymptotic Bahadur efficiencies or local Pitman efficiencies can also be computed for any two test statistics of the form (1.2) or for any test statistic of the form (1.2) with respect to the likelihood ratio statistic obtained if the X_i have a normal distribution with mean vector zero and covariance matrix Σ, which is assumed to be unknown. For a detailed discussion, we refer to the work of Baringhaus (1988), which points out that the results carry over to the broader class of test statistics obtained from (1.2) by replacing $\min(1 - (R_{ni} - 1)/n,\ 1 - (R_{nj} - 1)/n)$ by $k((R_{ni} - 1)/n, (R_{nj} - 1)/n)$, where $k(u, v)$ is a suitable kernel on $[0, 1] \times [0, 1]$. As a consequence, one is able to pick functions h and k providing local asymptotic Bahadur efficiency 1 with respect to the likelihood ratio statistic in the normal case.

REMARK 3. Testing for spherical symmetry can also be done by combining a test for uniformity of the directions Z_i and a test for independence of the directions Z_i and the radii $|X_i|$. Suitable tests for uniformity on S_p have been proposed by Giné (1975) and Prentice (1978). For a test of independence between the directions and radii, we refer to the paper of Jupp and Spurr (1985).

REMARK 4. We have assumed that the center of symmetry μ, say, is known. If μ is unknown, often it can be estimated consistently [see, e.g., Maronna (1976)]. If $\hat{\mu}$ is an estimator of μ, then putting $\hat{X}_i = X_i - \hat{\mu}$ and $\hat{Z}_i = \hat{X}_i/|\hat{X}_i|$ and denoting by $\hat{R}_{ni}$ the rank of $|\hat{X}_i|$, one is led to propose

$$\hat{T}_n = \frac{1}{n} \sum_{i,j=1}^{n} h\left(\hat{Z}_i'\hat{Z}_j\right) \min\left(1 - \frac{\hat{R}_{ni} - 1}{n}, 1 - \frac{\hat{R}_{nj} - 1}{n}\right)$$

as a test statistic for the hypothesis of spherical symmetry.

When the hypothesis of spherical symmetry is true, the distribution of $\hat{T}_n$ depends on the distribution F of the $|X_i - \mu|$. Thus, to give a critical value, one may possibly proceed by bootstrapping, that is, estimating F by the empirical distribution F_n of the $|\hat{X}_i|$ or by a continuous distribution $\hat{F}_n$ coinciding with F_n at the points of discontinuity of F_n and then calculating the critical value of $\hat{T}_n$ by assuming F_n or $\hat{F}_n$ to be the true distribution of the $|X_i - \mu|$. Since the calculation of this critical value is usually difficult, it must in turn be estimated. Typically, this is done by Monte Carlo simulation.

It should be mentioned that the hypothesis of elliptical symmetry can be treated in an analogous manner [Baringhaus (1988)]. For another treatment of this hypothesis we refer to the work of Beran (1979). Additionally, one is let to a test for the hypothesis of normality of a multivariate distribution [Baringhaus (1988)].

REMARK 5. Smith (1977) mentioned that tests for the hypothesis of circular symmetry apply to certain problems in animal navigation. Another application may be to samples of wind data consisting of simultaneous measurements of wind speed and wind direction on certain days at a certain site, where the hypothesis of circular symmetry means independence of the direction and speed with the direction being uniformly distributed. Samples of wind data are studied by Jensen (1981). Assuming that the samples are taken from a hyperboloid distribution, he also proposes a test for the hypothesis of circular symmetry.

In paleomagnetic studies it is of importance to determine whether the natural remanent magnetism of rock units is stable for long periods of time. Adapting the conglomerate test [see Irving (1964)], we suggest applying a test for the hypothesis of spherical symmetry to samples representing measurements of direction and intensity of the remanent magnetism in specimens taken from a conglomerate of the rock unit to be tested. Rejection of the hypothesis indicates that magnetization occurred after the deposition of the conglomerate. If the hypothesis cannot be rejected, the magnetization of the rock unit is assumed to be stable.

Acknowledgments. This work is based on a part of the author's Habilitationsschrift [Baringhaus (1988)] written at the University of Hannover. The author is most grateful to the Editor, the Associate Editor and the referees for their suggestions on an earlier draft of this paper.

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