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# $A$-OPTIMAL WEIGHING DESIGNS WHEN $N \equiv 3 \pmod 4$

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In this paper we consider the problem of $A$-optimal weighing designs for $n$ objects in $N$ weighings on a chemical balance when $N \equiv 3 \pmod 4$. Let $D(N, n)$ denote the class of $N \times n$ design matrices $X_d$ whose elements are $+1$ and $-1$. It is shown that if $X_d$ is such that $X_d'X_d$ is a block matrix having a specified block structure, then $X_d$ is $A$-optimal in $D(N, n)$. It is found that in some cases the $A$-optimal design in $D(N, n)$ is not unique. A larger class of chemical balance weighing designs is $D^0(N, n)$, where $X_d$ may have some elements equal to zero. It is observed that the designs which are $A$-optimal in $D(N, n)$ are not necessarily $A$-optimal in $D^0(N, n)$.

**1. Introduction.** Let $n$ and $N$ be positive integers, with $n \le N$. For convenience, we will denote the set of all $N \times n$ matrices $X_d = (x_{dij})$ whose elements are $+1$ or $-1$ [$+1$, $-1$ or 0] by $D(N, n)$ [$D^0(N, n)$]. In the chemical balance weighing design in which each of the $n$ objects appears in each of the $N$ weighings, the use of the design matrix $X_d$ means that the $j$th object appears on the left or the right pan of the balance according as $x_{dij} = +1$ or $-1$. If the $j$th object is not present in the $i$th weighing, then $x_{dij} = 0$. If the observations are uncorrelated and have the same variance $\sigma^2$ and the $j$th object weighs $w_j$, then the measured weight of the left pan minus that of the right pan in the $i$th weighing has expectation equal to $\sum_{j=1}^{n} x_{dij} w_j$. We shall restrict our study to the weighing problem having nonsingular $X_d'X_d$ and in which case the best linear unbiased estimate of the weight of every object can be obtained and their covariance matrix is $(X_d'X_d)^{-1}\sigma^2$.

In this paper we consider the $A$-optimality criterion. If $X_d^*$ minimizes $\operatorname{tr}(X_d'X_d)^{-1}$ over $D(N, n)$ [$D^0(N, n)$], then $X_d^*$ is said to be $A$-optimal in $D(N, n)$ [$D^0(N, n)$]. It is well known that when $N \equiv 0 \pmod 4$, any $X_d^* \in D^0(N, n)$ such that $X_d^{*\prime}X_d^* = NI_n$ is $A$-optimal in $D^0(N, n)$, where $I_n$ is the identity matrix. When $N \equiv 1 \pmod 4$, Cheng (1980) has shown that any $X_d^* \in D^0(N, n)$ such that $X_d^{*\prime}X_d^* = (N-1)I_n + J_{n,n}$ is optimal in $D^0(N, n)$ for a general class of criteria which includes, in particular, the $A$-, $D$- and $E$-optimality criteria, where $J_{n,n}$ is a square matrix of order $n$ with all elements equal to unity. Later Jacroux, Wong and Masaro (1983) showed that when $N \equiv 2 \pmod 4$, any $X_d^* \in D(N, n)$, such that $X_d^{*\prime}X_d^*$ consists of two diagonal block matrices of the form $(N-2)I_{n_i} + 2J_{n_i, n_i}$, $i = 1, 2$, where $n_1 = n_2 = n/2$ if $n$ is even and $n_1 = n_2 + 1 = (n+1)/2$ if $n$ is odd, and off-diagonal block matrices are null, is optimal in $D(N, n)$ for a general class of criteria which includes, in particular, the $A$- and $D$-optimality criteria. When $N \equiv 3 \pmod 4$, a complete characteriza-

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tion of the $A$-optimal designs even in $D(N, n)$ class is not known so far. Some results have been obtained in Cheng, Masaro and Wong (1985), Wong and Masaro (1984a, b) and Masaro (1988). In this paper, we give a characterization of some $A$-optimal designs in $D(N, n)$. However, it was not possible to obtain $X_d^*$ for all values of $n$ and $N$.

Let $F(N, n)$ denote the class of $n \times n$ symmetric positive definite (p.d.) matrices $M_n = (m_{ij})$, $m_{ij} \equiv 3 \pmod 4$ $(i \neq j)$, $m_{ii} = N$ and $N \equiv 3 \pmod 4$. Further, let $M_n^* \in F(N, n)$ be such that

$$\operatorname{tr} M_n^{*-1} = \min_{M_n \in F(N, n)} \operatorname{tr} M_n^{-1}.$$

Since every $X_d$ in $D(N, n)$ can be transformed, by multiplying certain of its columns by $-1$ (if necessary), into $\tilde{X}_d$ such that all elements of $\tilde{X}_d'\tilde{X}_d$ are of the form $4k + 3$, where $k$ is an integer, therefore, we may assume $\{X_d'X_d\colon X_d \in D(N, n)\} \subseteq F(N, n)$ and hence

$$\min_{X_d \in D(N, n)} \operatorname{tr}(X_d'X_d)^{-1} \geq \operatorname{tr} M_n^{*-1}.$$

Therefore, if one determines $M_n^*$ and finds $X_d$ such that $X_d'X_d = M_n^*$, then that $X_d$ will be $A$-optimal in $D(N, n)$. It is shown here that $M_n^*$ has all off-diagonal elements $-1$ or 3 and $M_n^*$ has a block structure which is defined below.

A block matrix of size $r_i$ is an $r_i \times r_i$ matrix with diagonal elements $N$ and off-diagonal elements 3 and can be written as

$$B_{r_i} = (N - 3)I_{r_i} + 3J_{r_i, r_i}.$$

A block matrix in $F(N, n)$, with block sizes $r_1, r_2, \dots, r_b$ satisfying $\sum_{i=1}^{b} r_i = n$, is an $n \times n$ matrix denoted by $C_b$, with diagonal blocks of those sizes and all other elements equal to $-1$. Any such matrix $C_b$ in $F(N, n)$ can be written as

$$C_b = \operatorname{diag}\{(B_{r_1} + J_{r_1, r_1}), \dots, (B_{r_b} + J_{r_b, r_b})\} - J_{n, n}$$

and

$$\text{(1.1)} \quad \operatorname{tr} C_b^{-1} = \sum_{i=1}^{b} L_i^{-1} + (n - b)(N - 3)^{-1} + \sum_{i=1}^{b} r_i L_i^{-2} \Big/ \left(1 - \sum_{i=1}^{b} r_i L_i^{-1}\right),$$

where $L_i = N - 3 + 4r_i$, $i = 1, 2, \dots, b$.

The idea of finding $A$-optimum weighing designs in the present paper is analogous to that of finding $D$-optimum designs originated by Ehlich (1964) and further developed by Galil and Kiefer (1980a, b; 1982a, b).

**2. Main results.** Hereafter we assume $N > 3$.

THEOREM 2.1. $\operatorname{tr} M_n^{*-1} < \operatorname{tr} M_{n-1}^{*-1} + (N - 3)^{-1}$.

This enables us to prove

THEOREM 2.2. *$M_n^*$ is a block matrix.*

Further, using Theorem 2.1(b) in Masaro (1988), we obtain

THEOREM 2.3. *$M_n^*$ is a block matrix having blocks of only one size or of two contiguous sizes.*

THEOREM 2.4. *$M_n^* = (N+1)I_n - J_{n,n}$ if and only if $N \geq N_0(n) = [7n - 16 + \sqrt{(n-4)(17n-36)}\,]/4$ and $n \geq 4$.*

Further, when $N < N_0(n)$ the procedure to determine $M_n^*$ is described in Section 4. The proofs of the main results are given in the following section.

**3. Proofs of the theorems.** Let

$$M_n = \begin{bmatrix} N & c & u_i' \\ c & N & u_j' \\ u_i & u_j & M_{n-2} \end{bmatrix} \quad \text{and} \quad M_{n-1}(s) = \begin{bmatrix} N & u_s' \\ u_s & M_{n-2} \end{bmatrix} \quad \text{for } s = i, j$$

be p.d. matrices.

Further, let $a_{st} = u_s' M_{n-2}^{-1} u_t$, $A_{st} = (N - a_{st})$, $b_{st} = u_s' M_{n-2}^{-2} u_t$ and $z_{st}(c) = c - a_{st}$ for $s = i, j$ and $t = i, j$.

LEMMA 3.1.

(a) $\operatorname{tr} M_{n-1}^{-1}(s) = \operatorname{tr} M_{n-2}^{-1} + A_{ss}^{-1}(1 + b_{ss})$ *for* $s = i, j$.

(b) $\operatorname{tr} M_n^{-1} = \operatorname{tr} M_{n-2}^{-1} + f_{ij}(c)$, *where*

$$f_{ij}(c) = \frac{(A_{ii} + A_{jj}) + A_{ii}b_{jj} + A_{jj}b_{ii} - 2b_{ij}z_{ij}(c)}{A_{ii}A_{jj} - z_{ij}^2(c)}. \tag{3.1}$$

(c) $A_{ii}A_{jj} - z_{ij}^2(c) > 0$ *and* $f_{ij}(c) > 0$.

(d) $\operatorname{tr} M_n^{-1} \geq \operatorname{tr} M_{n-1}^{-1}(j) + (A_{ii} - z_{ij}^2(c) A_{jj}^{-1})^{-1}$.

PROOF. For any p.d. matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where $M_{11}$ and $M_{22}$ are square matrices,

$$\operatorname{tr} M^{-1} = \operatorname{tr} M_{22}^{-1} + \operatorname{tr} V_{11}(I + M_{12}M_{22}^{-2}M_{21}),$$

where $V_{11} = (M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1}$.

(a) is proved by choosing $M = M_{n-1}(s)$ and $M_{11} = N$.

(b) is proved by choosing $M = M_n$ and $M_{11} = \begin{bmatrix} N & c \\ c & N \end{bmatrix}$.

(c) $A_{ii}A_{jj} - z_{ij}^2(c)$ is the determinant of the p.d. matrix $V_{11}^{-1}$ and $f_{ij}(c)$ is the trace of the p.d. matrix $V_{11}(I + M_{12}M_{22}^{-2}M_{21})$, where $M$ is chosen as in (b). Hence the result.

(d) Choosing $M = M_n$, $M_{11} = N$, $M_{22} = M_{n-1}(j)$ and $M_{12} = v_i' = (c, u_i')$, we get $\operatorname{tr} M_n^{-1} \geq \operatorname{tr} M_{n-1}^{-1}(j) + (N - v_i' M_{n-1}^{-1}(j) v_i)^{-1}$. Now $v_i' M_{n-1}^{-1}(j) = [A_{jj}^{-1} z_{ij}(c), u_i' M_{n-2}^{-1} - A_{jj}^{-1} z_{ij}(c) u_j' M_{n-2}^{-1}]$. On simplification we get the result. □

**Proof of Theorem 2.1.** For $n = 2$, $M_n^* = \begin{bmatrix} N & -1 \\ -1 & N \end{bmatrix}$ and $\operatorname{tr} M_n^{*-1} = (N+1)^{-1} + (N-1)^{-1} < N^{-1} + (N-3)^{-1}$. Thus, the theorem is true for $n = 2$. Suppose it is true for $n - 1$ and let

$$M_{n-1}^* = \begin{bmatrix} N & u_j' \\ u_j & M_{n-2} \end{bmatrix}.$$

Lemma 3.1(a) gives

$$\begin{aligned} A_{jj}^{-1}(1 + b_{jj}) &= \operatorname{tr} M_{n-1}^{*-1} - \operatorname{tr} M_{n-2}^{-1} \leq \operatorname{tr} M_{n-1}^{*-1} - \operatorname{tr} M_{n-2}^{*-1} \\ &< (N-3)^{-1}. \end{aligned} \tag{3.2}$$

Hence, we get $z_{jj}(3) > 0$.

Let

$$M_n(j) = \left[\begin{array}{c|cc} N & 3 & u_j' \\ \hline 3 & & \\ u_j & M_{n-1}^* & \end{array}\right],$$

$v_j' = (3, u_j')$ and $y' = (y_1, y_{n-1}')$, where $y_{n-1}' = (y_2, \ldots, y_n)$ is an $(n-1) \times 1$ vector. Now $y' M_n(j) y$ can be written as

$$y_1'\left(N - v_j' M_{n-1}^{*-1} v_j\right) y_1 + \left(y_{n-1} + y_1 M_{n-1}^{*-1} v_j\right)' M_{n-1}^*\left(y_{n-1} + y_1 M_{n-1}^{*-1} v_j\right).$$

Since $z_{jj}(3) > 0$,

$$N - v_j' M_{n-1}^{*-1} v_j = A_{jj} - z_{jj}^2(3) A_{jj}^{-1} = A_{jj}^{-1}(N-3)\left(N - 3 + 2 z_{jj}(3)\right) > 0,$$

therefore, $y' M_n(j) y > 0$ and $M_n(j) \in F(N, n)$.

Replacing $i$ by $j$ and $c$ by 3 in (3.1), and using Lemma 3.1(a) and (b), we get

$$\begin{aligned} \operatorname{tr} M_n^{-1}(j) &= \operatorname{tr} M_{n-1}^{*-1} - \frac{1 + b_{jj}}{A_{jj}} + \frac{2\left[z_{jj}(3) + (N-3)(1 + b_{jj})\right]}{(N-3)\left(N - 3 + 2 z_{jj}(3)\right)} \\ &= \operatorname{tr} M_{n-1}^{*-1} + \frac{2 z_{jj}(3) A_{jj} + (N-3)^2(1 + b_{jj})}{A_{jj}(N-3)\left(N - 3 + 2 z_{jj}(3)\right)} \\ &< \operatorname{tr} M_{n-1}^{*-1} + (N-3)^{-1} \quad \text{by (3.2)}. \end{aligned}$$

Thus $\operatorname{tr} M_n^{*-1} \leq \operatorname{tr} M_n^{-1}(j) < \operatorname{tr} M_{n-1}^{*-1} + (N-3)^{-1}$. Hence, the result is true for $n$. Since it is true for $n = 2$, it is true for all $n \geq 2$. □

LEMMA 3.2. *Let*

$$M_n^* = \begin{bmatrix} N & c & u_i' \\ c & N & u_j' \\ u_i & u_j & M_{n-2} \end{bmatrix} \quad \textit{and} \quad |c| \geq 3.$$

*Then*:

(a) $z_{ss}(3) > 0$ *for* $s = i, j$.
(b) $f_{ij}(c) < 2(N-3)^{-1}$.
(c)

$$M_n(s) = \left[\begin{array}{c|c} N & 3 \quad u_s' \\ \hline \begin{matrix} 3 \\ u_s \end{matrix} & M_{n-1}(s) \end{array}\right] \in F(N, n) \quad \textit{for } s = i, j.$$

(d) $|z_{ij}(c)| \geq z_a(3) \geq z_g(3)$, *where* $z_a(3) = (z_{ii}(3) + z_{jj}(3))/2$ *and* $z_g(3) = \sqrt{z_{ii}(3) z_{jj}(3)}$ *and where equality holds if and only if* $c = 3$ *and* $u_i = u_j$.
(e) $\sqrt{A_{ii} A_{jj}} \geq [N - 3 + z_g(3)]$.

PROOF. (a) Without loss of generality we may assume that $z_{ii}(3) \leq z_{jj}(3)$. Using Theorem 2.1 and Lemma 3.1(d), we get $(N-3) \leq A_{ii} - z_{ij}^2(c) A_{jj}^{-1} < A_{ii}$ and the result follows.

(b) From Lemma 3.1(b) and Theorem 2.1, we have

$$f_{ij}(c) = \operatorname{tr} M_n^{*-1} - \operatorname{tr} M_{n-2}^{-1} \leq \operatorname{tr} M_n^{*-1} - \operatorname{tr} M_{n-2}^{*-1} < 2(N-3)^{-1}.$$

(c) $z_{ss}(3) > 0$ implies $M_n(s) \in F(N, n)$ as proved in Theorem 2.1.

(d) $|z_{ij}(c)| \geq |c| - |a_{ij}| \geq 3 - \sqrt{a_{ii} a_{jj}} \geq 3 - (a_{ii} + a_{jj})/2 = z_a(3) \geq z_g(3)$. The second inequality is strict if $|c| > 3$ or $u_i \neq u_j$.

(e) $A_{ii} A_{jj} - (N - 3 + z_g(3))^2 = 2(N-3)(z_a(3) - z_g(3)) \geq 0$. Hence the result. □

PROOF OF THEOREM 2.2. Let

$$M_n^* = \begin{bmatrix} N & c & u_i' \\ c & N & u_j' \\ u_i & u_j & M_{n-2} \end{bmatrix} \quad \text{and} \quad |c| \geq 3.$$

By Lemma 3.2(c), $M_n(s) \in F(N, n)$ for $s = i, j$ and, by Lemma 3.1(b), $\operatorname{tr} M_n^{-1}(s) = \operatorname{tr} M_{n-2}^{-1} + f_{ss}(3)$, where

$$f_{ss}(3) = \frac{2[A_{ss} + (N-3) b_{ss}]}{(N-3)[A_{ss} + z_{ss}(3)]} \quad \text{for } s = i, j. \tag{3.3}$$

By Lemma 3.2(d), we have $|z_{ij}(c)| \geq z_g(3)$. Suppose $|z_{ij}(c)| > z_g(3)$. Using (3.3) in (3.1), we get

$$\begin{aligned}\left(A_{ii}A_{jj} - z_{ij}^2(c)\right)f_{ij}(c) \\ &= 2^{-1}(N-3)\left[\left(A_{ii} + z_{ii}(3)\right)f_{ii}(3) + \left(A_{jj} + z_{jj}(3)\right)f_{jj}(3)\right] \\ &\quad + z_{ii}(3)b_{jj} + z_{jj}(3)b_{ii} - 2b_{ij}z_{ij}(c).\end{aligned}$$

Since $z_{ii}(3)b_{jj} + z_{jj}(3)b_{ii} \geq 2\sqrt{b_{ii}b_{jj}}\,z_g(3) \geq 2|b_{ij}|z_g(3)$ and, from Lemma 3.1, $f_{ij}(c) \leq \min\{f_{ii}(3), f_{jj}(3)\}$, we get

$$\begin{aligned}&\left(A_{ii}A_{jj} - z_{ij}^2(c)\right)f_{ij}(c) \\ &\quad \geq 2^{-1}(N-3)\left[A_{ii} + z_{ii}(3) + A_{jj} + z_{jj}(3)\right]f_{ij}(c) + 2|b_{ij}|\left(z_g(3) - |z_{ij}(c)|\right) \\ &\quad = \left(A_{ii}A_{jj} - z_g^2(3)\right)f_{ij}(c) + 2|b_{ij}|\left(z_g(3) - |z_{ij}(c)|\right).\end{aligned}$$

On simplification we get

$$2|b_{ij}| \geq f_{ij}(c)\left[|z_{ij}(c)| + z_g(3)\right]. \tag{3.4}$$

Further, $A_{ii}b_{jj} + A_{jj}b_{ii} \geq 2\sqrt{A_{ii}A_{jj}}\,|b_{ij}|$. Hence from (3.1) we get

$$\begin{aligned}&\left(A_{ii}A_{jj} - z_{ij}^2(c)\right)f_{ij}(c) \\ &\quad \geq 2(N-3+z_a(3)) + 2|b_{ij}|\left[\sqrt{(A_{ii}A_{jj})} - |z_{ij}(c)|\right] \\ &\quad \geq 2(N-3+z_a(3)) + f_{ij}(c)\left(|z_{ij}(c)| + z_g(3)\right)\left(N-3+z_g(3) - |z_{ij}(c)|\right) \\ &\quad \geq 2(N-3+z_a(3)) + f_{ij}(c)\left[(N-3)z_a(3) + z_g^2(3) - z_{ij}^2(c)\right].\end{aligned} \tag{3.5}$$

The second inequality follows from (3.4) and Lemma 3.2(e). The third inequality follows from Lemma 3.2(a) and Lemma 3.2(d). On simplifying (3.5) we get $f_{ij}(c) > 2(N-3)^{-1}$, which contradicts Lemma 3.2(b). Hence $|z_{ij}(c)| = z_g(3)$. Therefore, we get $c = 3$, $u_i = u_j$, $f_{ij}(c) = \min\{f_{ii}(3), f_{jj}(3)\}$ and $M_n^* = M_n(i)$ or $M_n(j)$.

If $M_n^*$ is not a block matrix, then applying the above results for any two rows and the corresponding columns of $M_n^*$ we get a block matrix, after permuting the rows and the corresponding columns if necessary. □

A matrix $C_s$ in $F(N, n)$ with $u$ blocks of size $r$ and $v$ blocks of size $r+1$ satisfying $u + v = s$ and $sr + v = n$ is denoted by $C_s^*$. For given $s < n$, the parameters $r$, $u$ and $v$ of $C_s^*$ are uniquely determined by the conditions $r = [n/s]$, $u = s(r+1) - n$, $v = s - u$, $u \geq 1$ and $v \geq 1$ except when $s|n$. In this case the matrix $C_s^*$ with $r = r_0$, $u = u_0$ and $v = 0$ is identical to that with $r = r_0 - 1$, $u = 0$ and $v = u_0$, and either one yields the same value for $\operatorname{tr} C_s^{*-1}$. Hence we may assume $v \geq 1$.

PROOF OF THEOREM 2.4. The parameters $u, v, r, s$ hereafter refer to $C_s^*$ satisfying $s < n$.

Let $C^{**}$ be obtained from $C_s^*$ by replacing one block of length $r+1$ (since $v \geq 1$) by a block of length $r$ and a block of length 1. This may result in blocks of three lengths in $C^{**}$. Let $L = N - 3 + 4r$ and $g(v) = (L+4)(L-n) + 4(r+1)v$. Using (1.1), we get

$$\operatorname{tr} C_s^{*-1} = \frac{n-s}{N-3} + \frac{u-1}{L} + \frac{v-1}{L+4} + \frac{2L+4-n}{g(v)}$$

and

$$\operatorname{tr} C^{**-1} = \frac{n-s-1}{N-3} + \frac{u}{L} + \frac{v-2}{L+4} + \frac{1}{N+1} + \frac{(2L+4-n) + 16r(r-1)(N+1)^{-2}}{g(v) - 4r(N+1+L)(N+1)^{-1}}.$$

It can be seen easily that for fixed $r$, $\operatorname{tr} C_s^{*-1} - \operatorname{tr} C^{**-1}$ is an increasing function of $g(v)$. Since $g(v) \geq g(1) = (L+4)(L-n) + 4(r+1)$, therefore, we get

$$(3.6)\quad \begin{aligned} &\operatorname{tr} C_s^{*-1} - \operatorname{tr} C^{**-1} \\ &\geq \frac{4r}{L(N-3)} + \frac{2N-2-n+8r}{(L+4)(L-n)+4(r+1)} - \frac{2N-2-n+4r}{(N+1)(L-n)-4(r-1)} \\ &= \frac{4r[12(r-1)-(n-4)(N+1)]}{L(N-3)\{(N+1)(L-n)-4(r-1)\}} + \frac{4r[n(N+1)+4(n+2)(r-1)-(n+4)(n-2)]}{\{(L+4)(L-n)+4(r+1)\}\{(N+1)(L-n)-4(r-1)\}} \\ &= \frac{16r(r-1)G_1(r) + 8r((N+1)+12(r-1))G_0}{L(N-3)[(L+4)(L-n)+4(r+1)][(N+1)(L-n)-4(r-1)]} \quad \text{if } r > 1 \\ &= \frac{8G_0}{(N-3)(N+1)(N+1-n)\{(N+5)(N+1-n)+8\}} \quad \text{if } r = 1, \end{aligned}$$

where $G_1(r) = 48(r-1)^2 + 4(12N - 7n + 19)(r-1) + (N+1)^2 + 4(n-2)(5N - 5n + 6)$, $G_0 = 2N^2 - (7n - 16)N + (n-2)(4n-7)$ and equality holds in (3.6) if $v = 1$. Now for $r > 1$, $G_1(r) > 0$ and $G_0 \geq 0$, if $N \geq N_0(n)$ and $n \geq 4$. Therefore, if $N \geq N_0(n)$ and $n \geq 4$, then $\operatorname{tr} C_s^{*-1} - \operatorname{tr} C^{**-1} \geq 0$ and equality holds if and only if $v = r = 1$ and $N_0(n) = N$. Moreover, when $N < N_0(n)$, $G_0 < 0$ and therefore, from (3.6), if $v = r = 1$, we get $\operatorname{tr} C_s^{*-1} - \operatorname{tr} C^{**-1} < 0$. In this case $C^{**} = (N+1)I_n - J_{n,n}$ and $C_s^* = C_{n-1}^*$. □

REMARK. For $n \le 100$, equality holds in (3.6), if $n = 8$, $N = 15$ and $n = 54$, $N = 143$.

**4. Construction of $X_d^* \in D(N, n)$.** When $N \ge N_0(n)$ and $n \ge 4$, by Theorem 2.4, $M_n^* = (N+1)I_n - J_{n,n}$ and the method of constructing $X_d^* \in D(N, n)$ such that $X_d^{*\prime} X_d^* = M_n^*$ is the same as that given in Case 3 in Galil and Kiefer (1980a). For $N \le N_0(n)$ by Theorem 2.3, $M_n^* = C_{s_0}^*$, where $s_0$ is chosen such that $\operatorname{tr} C_s^{*-1}$ is minimized. To determine $s_0$ we first fix $r$ and minimize $T(N, n, s) = \operatorname{tr} C_s^{*-1}$ with respect to $s$ such that $b_r \le s \le a_r$, where $a_r = [n/r]$, $b_r = [n/(r+1)] + 1$ and $[\cdot]$ denotes the integer part. Let the corresponding $s$ be denoted by $s_r^*$. Now,

$$T(N, n, s+1) \ge T(N, n, s) \quad \text{if } \max\{b_r, [s_0^r]\} \le s \le a_r$$

and

$$T(N, n, s-1) \ge T(N, n, s) \quad \text{if } b_r \le s \le \min\{[s_0^r], a_r\},$$

where

$$s_0^r = \Big\{2(L+4)(L-n) + 4(2n+r)(r+1) - \sqrt{L(L+4)(N-3)(2L+4-n) + 16r^2(r+1)^2}\Big\} \div \{8r(r+1)\}.$$

Depending on the values of $[s_0^r]$, we have to consider three cases.

CASE (i) If $b_r \ge [s_0^r]$, then $s_r^* = b_r$.

CASE (ii) If $b_r < [s_0^r] < a_r$, then $s_r^* = [s_0^r]$.

CASE (iii) If $[s_0^r] \ge a_r$, then $s_r^* = a_r$.

Minimizing $T(N, n, s_r^*)$ over values of $r$ which satisfy $b_r \le a_r$, we get $T(N, n, s_0) = \operatorname{tr} C_{s_0}^{*-1}$. If $s_0^r$ is an integer for some $r$ which satisfies $b_r \le a_r$ and $s_0 = s_0^r$, then we get two matrices $C_{s_0}^*$ and $C_{s_0-1}^*$ having the same minimum. This happens for $n = 20$, $N = 27$. In Table 1, the values of $s_0$ and $\operatorname{tr} M_n^{*-1}$ are given for $6 \le n \le 10$ and $N \le N_0(n)$. (A table for larger values of $n$ and $N$ is also available.) The construction of $X_d^*$ corresponding to designs in Table 1 is given below.

(i) Consider the $8 \times 8$ block matrix

$$X = \begin{bmatrix} X_1 & X_2 & X_2 & X_2 \\ -X_2 & X_1 & -X_2 & X_2 \\ -X_2 & X_2 & X_1 & -X_2 \\ -X_2 & -X_2 & X_2 & X_1 \end{bmatrix},$$

where $X_1 = J_{2,2} - 2I_2$ and $X_2 = J_{2,2}$. We get $X'X = 4(I_2 + J_{2,2}) \otimes I_4$ where $\otimes$ denotes the Kronecker product. Delete the first and the third (the first) columns and then delete the first row of $X$. The resulting matrix gives design number 1 (2).

TABLE 1

| Design number | $n$ | $N$ | $s_0$ | $\operatorname{tr} M_n^{*-1}$ |
|---|---|---|---|---|
| 1 | 6 | 7 | 4 | 1.058333 |
| 2 | 7 | 7 | 4 | 1.277778 |
| 3 | 7 | 11 | 6 | 0.696970 |
| 4 | 8 | 11 | 5 | 0.810606 |
| 5 | 8 | 15 | 7 | 0.562500 |
| 6 | 9 | 11 | 5 | 0.925000 |
| 7 | 9 | 15 | 6 | 0.639634 |
| 8 | 10 | 11 | 5 | 1.041667 |
| 9 | 10 | 15 | 5 | 0.716667 |
| 10 | 10 | 19 | 8 | 0.549479 |

(ii) For design numbers 5, 7 and 9 (3, 4, 6 and 8) choose appropriate number of columns having 1's in the first row and then delete the first row of the matrix $H$ ($R$) of Example 2.1 (2.2) in Kounias and Chadjipantelis (1983).

(iii) Design number 10 is constructed by using the method given in (2.8) of Galil and Kiefer (1982b).

COMMENTS. (i) Design numbers 3 and 5 can also be obtained by the method given in Galil and Kiefer (1980b).

(ii) Design number 9 was also constructed by Mitchell (1974) via a computer routine.

We now give an example illustrating the fact that a design which is $A$-optimal in $D(N, n)$ is not $A$-optimal in $D^0(N, n)$.

EXAMPLE 4.1. Consider the following $X_d \in D^0(7,6)$,

$$X_d = \begin{bmatrix} -1 & -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Then we get

$$X_d'X_d = \begin{bmatrix} 8I_5 - J_{5,5} & 2i_5 \\ 2i_5' & 6 \end{bmatrix},$$

where $i_5$ is the fifth column of the identity matrix $I_5$. $\operatorname{Tr}(X_d'X_d)^{-1} = 1.046875$, hence design number 1 is not $A$-optimal in $D^0(7,6)$.

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